IRREDUCIBLE TENSOR PRODUCTS OVER ALTERNATING GROUPS

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ABSTRACT. We classify irreducible tensor products of modular representations of the alternating group over an algebraically closed field of characteristic $p > 5$.

INTRODUCTION

Let F be an algebraically closed field of characteristic $p \geq 0$, and A_n be the alternating group on n letters. In this paper we study *tensor decomposable* irreducible FA_n-modules, i.e. irreducible modules E which can be written as $E \cong E_1 \otimes E_2$ for *non-trivial* FA_n-modules E_1 and E_2. We refer the reader to [28, 2, 14, 3, 15] for results on the similar problem for the symmetric group Σ_n. For example, the following theorem is part of [3, Main Theorem].

Theorem A. *Let D_1 and D_2 be $F\Sigma_n$-modules of dimensions greater than one. Then $D_1 \otimes D_2$ is reducible unless $p = 2$ and n is even.*

Of course, the same result will follow for FA_n-modules E_1 and E_2, which lift to the symmetric group Σ_n. However, some FA_n-modules do not lift, and we need further investigation to complete the problem.

To describe our main result we first explain how the irreducible FA_n-modules can be parametrised. We start from the symmetric group, referring the reader to [16] for the standard facts on its representation theory. In particular, to every *p-regular* partition λ of n one can associate the irreducible $F\Sigma_n$-module D^λ. Assume that $p > 2$ and denote by $\mathbf{sgn}$ the 1-dimensional sign representation of Σ_n. Then $D^\lambda \otimes \mathbf{sgn}$ is irreducible so there should exist a p-regular partition $\lambda^{\mathbf{M}}$ with

$$D^\lambda \otimes \mathbf{sgn} \cong D^{\lambda^{\mathbf{M}}}. \tag{1}$$

The bijection $\lambda \mapsto \lambda^{\mathbf{M}}$ on the set of p-regular partitions is called the *Mullineux bijection*. This bijection can be described explicitly using a combinatorial algorithm suggested by Mullineux, see [22, 13, 4, 27].

Moreover, it follows easily from Clifford theory (see e.g. [12]) that the restriction $D^\lambda{\downarrow}_{A_n}$ is irreducible if and only if $D^\lambda \otimes \mathbf{sgn} \not\cong D^\lambda$. If this is the case, we denote this irreducible restriction by E^λ. Of course, $E^\lambda \cong E^{\lambda^{\mathbf{M}}}$. On the other hand, if $D^\lambda \otimes \mathbf{sgn} \cong D^\lambda$, then the restriction $D^\lambda{\downarrow}_{A_n}$ splits as a direct sum $E^\lambda_+ \oplus E^\lambda_-$ of two irreducible FA_n-modules. Finally,

$$\{E^\lambda \mid \lambda \neq \lambda^{\mathbf{M}}\} \cup \{E^\lambda_+, E^\lambda_- \mid \lambda = \lambda^{\mathbf{M}}\}$$

is a complete set of irreducible FA_n-modules, and distinct modules L and M from this set are isomorphic if and only if $L \cong E^\lambda$, $M \cong E^{\lambda^{\mathbf{M}}}$ for some p-regular partition λ with $\lambda \neq \lambda^{\mathbf{M}}$. If $\lambda = \lambda^{\mathbf{M}}$ we say that λ is *Mullineux-fixed.*

Gathering together equal parts of λ we can represent it in the form $\lambda = (l_1^{a_1}, l_2^{a_2}, \dots, l_k^{a_k})$ where $l_1 > l_2 > \cdots > l_k > 0$ and all $a_i > 0$. Then the partition λ is called a *Jantzen-Seitz* partition (or JS-partition for short) if it is p-regular and p divides $l_i - l_{i+1} + a_i + a_{i+1}$ for all i with $1 \leq i < k$. These partitions are important because the restriction $D^\lambda{\downarrow}_{\Sigma_{n-1}}$ is irreducible if and only if λ is a JS-partition, cf. [19, 20, 11].

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The number $a_1 + a_2 + \cdots + a_k$ is denoted $h(\lambda)$ and called the *height* of λ.

Now we can state our main result, which describes tensor decomposable FA_n-modules in characteristic $p > 5$. The case $p = 0$ has been treated in [2] (see also [28]), but the cases $p = 2, 3$ and 5 remain open.

Main Theorem. *Let $p > 5$ and E_1, E_2 be FA_n-modules of dimensions greater than one. Then $E_1 \otimes E_2$ is reducible with the only exception of $E^{(n-1,1)} \otimes E^\lambda_\pm$ where $p \not| n$ and λ is a JS-partition with $p \not| h(\lambda)$. In the exceptional case we have $E^{(n-1,1)} \otimes E^\lambda_\pm \cong E^\nu$, where the Young diagram of ν is obtained from that of λ by removing the top removable node and adding the bottom addable one.*

Example. (i) Let $p = 7$. Then $E^{(28,1)} \otimes E^{(15,3,2^5,1)}_\pm \cong E^{(14,3,2^5,1^2)}$.

(ii) To emphasize that the situation is really exceptional for small p, we note using [18, p. 2] that $E^{(3,2)}_+ \otimes E^{(3,2)}_- \cong E^{(4,1)}$ for $p = 2$, and $E^{(4,1^2)}_+ \otimes E^{(4,1^2)}_- \cong E^{(4,2)}$ for $p = 3$. But we believe that the main theorem should still hold for $p = 5$, and, with few exceptions, for $p = 3$. We refer the reader to [14] and [15] for some exceptional phenomena in characteristic 2.

The proof of the Main Theorem is given in section 3. Note that in view of Theorem A, we only need to consider tensor products of the form $E^\lambda \otimes E^\mu_\pm$ and $E^\lambda_\pm \otimes E^\mu_\pm$.

1. Preliminaries

Throughout the paper we assume that $\operatorname{char} F = p > 2$.

Let G be a group. We write $\mathbf{1}_G$ for the trivial FG-module. If M is an FG-module and $D_1, \dots, D_k$ are irreducible FG-modules then the notation $M = D_1 | \dots | D_k$ means that M is a *uniserial* FG-module with composition factors $D_1, \dots, D_k$ counted from bottom to top.

