The Pentagonal Number Theorem and All That

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1 Introduction

When I was a sophomore, my advanced calculus midterm was given in Memorial Hall, Harvard's grotesque memorial to students who died in the Civil War. The interior of the building is a room shaped like a cathedral, with a stained glass window at the front. But where the pews should be there are instead gigantic tables. College midterms are given in that room, with a table reserved for each class. Over there — philosophy, and there — German, and at our table — advanced calculus.

Our exam had four questions. At the end of the hour I realized that I was not going to answer any question. The saving grace was that the other students had the same problem. At the next class meeting, our professor David Widder said "You don't know anything! You don't even know the series expansion of $\frac{1}{1-x}$."

Since then, I have known that series cold. Wake me from sleep and I can recite it.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

I'm giving this lecture in thanks to David Widder.

2 Euler

Newton and Leibniz invented calculus in the seventeenth century, and their immediate successors the Bernoulli's worked at the start of the eighteenth century. But most of that century was dominated by a single man, Leonard Euler. Euler laid the foundations for mathematics of later centuries, and he is remembered with particular fondness as the master of beautiful formulas. Today I'm going to show you one of his most spectacular discoveries.

3 Partitions

A *partition* of a number n is a representation of n as a sum of positive integers. Order does not matter. For instance, there are 5 partitions of 4: 4, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$.

Let p_n be the number of partitions of n. Easily, $p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 5$, $p_5 = 7$. Unfortunately, there is no formula for p_n, and just writing the possibilities down and counting is not a good idea because p_n gets large fast: there are almost four trillion partitions of 200.

Euler discovered, however, an indirect way to compute p_n. I'll describe his method, and use it to show that $p_{1000} = 24,061,467,864,032,622,473,692,149,727,991$.

4 Euler's First Formula

Euler's technique proceeds in two steps. The first allows us to compute the p_n, but slowly. The second dramatically speeds up the process. Here is his first formula:

Theorem 1

$$(1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)(\ldots) = \sum p_n\, x^n$$

Proof: Recall the distributive law, which I like to call the "Chinese menu formula": to multiply $(a+b)(c+d)$, choose one of a and b from column A, and one of c and d from column B and multiply them, and then you add up all the possibilities, giving $ac+ad+bc+bd$.

This works for more complicated products as well. To compute the product

$$(1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)(\ldots)$$

we choose x^{k_1} from column A, x^{2k_2} from column B, x^{3k_3} from column C, etc., and multiply them to obtain $x^{k_1+2k_2+3k_3+\ldots+sk_s}$, and then add up the possibilities to get

$$\sum_{k_1,k_2,\ldots} x^{k_1+2k_2+3k_3+\ldots+sk_s}$$

The result is a sum of powers of x,

$$\sum_{k_1,k_2,\ldots} x^n$$

but the term x^n will occur as many times as we can write $n = k_1 + 2k_2 + 3k_3 + \ldots + sk_s$. However, such an expression is just a fancy way to write n as a partition, namely as a sum of k_1 1's, k_2 2's, k_3 3's, etc. So the final sum is $\sum p_n\, x^n$. QED.

Euler's first formula describes a way to organize a computation of p_n. This method can also be described in a manner that doesn't use algebra. Notice that the term $1+x+x^2+\ldots$ in Euler's product counts partitions containing only 1; each integer can be written as such a sum in only one way. The product

$$(1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)$$

counts partitions containing only 1's and 2's and thus equals

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + \ldots$$

because 0 and 1 have no extra partitions with 2's, 2 and 3 have one additional partition with 2's, 4 and 5 have two additional partitions with 2's, etc. (It is useful to think of zero as having exactly one partition, the partition with no 1's, no 2's, no 3's, etc. So we sometimes write $p_0 = 1$.)

In a similar way we can count partitions using 1's, 2's, and 3's, and then partitions using 1's, 2's, 3's, and 4's, and so forth. Let's consider one of these cases in detail. It turns out that

$$(1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)(1 + x^4 + x^8 + \ldots) =$$

$$1+x+2x^3+3x^3+5x^4+6x^5+9x^6+11x^7+15x^8+18x^9+23x^{10}+27x^{11}+34x^{12}+39x^{13}+\ldots$$

where the coefficient of x^n counts partitions of n containing only 1's, 2's, 3's, and 4's. Suppose we now want to count partitions of n containing only 1's, 2's, 3's, 4's, and 5's. Let's concentrate on the case $n = 13$. From the above product we see that there are 39 such partitions with no 5's. A partition containing exactly one 5 will contain 1's, 2's, 3's and 4's adding up to $13 - 5 = 8$, and the above product shows that there are 15 such partitions. A partition of 13 containing exactly two 5's will contain 1's, 2's, 3's, and 4's adding up to $13 - 5 - 5 = 3$ and there are 3 such partitions. There are no partitions of 13 with three or more 5's. So the total number of partitions of 13 with 1's, 2's, 3's, 4's, and 5's is $39 + 15 + 3 = 57$.

