CONNECTIONS BETWEEN THE REPRESENTATIONS OF THE SYMMETRIC GROUP AND THE SYMPLECTIC GROUP IN CHARACTERISTIC 2

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Introduction

Let K be a field of positive characteristic p, l a positive integer and E a vector space of dimension $2l$ over K. Let $b : E \times E \to K$ be a non-degenerate alternating bilinear form. The group of all isometries of b is the symplectic group $\mathrm{Sp}_{2l}(K)$. $\mathrm{Sp}_{2l}(K)$ acts on the exterior powers $\wedge^r E$ and the composition factors of these modules are known to be the fundamental irreducible modules, i.e. the irreducible modules labelled by the fundamental dominant weights of $\mathrm{Sp}_{2l}(K)$. If ω_r is a fundamental weight we denote the irreducible $\mathrm{Sp}_{2l}(K)$-module with highest weight ω_r by $L(\omega_r)$. Recursive formulae for the dimensions of the fundamental irreducible modules were given by Premet and Suprunenko in [26] when $p > 2$. These results were generalized to a field of an arbitrary characteristic by Adamovich [1, 2]. The multiplicities of the composition factors can also be found in [1, 2, 26]. The numerical results of Premet and Suprunenko depend on certain properties of the representations, corresponding to two-part partitions, of appropriate symmetric groups.

Suppose then that K has characteristic 2. It is well known that there is an embedding of the symmetric group Σ_{2l+1} into $\mathrm{Sp}_{2l}(K)$. In the first section of this paper we show that each composition factor for the action of $\mathrm{Sp}_{2l}(K)$ on $\wedge^r E$ remains irreducible on restriction to Σ_{2l+1}. More precisely, we prove that the irreducible module $L(\omega_r)$ labelled by the fundamental weight ω_r $(1 \le r \le l)$, is irreducible on restriction to Σ_{2l+1} and its restriction is isomorphic to the irreducible module $D^{(2l+1-r,r)}$ labelled by the partition $(2l+1-r, r)$ of $2l+1$. Thus the dimension problem for the fundamental irreducible modules of $\mathrm{Sp}_{2l}(K)$ is equivalent to the dimension problem for the irreducible modules $D^{(2l+1-r,r)}$ of Σ_{2l+1}, which has been investigated in some detail in [10].

In view of the fact that the l fundamental irreducible modules of $\mathrm{Sp}_{2l}(K)$ remain irreducible on restriction to Σ_{2l+1}, it seems reasonable to ask whether there any other irreducible modules of $\mathrm{Sp}_{2l}(K)$ with this property. We have not found any such examples, but the question raised does lead to some further interesting investigations. Provided $l \ge 2$, we actually have an embedding

$$\Sigma_{2l+1} < \Sigma_{2l+2} \le \mathrm{Sp}_{2l}(K),$$

compatible with the previous embedding for Σ_{2l+1}. The restriction of $L(\omega_r)$ to Σ_{2l+2} is of course irreducible for $1 \le r \le l$, with restriction isomorphic to the irreducible module $D^{(2l+2-r,r)}$ labelled by the partition $(2l+2-r, r)$ of $2l+2$. In section 2 of this paper, we show that if l is even, the irreducible module $L(\omega_1 + \omega_{2r})$ with highest weight $\omega_1 + \omega_{2r}$ is

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irreducible on restriction to Σ_{2l+2}, with restriction isomorphic to $D^{(2l+1-2r,2r,1)}$. The case when $r = l/2$ is particularly noteworthy. It is well known that $L(\omega_1+\omega_l)$ is expressible as the tensor product $L(\omega_1)\otimes L(\omega_l)$. $L(\omega_1)$ is the natural module E of dimension $2l$ and $L(\omega_l)$ is the spin module of dimension 2^l. The restrictions of $L(\omega_1)$ and $L(\omega_l)$ to Σ_{2l+2} are the irreducible modules $D^{(2l+1,1)}$ and $D^{(l+2,l)}$, respectively. We call $D^{(l+2,l)}$ the spin module for Σ_{2l+2}. Our result referred to above implies that $D^{(2l+1,1)}\otimes D^{(l+2,l)}$ is irreducible and isomorphic to $D^{(l+1,l,1)}$. Thus we obtain an example of what seems to be an uncommon phenomenon, the tensor product of two non-trivial irreducible modules for the symmetric group that is itself irreducible. Such non-trivial irreducible tensor products cannot occur for the symmetric group when the underlying field has characteristic 0, by [30]. Moreover, by [6], non-trivial tensor products can not even be homogeneous.

Following on the example of an irreducible tensor product involving the spin module, in Section 3 we investigate what can be said in general about tensoring with the spin module. We show that there is a connection here between tensoring with the spin module and the double of a spin regular partition, as defined by Benson in [4]. We find further irreducible tensor products using the spin modules and certain irreducible modules labelled by two-part partitions in the case of Σ_{10} and Σ_{14}. We also find examples of homogeneous tensor products. While our conclusions are far from definitive in most cases, we feel that the topic of tensoring with the spin module is worthy of further investigation.