We record two well-known general facts (for the first one see the explanations in [5, 5.1]).

Lemma 1.1. *Let G be a group, M be an FG-module and $H \lhd G$ be a normal subgroup of index $[G : H]$ prime to p. Then* $\operatorname{soc}(M{\downarrow_H}) \cong \operatorname{soc}(M){\downarrow_H}$.

Lemma 1.2. *Let X, Y, Z be FG-modules. Assume that $Z \subseteq X \oplus Y$ and that Z has a simple socle. Then X or Y (or both) contains an isomorphic copy of Z as a submodule.*

Proof. Let $f : Z \to X \oplus Y$ be an embedding, and π_X, π_Y be projections of $X \oplus Y$ to X, Y, respectively. Then $\pi_X \circ f$ or $\pi_Y \circ f$ must be an injection, since otherwise both maps annihilate the simple socle of Z, which is impossible as $\pi_X \circ f + \pi_Y \circ f = f$ is injective. $\square$

Let $\sigma \in \Sigma_n \setminus A_n$. Then $A_n \to A_n$, $g \mapsto \sigma g \sigma^{-1}$ is an (outer) automorphism of A_n. If M is an FA_n-module we can use this automorphism to twist the action of A_n on M: Let

$$g \cdot m := \sigma g \sigma^{-1} m, \quad g \in A_n, \ m \in M. \tag{2}$$

This defines a new FA_n-module denoted by ${}^\sigma M$. The following lemma and corollary follow from Clifford theory and the definitions:

Lemma 1.3. *Let λ be a p-regular partition of n with $\lambda = \lambda^{\mathbf{M}}$. Then ${}^\sigma E^\lambda_\pm \cong E^\lambda_\mp$.*

Corollary 1.4. *Let λ be a p-regular partition of n with $\lambda = \lambda^{\mathbf{M}}$. Then we have isomorphisms of FA_n-modules:*

$$\begin{aligned} {}^\sigma \operatorname{Hom}_F(E^\lambda_\pm, E^\lambda_\pm) &\cong \operatorname{Hom}_F(E^\lambda_\mp, E^\lambda_\mp), \\ {}^\sigma \operatorname{Hom}_F(E^\lambda_\pm, E^\lambda_\mp) &\cong \operatorname{Hom}_F(E^\lambda_\mp, E^\lambda_\pm). \end{aligned}$$

Now we show that partitions λ with few parts are usually not Mullineux-fixed.

Lemma 1.5. [24, 1.9] *Let $n \geq 5$, and λ be a partition of n.*

(i) *If $h(\lambda) \leq 2$ then $h(\lambda^{\mathbf{M}}) \neq h(\lambda)$. In particular, $\lambda \neq \lambda^{\mathbf{M}}$.*

(ii) *If $p > 3$ and $h(\lambda) = 3$ then $\lambda^{\mathbf{M}} \neq \lambda$, except for the cases where $p > 5$ and λ is one of the following* $(3, 1^2)$, $(3, 2, 1)$, $(3^2, 2)$, (3^3).

(iii) *If $p > 5$ and $h(\lambda) = 4$ then $\lambda^{\mathbf{M}} \neq \lambda$, except for the cases where $p > 7$ and λ is one of the following:* $(4, 1^3)$, $(4, 2, 1^2)$, $(4, 3, 2, 1)$, $(4, 3^2, 1)$, $(4^2, 2^2)$, $(4^2, 3, 2)$, $(4^3, 3)$, (4^4).

In the next two lemmas we investigate when 'rectangular' and 'near rectangular' partitions are Mullineux-fixed:

Lemma 1.6. *Let $a < p$ and $\lambda = (l^a)$. Then $\lambda^{\mathbf{M}} = \lambda$ if and only if $l = a$ and $p > 2l - 1$.*

Proof. Note that λ is a p-core if and only if $l + a - 1 < p$. In this case $\lambda^{\mathbf{M}} = \lambda^t$, the transpose partition, see e.g. [6, 4.1] or [8, 2.1], and so $\lambda = \lambda^{\mathbf{M}}$ means $l = a$. If λ is not a p-core then the height of $\lambda^{\mathbf{M}}$ is $p - a \neq a = h(\lambda)$, so $\lambda \neq \lambda^{\mathbf{M}}$, see e.g. [8, 2.4]. $\square$

Lemma 1.7. *Let $a_1, a_2 < p$ and $\lambda = ((l+1)^{a_1}, l^{a_2})$. Then $\lambda^{\mathbf{M}} = \lambda$ if and only if $\lambda = (2, 1)$ and $p > 3$.*

Proof. The proof is similar to that of Lemma 1.6. If λ is a p-core, we must have $\lambda = \lambda^t$ by [6, 4.1] or [8, 2.1], which is only possible if $\lambda = (2, 1)$ and $p > 3$. Now, we may assume that λ is not a p-core. If $h(\lambda) \geq p - 1$, the result follows for example from [1, 2.2]. Otherwise the height of $\lambda^{\mathbf{M}}$ is $p - h(\lambda) \neq h(\lambda)$, so $\lambda \neq \lambda^{\mathbf{M}}$. $\square$

As in [16], we denote by S^λ the Specht module over the symmetric group Σ_n corresponding to a partition λ. By construction, S^λ is a submodule of the permutation module M^λ. The module M^λ is known to be self-dual, so $(S^\lambda)^*$ is naturally a quotient module of M^λ. We are especially interested in two row partitions. For such a partition $(n-k, k)$ denote by $Y^{(n-k,k)}$ the block component of $M^{(n-k,k)}$ containing the Specht module $S^{(n-k,k)} \subseteq M^{(n-k,k)}$. We will use the description of the blocks of the symmetric group known as 'Nakayama's conjecture', see for example [17, 2.7.41, 6.1.21]. Now we describe the submodule structure of $Y^{(n-k,k)}$:

Lemma 1.8. [8, 3.3] *Let $k \geq 0$, $p > k$ and $n \geq 2k$. If there exists l such that $0 \leq l < k$ and $n \equiv k + l - 1 \pmod{p}$ then*

$$\begin{aligned} S^{(n-k,k)} &= D^{(n-l,l)} | D^{(n-k,k)}, \\ Y^{(n-k,k)} &= D^{(n-l,l)} | D^{(n-k,k)} | D^{(n-l,l)}, \text{ and} \\ Y^{(n-k,k)} / D^{(n-l,l)} &\cong (S^{(n-k,k)})^*. \end{aligned}$$

Otherwise $Y^{(n-k,k)} = S^{(n-k,k)} \cong (S^{(n-k,k)})^* \cong D^{(n-k,k)}$.

