

# The Long Line

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## 1 Introduction

Before this class began, I asked several topologists what I should cover in the first term. Everyone told me the same thing: go as far as the classification of compact surfaces. Having done my duty, I feel free to talk about more general results and give details about one of my favorite examples.

We classified all compact connected 2-dimensional manifolds. You may wonder about the corresponding classification in other dimensions. It is fairly easy to prove that the only compact connected 1-dimensional manifold is the circle  $S^1$ ; the book sketches a proof of this and I have nothing to add.

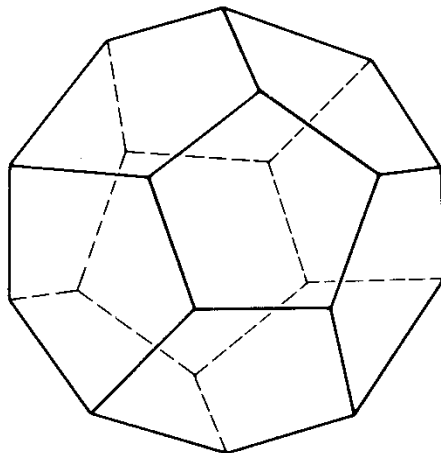
In dimension greater than or equal to four, it has been proved that a complete classification is impossible (although there are many interesting theorems about such manifolds). The idea of the proof is interesting: for each finite presentation of a group by generators and relations, one can construct a compact connected 4-manifold with that group as fundamental group. Logicians have proved that the word problem for finitely presented groups cannot be solved. That is, if I describe a group  $G$  by giving a finite number of generators and a finite number of relations, and I describe a second group  $H$  similarly, it is not possible to find an algorithm which will determine in all cases whether  $G$  and  $H$  are isomorphic. A complete classification of 4-manifolds, however, would give such an algorithm.

As for the theory of compact connected three dimensional manifolds, this is a very exciting time to be alive if you are interested in that theory. Three dimensional compact manifolds have been studied intensively since the work of Poincare around 1900. In the 1970's, Bill Thurston expanded this theory dramatically; roughly speaking, he showed that every such manifold can be canonically cut into pieces of a special nature, and then conjectured that each of these pieces can be given a particular kind of geometry. The geometry dramatically decreases the possible topologies of that piece. See for example, <http://mathworld.wolfram.com/ThurstonGeometrizationConjecture.html> for more details.

Around 1980, a possible approach to proving the Thurston conjecture was developed by R. S. Hamilton. Two faculty members at Oregon, Peng Lu and Jim Isenberg, are experts in this theory. In 2002, G. Perelman announced a proof of the full Thurston conjectures using Hamilton's technique. Experts are verifying his proof now; the experts believe that there is a good chance it is correct. Perelman's proof will not complete the classification, but it will push us astonishingly closer to that goal.

One special case of Perelman's result would give all possible compact 3-manifolds with finite fundamental group. In particular, it would show that  $S^3$  is the only compact 3-manifold with trivial fundamental group, so that every closed loop can be deformed to a point. Proving this special case is one of seven unsolved problems for which the Clay Institute offers a prize of \$1,000,000 for a solution.

One of the 3-manifolds with finite fundamental group is particularly interesting, and directly related to our polygonal construction of 2-manifolds. Start with a dodecahedron. Glue opposite faces together. Notice that opposite faces do not quite match; consistently rotate each face clockwise by the same amount so it matches its opposite before gluing. Then points inside the dodecahedron and points in the interior of a face clearly have open neighborhoods homeomorphic to open neighborhoods of  $R^3$ . It is not immediately clear that the same result holds for points on edges of the boundary or points on vertices of the boundary. By analyzing carefully how edges and vertices are glued together, you can show that the identification space is also locally Euclidean at these points. The resulting object is called *Poincare's dodecahedral space*.



In 2003, three physicists in France proposed that this space might model the actual universe. Their paper is titled *Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background*. Recent evidence makes the proposal unlikely.

## 2 Classification of Connected Manifolds in Dimension One

In dimension one, we can do even better and drop the compactness assumption. We obtain the following theorem, which I will not prove here:

**Theorem 1** *Every connected second countable 1-manifold is homeomorphic to  $S^1$  or  $R$*

However, you will notice that there is an extra hypothesis. This extra hypothesis is often used in advanced graduate courses, so let me discuss it briefly.

**Definition 1** *A topological space  $X$  is second countable if it is possible to find a countable number of open sets  $\{\mathcal{U}_i\}$  such that every open subset of  $X$  is a union of some of these sets.*

**Theorem 2**

- *The space  $R^n$  is second countable*
- *If  $A \subset R^n$  has the induced topology then  $A$  is second countable*
- *Every manifold with a countable coordinate cover is second countable*
- *Every compact manifold is second countable*

*Proof:* The last three points follow immediately from the first one; I'll let you fill in the details.

