LOWER BOUNDS FOR DEGREES OF IRREDUCIBLE BRAUER CHARACTERS OF FINITE GENERAL LINEAR GROUPS

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Introduction

Let G denote the finite general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ over the finite field with q elements. Associated to each partition λ of n, there is an irreducible unipotent complex character χ_λ of G. The degree of χ_λ is a polynomial in q given by Green's hook formula [8, p.444]; the polynomial is monic of degree $b(\lambda')$ where λ' is the transpose of λ and, for a partition $\mu = (m_1 \geq m_2 \geq \cdots \geq m_h > 0)$ of n, $b(\mu)$ denotes $\frac{n(n+1)}{2} - \sum_{i=1}^h im_i$. An easy consequence of Green's formula is that

$$\chi_\lambda(1) \geq q^{b(\lambda')}.$$

The purpose of this note is to prove similar lower bounds for the degrees of the irreducible p-modular Brauer characters of G when p is coprime to q.

We first state a special case of our main result. Let F be an algebraically closed field of characteristic $p > 0$ not dividing q. Then, for each partition λ of n, there is an associated irreducible unipotent FG-module $L(1, \lambda)$ (see [3, §3.5], or [11] where it is denoted D_λ). For an integer $N \geq 1$, we say that a partition is *N-regular* if it does not have N or more non-zero parts that are equal; in particular, the only 1-regular partition is the zero partition. Then we show:

Theorem A. *Let λ be an e-regular partition of n, where e is minimal such that $1 + q + \cdots + q^{e-1} \equiv 0 \pmod p$. Then, $\dim L(1, \lambda) \geq q^{b(\lambda')}$.*

Thus, for e-regular partitions, the same lower bound as in characteristic 0 can be used. If the regularity assumption is dropped, it is easy to find examples where the bound fails (e.g. take $\lambda = (1^e)$).

To formulate our main theorem in the general case, we need to recall a parametrization of the irreducible FG-modules. First, take $\sigma \in \bar{\mathbb{F}}_q^\times$ of degree d over $\mathbb{F}_q$. Define $\ell(d)$ to be the order of the image of q^d in $F^\times$. A partition λ will be called *σ-regular* if either σ is a p'-element and λ is $\ell(d)$-regular, or σ has order divisible by p and λ is p-regular. For $k \geq 1$ and each (not necessarily regular) partition λ of k, there is an associated irreducible $F\,\mathrm{GL}_{dk}(q)$-module $L(\sigma, \lambda)$ (see [3, §3.5], or [12] where it is denoted $D(\sigma, \lambda)$).

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Note that the precise labelling of $L(\sigma, \lambda)$ is not canonical: in the approach of [3], it depends ultimately on the choice of an embedding of $\bar{\mathbb{F}}_q^\times$ into the group of units of the algebraic closure of the p-adic field.

Choose a set Φ_0 of orbit representatives for the action of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ on $\bar{\mathbb{F}}_q^\times$. If $\sigma \in \Phi_0$ is a p'-element of degree d and $i \geq 0$, we say that τ is an *ith twist of* σ if $\tau \in \Phi_0$ has degree $d\ell(d)p^i$, order divisible by p and p'-part conjugate to σ. Now let Φ_p be some subset of Φ_0 containing all p'-elements of Φ_0, and exactly one ith twist of every p'-element of Φ_0 for every $i \geq 0$ (such a subset exists by [3, (2.1a)]). We should point out that $L(\sigma, \lambda) \cong L(\tau, \lambda)$ whenever $\sigma, \tau \in \bar{\mathbb{F}}_q^\times$ have the same degree and conjugate p'-parts, which ensures that the parametrization of irreducibles described in the next paragraph is really independent of this choice of Φ_p.

We also need the Harish-Chandra operator $\diamond$ as introduced originally by Green [8, p.403]; see [3, §5.2, §2.2] for its precise definition and properties in the modular case. Then, every irreducible FG-module L can be written as

$$L \cong L(\sigma_1, \lambda_1) \diamond L(\sigma_2, \lambda_2) \diamond \ldots \diamond L(\sigma_a, \lambda_a)$$

for some $a \geq 1$, distinct elements $\sigma_1, \ldots, \sigma_a \in \Phi_p$ of degrees $d_1, \ldots, d_a$, and σ_i-regular partitions λ_i of integers $k_i \geq 1$ (so n necessarily equals $n_1k_1 + \cdots + n_ak_a$). This labelling of L is unique up to reordering of the terms in the Harish-Chandra product, and the resulting parametrization of the irreducible FG-modules is the one that arises naturally from Harish-Chandra theory, see e.g. [4]. Its relationship to the more usual parametrization used in [12] and [3, §4.4] was originally explained in [6]; it is best understood as an application of the non-defining characteristic tensor product theorem, see [5] or [3, §4.3].

Our main result is as follows.

Theorem B. *Given an irreducible FG-module $L \cong L(\sigma_1, \lambda_1) \diamond \ldots \diamond L(\sigma_a, \lambda_a)$ for $a \geq 1$, distinct elements $\sigma_1, \ldots, \sigma_a \in \Phi_p$ of degrees $d_1, \ldots, d_a$, and σ_i-regular partitions λ_i of integers $k_i \geq 1$ for $i = 1, \ldots, a$, we have that*

$$\dim L \geq |\operatorname{GL}_n(q) : \operatorname{GL}_{k_1}(q^{d_1}) \times \cdots \times \operatorname{GL}_{k_a}(q^{d_a})|' q^{d_1 b(\lambda_1') + \cdots + d_a b(\lambda_a')},$$

where for an integer N, N' denotes its largest divisor that is coprime to q.