Lemma 1.8 immediately implies the following

Corollary 1.9. *Let $k \geq 0$, $p > k$ and $n \geq 2k$. If there exists l such that $0 \leq l < k$ and $n \equiv k + l - 1 \pmod{p}$ then*

$$\begin{aligned} &\dim \mathrm{End}_{F\Sigma_n}(Y^{(n-k,k)}) = 2, \\ &\dim \mathrm{Hom}_{F\Sigma_n}(S^{(n-k,k)}, Y^{(n-k,k)}) = 1, \\ &\dim \mathrm{Hom}_{F\Sigma_n}(Y^{(n-k,k)}, (S^{(n-k,k)})^*) = 1, \\ &\dim \mathrm{Hom}_{F\Sigma_n}(S^{(n-k,k)}, (S^{(n-k,k)})^*) = 1. \end{aligned}$$

Otherwise, all four homomorphism spaces above are 1-dimensional.

Corollary 1.10. *The socles of $Y^{(n-k,k)}{\downarrow_{A_n}}$ and $(S^{(n-k,k)}{\downarrow_{A_n}})^*$ are simple.*

Proof. This follows from Lemmas 1.8, 1.1 and 1.5(i). $\square$

Sometimes we will denote $Y^{(n-k,k)}$ by X_1^k and $(S^{(n-k,k)})^*$ by X_2^k. The next result will be used to prove that certain homomorphism spaces are large enough.

Lemma 1.11. *Let $k = k_1 > k_2 > \cdots > k_r \geq 0$ be integers, $p > k$, $n \geq 2k$, and M, N be $F\Sigma_n$-modules. Assume that for some $i_j, i'_j \in \{1, 2\}$ the modules $(X_{i_j}^{k_j})^*$ are quotients of M and the modules $X_{i'_j}^{k_j}$ are submodules of N, $j = 1, 2, \dots, r$. Then* $\dim \operatorname{Hom}_{F\Sigma_n}(M, N) \geq r$.

Proof. For $r = 1$ the claim follows immediately from Corollary 1.9. Assume $r = 2$ and $n \equiv k_1 + k_2 - 1 \pmod p$. Then Lemma 1.8 implies $(X_{i_2}^{k_2})^* \cong D^{(n-k_2,k_2)}$, and $(X_{i_1}^{k_1})^* \cong S := D^{(n-k_2,k_2)} | D^{(n-k_1,k_1)}$ or $(X_{i_1}^{k_1})^* \cong Y := D^{(n-k_2,k_2)} | D^{(n-k_1,k_1)} | D^{(n-k_2,k_2)}$. If $(X_{i_1}^{k_1})^* = S$ then $(X_{i_1}^{k_1})^*$ and $(X_{i_2}^{k_2})^*$ have different simple heads, so $(X_{i_1}^{k_1})^* \oplus (X_{i_2}^{k_2})^*$ is a quotient of M. Otherwise, $(X_{i_1}^{k_1})^* \cong Y$ is a quotient of M. Similarly, either $X_{i'_1}^{k_1} \oplus X_{i'_2}^{k_2}$ or Y is a submodule of N. By Corollary 1.9 again, we conclude that $\dim \operatorname{Hom}_{F\Sigma_n}(M, N) \geq 2$.

Now observe that for any $1 \leq s < t \leq r$ such that $n \not\equiv k_s + k_t - 1 \pmod p$ the modules $X_{i_s}^{k_s}$ and $X_{i_t}^{k_t}$ are in different blocks by virtue of Nakayama's conjecture. So the general case follows from the argument in the special cases considered above. □

Corollary 1.12. *Let $k = k_1 > k_2 > \cdots > k_r \geq 0$ be integers, $p > k$, $n \geq 2k$, and M, N be FA_n-modules. Assume that for some $i_j, i'_j \in \{1, 2\}$ the modules $(X_{i_j}^{k_j}\downarrow_{A_n})^*$ are quotients of M and the modules $X_{i'_j}^{k_j}\downarrow_{A_n}$ are submodules of N, $j = 1, 2, \dots, r$. Then* $\dim \operatorname{Hom}_{FA_n}(M, N) \geq r$.

Proof. Repeat the proof of Lemma 1.11 using Lemma 1.5 and Corollary 1.10. □

The following two results verify some assumptions of Lemma 1.11 for the $F\Sigma_n$-module $\operatorname{End}_F(D^\lambda)$.

Theorem 1.13. [24, 2.3, 2.4] *Let $p > k > 1$, $n \geq 2k$, and λ be a p-regular partition of n satisfying $h(\lambda), h(\lambda^{\mathbf{M}}) \geq k$. Then $Y^{(n-k,k)}$ or $(S^{(n-k,k)})^*$ (or both) is a submodule of the self-dual module* $\operatorname{End}_F(D^\lambda)$.

Proposition 1.14. *Let $p > 5$, $n \geq 8$, and $\lambda = (m, k)$ be a two row partition with $k \geq 2$.*

(i) *If $m \not\equiv k - 2 \pmod p$ and $m > k$ then $Y^{(n-3,3)}$ or $(S^{(n-3,3)})^*$ (or both) is a submodule of* $\operatorname{End}_F(D^\lambda)$.
(ii) *If $m \equiv k - 2 \pmod p$ or $m = k$ then $Y^{(n-4,4)}$ or $(S^{(n-4,4)})^*$ (or both) is a submodule of* $\operatorname{End}_F(D^\lambda)$.

Proof. This follows from [24, 2.4] and [8, 4.12]. □

In section 3 it will be convenient to assume that n is not too small, so we deal with small cases here.

