Review 2

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1 Hausdorff Spaces, Continued

Theorem 1 *A compact subset of a Hausdorff space is closed.*

Theorem 2 *Let $\pi : X \to Y$ be onto; give Y the quotient topology. If X is compact Hausdorff and π is a closed map, then Y is Hausdorff.*

Remark: Almost all of our constructions preserve topological properties. For example if X and Y are compact, so is $X \times Y$; if X and Y are connected, so is $X \times Y$. But quotient spaces are trickier and the main thing that goes wrong is that X is Hausdorff and Y is not. You should know an example.

Theorem 2 is often used to show that a quotient space is Hausdorff.

- Suppose X is formed from a polygon by gluing pairs of sides together. Explain why X is Hausdorff.
- Recall that RP^n is S^n with opposite points glued together. Explain why RP^n is Hausdorff
- Recall that CP^n is $S^{2n+1} \subseteq C^{n+1}$ with $p \sim \lambda p$ whenever $|\lambda| = 1$. Explain why CP^n is Hausdorff

2 Connected Spaces

Definition 1 *A space X is* connected *if it is impossible to write $X = \mathcal{U} \cup \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ are nonempty, disjoint, open sets*

Definition 2 *A space X is* pathwise connected *if whenever $p, q \in X$, there is a continuous path $\gamma : [0, 1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$.*

Theorem 3

- $[0, 1]$ *is connected*
- *if* $f : X \to Y$ *is continuous and* X *is connected, then* $f(X) \subseteq Y$ *is connected*
- *if* $X_\alpha \subseteq X$ *are connected subsets and* $\cap X_\alpha \neq \emptyset$*, then* $\cup X_\alpha$ *is connected*
- *every pathwise connected space is connected*

Remark: You should be able to prove that S^n, RP^n, and $R^n - \{0\}$ are connected (modulo a few exceptions for small n which you should be able to list). You should also know an example of a space which is connected but not pathwise connected.

3 Brouwer and Pancakes

Theorem 4 (Brouwer) *Let* $f : D^n \to D^n$ *be continuous. Then there is a point* $x_0 \in D^n$ *such that* $f(x_0) = x_0$.

Lemma 1 *There does not exist a continuous map* $g : D^n \to S^{n-1}$ *with the property that* $S^{n-1} \to D^n \to S^{n-1}$ *is the identity map.*

Remark: You should be able to prove Brouwer's theorem if you already know that the lemma is true. You should also be able to prove the lemma when $n = 1$. The proof of the lemma when $n = 2$ requires the fundamental group from the second term of our course. The proof of the lemma when $n > 2$ usually requires homology theory, although there are also elementary proofs from scratch.

Theorem 5 *Let* A *and* B *be bounded subsets of* R^2*. There is a line in the plane which divides each of* A *and* B *exactly in half. There are a pair of perpendicular lines which divide* A *into four equal pieces.*

Remark: You should know the idea of the proof. There is one tricky point. The line dividing just A into equal pieces is not unique. You should know an example. If care is not taken, the remaining construction in the proof will lead to a discontinuous function. You should understand the difficulty, and how we get around it.

4 Manifolds

I prefer my definition to the book's definition:

Definition 3 *An* n*-dimensional manifold is a Hausdorff topological space* X *which is locally-Euclidean: whenever* $p \in X$*, there is an open set* $p \in \mathcal{U} \subseteq X$ *and an open set* $\mathcal{V} \in R^n$ *and a homeomorphism* $\varphi : \mathcal{U} \to \mathcal{V}$*.*

Remark: You should know an example of a topological space which is locally-Euclidean but not Hausdorff. Such an example shows why we must explicitly add the Hausdorff condition.

Remark: You should know why S^n, RP^n, and CP^n are manifolds. You should know why a surface constructed from a polygon by gluing corresponding sides together is a manifold. You should understand the connected sum construction which gives a new manifold $X\#Y$ from existing manifolds X and Y of the same dimension.

5 Interesting Surface Constructions

Theorem 6
- *A Klein bottle can be obtained by gluing two Mobius bands together along their boundary, which is a circle*
- RP^2 *can be obtained by gluing a Mobius band to a disk along their boundary, which is a circle*
- $RP^2\#RP^2\#RP^2, T^2\#RP^2$*, and* $K\#RP^2$ *are homeomorphic*

6 The Classification of Surfaces

You should know the proof that every surface given by gluing corresponding sides of a polygon together is homeomorphic to a canonical surface. There is a handout sheet discussing this proof. The key points of the proof are

- the polygon can be dissected and reassembled so all vertices correspond to the same point in the surface
- each pair occurs in the form a, a or in the form a, a^{-1}
- the polygon can be dissected and reassembled so all a, a pairs are adjacent
- while maintaining this condition, the polygon can be further dissected and reassembled so all crossed pairs $a \dots b \dots a^{-1} \dots b^{-1} \dots$ are adjacent: $a_1 b_1 a_1^{-1} b_1^{-1} \dots$
- if both dissections have been done, then there are no remaining a, a^{-1} pairs and so no remaining edges
- thus if the original polygon only had a, a^{-1} pairs, then the polygon can be reduced to
$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$$

- if there is at least one a, a pair, then all crossed pairs can be eliminated and the polygon can be reduced to
$$a_1 a_1 a_2 a_2 \ldots a_g a_g$$

Remark: You should be able to identify these canonical polygons. There are two ways to write the final result. We can say that every compact surface is $T^2 \# T^2 \# \ldots \# T^2$ or $RP^2 \# RP^2 \# \ldots \# RP^2$. Or we can say that every compact surface is $T^2 \# T^2 \# \ldots \# T^2$ or $T^2 \# T^2 \# \ldots \# T^2 \# RP^2$ or $T^2 \# T^2 \# \ldots \# T^2 \# K$.