For  $R^n$ , let the countable collection  $\{\mathcal{U}_i\}$  of open sets be all open balls with rational radius and rational center (that is, all coordinates of the center are rational). This collection is easily seen to be countable. To prove that every open set is a union of such balls, it suffices to prove that whenever  $q \in B_\epsilon(p)$  is an arbitrary open ball, we can find one of our  $\mathcal{U}_i$  such that  $q \in \mathcal{U}_i \subseteq B_\epsilon(p)$ . To do this, let  $d = \epsilon - d(p, q)$ . Find a point  $x$  with rational coordinates such that  $d(x, q) < \frac{d}{3}$  and find a rational  $r$  such that  $\frac{d}{3} < r < \frac{d}{2}$ . Let  $\mathcal{U}_i$  be the disk of radius  $r$  centered at  $x$ . The disk  $\mathcal{U}_i$  contains  $q$  because  $d(q, x) < \frac{d}{3} < r$ . The disk  $\mathcal{U}_i$  is in  $B_\epsilon(p)$  because whenever  $s \in \mathcal{U}_i$  we have  $d(s, p) \leq d(s, x) + d(x, q) + d(q, p) < r + \frac{d}{3} + d(q, p) < \frac{d}{2} + \frac{d}{3} + (\epsilon - d) < \epsilon$ .

## 3 The Long Line

In the rest of these notes I will be rigorous and prove everything. The second countability hypothesis is necessary when classifying connected 1-manifolds because there is an astonishing connected 1-dimensional manifold which looks locally like a line but is not homeomorphic to  $R$ . Instead it is much, much longer. This manifold is called the *long line* and we will construct it in these notes. The long line  $L$  has the following properties:

- $L$  is not second countable
- $L$  is totally ordered; if  $r$  and  $s$  are points in  $L$  then the interval defined by  $(r, s) = \{ t \in L \mid r < t < s \}$  is open and homeomorphic to  $(0, 1)$  in the ordinary line
- every open set in the long line is a union of such open intervals; however many open sets are too large to be written as a *countable* union of open intervals
- the closed interval  $[r, s] = \{ t \in L \mid r \leq t \leq s \}$  in the long line is homeomorphic to  $[0, 1]$  in the ordinary line

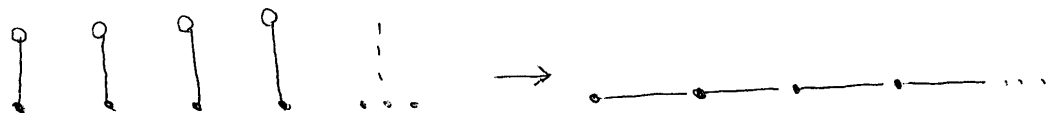
The ordinary line  $R$  is homeomorphic to an open interval  $(-1, 1)$ . But the long line is not homeomorphic to any subset of  $R^n$  because it is not second countable. If you try to examine the long line by selecting a piece of it, say  $[a, b]$ , then this piece looks like an ordinary interval but *almost all* of the line is outside this piece. Said poetically, it isn't possible to understand what is unusual in  $L$  by chopping off a piece because the piece you chop off is always conventional and the unusually large stuff is outside that piece.

There is one other amazing feature of  $L$ . I claim that no mathematician has ever been able to visualize it. The long line is too complicated for that. On the other hand, we definitely know that it exists and by the end of these notes you will also be certain of its existence.

## 4 Motivation for the Construction of $L$

In these notes I'll actually construct *the long ray*; it is an analogue of the ordinary ray  $[0, \infty)$ . This long ray will be ordered and have a first element which we can call zero. To get the long line, glue two long rays together at their initial points.

To motivate the construction, let me construct an ordinary ray  $[0, \infty)$  in an unusual way. Start with the index set  $I = \{1, 2, \dots\}$  ordered in the conventional way. Attach a copy of the interval  $[0, 1)$  to each of these elements as indicated in the picture below. Notice that the union of these intervals looks the ray  $[0, \infty)$  as also shown below.



We can make the topology on the union of these intervals more explicit. Call the union of the intervals  $L$ . An element in  $L$  can be described by a pair  $(m, s)$  where  $m \in I$  and

$s \in [0, 1)$ . Put a total ordering on this set by writing  $(m, s) < (n, t)$  if  $m < n$  or else if  $m = n$  and  $s < t$ . Notice that  $L$  has an initial point  $(1, 0)$ . We will call this point 0 for simplicity.

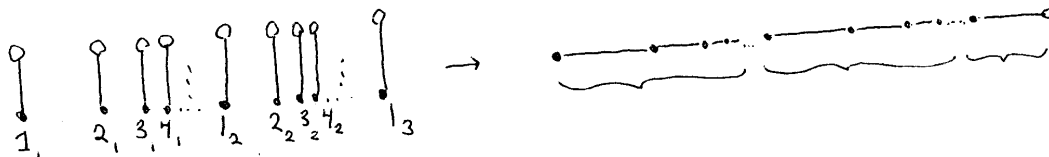
Call a subset  $A$  of  $L$  an *open interval in  $L$*  if  $A$  has the form  $[0, b)$  or  $(a, b)$  in  $L$ . That is,  $A$  is either all points greater than or equal to zero and less than  $b$  in the ordering of  $L$ , or else all the points greater than  $a$  and less than  $b$  in this ordering. Define an open set to be an arbitrary union of such open intervals in  $L$ . It is easy to see that this is a topology, and easy to prove that  $L$  with this topology is homeomorphic to  $[0, \infty) \subset \mathbb{R}$ .