Lemma 1.15. *Let $p > 5$.*

(i) *For $n \leq 9$ the only (non-trivial) irreducible tensor products are $E^{(8,1)} \otimes E_{\pm}^{(3^3)} \cong E^{(4,3,2)}$.*
(ii) *For $10 \leq n \leq 16$ there are no irreducible tensor products of the form $E_{\pm}^{\lambda} \otimes E_{\pm}^{\mu}$.*

Proof. (i) Follows for example from [9, 18], (ii) follows by dimensions using GAP. □

2. Restriction and induction

In this section we deal with results on induction $\operatorname{ind}_{\Sigma_n}^{\Sigma_{n+1}}$ and restriction $\operatorname{res}_{\Sigma_{n-1}}^{\Sigma_n}$. The list of notions defined for example in [22], [7] and used here is as follows.

$\lambda = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid j \leq \lambda_i\}$ is the *Young diagram* of the partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ (we do not distinguish between partitions and their Young diagrams);

$(i, j) \in \mathbb{N} \times \mathbb{N}$ is called a *node*;

$(i, \lambda_i) \in \lambda$ is called a *removable node* (of λ) if $\lambda_i > \lambda_{i+1}$;

$(i, \lambda_i + 1)$ is called an *addable node* (for λ) if $i = 1$ or $i > 1$ and $\lambda_i < \lambda_{i-1}$;

$\lambda_A = \lambda \setminus \{A\} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots)$ is a partition of $n-1$ obtained by removing a removable node $A = (i, \lambda_i)$ from λ;

$\lambda^B = \lambda \cup \{B\} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots)$ is a partition of $n+1$ obtained by adding an addable node $B = (i, \lambda_i + 1)$ to λ;

$\operatorname{res} A = j - i \pmod p$ is the (p)-*residue* of a node $A = (i, j)$.

A removable node A of λ is called *normal* if for every addable node B above A with $\operatorname{res} B = \operatorname{res} A$ there exists a removable node $C(B)$ strictly between A and B with $\operatorname{res} C(B) = \operatorname{res} A$, and such that $B \neq B'$ implies $C(B) \neq C(B')$. A removable node is called *good* if it is the lowest among the normal nodes of a fixed residue. An addable node B is called *conormal* if for every removable node A below B with $\operatorname{res} A = \operatorname{res} B$ there exists an addable node $C(A)$ strictly between B and A with $\operatorname{res} C(A) = \operatorname{res} B$, and such that $A \neq A'$ implies $C(A) \neq C(A')$. An addable node is called *cogood* if it is the highest among the conormal nodes of a fixed residue.

Theorem 2.1. *Let D^λ be an irreducible $F\Sigma_n$-module.*

(i) [20, Theorem D],[21, 0.6]. *The restriction $D^\lambda{\downarrow}_{\Sigma_{n-1}}$ is irreducible if and only if λ is a JS-partition, which is equivalent to the fact that the top node A of λ is its only normal node. In this case $D^\lambda{\downarrow}_{\Sigma_{n-1}} \cong D^{\lambda_A}$.*

(ii) *The socle of $D^\lambda{\downarrow}_{\Sigma_{n-1}}$ (resp. $D^\lambda{\uparrow}^{\Sigma_{n+1}}$) is isomorphic to $\oplus D^{\lambda_A}$ (resp. $\oplus D^{\lambda^B}$) where the sum is over all good (resp. cogood) nodes A (resp. B) of λ.*

(iii) [7, Theorem E(ii)] *The induced module $D^\lambda{\uparrow}^{\Sigma_{n+1}}$ is semisimple if and only if all conormal nodes of λ have different residues. In this case we have $D^\lambda{\uparrow}^{\Sigma_{n+1}} \cong \oplus D^{\lambda^B}$, where the sum is over all conormal nodes B of λ.*

(iv) [7, Theorem E(iv)] *For any addable node B such that λ^B is p-regular,*

$$[D^\lambda{\uparrow}^{\Sigma_{n+1}} : D^{\lambda^B}] = \begin{cases} d_B & \text{if } B \text{ is conormal for } \lambda, \\ 0 & \text{otherwise} \end{cases}$$

where d_B denotes the number of conormal nodes C below B (counting B itself) such that $\operatorname{res} C = \operatorname{res} B$.

(v) [7, Theorem E′(iv)] *For any removable node A such that λ_A is p-regular,*

$$[D^\lambda{\downarrow}_{\Sigma_{n-1}} : D^{\lambda_A}] = \begin{cases} f_A & \text{if } A \text{ is normal for } \lambda, \\ 0 & \text{otherwise} \end{cases}$$

where f_A denotes the number of normal nodes D above A (counting A itself) such that $\operatorname{res} D = \operatorname{res} A$.

(vi) [7, Theorem E(v)] *The dimension* $\dim \operatorname{End}_{F\Sigma_{n+1}}(D^\lambda{\uparrow}^{\Sigma_{n+1}})$ *is equal to the number of conormal nodes for λ.*

The following relation between the Mullineux bijection and (co)good nodes is known (see [22, §4] or [4, 4.12]):

Lemma 2.2. *Let λ be a p-regular partition, and A (resp. B) be a good (resp. cogood) node for λ of residue α. Then there exists a unique good (resp. cogood) node A' (resp. B') for $\lambda^{\mathbf{M}}$ of residue $-\alpha$ such that $(\lambda_A)^{\mathbf{M}} = (\lambda^{\mathbf{M}})_{A'}$ (resp. $(\lambda^B)^{\mathbf{M}} = (\lambda^{\mathbf{M}})^{B'}$).*

Proof. We prove the result for good nodes, the proof for the cogood ones being similar. By Theorem 2.1(ii), the socle of $D^{\lambda^{\mathbf{M}}}{\downarrow}_{\Sigma_{n-1}}$ is isomorphic to $\oplus D^{(\lambda^{\mathbf{M}})_{A'}}$ where the sum is over the good nodes A' of $\lambda^{\mathbf{M}}$. On the other hand, since $D^{\lambda^{\mathbf{M}}} \cong D^\lambda \otimes \mathbf{sgn}$ the same socle is isomorphic to $\oplus D^{(\lambda_A)^{\mathbf{M}}}$ where the sum is over the good nodes A of λ. By [25], the number of nodes in λ of residue α is equal to the number of nodes in $\lambda^{\mathbf{M}}$ of residue $-\alpha$, and the result follows. ∎

