Review 1

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1 Introduction

It is important to know all of the definitions listed in this review sheet.

I've also listed the important theorems. Any listed theorem might appear on the midterm, but many of these theorems have very easy proofs which you could construct on the spot if necessary. So I starred theorems which I think are harder and interesting enough to appear on the midterm.

The midterm will also contain gluing constructions like those we used to construct the Klein bottle and projective plane. Rather than inventing new examples, I have listed those exercises which I think best illustrate this aspect of our course. Please check my answers on the web if you are uncertain about any of these exercises.

Because the review sheet is mostly formal, you might make the mistake of skimping on the exercises. Actually the midterm will be about half and half between formal results and constructions discussed in exercises. Please consult my answer sheets or the last section of this review sheet for a more balanced approach.

2 Definitions

It is important to have the various definitions down cold. Here are the important ones from our course:

Definition 1 *A metric space is a set M and distance function $d(x, y)$ on M such that*

- $d(p, q) \geq 0$*;* $d(p, q) = 0$ *if and only if* $p = q$
- $d(p, q) = d(q, p)$
- $d(p, r) \leq d(p, q) + d(q, r)$

Definition 2 *In a metric space, a set $\mathcal{U}$ is open if whenever $p \in \mathcal{U}$ there is an $\epsilon > 0$ such that $B_\epsilon(p) \subseteq \mathcal{U}$.*

Definition 3 *If $f : M \to N$ is a map from one metric space to another, the map is continuous at $p \in M$ if for each $\epsilon > 0$ there is a $\delta > 0$ such that whenever $d(q,p) < \delta$ then $d(f(q), f(p)) < \epsilon$. If f is continuous at each point of M, we say it is continuous.*

Definition 4 *If x_n is a sequence in a metric space M, we say that $x_n \to x$ if for each $\epsilon > 0$ there is an N such that if $n > N$ then $d(x_n, x) < \epsilon$.*

Remark: In the exercises, we discussed unusual metric spaces. For the examination you should concentrate on the discrete metric on an arbitrary X and on two metrics on R^n: the standard $d(x,y) = \sqrt{\sum (x_i - y_i)^2}$ and the useful $d(x,y) = \max |x_i - y_i|$. You should be able to prove that these last two metrics on R^n yield the same open sets.

3 More Definitions

Definition 5 *A topological space is a set X and a collection of subsets called the open sets, such that*

- *the empty set and the entire set X are open*
- *if $\{\mathcal{U}_\alpha\}$ are open sets, then $\cup\, \mathcal{U}_\alpha$ is open*
- *if $\mathcal{U}$ and $\mathcal{V}$ are open, then $\mathcal{U} \cap \mathcal{V}$ is open*

Definition 6 *If $f : X \to Y$ is a map from one topological space to another, the map is continuous if whenever $\mathcal{U} \subseteq Y$ is open then $f^{-1}(\mathcal{U}) \subseteq X$ is open*

Definition 7 *If x_n is a sequence in a topological space X and $x \in X$, we say $x_n \to x$ if whenever $x \in \mathcal{U}$ with $\mathcal{U}$ open, there is an N such that whenever $n > N$ then $x_n \in \mathcal{U}$.*

Remark: Notice that every metric space is a topological space in a natural way. You should be able to prove that these definitions are equivalent to the earlier metric definitions for such spaces. The discrete metric on a metric space yields the discrete topology on the associated topological space. You should know several examples of topological spaces which are not metrizable.

Definition 8 *A set is closed if its complement is open. The closure of an arbitrary $A \subseteq X$ is the smallest closed set containing A. The interior of an arbitrary $A \subseteq X$ is the largest open set inside A.*

Remark: This definition assumes that there *actually is* a smallest closed set containing A and a largest open set inside A. You should know why.

4 Some Beginning Theorems

Theorem 1 (Star) *Let* $A \subseteq X$*.*

- *Suppose* A *is closed and* a_n *is a sequence in* A*. If* $a_n \to a$*, then* $a \in A$*.*
- *If* X *is metrizable, the converse is true: if* $a_n \in A$ *and* $a_n \to a$ *implies* $a \in A$*, then* A *is closed*
- *A counterexample shows that this converse can be false for some topological spaces*

Theorem 2 *If* $f : X \to Y$ *and* $g : Y \to Z$ *are continuous, then* $g \circ f : X \to Z$ *is continuous.*

5 New Spaces from Old Ones

Definition 9 *Let* $A \subseteq X$ *where* X *is a topological space. The induced topology on* A *is obtained by calling* $\mathcal{U}$ *open in* A *whenever there is an open* $\mathcal{V} \in X$ *such that* $\mathcal{U} = \mathcal{V} \cap A$*.*

Definition 10 *Let* $\pi : X \to Y$ *be an onto map from a topological space* X *to a set* Y*. The quotient topology on* Y *is obtained by calling* $\mathcal{U} \subseteq Y$ *open if* $\pi^{-1}(\mathcal{U})$ *is open in* X*.*

Definition 11 *Let* X *and* Y *be topological spaces. The product topology on* $X \times Y$ *is obtained by calling* $\cup(\mathcal{U}_\alpha \times \mathcal{V}_\alpha)$ *open in* $X \times Y$ *whenever* $\mathcal{U}_\alpha$ *are open in* X *and* $\mathcal{V}_\alpha$ *are open in* Y*.*

Remark: If $A \subseteq M$ where M is a metric space, then A automatically has a distance function and becomes a metric space and so a topological space. You should check that the induced topology gives the topology obtained from this metric structure on A.

Remark: The most powerful of these constructions is the quotient construction. You should know an example where X is a metric space, but Y is not metrizable.