## 5 More Motivation

The previous construction still works if we replace  $I$  by a more complicated ordered set. For example, let  $I$  consist of two copies of the positive integers and one additional integer, ordered as indicated below:

$$I = \{1_1, 2_1, 3_1, 4_1, \dots, 1_2, 2_2, 3_2, 4_2, \dots, 1_3\}$$

If we attach a copy of  $[0, 1)$  to each element of this ordered set, we easily assign an order to the union of the intervals and a topology to this union exactly as above. The new object is still homeomorphic to  $[0, \infty)$  as indicated by the picture below.



The key property of our index set  $I$  is given by

**Definition 2** Let  $I$  be a set which is totally ordered. We say  $I$  is well-ordered if every nonempty subset of  $I$  has a smallest element.

*Remark:* For example, the set  $I = \{1, 2, 3, \dots\}$  is well ordered. The set

$$I = \{1_1, 2_1, 3_1, 4_1, \dots, 1_2, 2_2, 3_2, 4_2, \dots, 1_3\}$$

is also well ordered. But the real numbers are not well ordered because, for instance,  $(1, 2)$  has no smallest element.

*Remark:* If a set is well-ordered, we can get a very concrete feeling about what the ordering looks like, at least at the start. Suppose  $I$  is well ordered. Then  $I$  must have a smallest element. Call this element 1. The set  $I - \{1\}$  has a smallest element. Call this element 2. The set  $I - \{1, 2\}$  has a smallest element. Call this element 3. Continue. If our set is finite, this process will eventually stop and our ordered set is order isomorphic to  $\{1, 2, \dots, n\}$ . But if our set is infinite, the process will continue and we will wind up with a copy of the positive integers.

However, these integers may not exhaust the set. In that case, the set  $I - \{1, 2, 3, \dots\}$  has a smallest element. Call this element  $1_2$ . Continue. At any moment we may exhaust the set, but if not we will continue until we have

$$\{1, 2, 3, \dots, 1_2, 2_2, 3_2, \dots\}$$

This still may not exhaust the set. If not, we can obtain yet another sequence. If none of these sequences exhaust the set, we will wind up with

$$\{1, 2, 3, \dots, 1_2, 2_2, 3_2, \dots, 1_3, 2_3, 3_3, \dots, \dots \}$$

where the set contains *all* symbols of the form  $m_n$ .

However, this still may not exhaust the set. So after all of these elements, there is a next one. Let us revise our naming scheme slightly. Call the first elements in the set  $1_1, 2_1, 3_1, \dots$ . Then perhaps the next element after all of the above elements could be called  $1_{11}$  to indicate that it comes after all single digit indices. Continue.

It is possible to push this naming scheme considerably further. But eventually the head starts to swim and we lose track of elements. Unfortunately, this always happens *before* we have listed uncountably many elements. Said another way, it is possible to write down many different well-orderings on a countable set, but nobody has written down an explicit well-ordering on an uncountable set.

Given this state of affairs, it is amazing that Cantor conjectured in his work on set theory that *every set can be given a well-ordering*. However, Cantor could not prove this result.

Let us suppose this result is true. Then we could produce a well-ordering on the set of real numbers and so we could find an uncountable well-ordered index set  $I$ . But in that case, we can obtain something even more remarkable. Let  $A$  be the set of all  $a \in I$  such that the set of all elements smaller than  $a$  is uncountable. This set would have a smallest element. If we throw away this smallest element and everything larger in  $I$ , we have a new index set  $J$  which is still uncountable. But this index set has the remarkable property that whenever we choose a particular element  $j \in J$ , then the set of all elements of  $J$  less than or equal to  $j$  is countable. It is possible to prove that such an index set is unique up to

order isomorphism, but we don't need to know that. This  $J$  is called *the first uncountable ordinal number*.

One summer I worked as a programmer in Norway. The head of the project had earned his PhD at MIT working on Lisp, but he was a mathematics fan. He said to me "I believe I can *almost* see the first uncountable ordinal number." If so, he is better than most of the rest of us.

The long ray is constructed exactly as above by starting with the index set  $J$  corresponding to the first uncountable ordinal, attaching a copy of  $[0, 1)$  to each element of  $J$ , and topologizing the union of these intervals as indicated earlier.

## 6 The Hilbert Problems

In 1900 Hilbert gave a famous lecture listing twenty-seven unsolved problems which he predicted would be important for the future. Much of the history of mathematics in the twentieth century consists of theories invented to solve these problems.

The very first problem was about set theory. Here is Hilbert's complete description:

### 1. Cantor's problem of the cardinal number of the continuum

Two systems, i. e., two assemblages of ordinary real numbers or points, are said to be (according to Cantor) equivalent or of equal cardinal number, if they can be brought into a relation to one another such that to every number of the one assemblage corresponds one and only one definite number of the other. The investigations of Cantor on such assemblages of points suggest a very plausible theorem, which nevertheless, in spite of the most strenuous efforts, no one has succeeded in proving. This is the theorem:

Every system of infinitely many real numbers, i. e., every assemblage of numbers (or points), is either equivalent to the assemblage of natural integers, 1, 2, 3, ... or to the assemblage of all real numbers and therefore to the continuum, that is, to the points of a line; as regards equivalence there are, therefore, only two assemblages of numbers, the countable assemblage and the continuum.