We make further remarks on restriction, induction and related combinatorics. The next lemma follows from the intertwining number theorem [10, 44.5], cf. [23, 4.1]:

Lemma 2.3. *Let V be an $F\Sigma_n$-module. Then*

$$\dim \mathrm{End}_{F\Sigma_{n+1}}(V\!\uparrow^{\Sigma_{n+1}}) = \dim \mathrm{End}_{F\Sigma_n}(V) + \dim \mathrm{End}_{F\Sigma_{n-1}}(V\!\downarrow_{\Sigma_{n-1}}).$$

Lemma 2.4. *Let D^λ be an irreducible $F\Sigma_n$-module. Then*
(i) *$D^\lambda\!\uparrow^{\Sigma_{n+1}}$ is reducible;*
(ii) *$D^\lambda\!\uparrow^{\Sigma_{n+1}}$ has at least three composition factors unless $D^\lambda\!\downarrow_{\Sigma_{n-1}}$ is irreducible.*

Proof. This follows from Lemma 2.3, using Schur's lemma and self-duality of irreducible modules over symmetric groups. $\square$

The following fact is proved in [24] using modular branching rules:

Proposition 2.5. [24, 3.6] *Let $\lambda = \lambda^{\mathbf{M}}$ be a JS-partition. Then $D^\lambda\!\downarrow_{\Sigma_{n-2}} \cong D^\nu \oplus D^{\nu^{\mathbf{M}}}$ for some $\nu \neq \nu^{\mathbf{M}}$.*

Corollary 2.6. *Let $\lambda = \lambda^{\mathbf{M}}$ be a JS-partition, and A be the top removable node of λ. Then* $\dim \mathrm{End}_{F\Sigma_n}((D^\lambda\!\downarrow_{\Sigma_{n-1}})\!\uparrow^{\Sigma_n}) = 3$. *In particular, λ_A has exactly three conormal nodes.*

Proof. For the first claim apply Lemma 2.3 to the $F\Sigma_{n-1}$-module $V = D^\lambda\!\downarrow_{\Sigma_{n-1}}$ using Proposition 2.5. The second claim follows from the first and Theorem 2.1(i)(vi). $\square$

The next observation on conormal nodes follows immediately from the definitions.

Lemma 2.7. *Let λ be a p-regular partition of n. The two bottom addable nodes of λ are conormal.*

Lemma 2.8. *Let λ be a JS-partition, and A be the top removable node of λ. Then A is a conormal addable node for the partition λ_A.*

Proof. Let $A = A_1, A_2, \dots, A_l$ (resp. $B_1, B_2, \dots, B_{l+1}$) be the removable (resp. addable) nodes of λ counted from top to bottom. By definition of JS-partitions we have $\mathrm{res}\, A_i = \mathrm{res}\, B_{i-1}$ for $i = 2, 3, \dots l$. Moreover, $\mathrm{res}\, A_1 \neq \mathrm{res}\, B_1 = \mathrm{res}\, A_2$, as λ is p-regular. Now the result follows from the definition of conormal. $\square$

Now we prove our main result on conormal nodes of Mullineux-fixed JS-partitions.

Lemma 2.9. *Let $n > 3$, $\lambda = \lambda^{\mathbf{M}}$ be a Mullineux-fixed JS-partition, A be the top removable node of λ, and B,C be the two bottom addable nodes of λ. Then* $\mathrm{res}\, A = 0$, $\mathrm{res}\, B = -\mathrm{res}\, C$, *and A, B, C are the only three conormal nodes for λ_A.*

Proof. Assume for certainty that B is below C. By Theorem 2.1(i), A is the only normal and hence the only good node of λ. Since $\lambda = \lambda^{\mathbf{M}}$, Lemma 2.2 implies $\mathrm{res}\, A = 0$. By Lemma 2.7, B and C are conormal nodes for λ. Moreover, it follows from Lemma 2.3 and Theorem 2.1(vi) that B and C are the only two conormal nodes for λ. If B and C have distinct residues then they are cogood and $\mathrm{res}\, B = -\mathrm{res}\, C$ by virtue of Lemma 2.2. Otherwise C is the only cogood node of λ, and this time Lemma 2.2 implies $\mathrm{res}\, C = 0$. Thus $\mathrm{res}\, B = \mathrm{res}\, C = 0$, and so $\mathrm{res}\, B = -\mathrm{res}\, C$ anyway.

By Corollary 2.6 and Theorem 2.1(vi), λ_A has exactly three conormal nodes. Since A is one of them by Lemma 2.8, it remains to prove that B and C are conormal addable nodes for λ_A. Observe that if λ has at least three removable nodes then B and C are the bottom addable nodes of λ_A, and so they are conormal by Lemma 2.7. If $\lambda = (l_1^{a_1}, l_2^{a_2})$ has exactly two removable nodes then $l_1 - l_2 > 1$ by Lemma 1.7, in which case B and C are the bottom addable nodes of λ_A, and we apply Lemma 2.7 again. Finally, the case where λ has only one removable node is treated using Lemma 1.6. $\square$

Finally, we prove a couple of very special combinatorial facts which will be used only once.

Lemma 2.10. *There are no p-regular Mullineux-fixed partitions λ which have exactly two normal nodes A_1 and A_2, such that A_2 is below A_1, $\operatorname{res} A_1 = \operatorname{res} A_2$, λ_{A_1} is not p-regular, and λ_{A_2} is a JS-partition.*

Proof. By definition, the top removable node is always normal, so it must be A_1. Since λ_{A_1} is not p-regular, we have $\lambda = (l_1^{a_1}, l_2^{a_2}, \dots)$ where $l_1 - l_2 = 1$ and $a_2 = p - 1$. So the second top removable node of λ is normal, hence it must be A_2. By Lemma 1.7, λ must have at least one more removable node, so let A_3 be the third removable node from the top. If B is the node immediately above A_2 then B is a removable node of λ_{A_2}. If $a_1 > 1$ then B is normal, which contradicts the fact that λ_{A_2} is JS. Finally, if $a_1 = 1$ then A_3 is normal for λ_{A_2}, which again contradicts the fact that λ_{A_2} is JS. □