7 Second Countability and the Long Line

Definition 4 *A topological space is* second countable *if there is a countable collection of open sets $\{U_i\}$ such that every open set is a union of some of these.*

Theorem 7

- *R^n is second countable*
- *Any $A \subseteq R^n$ with the induced topology is second countable*
- *Any compact manifold is second countable*

Remark: I want you to be able to describe the long line, and to explain why it is a one-dimensional manifold which is not second countable. Intricate details will not be needed. You need not understand the proof that an appropriate index set I exists.

8 Interesting Exercises

The following exercises are particularly interesting. It might pay to review them.

From assignment 6, exercises 8.14b and c. According to the first, X is Hausdorff if and only if the diagonal in $X \times X$ is closed. According to the second, if $f : X \to Y$ is continuous and Y is Hausdorff , then $\{x \times y \in X \times X \mid f(x) = f(y)\}$ is closed. These are nice "final exercises" to see if you can play with our ideas.

From the same assignment, look at exercise 8.14j. It isn't necessary to master all of the technical details, but you should have a clear idea of the geometric meaning of $\mathcal{U}^\infty$ and $X/(X - \mathcal{U})$, of the topology on these two sets, and of a map from one to the other. This is a good time to look at an exercise in a later homework set proving that any compact manifold can be embedded in some R^N.

While you are at it, look also at the exercise giving an explicit embedding of RP^2 into R^4. Recall that the map is $(x_1, x_2, x_3) \to (x_1^2 - x_2^2, x_1 x_2, x_1 x_3, x_2 x_3)$. Some people proved this map one-to-one without using the first coordinate. If that proof was correct, we'd have an embedding of RP^2 in R^3, and this is known to be impossible. Explain just what goes wrong if we omit the first component of the map. Show that the resulting map is "almost" one-to-one.

As you look at the previous exercise, think carefully about how we prove that the map is a homeomorphism onto its image. The argument uses compactness and a Hausdorff property.

From assignment 7, show that Q is not connected. Some people asserted that the induced topology on Q is the discrete topology; explain why that is false. On the other hand, what are the connected components of Q? Note that connected components are closed, but not necessarily open.

Exercise 9.8f in this assignment is interesting. A special case states that if A is a connected subset of X, then $\overline{A}$ is also connected.

Exercise 9.8h asks for a proof that $R^n - \{0\}$ is connected. This is a good time to think about giving a precise proof without handwaving. There are many ways to do this. Some people said "we'll prove it pathwise connected; if p and q are points, draw a path from p to q; if this path goes through the origin, then move it a little to avoid the origin." This argument can be made complete rigorous. For instance, you could introduce the two dimensional subspace of R^n spanned by p and q and argue that it therefore suffices to prove the result when $n = 2$. If p and q aren't linearly independent, add an independent r and look at the plane generated by p, q, r. Etc.

From assignment 8, look at 11.2. The proof is harder if you use the book's definition of a manifold than if you use my definition. (Most people handled this without sweat.)

It pays to look at CP^n in detail. This is a beautiful explanation to show how we prove a space locally-Euclidean when it is globally complicated.

From assignment 9, look at exercise 12.10f giving a space which is connected but not pathwise connected. Be sure to draw a picture of the set in question. Then see if you can explain the idea of the proof in a few sentences without elaborate details. You can fill in details if asked once you know the central idea.

All of the extra problems in this assignment are important. I got many elegant solutions for the last two problems. For example, consider the problem of showing that the objects below are not homeomorphic.

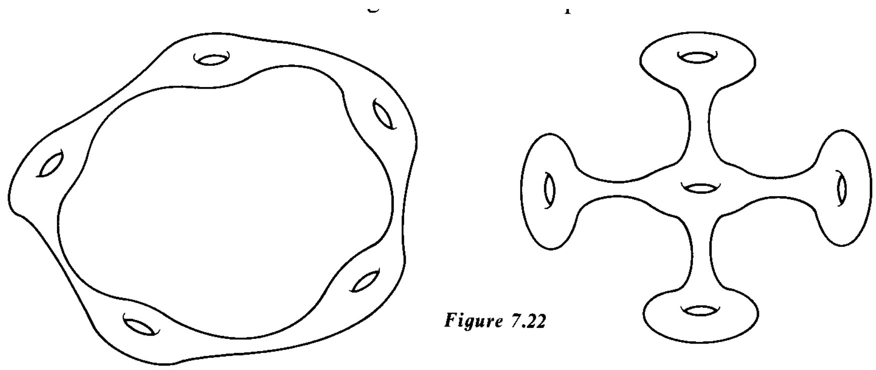

Figure 7.22

I was told to shrink the large hole in the center of the object on the left, and deform the object on the right, to obtain the first picture below. Then I was told to flatten the object as if I were rolling a loaf of French bread until the holes line up as in the second picture below. The number of holes if six in one case and five in the other case, so these objects aren't homeomorphic.