We remark that the lower bound in Theorem B is *exact* if and only if each partition λ_i has either just one row ("trivial") or just one column ("Steinberg"). Moreover, the lower bound is always a polynomial in q whose leading term is the same as the leading term in the generic degree for the ordinary irreducible character with the same labelling.

The remainder of the article is organized as follows. In §1, we prove the key auxiliary result, namely, an analogue of the Premet-Suprunenko theorem [16, 17] for quantum linear groups. Our proof of this follows the original arguments of [16, 17] closely. The main result is proved in §2, ultimately as a consequence of the Premet-Suprunenko theorem and [3, Theorem 5.5d]. Finally, in §3 we apply the theorem to improve on results of Guralnick and Tiep [9] determining the irreducible FG-modules of small dimension. In particular, in Theorem 3.4, we list all irreducible FG-modules of dimensions $\leq q^{3n-9}$ explicitly.

1 A q-analogue of the Premet-Suprunenko theorem

In this section, F denotes an arbitrary algebraically closed field and q is a primitive ℓth root of unity in F. For convenience, we *exclude* the possibility that $q = 1$, since the main result below is already known [16, 17] in the classical case. Choose a square root v of q in F so that if ℓ is odd, then v is also a primitive ℓth root of unity. We are concerned with the divided power version of the quantized enveloping algebra of $\mathfrak{gl}_n$ over F at the parameter v, as defined originally by Lusztig [14, 15] and Du [7, §2] (who extended Lusztig's construction from $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$). We also cite [1, 2] as general references for the rational representation theory of quantum groups at roots of unity.

To recall some definitions, let t be an indeterminate. Then, the quantized enveloping algebra $U_{\mathbb{Q}(t)}$ associated to $\mathfrak{gl}_n$ is the $\mathbb{Q}(t)$-algebra with generators $\{E_i, F_i, K_j^{\pm 1} \mid 1 \le i < n, 1 \le j \le n\}$ subject to the relations

$$\begin{array}{ll} K_iK_j = K_jK_i, & K_iK_i^{-1} = K_i^{-1}K_i = 1, \\ K_iE_j = t^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i, & K_iF_j = t^{\delta_{i,j+1}-\delta_{i,j}}F_jK_i, \end{array}$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_{i,i+1} - K_{i,i+1}^{-1}}{t - t^{-1}},$$

$$\begin{array}{ll} E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i & \text{if } |i-j| > 1, \\ E_i^2E_j - (t+t^{-1})E_iE_jE_i + E_jE_i^2 = 0, & \\ F_i^2F_j - (t+t^{-1})F_iF_jF_i + F_jF_i^2 = 0 & \text{if } |i-j| = 1. \end{array}$$

Here, for any $1 \le i < j \le n$, $K_{i,j}$ denotes $K_iK_j^{-1}$. For $a, b \in \mathbb{N}, X \in U_{\mathbb{Q}(t)}$ and $1 \le j \le n$, define

$$[a]! := \prod_{c=1}^{a} \frac{t^c - t^{-c}}{t - t^{-1}}, \qquad \begin{bmatrix} a \\ b \end{bmatrix} := \prod_{c=1}^{b} \frac{t^{a-c+1} - t^{-a+c-1}}{t^c - t^{-c}},$$

$$X^{(a)} := \frac{X^a}{[a]!}, \qquad \begin{bmatrix} K_j \\ b \end{bmatrix} := \prod_{c=1}^{b} \frac{K_jt^{-c+1} - K_j^{-1}t^{c-1}}{t^c - t^{-c}}.$$

Let $U_{\mathbb{Z}[t,t^{-1}]}$ be the $\mathbb{Z}[t,t^{-1}]$-subalgebra of $U_{\mathbb{Q}(t)}$ generated by the elements $E_i^{(a)}, F_i^{(a)}, K_j^{\pm 1}$ and $\begin{bmatrix} K_j \\ a \end{bmatrix}$ for $a \ge 0, 1 \le i < n, 1 \le j \le n$. We then obtain the F-algebra

$$U := F \otimes_{\mathbb{Z}[t,t^{-1}]} U_{\mathbb{Z}[t,t^{-1}]}$$

on change of rings, where we are regarding F as a $\mathbb{Z}[t,t^{-1}]$-module by letting $t \in \mathbb{Z}[t,t^{-1}]$ act on F by multiplication by $v \in F$. ¿From now on, we only work with U, so can denote the images of $F_i^{(a)}, E_i^{(a)}, K_i^{\pm 1}, \begin{bmatrix} K_i \\ b \end{bmatrix} \in U_{\mathbb{Z}[t,t^{-1}]}$ in U by the same names without confusion. Finally, let U^-, U^+ and U^0 denote the subalgebras of U generated by $\left\{F_i^{(a)} \mid 1 \le i < n, a \ge 0\right\}, \left\{E_i^{(a)} \mid 1 \le i < n, a \ge 0\right\}$ and $\left\{K_i^{\pm 1}, \begin{bmatrix} K_i \\ b \end{bmatrix} \,\middle|\, 1 \le i \le n, b \ge 0\right\}$ respectively.