Lemma 2.11. *There are no p-regular Mullineux-fixed partitions λ which have exactly two normal nodes A_1 and A_2 such that $\operatorname{res} A_1 \neq \operatorname{res} A_2$, λ_{A_1} and λ_{A_2} are JS-partitions, and $(\lambda_{A_1})_{B_1} = (\lambda_{A_2})_{B_2}$, where B_i is the top removable node of λ_{A_i}.*

Proof. Let A_1 be above A_2, and let $\lambda = (l_1^{a_1}, l_2^{a_2}, \dots, l_k^{a_k})$. Then A_1 is the top removable node of λ and of λ_{A_2}. So $A_1 = B_2$ and $(\lambda_{A_2})_{B_2} = (\lambda_{A_2})_{A_1}$. Now $(\lambda_{A_1})_{B_1} = (\lambda_{A_2})_{B_2}$ implies that $A_2 = B_1$, which is only possible if $a_1 = 1$ and $l_1 - l_2 = 1$. By Lemma 1.7, we have $k \geq 3$ so we can pick the third removable node from the top, A_3, say. Since λ_{A_1} is a JS-partition, we have $\operatorname{res} A_3 = \operatorname{res} A_1$, hence A_3 is normal for λ_{A_2}, which is impossible as λ_{A_2} is a JS-partition. □

3. Proof of the main theorem

'Split-non-split' case. Throughout this subsection λ and μ are p-regular partitions of n satisfying $\lambda \neq \lambda^{\mathbf{M}}$, $\lambda \neq (n), (1^n)$, $\mu = \mu^{\mathbf{M}}$. We are interested in tensor products of the form $E^\lambda \otimes E^\mu_\pm$. Note that in view of Lemma 1.3, $E^\lambda \otimes E^\mu_+$ is irreducible if and only if $E^\lambda \otimes E^\mu_-$ is irreducible.

Lemma 3.1. *The product $E^\lambda \otimes E^\mu_\pm$ is irreducible if and only if over Σ_n we have $D^\lambda \otimes D^\mu \cong D^\nu \oplus (D^\nu \otimes \mathbf{sgn})$ with $D^\nu \not\cong D^\nu \otimes \mathbf{sgn}$. In this case, $E^\lambda \otimes E^\mu_+ \cong E^\lambda \otimes E^\mu_- \cong E^\nu$.*

Proof. If $E^\lambda \otimes E^\mu_\pm$ is irreducible then $(D^\lambda \otimes D^\mu){\downarrow_{A_n}} \cong (E^\lambda \otimes E^\mu_+) \oplus (E^\lambda \otimes E^\mu_-)$ is semisimple, and so $D^\lambda \otimes D^\mu$ is also semisimple, thanks to Lemma 1.1. Moreover, since $D^\mu \otimes \mathbf{sgn} \cong D^\mu$, we have $D^\lambda \otimes D^\mu \otimes \mathbf{sgn} \cong D^\lambda \otimes D^\mu$. So either the tensor product $D^\lambda \otimes D^\mu$ has one composition factor D^κ with $D^\kappa \otimes \mathbf{sgn} \cong D^\kappa$ or it has two composition factors D^ν and $D^\nu \otimes \mathbf{sgn}$, with $D^\nu \not\cong D^\nu \otimes \mathbf{sgn}$. But the former option is impossible by Theorem A. The rest is clear. □

Theorem 3.2. *Let $p > 5$, $n \geq 10$, and $E^\lambda \not\cong \mathbf{1}_{A_n}, E^{(n-1,1)}$. Then $E^\lambda \otimes E^\mu_\pm$ is reducible.*

Proof. If $E^\lambda \otimes E^\mu_\pm$ is irreducible then Lemma 3.1 implies

$$\dim \operatorname{Hom}_{F\Sigma_n}(\operatorname{End}_F(D^\lambda), \operatorname{End}_F(D^\mu)) = \dim \operatorname{End}_{F\Sigma_n}(D^\lambda \otimes D^\mu) = 2.$$

We use Lemma 1.11 (and notation therein) to show that the first Hom-space above actually has a larger dimension. Indeed, by Lemma 1.11, it suffices to show that the following two conditions are satisfied:

(a) the modules $(X^0_{i_0})^*$, $(X^2_{i_2})^*$, and one of the modules $(X^3_{i_3})^*$, $(X^4_{i_4})^*$ for some $i_j \in \{1, 2\}$ are quotients of $\operatorname{End}_F(D^\lambda)$, and

(b) the modules $X^k_{i'_k}$ are submodules of $\operatorname{End}_F(D^\mu)$ for $k = 0, 2, 3, 4$ and some $i'_j \in \{1, 2\}$.

But (a) and (b) hold in view of Lemma 1.5(i),(ii), Theorem 1.13 and Proposition 1.14. □

Tensor products involving the natural module. Now we study products of the form $E^{(n-1,1)} \otimes E^{\mu}_{\pm}$ where μ is a Mullineux-fixed partition.

Theorem 3.3. *The product $E^{(n-1,1)} \otimes E^{\mu}_{\pm}$ is irreducible if and only if $p \not| n$ and μ is a JS-partition with $p \not| h(\mu)$. In this case $E^{(n-1,1)} \otimes E^{\mu}_{\pm} \cong E^{\mu_A^B} \cong E^{\mu_A^C}$, where A is the top removable node of μ and B, C are the two bottom addable nodes for μ_A.*

Proof. By Lemma 3.1, $E^{(n-1,1)} \otimes E^{\mu}_{\pm}$ is irreducible if and only if

$$D^{(n-1,1)} \otimes D^{\mu} \cong D^{\nu} \oplus D^{\nu^{\mathbf{M}}}$$

for some $\nu \neq \nu^{\mathbf{M}}$. To find when this happens, note that

$$M^{(n-1,1)} \otimes D^{\mu} \cong (\mathbf{1}_{\Sigma_{n-1}})\uparrow^{\Sigma_n} \otimes D^{\mu} \cong (D^{\mu}\downarrow_{\Sigma_{n-1}})\uparrow^{\Sigma_n}.$$

Assume first that $p \not| n$. Then $M^{(n-1,1)} \cong D^{(n-1,1)} \oplus \mathbf{1}_{\Sigma_n}$. So

$$(D^{\mu}\downarrow_{\Sigma_{n-1}})\uparrow^{\Sigma_n} \cong D^{\mu} \oplus (D^{(n-1,1)} \otimes D^{\mu}).$$

Thus, we have to find when $(D^{\mu}\downarrow_{\Sigma_{n-1}})\uparrow^{\Sigma_n}$ is a direct sum of three irreducible modules. By Lemma 2.4, this can only happen if $D^{\mu}\downarrow_{\Sigma_{n-1}}$ is irreducible. In view of Theorem 2.1(i), this means that μ is a JS-partition or, equivalently, the top removable node A of μ is its only normal node. So, in view of Corollary 2.6 and Theorem 2.1(iii), it remains to find when the three conormal nodes of μ_A have different residues. By Lemma 2.9 this happens if and only the bottom addable node of μ has residue different from 0. But the residue of this node is $-h(\mu)$, which implies the required result.

