Assignment 7; Due Friday, November 11

Read section nine on connected spaces, and the beginning of section ten on pancake problems. Then do the following problems:

- 9.8 abd
- 9.8 ef (graduate students only)
- 9.8 h
- 9.8 i (graduate students only)
- 10.7 a
- Let $p \in X$. Prove that the union of all connected subsets of X containing p is itself connected. This union is thus the *largest* connected subset of X containing p. It is called the *connected component* of p.
- If $p, q \in X$, prove that the connected component of p and the connected component of q are either equal or else disjoint. Conclude that X can be written uniquely as a disjoint union of connected components.
- What are the connected components of $\{(x, y) \in R^2 \mid y \neq 0\}$? What are the connected components of Q?
- (graduate students only) Show that each connected component is closed. Show by example that connected components may or may not be open.
- (graduate students only) Let $O(n)$ be the set of all linear transformations

  $$A : R^n \to R^n$$

  which preserve distance, so $||Av|| = ||v||$. This set is a group, the *orthogonal group*.

  Denote the standard dot product on R^n by $< v, w >$. Prove that $||v + w||^2 - ||v||^2 - ||w||^2 = 2 < v, w >$. Conclude that a linear transformation A is in $O(n)$ if and only if $< Av, Aw >=< v, w >$ for all vectors v and w. Using this result, prove that a matrix A represents an element of $O(n)$ if and only if $A^t A = I$.

  Give $O(n)$ a topology by noticing that each $n \times n$ matrix has n^2 components, so $O(n) \subseteq R^{n^2}$. Prove that $O(n)$ is compact.

  Find the connected components of $O(n)$. I'll give two hints:

  Suppose $A \subseteq X$ and suppose that whenever p and q are points in A, there is a continuous map $\gamma : [0, 1] \to A$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Then A is connected. This result is easy to prove.

You may use without proof the following result from linear algebra: If $A \in O(n)$, there exist matrices B and C in $O(n)$ such that $A = BCB^{-1}$ where C has 1×1 and 2×2 blocks on the diagonal and is otherwise zero, and the 1×1 blocks are (± 1) and the 2×2 blocks are

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$