Euler's first formula is just a fancy way to summarize this technique. To count all partitions of n, first count all partitions of all $k <= n$ containing only 1's, and then count all partitions of all $k <= n$ containing only 1's and 2's, and then count all partitions of all $k <= n$ containing only 1's, 2's, and 3's, and continue in this way up to partitions containing only 1's, 2's, 3's, $\ldots$, n's. If $a_0, a_1, a_2, \ldots$ are counts of partitions with summands less than t, then the $b_0, b_1, b_2, \ldots$ counting partitions with summands less than or equal to t are given by the formula

$$b_k = a_k + a_{k-t} + a_{k-2t} + \ldots$$

5 For Programmers

If you are a programmer, you'll understand this much better by writing a program to do the calculation. Almost any language will do, but you have to remember that the numbers will get large and might overflow.. I'll write a program in *Mathematica* because I have it handy and it can deal with arbitrarily large integers.

```
F[limit_] :=
Block[{N, f, i, j, k, list1, list2},(* local variables *)
N = limit + 1;(* number of series coefficients *)
f[s_] := 1;
list1 = Array[f, N];(* series coefficients *)
list2 = Array[f, N]; (* fill initial list with 1's *)

k = 2;
While[k <= limit,
     Print[" "];   Print["Partitians using 1 through ", k];
     For[i = 1, i <= N, i++,
          sum = list1[[i]];
          For[j = i - k, j > 0, j = j - k,
               sum = sum + list1[[j]];];
               list2[[i]] = sum;
               ];
     For[i = 1, i <= N, i++, list1[[i]] = list2[[i]];];
     For[i = 1, i <= N, i++, Print[i  - 1, ":  ", list2[[i]]]];
     k++;
     ];
];
```

This program takes 50 seconds to compute the first fifty values of p_n. In particular, $p_{50} = 204,226$.

However, Euler discovered a much faster method. It takes my computer a little over one second to compute the first fifty values of p_n with Euler's second method. I'll explain his method in the next three sections.

6 Dealing with the Analysts

You old calculus teacher is probably whispering in your ear about convergence, rigor, and all that. We're going to tell the analysts to shut up by defining their objection away.

Definition 1 *Let*

$$\mathcal{U} = \{ \ 1 + a_1x + a_2x^2 + \dots \ | \ a_i \in Z \ \}$$

Define a product on this set by writing

$$(1 + a_1x + a_2x^2 + \dots)(1 + b_1x + b_2x^2 + \dots) = 1 + c_1x + c_2c^2 + \dots$$

where $c_k = a_k + a_{k-1}b_1 + a_{k-2}b_2 + \dots + b_k$*.*

Remark: Thus an element of $\mathcal{U}$ is a formal power series, and no convergence is required. This multiplication is well-defined because we can compute any particular element of the product in a finite amount of time. I need to warn you that $\mathcal{U}$ is my personal notation, not something any mathematician would recognize. To me, $\mathcal{U}$ stands for "units."

Theorem 2 *The set* $\mathcal{U}$ *with this product is a group.*

Proof: Only inverses are unclear. At first sight, it seems ridiculous to suppose that $\mathcal{U}$ is a group because

$$\frac{1}{1 + a_1x + a_2x^2 + a_3x^3 + \dots}$$

isn't itself a power series. But you have to remember what the axiom really says. It says the series $1 + a_1x + a_2x^2 + a_3x^3 + \dots$ has an inverse

$$1 + b_1x + b_2x^2 + b_3x^3 + \dots$$

such that the product of the two series is the identity:

$$(1 + a_1x + a_2x^2 + a_3x^3 + \dots)(1 + b_1x + b_2x^2 + b_3x^3 + \dots) = 1$$

Computing this product, we want to find b_i such that

$$1 + (a_1 + b_1)x + (a_2 + a_1b_1 + b_2)x^2 + (a_3 + a_2b_1 + a_1b_2 + b_3) + \dots = 1$$

Setting each coefficient of x^i to zero, we get a series of equations for the b_i, which have a unique inductive solution:

$$b_1 = -a_1$$

$$b_2 = -a_1b_1 - a_2$$

$$b_3 = -a_1b_2 - a_2b_1 - a_3$$

$$\dots$$

QED.

Remark: In particular, $\frac{1}{1-x} = 1 + x + x^2 + \dots$, as I learned from David Widder. Indeed

$$(1 - x)(1 + x + x^2 + x^3 + \dots) = (1 - x) + (x - x^2) + (x^2 - x^3) + (x^3 - x^4) + \dots = 1$$

7 Euler's Second Formula

Theorem 3

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\dots} = \sum p_n\, x^n$$

Proof: This follows immediately from Euler's first formula by taking inverses of the series on the left side.