Let E denote the Euclidean space with orthonormal basis $\varepsilon_1, \dots, \varepsilon_n$. For $0 \neq \lambda \in E$, $\lambda^\vee$ denotes $2\lambda/\langle \lambda, \lambda \rangle$, where $\langle .,. \rangle$ is the inner product on E. The root system of type A can be identified with the subset $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i,j \leq n, i \neq j\}$ of E, and a set of simple roots is given by the $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, n-1$. The weight lattice is then the $\mathbb{Z}$-submodule X of E generated by $\varepsilon_1, \dots, \varepsilon_n$, where we identify the element $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in X$ with the unique algebra homomorphism $\lambda : U^0 \to F$ such that

$$K_j \mapsto v^{\lambda_j}, \begin{bmatrix} K_j \\ a \end{bmatrix} \mapsto \begin{bmatrix} \lambda_j \\ a \end{bmatrix} \text{ for } 1 \leq j \leq n,\ a \geq 0.$$

A weight $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in X$ is dominant, relative to our choice of simple roots, if and only if $\lambda_1 \geq \dots \geq \lambda_n$; we let $X^+ \subseteq X$ denote the set of all such dominant weights. We also have the usual dominance ordering $\leq$ on X: $\lambda \leq \mu$ if and only if $\mu - \lambda$ is a sum of simple roots.

We say a vector v in a U-module V has weight λ if $Kv = \lambda(K)v$ for all $K \in U^0$, and is a highest weight vector if $E_i^{(a)} v = 0$ for all $1 \leq i < n$ and $a \geq 1$. For each $\lambda \in X^+$, there is a unique irreducible U-module $L(\lambda)$ generated by a non-zero highest vector v_λ of weight λ. We also recall that $L(\lambda)$ decomposes as the direct sum of its weight spaces, i.e. $L(\lambda) = \bigoplus_{\mu \in X} L(\lambda)_\mu$ where $L(\lambda)_\mu = \{v \in L(\lambda) \mid v \text{ has weight } \mu\}$. For $\lambda \in X^+$, $\Omega(\lambda)$ denotes the set of all weights $\mu \in X$ appearing with non-zero multiplicity in Weyl's character formula for the irreducible $\mathfrak{gl}_n(\mathbb{C})$-module of highest weight λ. So, $\mu \in \Omega(\lambda)$ if and only if μ and all its conjugates under the Weyl group are $\leq \lambda$ in the dominance order. Call $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in X^+$ *ℓ-restricted* if $\langle \lambda, \alpha_i^\vee \rangle = \lambda_i - \lambda_{i+1} < \ell$ for all $i = 1, \dots, n-1$. The first lemma is well known.

1.1. **Lemma.** *Let $\lambda \in X^+$ be ℓ-restricted and $0 \neq v \in L(\lambda)_\mu$ for some $\mu \in X$. If $E_i v = 0$ for all $1 \leq i < n$, then $\mu = \lambda$.*

Proof. This is equivalent to the fact that for ℓ-restricted λ, the module $L(\lambda)$ is irreducible over the (thickened) Frobenius kernel, i.e. the subalgebra of U generated by U^0 and $\{E_i, F_i\}_{1 \leq i < n}$ (see e.g. [2, 1.9]). $\square$

1.2. **Lemma.** *Let $\lambda \in X^+$ be ℓ-restricted and fix some $1 \leq i < n$. Given $\mu = \lambda - \sum_{j \neq i} m_j \alpha_j$ for integers $m_j \geq 0$, the restriction of the operator $F_i^{(a)}$ to the weight space $L(\lambda)_\mu$ is injective for all $0 \leq a \leq \langle \mu, \alpha_i^\vee \rangle$.*

Proof. We use induction on $M := \sum_{j \neq i} m_i$. Suppose first that $M = 0$, when $\mu = \lambda$ and we need to prove that $F_i^{(a)} v_\lambda \neq 0$ for all $0 \leq a \leq \langle \lambda, \alpha_i^\vee \rangle$. Using well known commutation relations (see e.g. [15, 6.5(a2)], [1, Lemma 1.1]),

$$E_i^{(a)} F_i^{(a)} v_\lambda = \begin{bmatrix} \langle \lambda, \alpha_i^\vee \rangle \\ a \end{bmatrix} v_\lambda$$

which is non-zero as $a \leq \langle \lambda, \alpha_i^\vee \rangle < \ell$. So certainly $F_i^{(a)} v_\lambda \neq 0$.

Now suppose that $M > 0$ and that the lemma has been proved for all smaller M. Take $0 \neq v \in L(\lambda)_\mu$. We need to show that $F_i^{(a)} v \neq 0$ for all $0 \leq a \leq \langle \mu, \alpha_i^\vee \rangle$. Noting that $E_i v = 0$ by weights, Lemma 1.1 implies that for some $h \neq i$, $E_h v \neq 0$. By induction, $F_i^{(a)} E_h v \neq 0$ for all $0 \leq a \leq \langle \lambda + \alpha_h - \sum_{j \neq i} m_j \alpha_j, \alpha_i^\vee \rangle$. So, since $F_i^{(a)}$ and E_h commute, we deduce that $F_i^{(a)} v \neq 0$ for all $0 \leq a < \langle \lambda - \sum_{j \neq i} m_j \alpha_j, \alpha_i^\vee \rangle$. Now consider the case $a = \langle \lambda - \sum_{j \neq i} m_j \alpha_j, \alpha_i^\vee \rangle$. Here,

$$E_i^{(a)} F_i^{(a)} v = \begin{bmatrix} a \\ a \end{bmatrix} v = v$$

so $F_i^{(a)} v \neq 0$ as required $\square$

1.3. **Lemma ([17, Lemma 3],[16, Lemma 3]).** *Let $\lambda \in X^+$ and $\mu = \lambda - \sum_j m_j \alpha_j$ for integers $m_j \geq 0$ such that μ is dominant. Choose i such that m_i is minimal among all the m_j. Then, $\lambda - \sum_{j \neq i} m_j \alpha_j$ is a weight in $\Omega(\lambda)$, and $m_i \leq \langle \lambda - \sum_{j \neq i} m_j \alpha_j, \alpha_i^\vee \rangle$.*