We note that both the problem on irreducible restrictions from symplectic groups to symmetric groups discussed in sections 1,2 and the problem on the irreducible tensor products discussed in section 3 are of importance for describing the maximal subgroups in finite classical groups, cf. [3, 18].

We complete this introduction with a conjecture.

Conjecture. Let D_1 and D_2 be two irreducible $F\Sigma_n$-module of dimensions greater than 1. Then $D_1\otimes D_2$ is irreducible if and only if $p=2$, $n=2+4l$ for some positive integer l, one of the modules corresponds to the partition $(2l+2,2l)$ and the other corresponds to a partition of the form $(n-2j-1,2j+1)$, $0\le j<l$. Moreover, in the exeptional cases one has

$$D^{(2l+2,2l)}\otimes D^{(n-2j-1,2j+1)}\cong D^{(2l+1-j,2l-j,j+1,j)}.$$

1. Restriction of the fundamental irreducible modules for $\mathrm{Sp}_{2l}(K)$ to Σ_{2l+1}

Let K be a field of characteristic 2 and let n be a positive integer. Let $x_1,\dots,x_n$ be commuting algebraically independent indeterminates over K. Let r be an integer satisfying $1\le r\le n$ and let P_r denote the set of monomials of the form $x_{i_1}x_{i_2}\dots x_{i_r}$ in $K[x_1,\dots,x_n]$, where $i_1, i_2,\dots, i_r$ are different integers satisfying $1\le i_1,i_2,\dots,i_r\le n$. We note that, since the indeterminates commute, $x_{i_1}x_{i_2}\dots x_{i_r}=x_{j_1}x_{j_2}\dots x_{j_r}$ if and only if the sets of indices $\{i_1,i_2,\dots,i_r\}$ and $\{j_1,j_2,\dots,j_r\}$ are equal.

The symmetric group Σ_n of degree n acts on P_r by the rule

$$\sigma\cdot(x_{i_1}x_{i_2}\cdots x_{i_r})=x_{\sigma(i_1)}x_{\sigma(i_2)}\dots x_{\sigma(i_r)}.$$

The corresponding permutation $K\Sigma_n$-module is denoted by $M^{(n-r,r)}$. We define the symmetric bilinear form f_r on $M^{(n-r,r)}\times M^{(n-r,r)}$ by the condition that the basis

$$\{x_{i_1}x_{i_2}\dots x_{i_r} : 1\le i_1<i_2<\dots<i_r\le n\}$$

of $M^{(n-r,r)}$ is orthonormal with respect to f_r.

Assume now that $1\le 2r\le n$ and let $a_1, b_1,\dots, a_r, b_r$ be different integers lying between 1 and n. Define a polynomial $z(a_1,b_1,\dots,a_r,b_r)$ by

$$z(a_1,b_1,\dots,a_r,b_r)=(x_{a_1}+x_{b_1})\dots(x_{a_r}+x_{b_r}).$$

Clearly, for $\sigma \in \Sigma_n$, we have

$$\sigma \cdot z(a_1, b_1, \dots, a_r, b_r) = z(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_r), \sigma(b_r)).$$

Thus all such $z(a_1, b_1, \dots, a_r, b_r)$ span a $K\Sigma_n$-submodule of $M^{(n-r,r)}$. This submodule is called the Specht module $S^{(n-r,r)}$. The following lemma is a special case of results proved by James, [14, 4.9 and 11.1].

Lemma 1.1. *Suppose that $2r < n$. Then, if $\operatorname{rad} S^{(n-r,r)}$ denotes the radical of $S^{(n-r,r)}$ with respect to f_r, the quotient*

$$D^{(n-r,r)} = S^{(n-r,r)}/\operatorname{rad} S^{(n-r,r)}$$

is non-zero and irreducible.

According to [14, 17.17], $M^{(n-r,r)}$ has a filtration with factors S^n, $S^{(n-1,1)}, \dots, S^{(n-r,r)}$. Since $\dim M^{(n-r,r)} = \binom{n}{r}$, it follows that

$$\dim S^{(n-r,r)} = \tbinom{n}{r} - \tbinom{n}{r-1}.$$

We define a map $\partial : K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]$ by

$$\partial = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}.$$

Given different integers i_0, i_1, $\dots$, i_r lying between 1 and n, we set

$$d(i_0, i_1, \dots, i_r) = \partial(x_{i_0} x_{i_1} \dots x_{i_r}).$$

We observe that for any $\sigma \in \Sigma_n$,

$$\sigma \cdot d(i_0, i_1, \dots, i_r) = d(\sigma(i_0), \sigma(i_1), \dots, \sigma(i_r)). \tag{1}$$

Let $L(n,r)$ denote the span of all such elements $d(i_0, i_1, \dots, i_r)$. Equation (1) shows that $L(n,r)$ is a $K\Sigma_n$-submodule of $M^{(n-r,r)}$.