Now assume that $p|n$. Then $M^{(n-1,1)} = \mathbf{1}_{\Sigma_n}|D^{(n-1,1)}|\mathbf{1}_{\Sigma_n}$. So $(D^{\mu}\downarrow_{\Sigma_{n-1}})\uparrow^{\Sigma_n}$ has a filtration with layers D^{μ}, $D^{(n-1,1)} \otimes D^{\mu}$, D^{μ}. In particular, it has four composition factors, exactly two of which are isomorphic to each other. Assume that $D^{(n-1,1)} \otimes D^{\mu} \cong D^{\nu} \oplus D^{\nu^{\mathbf{M}}}$ for some $\nu \neq \nu^{\mathbf{M}}$. By Lemma 2.4(i), $D^{\mu}\downarrow_{\Sigma_{n-1}}$ has at most two composition factors.

If $D^{\mu}\downarrow_{\Sigma_{n-1}} = D^{\mu_A}$ is irreducible then μ_A has three conormal nodes, see Lemma 2.9. If these nodes have different residues then $(D^{\mu}\downarrow_{\Sigma_{n-1}})\uparrow^{\Sigma_n}$ has three composition factors, giving a contradiction. So all three conormal nodes of λ have the same residue 0, see Lemma 2.9 again. The top of them, call it C, is cogood and so μ_A^C is p-regular and appears in $(D^{\mu}\downarrow_{\Sigma_{n-1}})\uparrow^{\Sigma_n}$ with multiplicity 3, see Theorem 2.1(ii)(iv). This contradiction shows that $D^{\mu}\downarrow_{\Sigma_{n-1}}$ can not be irreducible.

Thus, $D^{\mu}\downarrow_{\Sigma_{n-1}}$ has exactly two composition factors, say D_1 and D_2. In view of Lemma 2.4, the restrictions $D_i\downarrow_{\Sigma_{n-2}}$ must be irreducible. If $D_1 \cong D_2$ then by Theorem 2.1(i)(v), μ must have two normal nodes which satisfy the conditions of Lemma 2.10. Application of this lemma leads us to a contradiction, so $D_1 \not\cong D_2$. In this case Theorem 2.1 implies that μ has exactly two normal nodes, say A_1, A_2, such that $\operatorname{res} A_1 \neq \operatorname{res} A_2$, $D^{\mu}\downarrow_{\Sigma_{n-1}} \cong D^{\mu_{A_1}} \oplus D^{\mu_{A_2}}$, and we may assume that $D_i = D^{\mu_{A_i}}$. Observe also that

$$\dim \operatorname{Hom}_{F\Sigma_n}(D^{\mu}, (D^{\mu}\downarrow_{\Sigma_{n-1}})\uparrow^{\Sigma_n}) = \dim \operatorname{End}_{F\Sigma_{n-1}}(D^{\mu}\downarrow_{\Sigma_{n-1}}) = 2$$

This implies that

$$(D^{\mu}\downarrow_{\Sigma_{n-1}})\uparrow^{\Sigma_n} \cong D^{\mu} \oplus D^{\mu} \oplus D^{\nu} \oplus D^{\nu^{\mathbf{M}}},$$

whence $\dim \operatorname{End}_{F\Sigma_n}((D^{\mu}\downarrow_{\Sigma_{n-1}})\uparrow^{\Sigma_n}) = 6$. Now by Lemma 2.3, $\dim \operatorname{End}_{F\Sigma_{n-2}}(D^{\mu}\downarrow_{\Sigma_{n-2}}) = 4$. This shows that the irreducible restrictions $D^{\mu_{A_2}}\downarrow_{\Sigma_{n-2}}$ and $D^{\mu_{A_1}}\downarrow_{\Sigma_{n-2}}$ must be isomorphic to each other, which is impossible by Theorem 2.1 and Lemma 2.11. □

'Double-split case'. In this subsection λ and μ are Mullineux-fixed partitions of n. As our arguments for any product of the form $E^\lambda_\pm \otimes E^\mu_\pm$ are similar we consider only $E^\lambda_+ \otimes E^\mu_+$.

Lemma 3.4. *Assume $E^\lambda_+ \otimes E^\mu_+$ is irreducible. Then*

$$\begin{aligned}
\dim \mathrm{Hom}_{FA_n}(\mathrm{End}_F(E^\lambda_+), \mathrm{End}_F(E^\mu_+)) &= 1,\\
\dim \mathrm{Hom}_{FA_n}(\mathrm{End}_F(E^\lambda_+), \mathrm{Hom}_F(E^\mu_+, E^\mu_-)) &\leq 1,\\
\dim \mathrm{Hom}_{FA_n}(\mathrm{Hom}_F(E^\lambda_+, E^\lambda_-), \mathrm{Hom}_F(E^\mu_-, E^\mu_+)) &\leq 1,\\
\dim \mathrm{Hom}_{FA_n}(\mathrm{Hom}_F(E^\lambda_+, E^\lambda_-), \mathrm{Hom}_F(E^\mu_+, E^\mu_+)) &\leq 1.
\end{aligned}$$

Proof. Note that

$$\mathrm{Hom}_{FA_n}(\mathrm{End}_F(E^\lambda_+), \mathrm{End}_F(E^\mu_+)) \cong \mathrm{Hom}_{FA_n}(E^\lambda_+ \otimes E^\mu_+, E^\lambda_+ \otimes E^\mu_+),$$

which is 1-dimensional by assumptions and Schur's lemma. Moreover,

$$\mathrm{Hom}_{FA_n}(\mathrm{End}_F(E^\lambda_+), \mathrm{Hom}_F(E^\mu_+, E^\mu_-)) \cong \mathrm{Hom}_{FA_n}(E^\lambda_+ \otimes E^\mu_+, E^\lambda_+ \otimes E^\mu_-).$$