Proof. We claim more: for any $d \leq m_i$ and any root α, the weight $\mu + d\alpha$ lies in $\Omega(\lambda)$. To see this, let $\alpha_0 = \alpha_1 + \cdots + \alpha_{n-1}$ be the highest root and take w in the Weyl group such that $w(\mu + d\alpha)$ is dominant. Then, $w\mu + wd\alpha \leq w\mu + d\alpha_0 \leq \mu + d\alpha_0 \leq \lambda$ by choice of d. So, $w(\mu + d\alpha) \leq \lambda$, as required to show that $\mu + d\alpha \in \Omega(\lambda)$. The second statement of the lemma now follows easily on considering the α_i-string through $\lambda - \sum_{j \neq i} m_j \alpha_j$. $\square$

Now we can prove the main result of this section. The theorem asserts that for ℓ-restricted λ, the set of non-zero weights of $L(\lambda)$ is the same as the set $\Omega(\lambda)$ of non-zero weights of the corresponding Weyl module. Combining the theorem with the q-analogue of Steinberg's tensor product theorem, it allows one to determine the non-zero weights of $L(\lambda)$ for arbitrary $\lambda \in X^+$. The proof given here is essentially identical to Suprunenko's proof in the classical case [17].

1.4. **Theorem.** *Let $\lambda \in X^+$ be ℓ-restricted. Then, $\dim L(\lambda)_\mu \geq 1$ for all $\mu \in \Omega(\lambda)$.*

Proof. The theorem is clear in the case $n = 1$, so suppose that $n > 1$ and that the result has been proved for all smaller n. For a fixed ℓ-restricted $\lambda \in X^+$, we proceed by downward induction on the dominance order on $\Omega(\lambda)$, the result being clear for $\mu = \lambda$. It suffices to consider the case that $\mu \in \Omega(\lambda)$ is dominant. Write $\mu = \lambda - \sum_j m_j \alpha_j$ and choose i such that m_i is minimal.

Suppose first that $m_i = 0$. Then, we pass to the Levi subalgebra U' of U generated by U^0 and all $E_j^{(a)}, F_j^{(a)}$ for $j \neq i$. The vector v_λ generates a U'-submodule of $L(\lambda)$ which is a highest weight U'-module of highest weight λ (actually, the irreducible U'-module of highest weight λ though we do not need this much). By the hypothesis on n, this submodule has non-zero μ-weight space, as required.

If $m_i > 0$, $\lambda - \sum_{j \neq i} m_j \alpha_j > \mu$ also lies in $\Omega(\lambda)$ by Lemma 1.3. So, by the induction hypothesis, $L(\lambda)_{\lambda - \sum_{j \neq i} m_j \alpha_j} \neq 0$. By Lemma 1.3 again, $m_i \leq \langle \lambda - \sum_{j \neq i} m_j \alpha_j, \alpha_i^\vee \rangle$. So by Lemma 1.2, the operator $F_i^{(m_i)}$ gives an injection of $L(\lambda)_{\lambda - \sum_{j \neq i} m_j \alpha_j}$ into the required weight space $L(\lambda)_\mu$. Hence $L(\lambda)_\mu$ is non-zero too. $\square$

2 Main results

Let $n \geq 1$ and $\mu = (m_1, m_2, \ldots, m_h)$ be any composition of n (i.e. $m_1, \ldots, m_h$ are non-negative integers summing to n) with $m_h \neq 0$. We say that μ *has no gaps* if $m_i > 0$ for all $i = 1, 2, \ldots, h$. If μ has no gaps, we write $\nu \sim \mu$ if ν is a composition with no gaps obtained from μ by reordering its non-zero parts. We denote by μ^+ the unique partition obtained by ordering the parts of μ in decreasing order. Also, μ^{op} denotes the composition with no gaps obtained by reversing the order of the parts (e.g. $(2,5,7)^{\mathrm{op}} = (7,5,2)$).

For a composition $\mu = (m_1, \ldots, m_h)$ of n, we define

$$\mathrm{Int}(\mu) := \{1, 2, \ldots, n\} \setminus \{m_1, m_1 + m_2, \ldots, m_1 + \cdots + m_{h-1}, m_1 + \cdots + m_h\}.$$

If in addition λ is a partition of n and t is an indeterminate, the following polynomials in $\mathbb{Z}[t]$ were defined in [3, §5.5] by:

$$R_\mu(t) := \prod_{i \in \mathrm{Int}(\mu)} (t^i - 1), \qquad S_\lambda(t) := \sum_{\mu \sim \lambda} R_\mu(t).$$

We also let $\leq$ denote the dominance order on partitions as in the previous section, and for a partition λ we define the integer $b(\lambda)$ as in the introduction.