8 The Pentagonal Number Theorem

After writing this formula, Euler multiplied out the denominator by hand, hoping to find a pattern. One of my sources says he multiplied the first fifty terms, while another says he multiplied as many as one hundred terms.. Amazingly, he found that

Theorem 4 (The Pentagonal Number Theorem)

$$(1-x)(1-x^2)(1-x^3)\dots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \dots$$

Notice that pairs of terms with minus signs alternate with pairs with positive signs. Notice that the intervals between the exponents of pairs with the same sign increase by one, then two, then three, etc. Notice that the intervals between exponents of pairs with opposite signs increase by three, then five, then seven, etc.

The Pentagonal Number Theorem leads to a rapid method of computing the partition numbers. Indeed rewriting theorem 4 using theorem 3 gives

$$(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots)(1 + p_1 x + p_2 x^2 + p_3 x^3 + \dots) = 1$$

Consequently the coefficient of x^n in the product is zero, and so

$$p_n - p_{n-1} - p_{n-2} + p_{n-5} + p_{n-7} - p_{n-12} - p_{n-15} + \dots = 0$$

Each of these expressions is a finite sum because $p_0 = 1$ and $p_k = 0$ for negative k by definition. These formulas then allow us to compute the p_n inductively starting with the value $p_0 = 1$. Thus $p_1 - p_0 = 0$, so $p_1 = 1$. Then $p_2 - p_1 - p_0 = 0$, so $p_2 = 2$. Etc.

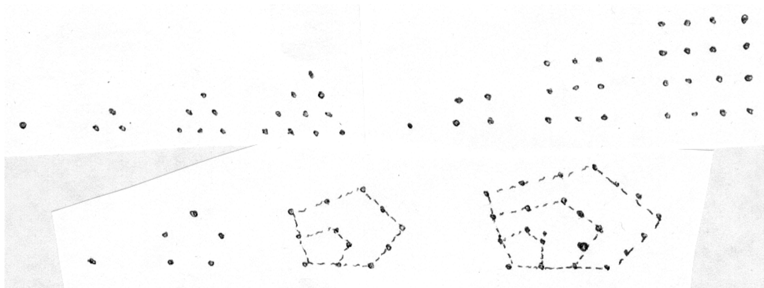
Using this revised formula, it takes my computer only a second to find the first 50 values of p_n and only 50 seconds to find the first 1000 values. And indeed, as promised the computer gives $p_{1000} = 24,061467,864,032,622,473,692,149,727,991$.

For the record, I'll show the Mathematica program I used to do this computation. You can easily rewrite this program in your favorite language, but keep in mind that the integers computed by the program will be very large.

```
FPentagonal[limit_] :=
Block[{N, f, P, Pinverse, k, index, n,  i},(* local variables *)
N = limit + 1; (* number of series coefficients *)
f[s_] := 0;
P = Array[f, N];(* the p(n), initially filled with zeros *)
P[[1]] = 1; (* p(0) = p(1) = 1; arrays in Mathematica are one-based *)
P[[2]] = 1;
Pinverse = Array[f, N];(* inverse of p(n) series *)
k = 1;(* now fill in Pinverse using the Pentagonal Number Theorem *)
index = k (3 k - 1) / 2;
While[index <= N,
     Pinverse[[index]] = (-1)^k;
     index = k (3 k + 1) / 2;
     If[ index <= N,  Pinverse[[index]] = (-1)^k];
     k = k + 1;
     index = k (3 k - 1) / 2;
     ];
For[n = 2, n < N, n = n + 1,(* compute p(n) as inverse of Pinverse *)
     P[[n ]] = 0;
     For[i = 1, i < n, i = i + 1,
          P[[n]] = P[[n]] -  P[[n - i]] Pinverse[[i]]
          ];
     P[[n]] = P[[n]] - Pinverse[[n]];
     Print[n , ":  ", P[[n]]];
     ]
]
```

I don't know how many values of p_n were computed by Euler. In 1918, MacMahon in England computed the first 200 values of p_n. This table was extended to 600 by Gupta in 1935, and to 1000 by Gupta, Gwyther and Miller in 1958. I don't know if a computer was used for this final table.

Why is the theorem called "the pentagonal number theorem"? The reason isn't very interesting mathematically, but here it is. Exponents in the pentagonal series with nonzero coefficients have the form $n = \frac{k(3k-1)}{2}$ and $n = \frac{k(3k+1)}{2}$. The numbers $\frac{k(3k-1)}{2}$ are called "pentagonal numbers" because they count the number of dots in a pentagonal pattern, just as the numbers $n = \frac{k(k+1)}{2}$ and $n = k^2$ are "triangular numbers" and "square numbers" because they count dots in triangular and square patterns.



9 Euler and Proofs of the Pentagonal Number Theorem

Jordan Bell wrote an interesting paper on the history of the Pentagonal Number Theorem. It can be found at `http://arxiv.org/pdf/math/0510054v2`.