From this theorem it would follow at once that the continuum has the next cardinal number beyond that of the countable assemblage; the proof of this theorem would, therefore, form a new bridge between the countable assemblage and the continuum.

Let me mention another very remarkable statement of Cantor's which stands in the closest connection with the theorem mentioned and which, perhaps, offers the key to its proof. Any system of real numbers is said to be ordered, if for every two numbers of the system it is determined which one is the earlier and

which the later, and if at the same time this determination is of such a kind that, if  $a$  is before  $b$  and  $b$  is before  $c$ , then  $a$  always comes before  $c$ . The natural arrangement of numbers of a system is defined to be that in which the smaller precedes the larger. But there are, as is easily seen infinitely many other ways in which the numbers of a system may be arranged.

If we think of a definite arrangement of numbers and select from them a particular system of these numbers, a so-called partial system or assemblage, this partial system will also prove to be ordered. Now Cantor considers a particular kind of ordered assemblage which he designates as a well ordered assemblage and which is characterized in this way, that not only in the assemblage itself but also in every partial assemblage there exists a first number. The system of integers  $1, 2, 3, \dots$  in their natural order is evidently a well ordered assemblage. On the other hand the system of all real numbers, i. e., the continuum in its natural order, is evidently not well ordered. For, if we think of the points of a segment of a straight line, with its initial point excluded, as our partial assemblage, it will have no first element.

The question now arises whether the totality of all numbers may not be arranged in another manner so that every partial assemblage may have a first element, i. e., whether the continuum cannot be considered as a well ordered assemblage – a question which Cantor thinks must be answered in the affirmative. It appears to me most desirable to obtain a direct proof of this remarkable statement of Cantor's, perhaps by actually giving an arrangement of numbers such that in every partial system a first number can be pointed out.

The resolution of the full problem is due to Godel (working around 1940) and Cohen (working around 1970). It turns out that the cardinal number of the continuum may or may not be the next infinite cardinal depending on the particular axioms of set theory used. But the second part of Hilbert's problem was resolved by Zermelo in 1904, in a remarkable paper proving

**Theorem 3 (Zermelo)** *Every set can be well-ordered.*

This proof brought Zermelo immediate fame; when it was published he was a lecturer at Gottingen but in December of 2005 he was promoted to a professorship there.

The proof also brought criticism. Zermelo's proof used Cantor's set theory, which many mathematicians found hard to stomach. In 1908 Zermelo published a second proof. This second proof used only elementary principles of set theory and soon afterward most mathematicians accepted the result. To be honest, the axiom of choice is used at the start of the proof, and some mathematicians down to the current time prefer to avoid this axiom.



## 7 Zermelo's Second Proof

**Proof 1:** Let  $X$  be a set. Begin by simultaneously choosing for each nonempty subset  $A \subseteq X$  an element  $\varphi(A) \in A$ . This can be done by the axiom of choice.

**Motivation 1:** We now motivate the proof. If  $X$  had been well-ordered to begin with, we could have defined  $\varphi(A)$  to be the smallest element of  $A$ . We are going to turn this around and define an ordering so  $\varphi(A)$  is given by this rule.

Let  $1_1$  be  $\varphi(X)$ . This will be the first element of our ordering. Let  $2_1 = \varphi(X - \{x_1\})$ . In general, suppose we have defined an ordering on a subset  $A \subset X$ . Let  $a = \varphi(X - A)$  and define an ordering on  $A \cup \{a\}$  by declaring that  $a$  is greater than every element of  $A$ .

Unfortunately, when we try to carry out this program we find that the ordering grows more and more complicated and it is not at all obvious that we eventually reach every element of  $X$ . For example, suppose  $X$  is not countable. Following the program above, we obtain  $1_1, 2_1, 3_1, \dots$ . Imagine that this has been done for countably many terms. But these terms will not exhaust  $X$ , so after they have *all* been chosen, we get a next element  $1_2$ . After that we get  $2_2$ , etc. So we will have

$$1_1, 2_1, 3_1, \dots, \quad 1_2, 2_2, 3_2, \dots, \quad 1_3, 2_3, 3_3, \dots, \quad 1_4, \dots$$

But this sequence is still countable, so after all of these elements, there is another. It is virtually impossible to see into this construction far enough to view an ordered uncountable subset. So this method of ordering  $X$  doesn't really work.

**Motivation 2:** To get at an ordering of  $X$ , Zermelo noticed that if  $X$  were well-ordered and  $\varphi$  were defined as above, we could talk about "final sequences" in the ordering; by definition, these would be subsets of the form

$$F(a) = \{ x \in X \mid a \leq x \}$$

By convention  $\emptyset$  also counts as a final sequence.

Suppose we knew the final sequences. Then we could easily reconstruct the ordering. Indeed given  $a \in X$  construct  $F(a)$ . Then  $a < b$  if and only if  $b \in F(a)$ .

But how do we get our hands on the set of all final sequences? Let  $\mathcal{C}$  be the set of all these final sequences. Zermelo noticed that this set has the following properties:

1.  $\emptyset \in \mathcal{C}$  and  $X \in \mathcal{C}$
2. If  $A_\alpha \in \mathcal{C}$  for each  $\alpha$ , then  $\cap A_\alpha \in \mathcal{C}$
3. If  $A \in \mathcal{C}$  and  $A' = A - \varphi(A)$ , then  $A' \in \mathcal{C}$ .

