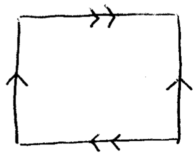
# Assignment 4; Due Friday, October 21

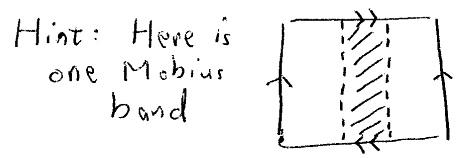
Read sections five and six on the quotient topology and on product spaces. The first of these sections is very important. Then do the following problems:

- 5.3bc
- 5.3e
- 5.3f This shows that projective space can be mapped continuously in a one-to-one manner into $R^4$.

  Perhaps you have already seen a little topology. In that case, you may have studied compact spaces and Hausdorff spaces. An easy theorem states that if $X \to Y$ is continuous, one-to-one, and onto, then the inverse is automatically continuous if $X$ is compact and $Y$ is Hausdorff. We will prove that $RP^2$ is compact. Any subspace of $R^n$ is Hausdorff. So it follows that your map is a homeomorphism from $RP^2$ to a subspace of $R^4$.
- 5.3h (graduate students only)
- 5.4b
- 6.2a
- 6.2b (graduate students only)
- 6.2d
- 6.6h
- Also do four additional problems. Here is the first. Let $M$ be a Mobius band. Notice that the boundary of this band consists of a single circle. Glue two Mobius bands together along their common circle. Prove that the result is homeomorphic to a Klein bottle.

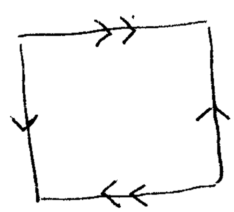
The book discusses this result. Please prove the result by considering the picture of the Klein bottle below from the first day of class. Show that this square contains two almost disjoint Mobius bands, and show that their intersection is just a circle where they are glued together.






- Repeat this problem for projective space. Notice that the boundary of a Mobius band is a circle and the boundary of a disk is also a circle. Glue a Mobius band to a disk along this circle. Show that the result is projective space.

  Please do this problem using the following picture. If we glue as indicated in the diagram, we obtain projective space. Show that this square contains almost disjoint subsets, one homeomorphic to a Mobius band and one homeomorphic to a disk. Show that the intersection is a circle along which these objects are glued together.



- This is an exercise for graduate students only. No doubt you know about $G/H$ where $G$ is a group and $H$ is a subgroup (not necessarily normal). Notice that $G/H$ consists of equivalence classes where $g_1 \sim g_2$ if $g_1^{-1} g_2 \in H$; equivalently $G/H$ consists of left cosets $gH$ which are either the same or else disjoint.

  Consider the special case when $G = SO(3)$, the group of rotations of $R^3$, and $H = SO(2)$, the group of rotations of $R^2$. We can consider $SO(2) \subseteq SO(3)$ by letting the rotations in $SO(2)$ act trivially on the $z$-axis. Said another way, we identify the two dimensional rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with the three dimensional rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Each element of $SO(3)$ is a $3 \times 3$ matrix. We can think of the nine entries of this matrix as the coefficients of a point in $R^9$. Give $SO(3)$ the induced topology.

Notice that $H$ cosets are equivalence classes in $SO(3)$. Consequently we can give $G/H$ the quotient topology from the map $SO(3) \to SO(3)/SO(2)$. The purpose of this exercise is to identify this quotient space with a familiar geometrical object.

Map $R^9$ to $R^3$ by sending $(a_1, \ldots, a_9)$ to the $3 \times 3$ matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

and then mapping $A$ to $A(e_3)$ where $e_3$ is the unit vector in the $z$-direction. This map can be written explicitly and is clearly continuous.

Explain why this map sends $SO(3)$ to $S^2$, the unit sphere inside $R^3$. Give both of these objects the induced topology. Explain why the resulting map is continuous. (This should involve general abstract nonsense about the induced topology. If you start talking about $\epsilon$ and $\delta$, you've missed the point.)

Explain why the above map is constant on $H$ cosets. Explain why the above map induces a map $G/H \to S^2$. Explain why this map is continuous. (Again, this last result should follow by abstract nonsense about the quotient topology.) Finally show that this map $G/H \to S^2$ is one-to-one and onto.

*Remark:* Read the comment on page one about 5.3f. One can prove that $SO(3)$ is a compact subset of $R^9$. It will then follow from results in the next two weeks that the map $G/H \to S^2$ is a homeomorphism.

- Last week as discussed two disks joined by three twisted strips. In the solution to this problem on the web, I point out that the resulting object looks like a doughnut with a small disk removed.

  Consider the corresponding object with just one twisted strip rather than three. Show that this object is homeomorphic to a sphere with a small disk removed. Consider the object with just two twisted strips rather than three. Show that this object is homeomorphic to a sphere with two small disks removed.

  Finally, consider the corresponding object with four twisted strips rather than three. Show that this object is homeomorphic to a doughnut with two small disks removed.