The first mention of the theorem is in a letter from Daniel Bernoulli to Euler on January 28, 1741. Bernoulli is replying to a (lost) letter from Euler about the expansion, and he writes "The other problem, to transform $(1-x)(1-x^2)(\ldots)$ into $1-x-x^2+x^5+\ldots$, follows easily by induction, if one multiplied many factors. The remainder of the series I do not see. This can be shown in a most pleasant investigation, together with tranquil pastime and the endurance of pertinacious labor, all three of which I lack."

Euler mentions the theorem many more times over the next few years, in letters we do possess to Niklaus Bernoulli, Christian Goldbach, d'Alembert, and others, and in the first publication of 1751. (This paper was written on April 6, 1741 and had no proof. Euler wrote so many papers that the publishers fell dramatically behind; they were publishing new papers many years after his death.) A typical entry, from a letter to Goldbach, reads "If these factors $(1-n)(1-n^2)(1-n^3)$ etc. are multiplied out onto infinity, the following series $1-n-n^2+n^5+n^7-$ etc is produced. I have however not yet found a method by which I could prove the identity of these two expressions. The Hr. Prof. Niklaus Bernoulli has also been able to prove nothing beyond induction." Here the word "induction" means "by experiment" rather than "a proof by induction".

Euler is not above a little trickery. Learning that d'Alembert wanted to leave mathematical research to regain his health, he wrote him "If in your spare time you should wish to do some research which does not require much effort, I will take the liberty to propose the expression $(1-x)(1-x^2)(1-x^3)(1-x^4)$ etc., which upon expansion by multiplication gives the series $1-x-x^2+x^5+x^7-$ etc., which would seem very remarkable to me because of the law which we easily discover within it, but I do not see how his law may be deduced without induction of the proposed expression." Eventually d'Alembert wrote back "regarding the series of which you have spoken, it is very peculiar, but I only see induction to show it. But no one is deeper and better versed on such matters than you."

Euler finally was able to prove the theorem on June 9, 1750, in a letter to Goldbach. His proof is algebraic. The proof was first published in 1760, and Euler gives more details about points which were vague in his letter to Goldbach. You can consult Bell's paper if you want to follow this original Euler proof.

10 Franklin's Proof

In 1881, the American mathematician Franklin gave a proof which involves no algebra at all. Hans Rademacher called this proof "the first major achievement of American mathematics." Here is Franklin's proof:

Proof: The basic idea is that the series $(1-x)(1-x^2)(1-x^3)(\ldots)$ can be interpreted as a sophisticated count of a certain restricted type of partitions. Let us begin with the following formula, which I've obtained inductively by multiplying out terms:

$$(1+x)(1+x^2)(1+x^3)(\ldots) = 1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+12x^{11}+\ldots$$

Looking back at the proof of theorem 1, we see that the product on the left is equal to

$$\sum_{k_1,k_2,\ldots} x^{k_1+2k_2+3k_3+\ldots+sk_s}$$

but this time each k_i is either zero or one. This means that in the partition $n = k_1 + 2k_2 + 3k_3 + \ldots + sk_s$, the number of 1's is either zero or one, the number of 2's is either zero or one, the number of 3's is either zero or one, etc. Thus the product is equal to

$$\sum q_n x^n$$

where q_n counts partitions of n as a sum of *distinct* positive numbers. For example, the coefficient of x^7 is 5 because there are only five partitions of 7 with distinct factors, namely 7, 6 + 1, 5 + 2, 4 + 3, and 4 + 2 + 1.

We aren't quite interested in this series, but instead in

$$(1-x)(1-x^2)(1-x^3)(\ldots) = \sum_{k_1,k_2,\ldots} (-1)^{k_1+k_2+k_3+\ldots+k_s} x^{k_1+2k_2+3k_3+\ldots+sk_s}$$

So this time when we count partitions with distinct summands, we count partitions with an even number of terms *positively*, but partitions with an odd number of terms *negatively*. The coefficient of x^n is thus "the number of distinct partitions of n with an even number of terms, minus the number of distinct partitions of n with an odd number of terms."

According to the pentagonal number theorem, this product is

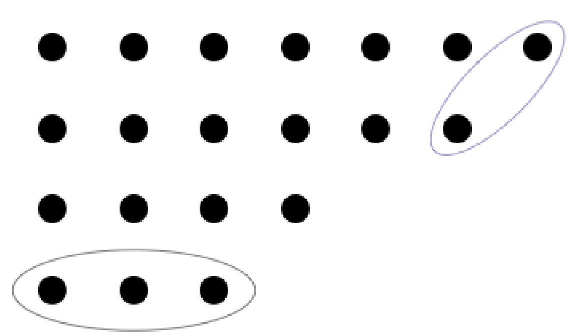
$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \ldots$$

Notice in particular that the coefficient of x^n is usually zero. Franklin concentrated on that fact, and tried to understand why the number of distinct partitions of n with an even number of terms is usually exactly the same as the number of distinct partitions of n with an odd number of terms. And the explanation he gave is that you can pair up each even partition with a corresponding odd partition. For example, there are 12 partitions of 11 into distinct summands, and we will see that the appropriate pairing is

$$\begin{aligned} 10+1 &\leftrightarrow 11 \\ 9+2 &\leftrightarrow 8+2+1 \\ 8+3 &\leftrightarrow 7+3+1 \\ 7+4 &\leftrightarrow 6+4+1 \\ 6+5 &\leftrightarrow 5+4+2 \\ 5+3+2+1 &\leftrightarrow 6+3+2 \end{aligned}$$

How is this pairing defined?