The first property is obvious. Property 2) is obvious if  $\cap A_\alpha$  is empty. Otherwise let  $a$  be the smallest element of  $\cap A_\alpha$ . Then  $a \in A_\alpha$  for all indices. Since  $A_\alpha$  is a final sequence,  $F(a) \subseteq A_\alpha$ . So  $F(a) \subseteq \cap A_\alpha$ . This intersection cannot have an element smaller than  $a$  since  $a$  was defined to be its smallest element, so  $F(a) = \cap A_\alpha$ . Property 3 is also clear, for if  $A = F(a)$  then  $a$  is the smallest element of  $A$  and the set  $A - \{a\}$  has a smallest element  $b$ , so  $A - \{a\} = F(b)$ .

**Motivation 3:** Zermelo's ingenious idea is to work backward by defining the set  $\mathcal{C}$  directly, and then using it to recover the ordering on  $X$ . He defines  $\mathcal{C}$  as follows:

**Proof 2:** Define  $\mathcal{C}$  to be the smallest collection of subsets of  $X$  with properties 1), 2), and 3) listed above. This definition makes sense because there certainly is one such collection, namely the set of all subsets of  $X$ , and the intersection of all collections satisfying these properties is also a collection satisfying the properties. For example,  $\emptyset$  and  $X$  would belong to all such collections and thus to their intersection. If  $A_\alpha$  are sets which belong to all such collections and thus to their intersection, then  $\cap A_\alpha$  is a set which belongs to all such collections and thus to their intersection. Finally if  $A$  belongs to all such collections and so to their intersection, then  $A'$  belongs to all such collections and so to their intersection.

**Motivation 4:** Let me stop and convince you that this technique has a chance of working. Assume again that  $X$  already has a well ordering as discussed earlier. This well ordering then starts

$$\{1_1, 2_1, 3_1, \dots, 1_2, 2_2, 3_2, \dots\}$$

Notice that the set  $F(1_1)$  is all of  $X$ ; this set belongs to  $\mathcal{C}$  by construction. The set  $F(2_1)$  is obtained from  $X$  by removing its first element, which is  $\varphi(X) = 1_1$ . So  $F(2_1) = X'$ . This set is again in  $\mathcal{C}$  by construction because whenever  $A \in \mathcal{C}$  then  $A' \in \mathcal{C}$ . Continue. We conclude that all of the sets  $F(n_1)$  belong to  $\mathcal{C}$ .

The intersection of these sets then belongs to  $\mathcal{C}$  by construction; this set is clearly  $F(1_2)$ . In this way we gradually build up all of the  $F(a)$ , although we will need to prove that they all belong in a more convincing way.

However, how do we know that  $\mathcal{C}$  only contains these  $F(a)$  and doesn't contain other junk subsets? Well, this is proved in the hardest step below, but the idea is rather simple. Consider the set  $F(a)$ . If we omit the first element, then there is a new first element and we'll symbolically call it  $F(a+1)$ . Now let us define a new candidate  $\mathcal{D}$  for  $\mathcal{C}$  as

$$\mathcal{D} = \{B \in \mathcal{C} \mid B \subseteq F(a+1)\} \cup \{B \in \mathcal{C} \mid F(a) \subseteq B\}$$

We will prove that  $\mathcal{D}$  satisfies all three properties required of  $\mathcal{C}$ . Since  $\mathcal{C}$  is the smallest set which satisfies all three properties,  $\mathcal{D} = \mathcal{C}$ . If  $a \notin B$ , this tells us that  $B$  only contains elements larger than  $a$ , although it doesn't tell us that  $B$  cannot have gaps. But if  $a \in B$ , it

says that the entire ray of larger elements is also in  $B$ . So  $B$  cannot have gaps further out. We rather easily conclude that elements of  $\mathcal{C}$  must be full rays rather than more general sets with gaps.

OK, enough motivation; on to the proof!

**Proof 3:**

**Lemma 1** *If  $A \in \mathcal{C}$  and  $B \in \mathcal{C}$ , then either  $A \subseteq B$  or  $B \subseteq A$ .*

*Proof:* This is the main step in the proof. To prove it, , define

$$\mathcal{D} = \{ A \in \mathcal{C} \mid \text{whenever } B \in \mathcal{C} \text{ we have } B \subseteq A \text{ or } A \subseteq B \}$$

The set  $\mathcal{D}$  is not empty because  $X \in \mathcal{D}$ . We are going to show that  $\mathcal{D}$  has properties 1), 2), and 3). Since  $\mathcal{C}$  is the smallest collection of subsets with these properties, it will follow that  $\mathcal{C} \subseteq \mathcal{D}$  and thus every element  $A \in \mathcal{C}$  has the required property.