2.1. **Lemma.** *Let λ be a partition of n. Then $R_{\lambda^{\mathrm{op}}}(t)$ is a monic polynomial of degree $b(\lambda)$, and $\deg R_\mu(t) < \deg R_{\lambda^{\mathrm{op}}}(t)$ for all $\mu \sim \lambda$ different from λ^{op}. In particular, $S_\lambda(t)$ is monic of degree $b(\lambda)$.*

Proof. Follows from the definitions. $\square$

2.2. **Lemma.** *If $\mu < \lambda$ are distinct partitions of n then $b(\lambda) < b(\mu)$.*

Proof. Let $\lambda = (l_1, \ldots, l_h)$ and $\mu = (m_1, \ldots, m_k)$ be the parts of λ and μ. Then, we have $\sum_{i=1}^j l_i \geq \sum_{i=1}^j m_i$ for all j, with strict inequality for at least one j. Hence $\sum_{j=1}^n \sum_{i=1}^j l_i > \sum_{j=1}^n \sum_{i=1}^j m_i$ or $\sum_{i=1}^n i l_i > \sum_{i=1}^n i m_i$, which implies the result. $\square$

2.3. **Lemma.** *Let μ be a composition of n with no gaps, $M := \mathrm{Int}(\mu^{\mathrm{op}})$, and N be any subset of M. Then there exists a composition ν of n with no gaps such that $\nu^+ \leq \mu^+$ and $\mathrm{Int}(\nu) = N$.*

Proof. Let $M = \{i_1, \ldots, i_r\}$ and $1 \leq j \leq r$. It suffices to prove the result for $N = \{i_1, \ldots, \hat{i_j}, \ldots, i_r\}$. Let us label the boxes of the Young diagram of μ with $1, 2, \ldots, n$ from left to right along the rows starting from the bottom row, as in the following picture:

$$\begin{array}{|c|c|c|}
\hline
7 & 8 & \multicolumn{1}{c}{} \\
\hline
4 & 5 & 6 \\
\hline
2 & 3 & \multicolumn{1}{c}{} \\
\cline{1-2}
1 & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array}$$

Then $i \in M$ if and only if i is not the last entry in a row. Now, let i_j be contained in row k and column l. Set $\nu := (\mu_1, \dots, \mu_{k-1}, l, \mu_k - l, \mu_{k+1}, \dots, \mu_h)$. For instance, if $i_j = 5$ in the example above, then

$$\nu = \ydiagram{2,2,1,2,1}$$

It is clear that $\mathrm{Int}(\nu) = \{i_1, \dots, \hat{i_j}, \dots, i_r\}$ and $\nu^+ \leq \mu^+$. $\square$

If $M \subseteq \{1, \dots, n\}$, we set

$$\varphi_M(t) := \prod_{i \in M} (t^i - 1).$$

In particular, $R_\nu(t) = \varphi_{\mathrm{Int}(\nu)}(t)$. Also set $\sigma(M) = \sum_{i \in M} i$.

2.4. **Lemma.** $\sum_{N \subseteq M} \varphi_N(t) = t^{\sigma(M)}$.

Proof. $\sum_{N \subseteq M} \prod_{i \in N} (t^i - 1) = \prod_{i \in M}((t^i - 1) + 1) = \prod_{i \in M} t^i = t^{\sigma(M)}$. $\square$

2.5. **Theorem.** *For a positive integer q and a partition λ of n,*

$$\sum_{\mu \leq \lambda} S_\mu(q) \geq q^{b(\lambda)},$$

summing over partitions μ of n, with equality if and only if $\lambda = (n)$ or (1^n).

Proof. For every $N \subseteq \mathrm{Int}(\lambda^{\mathrm{op}})$ pick ν_N with $(\nu_N)^+ \leq \lambda$ and $\mathrm{Int}(\nu_N) = N$. This is possible in view of Lemma 2.3. Then, we have

$$\sum_{\mu \leq \lambda} S_\mu(q) = \sum_{N \subseteq \mathrm{Int}(\lambda^{\mathrm{op}})} R_{\nu_N}(q) + (*)$$

where $(*)$ stands for a sum of certain terms of the form $R_\nu(q)$. Observe by the definition of $R_\nu(q)$, $(*)$ is a non-negative integer. Moreover, using Lemma 2.4 we get

$$\sum_{N \subseteq \mathrm{Int}(\lambda^{\mathrm{op}})} R_{\nu_N}(q) = \sum_{N \subseteq \mathrm{Int}(\lambda^{\mathrm{op}})} \varphi_{\mathrm{Int}(\nu_N)}(q) = \sum_{N \subseteq \mathrm{Int}(\lambda^{\mathrm{op}})} \varphi_N(q) = q^{\sigma(\mathrm{Int}(\lambda^{\mathrm{op}}))}$$

which equals $q^{b(\lambda)}$. This proves the inequality. To see that it is strict if $\lambda \neq (n), (1^n)$, observe that then there is some $1 \leq m < n$ not contained in $\mathrm{Int}(\lambda^{\mathrm{op}})$. Set $\nu = (1^{m-1}, 2, 1^{n-m-1})$; then, $\nu^+ \leq \lambda$ and $\mathrm{Int}(\nu) = \{m\}$ which is not contained in $\mathrm{Int}(\lambda^{\mathrm{op}})$. So the term $R_\nu(q)$ gives a non-zero contribution to $(*)$ which makes the inequality strict. Finally, for $\lambda = (n)$ or (1^n), one easily sees that the term $(*)$ is zero so equality holds in these cases. $\square$

Now we can prove Theorem B from the introduction. For partitions λ, μ of n and $d \geq 1$ define

$$m_{\lambda,\mu}(q^d) := \dim L(\lambda)_\mu,$$

the dimension of the μ-weight space of the irreducible highest weight module $L(\lambda)$ over the quantum $\mathfrak{gl}_n$ over F as in §1 but with the image of q^d in F replacing the parameter q there; all other notation is as in the introduction. We state [3, Theorem 5.5d]:

2.6. **Lemma.** *For $\sigma \in \Phi_p$ of degree d and a partition λ of $k \geq 1$,*

$$\dim L(\sigma, \lambda') = |\operatorname{GL}_{dk}(q) : \operatorname{GL}_k(q^d)|' \sum_{\mu \leq \lambda} m_{\lambda,\mu}(q^d) S_\mu(q^d)$$

where the sum is over partitions μ of n.