Franklin's trick is exposed in the next picture. You could finish the proof without reading further by thinking carefully about this picture. Draw a distinct partition as a pattern of rows of dots; for instance the picture below corresponds to $20 = 7 + 6 + 4 + 3$. Concentrate on the last row, and on the largest diagonal that can be drawn at the right. The idea is to move the bottom row up to form a new diagonal, or move the diagonal down to form a new row. In the picture below, the bottom row cannot be moved up because that would leave a hanging dot, but the diagonal can be moved down. Notice that moving converts a partition with an even number of terms into a partition with an odd number of terms. In the case illustrated below, it converts $7 + 6 + 4 + 3$ into $6 + 5 + 4 + 3 + 2$.

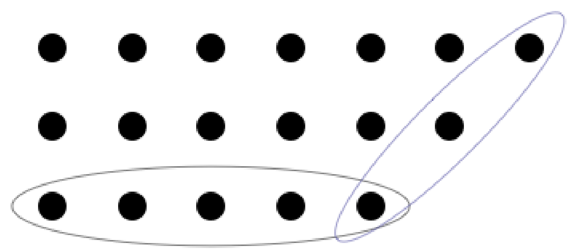


What is the rule for moving? Say the bottom row has a dots and the diagonal on the right has b dots. If we want to move the bottom row up without getting a hanging dot, we need $a \leq b$. If we want to move the diagonal down and get a shorter final row, we need $a > b$.

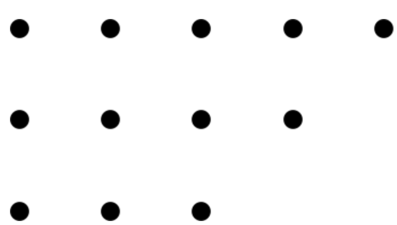
So when $a \leq b$, we can move up, but not down, and when $a > b$ we can move down but not up.

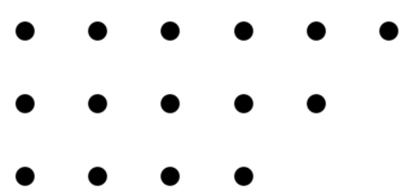
The one legal move for a given diagram produces a new diagram. This diagram also has a unique legal move. But certainly one thing we can do is to reverse the original move and return to the original diagram, so that must be the unique legal move. It follows that our diagrams are paired: the legal move for each leads to the other. Since the number of rows increases or decreases by one, even partitions are paired with odd partitions, as promised.

The only problem with this argument is that it seems to show that *all* of the coefficients of $(1-x)(1-x^2)(1-x^3)(\ldots)$ are zero. The truth is that there are "edge cases" where the analysis just given doesn't quite work. These edge cases occur when the row at the bottom and the diagonal strip on the right side share a common corner, as below.

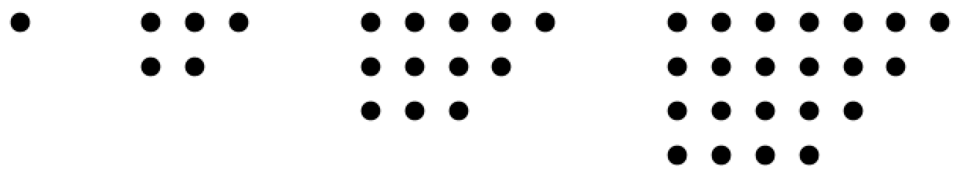


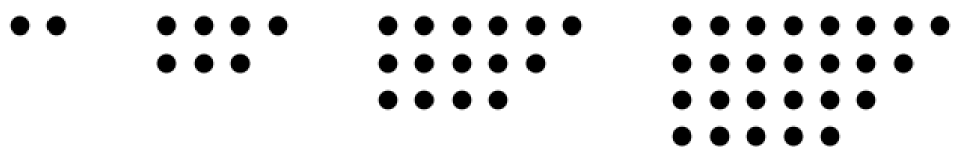
Let us again say that the bottom row has a dots and the diagonal on the right has b dots. If we want to move the bottom row up, then the diagonal length will decrease by one, and so to avoid a hanging dot we need $a \leq (b-1)$. If we want to move the diagonal down and get a shorter final row, then the existing final row length before the motion will decrease by one and we need $(a-1) > b$. So there are two troublesome cases where neither motion is legal: $a = b$ and $a = b+1$. Below are two samples, the first with $a = b$ and the second with $a = b+1$. Notice that neither motion is legal for these diagrams. The left diagram corresponds to an odd partition of 12 and the right corresponds to an odd partition of 15. Note that the pentagonal number theorem expansion contains $-x^{12} - x^{15}$.