Certainly the empty set and the entire set  $X$  are in  $\mathcal{D}$ . Suppose  $\{A_\alpha\}$  are sets in  $\mathcal{D}$ . If  $B$  is in  $\mathcal{C}$ , then either  $A_\alpha \subseteq B$  for some  $\alpha$  and so  $\cap A_\alpha \subseteq B$ , or else  $B \subseteq A_\alpha$  for all  $\alpha$  and then  $B \subseteq \cap A_\alpha$ . Finally, we must prove that  $A \in \mathcal{D}$  implies  $A' \in \mathcal{D}$ .

To see this, fix  $A \in \mathcal{D}$ . Then whenever  $B \in \mathcal{C}$  we have  $B \subseteq A$  or  $A \subseteq B$ . Let

$$\tilde{\mathcal{C}} = \{ B \in \mathcal{C} \mid B \subseteq A' \} \cup \{ B \in \mathcal{C} \mid A \subseteq B \}$$

We will show that  $\tilde{\mathcal{C}}$  satisfies properties 1), 2), 3). If so, we have  $\mathcal{C} \subseteq \tilde{\mathcal{C}}$  because  $\mathcal{C}$  is the smallest collection with these properties. But then whenever  $B \in \mathcal{C}$  we have  $B \subseteq A'$  or  $A \subseteq B$ . Consequently  $B \subseteq A'$  or  $A' \subseteq B$  as required.

Certainly  $\emptyset$  and  $X$  belong to  $\tilde{\mathcal{C}}$ . Suppose  $B_\alpha$  are elements in  $\tilde{\mathcal{C}}$ . If some element of  $B_\alpha \subseteq A'$ , then  $\cap B_\alpha \subseteq A'$ . Otherwise  $A \subseteq B_\alpha$  for all  $\alpha$ , so  $A \subseteq \cap B_\alpha$ .

We must finally show that  $B \in \tilde{\mathcal{C}}$  implies  $B' \in \tilde{\mathcal{C}}$ . This is clear if  $B \subseteq A'$  for then  $B' \subset B \subseteq A'$ . It is also obvious if  $A = B$  for then  $B' = A'$  and so  $B' \subseteq A'$ . Finally, suppose  $A \subset B$ . We know that either  $B' \subseteq A$  or  $A \subseteq B'$ . If  $A \subseteq B'$  we are done. So the only remaining case is  $B' \subseteq A$ . Thus we simultaneously have  $B' \subseteq A \subset B$ . But  $B' = B - \varphi(B)$ , so this can only be true if  $A = B'$ .

**Lemma 2** *If  $A$  and  $B$  are sets in  $\mathcal{C}$ , then either  $A \subseteq B'$  or  $A = B$  or  $B \subseteq A'$ .*

*Proof:* We know that  $A \subseteq B$  or  $B \subseteq A$ . Without loss of generality suppose  $A \subseteq B$ . We know that either  $A \subseteq B'$  or  $B' \subseteq A$ , and if  $A \subseteq B'$  we are done. So assume  $B' \subseteq A$  and thus  $B' \subseteq A \subseteq B$ . This can only be true if  $A = B$  or  $A = B'$ .

**Lemma 3** *Let  $A$  be an arbitrary subset of  $X$ . Then there is a unique  $C \in \mathcal{C}$  such that  $A \subseteq C$  and  $\varphi(C) \in A$ .*

*Remark:* In the motivating example for the proof,  $A$  has a smallest element  $a$  and we can take  $C = F(a)$ .

*Proof:* Let  $C$  be the intersection of all subsets of  $\mathcal{C}$  containing  $A$  and thus the smallest element of  $\mathcal{C}$  containing  $A$ . This element is in  $\mathcal{C}$  by property 2). We must have  $\varphi(C) \in A$  for otherwise  $C - \varphi(C)$  would be a smaller element of  $\mathcal{C}$  containing  $A$ .

We still must prove that this element is unique. Suppose  $C_1$  and  $C_2$  both work. Then  $C_1 \subseteq C_2'$  or  $C_2 \subseteq C_1'$  or  $C_1 = C_2$ . But the first two possibilities immediately lead to contradictions. For example, if  $C_1 \subseteq C_2'$  then  $A \subseteq C_1 \subseteq C_2'$  contradicting the requirement that  $\varphi(C_2) \in A$ .

**Lemma 4** *For each  $a \in A$ , there is a unique subset  $C \in \mathcal{C}$  such that  $\varphi(C) = a$ . Call this set  $F(a)$ . This assignment sets up a one-to-one correspondence between the set  $X$  and the elements of  $\mathcal{C}$ .*

*Proof:* Apply the previous lemma to the set  $A = \{a\}$ . According to the lemma,  $\mathcal{C}$  contains a unique set  $C$  such that  $a \in C$  and  $a = \varphi(C)$ . These conditions are redundant; we may as well just say that  $\mathcal{C}$  contains a unique set  $C$  such that  $a = \varphi(C)$ .

Every nonempty set in  $\mathcal{C}$  must have this form, for if  $C$  is such a set and  $a = \varphi(C)$  then  $C = F(a)$ .

*Remark:* Notice that we are beginning to recover rigorously the intuitive picture which began the proof.