Combining this with Theorem 1.4 (or [16, 17] if $q^d \equiv 1 \pmod p$) and Theorem 2.5, we deduce immediately that:

2.7. **Theorem.** *For $\sigma \in \Phi_p$ of degree d and a σ-regular partition λ of $k \geq 1$,*

$$\dim L(\sigma, \lambda') \geq |\operatorname{GL}_{dk}(q) : \operatorname{GL}_k(q^d)|' q^{db(\lambda)}$$

with equality if and only if $\lambda = (k)$ or (1^k).

Theorem B in the introduction now follows immediately from this and the definition of the Harish-Chandra operator $\diamond$. To deduce Theorem A, it is obviously a special case of Theorem B unless $q \equiv 1 \pmod p$, in which case there exists a 0th twist σ of 1, also of degree 1 over $\mathbb{F}_q$. Then, $L(1, \lambda) \cong L(\sigma, \lambda)$ and using this observation, the statement in Theorem A again follows as a special case of Theorem B.

3 Application: low-dimensional representations

Now we illustrate the usefulness of Theorem B by applying it to list all irreducible FG-modules of dimension $\leq q^{3n-9}$. For simplicity, we only consider $n \geq 5$ (for $n < 5$ it is an easy matter to explicitly list all irreducible FG-modules and their dimensions using [13]).

Given non-negative integers $n_1, \dots, n_a$ summing to n, define

$$\{n\}! := \frac{(q^n - 1)(q^{n-1} - 1) \dots (q - 1)}{(q - 1)^n}, \qquad \left\{ {n \atop n_1, \dots, n_a} \right\} := \frac{\{n\}!}{\{n_1\}!\{n_2\}! \dots \{n_a\}!},$$

$$\{k|d\} := \frac{\prod_{i=1}^{dk}(q^i - 1)}{\prod_{i=1}^{k}(q^{di} - 1)}.$$

Note that the index $|\operatorname{GL}_n(q) : \operatorname{GL}_{n_1}(q) \times \dots \times \operatorname{GL}_{n_a}(q)|'$ equals $\left\{ {n \atop n_1, \dots, n_a} \right\}$ and the index $|\operatorname{GL}_{dk}(q) : \operatorname{GL}_k(q^d)|' = \{k|d\}$.

3.1. **Lemma.** *For $a \geq 2$ and $n_1, \dots, n_a \geq 1$ with $n = n_1 + \cdots + n_a$,*

$$\left\{ {n \atop n_1, \dots, n_a} \right\} > q^{\sum_{1 \leq i < j \leq a} n_i n_j}.$$

Proof. Noting that $\frac{q^n - 1}{q^m - 1} > q^{n-m}$ for $n > m$, we have for $a = 2$ that

$$\left\{ {n \atop n_1, n_2} \right\} = \frac{(q^n - 1) \dots (q^{n-n_1+1} - 1)}{(q^{n_1} - 1) \dots (q - 1)} > (q^{n-n_1})^{n_1} = q^{n_1 n_2}.$$

Since $\left\{ {n \atop n_1, \dots, n_a} \right\} = \left\{ {n \atop n_1+n_2, n_3, \dots, n_a} \right\} \left\{ {n_1+n_2 \atop n_1, n_2} \right\}$, the general case $a > 2$ now follows easily by induction. $\square$

3.2. **Lemma.** *Suppose that $n = kd$ for integers $k \geq 1$, $d \geq 2$. Then,*

$$\{k|d\} > q^{\frac{1}{2}n(n-k)-1-\delta_{q,2}} \geq q^{\frac{n^2}{4}-2}.$$

Proof. Dividing both sides by $q^{\frac{1}{2}n(n-k)}$, the inequality is equivalent to proving that

$$\prod_{1 \leq i \leq n, d \nmid i} \left(1 - \frac{1}{q^i}\right) > \frac{1}{q^{1+\delta_{q,2}}}.$$

The left hand side is certainly greater than the infinite product $\prod_{i \geq 1} \left(1 - \frac{1}{q^i}\right)$. Using a theorem of Euler [10, Theorem 353], this is bounded below by

$$1 - \frac{1}{q} - \frac{1}{q^2} = \frac{q^2 - q - 1}{q^2} \geq \frac{1}{q^{1+\delta_{q,2}}},$$

which completes the proof of the first inequality. The second is obvious. $\square$

3.3. **Lemma.** *Suppose that $n \geq 5$, $a \geq 2$, $1 \leq n_1 \leq \cdots \leq n_a$ and $n = n_1 + \cdots + n_a$. If $\left\{ {n \atop n_1, \dots, n_a} \right\} \leq q^{3n-9}$ then $(n_1, \dots, n_a)$ is equal to either $(r, n-r)$ with $r \leq 2$ or $(1, 1, n-2)$.*