Remark: We have a series of theorems saying that these constructions behave well with respect to continuous maps, compactness, and the Hausdorff property. I've giving these theorems slightly out of order so you can see them all at once.

Theorem 3 *Suppose* $f : X \to Y$ *is continuous and the image of* f *is in* $A \subseteq Y$*. If* A *has the induced topology then* $f : X \to A$ *is continuous.*

Example: The map $R \to S^1$ given by $t \to (\cos t, \sin t)$ is continuous because the corresponding map $R \to R^2$ is continuous.

Theorem 4 (Star) *Suppose $\pi : X \to Y$ is onto and Y has the quotient topology. Let $f : X \to Z$ be continuous. If f induces a well-defined map $\tilde{f} : Y \to Z$ then $\tilde{f}$ is automatically continuous.*

Theorem 5 *Let $X \times Y$ have the product topology. Then*

- *$\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous*
- *If $f : Z \to X$ and $g : Z \to Y$ are continuous, then $f \times g : Z \to X \times Y$ is continuous.*

Theorem 6 *Let $A \subseteq X$ have the induced topology. If A is closed and X is compact, then A is compact.*

Theorem 7 (Star) *Let $\pi : X \to Y$ be onto and give Y the quotient topology. If X is compact then Y is compact.*

Theorem 8 (Star) *If X and Y are compact then $X \times Y$ is compact.*

Definition 12 *A topological space X is Hausdorff if whenever $x \neq y$ are points in X, we can find open sets $x \in \mathcal{U}$ and $y \in \mathcal{V}$ such that $\mathcal{U} \cap \mathcal{V} = \emptyset$.*

Remark: Any metric space is Hausdorff by a trivial proof, so any subset $A \subseteq R^n$ is Hausdorff.

Theorem 9 *Let $A \subseteq X$ with the induced topology. If X is Hausdorff then A is Hausdorff.*

Theorem 10 *Let X and Y be Hausdorff. Then $X \times Y$ is Hausdorff.*

Remark: There is no corresponding theorem for the quotient topology. If X is Hausdorff then Y may not be Hausdorff.

6 Compactness

This is our deepest notion. The key point here is that compactness is a topological property, so two topological spaces cannot be homeomorphic if one is compact and the other isn't.

Definition 13 *A topological space is compact if whenever $\{\mathcal{U}_\alpha\}$ is an open cover, that is, $X \subseteq \cup\, \mathcal{U}_\alpha$, then we can find a finite number of these open sets which already cover X, so $X \subseteq \mathcal{U}_{\alpha_1} \cup \ldots \cup \mathcal{U}_{\alpha_k}$.*

Theorem 11 (Star) *$[a, b] \subseteq R$ is compact*

Theorem 12 (Star) *A subset of R^n is compact if and only if it is closed and bounded.*

Theorem 13 *If $f : X \to R$ is continuous and X is compact, then f takes a maximum value.*

Theorem 14 *If $f : X \to Y$ is continuous, one-to-one, and onto, and X is compact and Y is Hausdorff, then f^{-1} is automatically continuous.*

7 Important Exercises

From exercise set 2, look at 1.8a proving that $B_\epsilon(p)$ is open, 1.8d showing that an infinite intersection of open sets need not be open, and exercise 2.2a showing that if a metric is discrete, then the associated topological space has the discrete topology.

From exercise set 3, look at exercise 2.9d showing that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Then pay very close attention to 4.2, parts one and two, and 4.3. These are the first "cut and paste" exercises. The first separates letters and numbers into homeomorphism equivalence classes, and the second shows that two disks joined by three twisted links is homeomorphic to a torus with a hole removed.

From exercise set 4, examine exercise 5.3f showing that RP^2 is homeomorphic to a subset of R^4. The proof that the given map is one-to-one is interesting, but pay particular attention to the definition of RP^2, to the proof that the map from RP^2 to R^4 is well-defined and continuous, and then to the proof that this map is a homeomorphism. The exercise didn't ask you to prove that f^{-1} is continuous, but you should be able to do that now using our compactness results.

In the same exercise set, exercise 5.3h is interesting. This exercise is about a quotient space $\pi : X \to Y$, about subsets $A \subseteq X$ and $B \subseteq Y$, and about whether B has the quotient topology induced by $\pi : A \to B$. I'd master the rest of this review set first, but if you have time, study this solution.

Study the cut and paste solution of 5.4b explaining what happens when a Mobius band is cut in the middle.

Study carefully extra problems one and two. The first proves that a Klein bottle can be obtained by gluing two Mobius bands together along thir boundaries (both have S^1 as boundary), and the second proves that RP^2 can be obtained by gluing a Mobius band and a disk together along their boundaries (both have S^1 as boundary).

From exercise set 5, examine 6.6a showing that if $f : X \times Y \to X$ and $g : X \times Y \to Y$ are continuous with an unknown topology on $X \times Y$, then at least the open sets of the product topology are open in $X \times Y$. If you have time, study the example in 6.6l of a non-discrete space X which is homeomorphic to $X \times X$.

Look at exercise 7.13h defining the one-point compactification of a topological space, in particular the definition of the open sets of this space and the proof that the space is compact.

Look at extra problem 3 showing that if A and B are nonintersecting closed sets in a compact Hausdorff space, then there are nonintersecting open sets $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{V}$. Finally be sure you understand the final exercise showing that the closed unit ball in R^n is not homeomorphic to the open unit ball in R^n. The proof is very easy, but the exercise illustrates the central philosophy of the course: to prove two spaces are not homeomorphic, we must find a purely topological property of one of the spaces which is not a property of the other space.