**Lemma 5** *Let  $a, b \in X$ ,  $a \neq b$ . Then either  $a \in F(b)$  or  $b \in F(a)$  but not both. If  $b \in F(a)$  we write  $a < b$ .*

*Proof:* According to a previous lemma, there is a unique  $C \in \mathcal{C}$  such that  $\{a, b\} \subseteq C$  and  $\varphi(C) \in \{a, b\}$ . If  $\varphi(C) = a$  then  $C = F(a)$  and thus  $b \in F(a)$ . Otherwise  $C = F(b)$  and so  $a \in F(b)$ . We cannot have both statements true, for then  $\varphi(C)$  would equal both  $a$  and  $b$  and yet  $a \neq b$ .

**Lemma 6** *The relation  $a < b$  is a total ordering on  $X$ .*

*Proof:* We know that exactly one of  $a < b, a = b, b < a$  is true. We must prove that if  $a < b$  and  $b < c$ , then  $a < c$ . We know that there is a unique element  $C \in \mathcal{C}$  such that  $\{a, b, c\} \subseteq C$  and  $\varphi(C) \in \{a, b, c\}$ . If  $\varphi(C) = a$ , then  $C = F(a)$  and so  $c \in F(a)$  implies  $a < c$  and we are done. If  $\varphi(C) = b$ , then  $C = F(b)$  and so  $a \in F(b)$  implies  $b < a$  contradicting our assumption that  $a < b$ . If  $\varphi(C) = c$  then  $C = F(c)$  and so  $b \in F(c)$  implies  $c < b$  contradicting our assumption that  $b < c$ .

**Lemma 7** *The total ordering on  $X$  is a well ordering.*

**Proof:** Let  $A$  be a subset of  $X$ . By a previous lemma, there is a unique  $C \in \mathcal{C}$  such that  $A \subseteq C$  and  $\varphi(C) \in A$ . Let  $c = \varphi(C)$  so  $C = F(c)$ . I claim that  $c$  is the smallest element of

A. Indeed  $c \in A$ ; if  $a \in A$  and  $a \neq c$ , then  $a \in F(c)$  and so  $c < a$ .

## 8 The First Uncountable Ordinal

**Theorem 4** *There is a totally ordered and well-ordered set  $I$  which is not countable, but has the property that whenever  $a \in I$  then  $\{i \in I \mid i \leq a\}$  is countable.*

*Proof:* Give  $R$  a well-ordering. This set is uncountable. If it has the above property, we are done. Otherwise let  $A$  be the set of all elements  $a$  of  $R$  such that the set of elements less than  $a$  is not countable. This set has a smallest element,  $b$ . Remove  $b$  and all larger elements from  $R$ . The remaining set  $I$  is not countable. But if  $i \in I$ , then  $i$  is smaller than  $b$  and so the set of elements smaller than  $i$  is countable.

## 9 The Long Ray

Just to repeat our earlier construction, let  $I$  be a well-ordered index set which is not countable but such that whenever  $a \in I$  then  $\{i \in I \mid i \leq a\}$  is countable. Attach a copy of  $[0, 1)$  to each index and call the union of these intervals  $L$ . Thus an element of  $L$  has the form  $(i, s)$  where  $i \in I$  and  $s \in [0, 1)$ . Define a total ordering on  $L$  by writing  $(i, s) < (j, t)$  if  $i < j$  or  $i = j$  and  $s < t$ . Call a subset  $\mathcal{U} \subseteq L$  open if every point  $p \in \mathcal{U}$  is in an open interval which is contained in  $\mathcal{U}$ . By definition an open interval is a set of the form  $[0, b) = \{l \in L \mid 0 \leq l < b\}$  or a set of the form  $(a, b) = \{l \in L \mid a < l < b\}$ . Here  $0 \in L$  is the element  $(i, s)$  where  $i$  is the first element of  $I$  and  $s = 0 \in [0, 1)$ .

It is easy to check that this is a topology. The resulting space is not second countable because we easily find an uncountable disjoint family of open sets in  $L$ : namely, for each index  $i$  consider the interval between  $(i, 0)$  and  $(i, 1)$ . We must still prove that this space is connected and that each point in the space (except the first point  $0$ ) has an open neighborhood homeomorphic to an open neighborhood of  $R$ . We will actually prove more. We will prove that if  $a$  and  $b$  are in  $L$ , the interval  $[a, b]$  in  $L$  is homeomorphic to  $[0, 1]$ . It immediately follows that  $L$  is pathwise connected and locally Euclidean away from the first element. Gluing two copies of this long ray together at the origin gives the long line.

## 10 The Long Line is Locally Euclidean

**Theorem 5** *Let  $a$  and  $b$  be points in the long ray. Then the set  $[a, b]$  of all points in the long ray between  $a$  and  $b$  is homeomorphic to  $[0, 1]$ .*

**Theorem 6** *Let  $a$  and  $b$  be points in the long ray. Then the set  $(a, b)$  of all points in the long ray between  $a$  and  $b$  is homeomorphic to  $(0, 1)$ .*

**Theorem 7** *Let  $\tilde{0}$  be the smallest point in the long ray and let  $b$  be a point in the long ray greater than  $\tilde{0}$ . Then the set  $[\tilde{0}, b)$  of all points in the long ray between its first point and  $b$  is homeomorphic to  $[0, 1)$ , and the homeomorphism can be chosen to be an order preserving map which sends the point  $\tilde{0}$  to 0 and the point  $b$  to 1.*

*Proof:* Notice that theorem 5 implies that the long ray is pathwise connected and so connected. Theorem 6 shows that at each point except its first point, the long ray is locally-homeomorphic to  $R$ . It follows immediately that the long line, formed by gluing two long rays together at their first points, is a 1-manifold.