The first list below contains diagrams with $a = b$. The second list contains diagrams with $a = b + 1$. Each list extends infinitely to the right. The number of dots in diagrams on the first list is 1, 5, 12, 22, ... and the number of dots in diagrams on the second list is 2, 7, 15, 26, ... and these are exactly the exponents which occur in the pentagonal number expansion! The reader can check that the numbers from the first list equal $\frac{k(3k-1)}{2}$ and the numbers from the second list equal $\frac{k(3k+1)}{2}$. QED.





11 Ramanujan

Do you know the story of Ramanujan?

The following information comes from G. H. Hardy's book on Ramanujan. Ramanujan was born in India, where his father was a clerk in a cloth-merchant's office; all of his relatives were very poor. His mathematical abilities became apparent in school, and by the time he was thirteen he was recognized as a quite abnormal boy. In 1903 he passed the Matriculation Examination of the University of Madras and won the Subrahmanyam scholaship. But then he concentrated so heavily on mathematics that he failed to secure promotion to the senior class and his scholarship was discontinued. In 1906 he appeared as a private student for the F. A. examination, but failed. In 1909 he married, but had no definite occupation until 1912, when he became a clerk in he Port Trust in Madras, at a salary of thirty pounds a year. In 1913 he wrote to G. H. Hardy in Cambridge, listing around one hundred formulas he had discovered, with no proofs. I have listed some of those formulas at the end of these notes. Ramanujan had written to other English mathematicians earlier that year, making no impression. But Hardy, one of the greatest English mathematicians at that time, recognized his genius, and brought him to England. Here are some of Hardy's comments about the formulas in his letter:

> I thought that, as an expert in definite integrals, I could prove 1.6, and did so, though with a good deal more trouble than I had expected. The series formulas 1.1 - 1.4 I found much more intriguing, and it soon became obvious that Ramanujan must possess much more general theorems and was keeping a good deal up his sleeve. The second is a formula well known in the theory of Legendre series, but the others are much harder than they look.
>
> The formulae 1.10 - 1.11 are on a different level and obviously both difficult and deep. Indeed 1.10 - 1.11 defeated me completely; I had never seen anything in the least like them. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.
>
> The last formulae stands apart because it is not right and shows Ramanujan's limitations, but that does not prevent it from being additional evidence of his extraordinary powers. The function in 1.14 is a genuine approximation to the coefficient, thought not at all so close as Ramanujan imagined, and Ramanujan's false statement is one of the most fruitful he ever made, since it ended by leading us to all our joint work on partitions.

Ramanujan is often considered the greatest inventor of beautiful formulas since Euler. But his mathematical training was spotty. So Hardy was faced with the task of filling the mathematical gaps in his education without destroying his natural gifts. Ramanujan died young, in 1920.

Ramanujan was in England when Macmahon computed the first 200 values of p_n using Euler's formula. Looking at this table, Ramanujan noticed some remarkable patterns. I've printed the table of the first 200 values on the last pages. Before reading further, look at the table and see if you can find obvious patterns.

(As a hint, look at cases when p_n is divisible by 5. There are some distracting values when a term not part of the pattern is divisible by 5, and you'll have to ignore the distractions. If you find a pattern, congratulations, but keep looking. Ramanujan found two other patterns using other divisors, plus more subtle patterns I won't discuss.)

Notice that $p_4 = 5$. Ramanujan noticed that every fifth entry in the table after that was also divisible by 5. In modern notation

$$p_{5m+4} \equiv 0 \pmod 5$$

Notice that $p_5 = 7$. Ramanujan also noticed that every seventh entry after that was also divisible by 7. Similarly $p_6 = 11$ and Ramanujan noticed that every eleventh entry after that was also divisible by 11. So

$$p_{7m+5} \equiv 0 \pmod 7$$
$$p_{11m+6} \equiv 0 \pmod{11}$$

But don't get your hopes up. The generalization to 13 fails.

Eventually Ramanujan managed to prove all three observations. The proofs are not easy, and the result for 11 is decidedly harder than the results for 5 and 7. There are now many different proofs, some involving remarkable variations on the pentagonal number theorem. For example, one proof of the mod 5 congruence depends on proving the following amazing formula asserted (but not proved) by Ramanujan:

Theorem 5 *Let*

$$f(x) = (1-x)(1-x^2)(1-x^3)(\dots) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots$$

Then

$$\sum p_{5n+4}\, x^n = 5\, \frac{f(x^5)^5}{f(x)^6}$$

Remark: Notice that $f(x) \in \mathcal{U}$ and $f(x^5) \in \mathcal{U}$. So the quotient on the right is a formal series with integer coefficients. After multiplying by 5, all coefficients are divisible by 5.