If the last space is at least 2-dimensional then the simple module $E^\lambda_+ \otimes E^\mu_+$ appears at least twice in the socle of $E^\lambda_+ \otimes E^\mu_-$, which is impossible as $\dim(E^\lambda_+ \otimes E^\mu_+) = \dim(E^\lambda_+ \otimes E^\mu_-)$. Next,

$$\mathrm{Hom}_{FA_n}(\mathrm{Hom}_F(E^\lambda_+, E^\lambda_-), \mathrm{Hom}_F(E^\mu_-, E^\mu_+)) \cong \mathrm{Hom}_{FA_n}(E^\lambda_- \otimes E^\mu_-, E^\lambda_+ \otimes E^\mu_+),$$

which is at most 1-dimensional as both modules $E^\lambda_+ \otimes E^\mu_+$ and $E^\lambda_- \otimes E^\mu_- \cong {}^\sigma(E^\lambda_+ \otimes E^\mu_+)$ are irreducible by assumption. Finally,

$$\mathrm{Hom}_{FA_n}(\mathrm{Hom}_F(E^\lambda_+, E^\lambda_-), \mathrm{Hom}_F(E^\mu_+, E^\mu_+)) \cong \mathrm{Hom}_{FA_n}(E^\lambda_- \otimes E^\mu_+, E^\lambda_+ \otimes E^\mu_+),$$

which is at most 1-dimensional as $E^\lambda_+ \otimes E^\mu_+$ is irreducible, and $\dim E^\lambda_- \otimes E^\mu_+ = \dim E^\lambda_+ \otimes E^\mu_+$. ☐

Theorem 3.5. *Let $p > 5$ and $n > 16$. Then $E^\lambda_+ \otimes E^\mu_+$ is reducible.*

Proof. Assume $E^\lambda_+ \otimes E^\mu_+$ is irreducible. As $n > 16$, we have $h(\lambda), h(\mu) \geq 5$ by Lemma 1.5. So by Theorem 1.13 and self-duality of $\mathrm{End}_F(D^\lambda)$, the modules $(X^k_{j_k})^*$ are quotients of $\mathrm{End}_F(D^\lambda)$ and the modules $X^k_{j'_k}$ are submodules of $\mathrm{End}_F(D^\mu)$ for $k = 0, 2, 3, 4, 5$, and some $j_k, j'_k \in \{1, 2\}$, see the notation of Lemma 1.11. So the modules $(X^k_{j_k}{\downarrow}_{A_n})^*$ are quotients of $\mathrm{End}_F(D^\lambda){\downarrow}_{A_n}$ and the modules $X^k_{j'_k}{\downarrow}_{A_n}$ are submodules of $\mathrm{End}_F(D^\mu){\downarrow}_{A_n}$. Note that

$$\mathrm{End}_F(D^\nu){\downarrow}_{A_n} \cong \mathrm{End}_F(E^\nu_+) \oplus \mathrm{End}_F(E^\nu_-) \oplus \mathrm{Hom}_F(E^\nu_+, E^\nu_-) \oplus \mathrm{Hom}_F(E^\nu_-, E^\nu_+)$$

for $\nu = \lambda$ or μ. Let us denote

$$M_1(\nu) := \mathrm{End}_F(E^\nu_+) \oplus \mathrm{End}_F(E^\nu_-), \quad M_2(\nu) := \mathrm{Hom}_F(E^\nu_+, E^\nu_-) \oplus \mathrm{Hom}_F(E^\nu_-, E^\nu_+).$$

By Lemma 1.2, Corollary 1.10 and the remarks above, there are numbers $r, s \in \{1, 2\}$ and $m, l \in \{0, 2, 3, 4, 5\}$, $m \neq l$, such that $(X^m_{j_m}{\downarrow}_{A_n})^*$, $(X^l_{j_l}{\downarrow}_{A_n})^*$ are quotients of $M_r(\lambda)$ and $X^m_{j'_m}{\downarrow}_{A_n}$, $X^l_{j'_l}{\downarrow}_{A_n}$ are submodules of $M_s(\mu)$. We will show that this contradicts Lemma 3.4. Indeed, consider for example the case $r = 1, s = 2$, the remaining three cases being similar.

First, we claim that $X^m_{j'_m}{\downarrow}_{A_n}$ and $X^l_{j'_l}{\downarrow}_{A_n}$ are submodules of $\mathrm{Hom}_F(E^\mu_+, E^\mu_-)$. Indeed, by Corollary 1.4,

$$M_2(\mu) = \mathrm{Hom}_F(E^\mu_+, E^\mu_-) \oplus \mathrm{Hom}_F(E^\mu_-, E^\mu_+) \cong \mathrm{Hom}_F(E^\mu_+, E^\mu_-) \oplus {}^\sigma\,\mathrm{Hom}_F(E^\mu_+, E^\mu_-).$$

As $X^k_{j'_k}{\downarrow}_{A_n} \cong {}^\sigma(X^k_{j'_k}{\downarrow}_{A_n})$, we have $X^k_{j'_k}{\downarrow}_{A_n}$ is a submodule of $\mathrm{Hom}_F(E^\mu_+, E^\mu_-)$ if and only if it is a submodule of ${}^\sigma\,\mathrm{Hom}_F(E^\mu_+, E^\mu_-)$. Now the claim follows from Lemma 1.2 and Corollary 1.10.

Similarly, $(X^m_{j_m}{\downarrow}_{A_n})^*$ and $(X^l_{j_l}{\downarrow}_{A_n})^*$ are quotients of $\mathrm{End}_F(E^\lambda_+)$. So, by Corollary 1.12, we have $\dim \mathrm{Hom}_{FA_n}(\mathrm{End}_F(E^\lambda_+), \mathrm{Hom}_F(E^\mu_+, E^\mu_-)) \geq 2$, giving the desired contradiction with Lemma 3.4. ☐

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