It suffices to prove theorem 7 because theorems 5 and 6 follow immediately. For instance, if  $a < b$ , find a homeomorphism from  $[\tilde{0}, b)$  to  $[0, 1)$ . Let this homeomorphism map  $a$  to  $A \in [0, 1)$ . Then it maps  $(a, b)$  to  $(A, 1)$ , but  $(A, 1)$  is homeomorphic to  $(0, 1)$  by an obvious stretching map.

We now prove theorem 7. We can assume that  $b$  has the form  $(i, 0)$  where  $i \in I$  is a given index. Indeed if the set of all points in the long line from the first point in the line to  $(i, 0)$  is homeomorphic to  $[0, 1)$ , we can find a homeomorphism from  $[(i, 0), (i, t))$  to  $[1, 2)$ , and then glue these homeomorphisms together to get a homeomorphism from  $[\tilde{0}, (i, t))$  to  $[0, 1) \cup [1, 2) = [0, 2)$ . But this set is in turn homeomorphic to  $[0, 1)$ .

**Lemma 8** *If  $i \in I$  is an index greater than the first index, then there is a homeomorphism from  $[\tilde{0}, (i, 0))$  to  $[0, 1) \subset R$ . This homeomorphism can be chosen to be order preserving.*

*Proof:* If this lemma is false, let  $J$  be the set of all indices  $i \in I$  greater than the first index for which such a homeomorphism does not exist. Since  $I$  is well-ordered, there is a smallest element  $j_0 \in J$ .

If  $j_0$  has an immediate predecessor in the index set, call this predecessor  $i_0$ . Then the interval  $[(i_0, 0), (j_0, 0))$  is just the set of all  $(i_0, t)$  for  $t \in [0, 1)$ , and this interval is homeomorphic to  $[1, 2)$ . Moreover,  $i_0 < j_0$ , so we can find a homeomorphism from  $[\tilde{0}, (i_0, 0))$  to  $[0, 1)$ . Gluing these two homeomorphisms together gives a homeomorphism from  $[\tilde{0}, (j_0, 0))$  to  $[0, 2)$ . This set is homeomorphic to  $[0, 1)$ , so we obtain a homeomorphism from  $[\tilde{0}, (j_0, 0))$  to  $[0, 1)$ , contradicting the definition of  $j_0$ .

Thus  $j_0$  must be a limiting index with no immediate predecessor. For example, in the ordering

$$\{1_1, 2_1, 3_1, \dots, 1_2, 2_2, 3_2, \dots, 1_3, 2_3\},$$

the indices  $1_2$  and  $1_3$  are limiting indices.

In this situation, I claim that we can find indices  $i_0 < i_1 < i_2 < \dots$  which “converge to  $j_0$ ” in the sense that the  $i_k$  are all smaller than  $j_0$  and whenever  $i$  is an index smaller than  $j_0$

then there is an  $i_n$  such that  $i < i_n$ . We can also insure that  $i_0$  is the first element of the index set. Suppose all of this is possible for a moment.

Since  $j_0$  is the smallest index for which no appropriate homeomorphism exists, we can find a homeomorphism from  $[\tilde{0}, (i_{k+1}, 0))$  to  $[0, 1)$ . This homeomorphism maps the element  $(i_k, 0)$  to a point  $A \in [0, 1)$ , so the homeomorphism maps  $[(i_k, 0), (i_{k+1}, 0))$  to  $[A, 1)$ .

Rescale these homeomorphisms so  $[\tilde{0}, (i_1, 0))$  maps to  $[0, 1/2)$  and  $[(i_1, 0), (i_2, 0))$  maps to  $[1/2, 3/4)$  and  $[(i_2, 0), (i_3, 0))$  maps to  $[3/4, 7/8)$ , etc. Glue these homeomorphisms together to get a homeomorphism from  $\cup[(i_k, 0), (i_{k+1}, 0)) \rightarrow [0, 1)$ . Since every index  $i < j_0$  is eventually  $i < i_n$ , this union is exactly  $[\tilde{0}, (j_0, 0))$ , so this last interval in the long ray is homeomorphic to  $[0, 1)$ , contradicting the definition of  $j_0$ .

We conclude that  $j_0$  cannot exist and thus that the lemma is true.

To finish the proof, we must show that the sequence  $i_0 < i_1 < i_2 < \dots$  exists. In our situation, the set of all indices smaller than  $j_0$  is countable. List these indices as  $k_1, k_2, k_3, \dots$ ; in this list, the elements are no longer ordered. Choose  $i_0$  to be the smallest element of the index set. Choose  $i_1$  by starting at  $k_1$  and picking the first index larger than  $i_0$ . Choose  $i_2$  by picking the first element of the  $k$ -list larger than  $i_1$ . Etc. Such elements always exist because  $j_0$  is a limit index which must be preceded by infinitely many elements. Since the  $k$ -list contains all indices smaller than  $j_0$ , eventually  $i_n$  will be larger than any particular index  $i$  in the list.

This completes the argument.