Remark: Similarly the mod 7 congruence follows from the following Ramanujan formula:

$$\sum p_{7n+5}\, x^n = 7\, \frac{f(x^7)^3}{f(x)^4} + 49\, x \frac{f(x^7)^7}{f(x)^8}$$

12 The Size of p_n

Hardy and Ramanujan also provided an estimate for the size of p_n. The argument, developed further by Hardy, Littlewood, and Rademacher, uses what is called the *circle method* and involves relating the pentagonal number series with the theory of modular forms.

Theorem 6

$$p_n \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

Examples: When $n = 200$, this approximation is 4.10025×10^{12}, compared to the exact value of 3.97299×10^{12}. When $n = 1000$, the approximation is 2.4402×10^{31} and the actual value is 2.4061×10^{31}. When $n = 10000$, the approximation is 3.6328×10^{106}, far larger than the expected number of atoms in the universe, and p_{10000} is thus a number with around 106 digits.

13 Freeman Dyson

In 1944, an Oxford undergraduate named Freeman Dyson wrote a paper on Ramanujan's congruences. Dyson switched to physics in graduate school and became one of the most important physicists of the twentieth century. A few weeks ago, he gave a talk in Eugene.

Dyson's undergraduate paper sketched an alternate proof of Ramanujan's mod 5 result, although he couldn't prove the central observation he made (it was proved ten years later by Atkin and Swinnerton-Dyer.) Suppose we have a partition $\lambda_1 + \lambda_2 + \ldots + \lambda_k$ with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_5$. Dyson defined the *rank* of the partition to be $\lambda_k - k$. For a fixed n, Dyson divided all partitions of $5n + 4$ into five classes, namely those with rank divisible by 5, those with rank congruent to one modulo 5, those congruent to two modulo 5, ..., and those congruent to four modulo five. Purely empiracally, Dyson noticed that these five classes always have the same number of elements, and consequently the total number of partitions of $5n + 4$ must be divisible by five. The same empirical method worked for Ramanujan's mod 7 result.

But curiously, Dyson noticed that his method did *not* work in the mod 11 case. So he conjectured in his paper that there was a better function from partitions to integers, replacing the rank, and working for all of 5, 7, and 11. Dyson named this unknown function *the crank*, and he wrote "Whatever the final verdict of posterity may be, I believe the 'crank' is unique among arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!" Forty years later, Andrews and Garvan discovered just such a crank function.

14 Ken Ono

John Leahy is the Oregon mathematician who represented the University of Oregon at Bend when Cascade College was founded. John and George Andrews (one of the discoverers of the crank) were fellow graduate students at the University of Pennsylvania in the 1960's. Andrews later became one of the world's experts on Ramanujan. When Andrews visited John in Eugene, he knew fascinating stories about Ramanujan and his notebooks.

I met John at Penn, and we knew a faculty member there named Takashi Ono. Ono's son, Ken Ono, also became a mathematician; I am unhappy to report that Ken hadn't been born when John and I were in Philadelphia. Ken left college early to race bicycles as a member of the Pepsi-Miyata cycling team, and to become a graduate student at Chicago.

For most of the twentieth century, mathematicians working on partitions tried to understand the special significance of the primes 5, 7, and 11. Thus it came as an enormous surprise when Ken Ono proved in 2001 that there are similar congruence relations for *every prime greater than 3*. Indeed, in 2006, Ono proved that whenever N is an integer whose prime factorization does not contain 2 or 3, then there are integers a and b such that

$$p_{am+b} \equiv 0 \pmod{N}$$

for all m. This is by far the greatest advance in the theory since Ramanujan.

The status of the primes 2 and 3 is still unknown. In particular, nobody knows how to predict when p_n is even, or even how to predict an infinite number of cases when p_n is even.

Why didn't Ramanujan discover these additional congruences, and why weren't they found in the century after Ramanujan? You'll understand as soon as I show your the next simplest cases, for the primes 13 and 17:

$$p_{17303m+237} \equiv 0 \pmod{13}$$

$$p_{48037937m+1122838} \equiv 0 \pmod{17}$$

It is just possible that you'd notice that 13 divides p_{237} from the tables. But what are the chances that you'd notice that it also divides $p_{237+17303} = p_{17440}$? And if you noticed that, what are the changes that you'd leap to the conclusion that after another 17303 steps, it would divide $p_{17540+17303}$, and so on forever?