Proof. First suppose that $a = 2$. It is enough to show that $\left\{ {n \atop 3, n-3} \right\} > q^{3n-9}$, which follows from Lemma 3.1. Now suppose that $a = 3$. Since $\left\{ {n \atop n_1, n_2, n_3} \right\} \geq \left\{ {n \atop n_1+n_2, n_3} \right\}$, one can only have $(n_1, n_2, n_3) = (1, 1, n-2)$ by what we have proved so far. Similarly, $a > 3$ cannot occur. $\square$

Now we prove the main result of the section:

3.4. **Theorem.** *Let $n \geq 5$ and L be an irreducible FG-module. Then,*

$$\dim L \leq q^{3n-9}$$

if and only if L is isomorphic to one of the modules in the table below. In the table, δ_n, ε are defined by

$$\delta_n = \begin{cases} 1 & \text{if } e|n, \\ 0 & \text{otherwise,} \end{cases} \qquad \varepsilon = \begin{cases} 1 & \text{if } e > 2 \text{ and } e|n-1, \text{ or } e = 2 \text{ and } 2p|(n-1), \\ -1 & \text{if } e = 2,\ 2|n \text{ and } 2p \nmid (n-2), \\ 0 & \text{otherwise} \end{cases}$$

where e is the smallest positive integer such that $1 + q + \cdots + q^{e-1} \equiv 0 \,(\mathrm{mod}\, p)$.

L	$\dim L$	*Conditions*
$L(\sigma,(n))$	1	$\deg(\sigma)=1$
$L(\sigma,(n-1,1))$	$\frac{q^n-q}{q-1} - \delta_n$	$\deg(\sigma)=1$
$L(\sigma,(n-2,2))$	$\frac{(q^n-1)(q^{n-1}-q^2)}{(q^2-1)(q-1)} - \delta_{n-2}\frac{q^n-q}{q-1} - \varepsilon$	$\deg(\sigma)=1, n>5,$ $(n,q) \neq (6,2)$
$L(\sigma,(n-2,1^2))$	$\frac{(q^n-q^2)(q^n-q)}{(q^2-1)(q-1)} - \delta_n \frac{q^n-2q+1}{q-1}$	$\deg(\sigma)=1, n>6,$ $(n,q) \neq (7,2)$
$L(\sigma,(3))$	$(q-1)(q^3-1)(q^5-1)$	$\deg(\sigma)=2, n=6$
$L(\sigma,(4))$	27559	$\deg(\sigma)=2, n=8, q=2$
$L(\sigma_1,(1)) \diamond L(\sigma_2,(n-1))$	$\frac{q^n-1}{q-1}$	$\deg(\sigma_i)=1, \sigma_1 \neq \sigma_2$
$L(\sigma_1,(1)) \diamond L(\sigma_2,(n-2,1))$	$\frac{q^n-1}{q-1}\left(\frac{q^{n-1}-q}{q-1} - \delta_{n-1}\right)$	$\deg(\sigma_i)=1, \sigma_1 \neq \sigma_2, n>6,$ $(n,q) \neq (7,2)$
$L(\sigma_1,(2)) \diamond L(\sigma_2,(n-2))$	$\frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}$	$\deg(\sigma_i)=1, \sigma_1 \neq \sigma_2, n>5,$ $(n,q) \neq (6,2)$
$L(\sigma_1,(1^2)) \diamond L(\sigma_2,(n-2))$	$\frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}(q-\delta_2)$	$\deg(\sigma_i)=1, \sigma_1 \neq \sigma_2, n>6,$ *if* $e \nmid 2$ *then* $(n,q) \neq (7,2)$
$L(\sigma_1,(1)) \diamond L(\sigma_2,(n-2))$	$\frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)}$	$\deg(\sigma_1)=2, \deg(\sigma_2)=1, n>6,$
$L(\sigma_1,(1)) \diamond L(\sigma_2,(1))$ $\diamond L(\sigma_3,(n-2))$	$\frac{(q^n-1)(q^{n-1}-1)}{(q-1)^2}$	$\deg(\sigma_i)=1, \sigma_i \neq \sigma_j, n>6,$ $(n,q) \neq (7,2)$

Proof. First, we show that if $\dim L \leq q^{3n-9}$, then L is one of the modules in the table. Take

$$L \cong L(\sigma_1, \lambda_1) \diamond \ldots \diamond L(\sigma_a, \lambda_a)$$

where the σ_i of degree d_i and the σ_i-regular partitions λ_i of k_i are as in Theorem B. Also let x_i be the number of nodes in the diagram of λ_i outside of the first row and set $n_i = d_i k_i$, for each $i = 1, \ldots, a$. By Theorem B and Lemma 3.3, we may assume that $(n_1, \ldots, n_a) = (n), (r, n-r)$ with $r \leq 2$ or $(1,1,n-2)$. Consider the case $(1,1,n-2)$, when we need to show that $d_3 = 1$ and $x_3 = 0$. Well, if $d_3 > 1$ then $\dim L > q^{2n-3} q^{\frac{(n-2)^2}{4}-2}$ using Lemmas 3.1–3.2, which for $n \geq 5$ is $> q^{3n-9}$. So, $d_3 = 1$. Now suppose $x_3 > 0$. Then, by Lemma 3.1 and Theorem B, $\dim L > q^{2n-3}q^{n-3} > q^{3n-9}$, so $x_3 = 0$. For the case $(2, n-2)$, the same argument forces $d_2 = 1, x_2 = 0$. For $(1, n-1)$, first suppose that $d_2 > 1$. Then, $\dim L > q^{n-1}q^{\frac{(n-1)^2}{4}-2}$ which is easily checked to be $\geq q^{3n-9}$ for $n \geq 5$. So $d_2 = 1$. Also if $x_2 > 1$, then $\dim L > q^{n-1}q^{2n-6} > q^{3n-9}$ using Lemma 3.1 and Theorem B. So $x_2 \leq 1$. Finally, consider the case $a = 1$. If $d_1 \geq 2$, we get $\dim L > q^{\frac{n^2}{4}-2}$ which easily forces $n \leq 8$. A little further direct calculation reveals the only possibilities are then $d_1 = 2$ and $n = 6$, or $d_1 = 2, n = 8$ and $q = 2$. Finally, if $d_1 = 1$, and $x_1 \geq 3$, then $\dim L > q^{3n-9}$ applying Theorem 2.7 (note the equality there is certainly strict for $x_1 \geq 3$).