Formulas from Ramanujan's Letter to Hardy

1.1

$$(1 - \frac{3!}{(1!2!)^3}x^2 + \frac{6!}{(2!4!)^3}x^4 - \ldots = \left(1 + \frac{x}{(1!)^3} + \frac{x^2}{(2!)^3} + \ldots\right)\left(1 - \frac{x}{(1!)^3} + \frac{x^2}{(2!)^3} - \ldots\right)$$

1.2

$$1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 - 13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \ldots = \frac{2}{\pi}$$

1.3

$$1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1\cdot 5}{4\cdot 8}\right)^4 + 25\left(\frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}\right)^4 + \ldots = \frac{2^{\frac{3}{2}}}{\pi^{\frac{1}{2}}\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}$$

1.4

$$1 - 5\left(\frac{1}{2}\right)^5 + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^5 - 13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^5 + \ldots = \frac{2}{\{\Gamma\left(\frac{3}{4}\right)\}^4}$$

1.6

$$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2+\ldots} = \frac{\pi}{2\,(1+r+r^3+r^6+r^{10}+\ldots)}$$

1.10

If $u = \frac{x}{1+}\,\frac{x^5}{1+}\,\frac{x^{10}}{1+}\,\frac{x^{15}}{1+\ldots}, v = \frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\,\frac{x^4}{1+\ldots},$ then $v^5 = u\,\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}$

1.11

$$\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+\ldots} = \left\{\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)} - \frac{\sqrt{5}+1}{2}\right\} e^{\frac{2}{5}\pi}$$

1.14

The coefficient of x^n in $(1-2x+2x^4-2x^9+\ldots)^{-1}$ is the integer nearest to

$$\frac{1}{4n}\left(\cosh\pi\sqrt{n} - \frac{\sinh\pi\sqrt{n}}{\pi\sqrt{n}}\right)$$

Partition Table

1: 1	26: 2436	51: 239943	76: 9289091
2: 2	27: 3010	52: 281589	77: 10619863
3: 3	28: 3718	53: 329931	78: 12132164
4: 5	29: 4565	54: 386155	79: 13848650
5: 7	30: 5604	55: 451276	80: 15796476
6: 11	31: 6842	56: 526823	81: 18004327
7: 15	32: 8349	57: 614154	82: 20506255
8: 22	33: 10143	58: 715220	83: 23338469
9: 30	34: 12310	59: 831820	84: 26543660
10: 42	35: 14883	60: 966467	85: 30167357
11: 56	36: 17977	61: 1121505	86: 34262962
12: 77	37: 21637	62: 1300156	87: 38887673
13: 101	38: 26015	63: 1505499	88: 44108109
14: 135	39: 31185	64: 1741630	89: 49995925
15: 176	40: 37338	65: 2012558	90: 56634173
16: 231	41: 44583	66: 2323520	91: 64112359
17: 297	42: 53174	67: 2679689	92: 72533807
18: 385	43: 63261	68: 3087735	93: 82010177
19: 490	44: 75175	69: 3554345	94: 92669720
20: 627	45: 89134	70: 4087968	95: 104651419
21: 792	46: 105558	71: 4697205	96: 118114304
22: 1002	47: 124754	72: 5392783	97: 133230930
23: 1255	48: 147273	73: 6185689	98: 150198136
24: 1575	49: 173525	74: 7089500	99: 169229875
25: 1958	50: 204226	75: 8118264	100: 190569292

101: 214481126	126: 3519222692	151: 45060624582	176: 476715857290
102: 241265379	127: 3913864295	152: 49686288421	177: 522115831195
103: 271248950	128: 4351078600	153: 54770336324	178: 571701605655
104: 304801365	129: 4835271870	154: 60356673280	179: 625846753120
105: 342325709	130: 5371315400	155: 66493182097	180: 684957390936
106: 384276336	131: 5964539504	156: 73232243759	181: 749474411781
107: 431149389	132: 6620830889	157: 80630964769	182: 819876908323
108: 483502844	133: 7346629512	158: 88751778802	183: 896684817527
109: 541946240	134: 8149040695	159: 97662728555	184: 980462880430
110: 607163746	135: 9035836076	160: 107438159466	185: 1071823774337
111: 679903203	136: 10015581680	161: 118159068427	186: 1171432692373
112: 761002156	137: 11097645016	162: 129913904637	187: 1280011042268
113: 851376628	138: 12292341831	163: 142798995930	188: 1398341745571
114: 952050665	139: 13610949895	164: 156919475295	189: 1527273599625
115: 1064144451	140: 15065878135	165: 172389800255	190: 1667727404093
116: 1188908248	141: 16670689208	166: 189334822579	191: 1820701100652
117: 1327710076	142: 18440293320	167: 207890420102	192: 1987276856363
118: 1482074143	143: 20390982757	168: 228204732751	193: 2168627105469
119: 1653668665	144: 22540654445	169: 250438925115	194: 2366022741845
120: 1844349560	145: 24908858009	170: 274768617130	195: 2580840212973
121: 2056148051	146: 27517052599	171: 301384802048	196: 2814570987591
122: 2291320912	147: 30388671978	172: 330495499613	197: 3068829878530
123: 2552338241	148: 33549419497	173: 362326859895	198: 3345365983698
124: 2841940500	149: 37027355200	174: 397125074750	199: 3646072432125
125: 3163127352	150: 40853235313	175: 435157697830	200: 3972999029388