To calculate the dimensions of the entries in the table, one uses the hook formula and the following fragment of the unipotent decomposition matrix of $\mathrm{GL}_n(\mathbb{F}_q)$ (in the notation of [13]):

	(n)	$(n-1,1)$	$(n-2,2)$	$(n-2,1^2)$
(n)	1	0	0	0
$(n-1,1)$	δ_n	1	0	0
$(n-2,2)$	κ	δ_{n-2}	1	0
$(n-2,1^2)$	0	δ_n	0	1

where

$$\kappa = \begin{cases} 1 & \text{if } e = 2 \text{ and } n \equiv 1 \text{ or } 2 \pmod{2p}, \\ \delta_{n-1} & \text{if } e > 2, \\ 0 & \text{otherwise.} \end{cases}$$

The entries in this decomposition matrix follow at once from [13, Theorem 6.22] (for hook partitions) or [11, Theorem 20.6] (for two row partitions).

Finally, we need to determine which of the remaining possibilities do indeed satisfy the bound. Certainly, the largest dimension of any of our modules is

$$\frac{(q^n-1)(q^{n-1}-1)}{(q-1)^2} < q^{2n-2+\delta_{q,2}},$$

so for $n \geq 7 + \delta_{q,2}$, all entries do definitely satisfy the bound $\dim L \leq q^{3n-9}$. Further calculation for $n < 7 + \delta_{q,2}$ gives rise to some extra exclusions for small n and q, as in the table. □

3.5. **Remark.** There are several irreducible FG-modules with $q^{3n-9} < \dim L \leq q^{3n-8}$. The smallest of these is the module $L(\sigma, (n-3,3))$ for σ of degree 1, whose dimension generically is $\frac{(q^n-1)(q^{n-1}-q)(q^{n-2}-q^2)}{(q^3-1)(q^2-1)(q-1)}$.

References

[1] H. Andersen, P. Polo, and Wen K., Representations of quantum algebras, *Invent. Math.* **104** (1991), 1–59.

[2] H. Andersen and Wen K., Representations of quantum algebras: the mixed case, *J. reine ang. Math.* **427** (1992), 35–50.

[3] J. Brundan, R. Dipper and A. Kleshchev, Quantum linear groups and representations of $\mathrm{GL}_n(\mathbb{F}_q)$, *preprint*, University of Oregon, 1999.

[4] R. Dipper, On the decomposition numbers of finite general linear groups I: *Trans. Amer. Math. Soc.* **290** (1985), 315–344; II: *Trans. Amer. Math. Soc.* **292** (1985), 123–133.

[5] R. Dipper and J. Du, Harish-Chandra vertices and Steinberg's tensor product theorem for general linear groups in non-describing characteristic, *Proc. London Math. Soc.* **75** (1997), 559–599.

[6] R. Dipper and G. D. James, Identification of the irreducible modular representations of $\mathrm{GL}_n(q)$, *J. Algebra* **104** (1986), 266–288.

[7] J. Du, A note on quantized Weyl reciprocity at roots of unity, *Alg. Colloq.* **2** (1995), 363–372.

[8] J. A. Green, The characters of the finite general linear groups, *Trans. Amer. Math. Soc.* **80** (1955), 402–447.

[9] R. Guralnick and Pham Huu Tiep, Low-dimensional representations of special linear groups in cross characteristics, *Proc. London Math. Soc.* **78** (1999), 116–138.

[10] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th edition, Oxford University Press, 1960.

[11] G. D. James, *Representations of general linear groups*, London Math. Soc. Lecture Notes 94, Cambridge University Press, 1984.

[12] G. D. James, The irreducible representations of the finite general linear groups, *Proc. London Math. Soc.* **52** (1986), 236–268.

[13] G. D. James, The decomposition matrices of $\mathrm{GL}_n(q)$ for $n \leq 10$, *Proc. London Math. Soc.* **60** (1990), 225–265.

[14] G. Lusztig, Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, *J. Am. Math. Soc.* **3** (1990), 257–297.

[15] G. Lusztig, Quantum groups at roots of 1, *Geom. Ded.* **35** (1990), 89–114.

[16] A. A. Premet, Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic, *Math. USSR Sbornik* **61** (1988), 167–183.

[17] I. D. Suprunenko, The invariance of the set of weights of irreducible representations of algebraic groups and Lie algebras of type A_l with restricted highest weight under reduction modulo p, *Vesci Akad. Navuk BSSR, ser. fiz.-mat.* **2** (1983), 18–22. (Russian)

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