Lemma 1.2. *Let i_0, i_1, $\dots$, i_r be different integers with $1 < i_0, i_1, \dots, i_r \le n$. Then*

$$d(i_0, i_1, \dots, i_r) = \sum_{s=0}^{r} d(1, i_0, \dots, \hat{i}_s, \dots, i_r). \tag{2}$$

(The symbol $\hat{i}_s$ means that the term i_s is omitted in the formula.)

Proof. Note that

$$\sum_{2 \le k,j \le n} \frac{\partial^2}{\partial x_k \partial x_j}(x_1 x_{i_0} \dots x_{i_r}) = 0,$$

since K has characteristic 2. Therefore,

$$\begin{aligned}
& \sum_{s=0}^{r} d(1, i_0, \dots, \hat{i}_s, \dots, i_r) = \sum_{s=0}^{r} \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(x_1 x_{i_0} \dots \hat{x}_{i_s} \dots x_{i_r}) \\
= \; & \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \sum_{s=0}^{r} (x_1 x_{i_0} \dots \hat{x}_{i_s} \dots x_{i_r}) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \sum_{j=2}^{n} \frac{\partial}{\partial x_j}(x_1 x_{i_0} \dots x_{i_r}) \\
= \; & \frac{\partial}{\partial x_1} \sum_{j=2}^{n} \frac{\partial}{\partial x_j}(x_1 x_{i_0} \dots x_{i_r}) + \sum_{2 \le k,j \le n} \frac{\partial^2}{\partial x_k \partial x_j}(x_1 x_{i_0} \dots x_{i_r}) \\
= \; & \sum_{j=2}^{n} \frac{\partial}{\partial x_j}(x_{i_0} x_{i_1} \dots x_{i_r}) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(x_{i_0} x_{i_1} \dots x_{i_r}) = d(i_0, i_1, \dots, i_r),
\end{aligned}$$

as demanded. ∎

Corollary 1.3. *The subset*

$$\mathcal{B}_r = \{d(1, i_1, \ldots, i_r) : 1 < i_1 < \ldots < i_r \le n\}$$

is a basis for $L(n, r)$*.*

Proof. It follows from equation (2) of Lemma 1.2 that $\mathcal{B}_r$ generates $L(n, r)$. To prove the linear independence of the elements of $\mathcal{B}_r$, it suffices to note that the only element of $\mathcal{B}_r$ involving the term $x_{i_1} x_{i_2} \ldots x_{i_r}$ is $d(1, i_1, \ldots, i_r)$. □

The module $M^{(n-1,1)}$ is the natural permutation module for $K\Sigma_n$. The Specht module $S^{(n-1,1)}$ is the submodule of $M^{(n-1,1)}$ which consists of sums $\alpha_1 x_1 + \cdots + \alpha_n x_n$, where the $\alpha_i \in K$ and $\alpha_1 + \cdots + \alpha_n = 0$. We will denote the element

$$(x_{i_1} + x_{i_0}) \wedge (x_{i_2} + x_{i_0}) \wedge \ldots \wedge (x_{i_r} + x_{i_0}) \in \wedge^r S^{(n-1,1)}$$

by $v(i_0, i_1, \ldots, i_r)$. Here, the integers i_0, i_1, $\ldots$, i_r are all different and satisfy $1 \le i_0, i_1, \ldots, i_r \le n$. Note that the exterior power $\wedge^r S^{(n-1,1)}$ is a $K\Sigma_n$-module in a natural way, and

$$\sigma \cdot v(i_0, i_1, \ldots, i_r) = v(\sigma(i_0), \sigma(i_1), \ldots, \sigma(i_r)). \tag{3}$$

Moreover,

$$v(i_0, i_1, \ldots, i_r) = \sum_{s=0}^{r} v(1, i_0, \ldots, \hat{i}_s, \ldots, i_r). \tag{4}$$

Indeed, this follows from the equality

$$\begin{aligned} & (x_{i_1} + x_{i_0}) \wedge \ldots \wedge (x_{i_r} + x_{i_0}) \\ = \ & ((x_{i_1} + x_1) + (x_{i_0} + x_1)) \wedge \ldots \wedge ((x_{i_r} + x_1) + (x_{i_0} + x_1)) \,. \end{aligned}$$

The following result is proved by Peel in [25, pp.146-147]. Peel's notation differs from ours, in that he uses the Specht module $S^{(n-r,1^r)}$, corresponding to the partition $(n-r, 1^r)$ of n, in place of $\wedge^r S^{(n-1,1)}$. The $K\Sigma_n$-modules $S^{(n-r,1^r)}$ and $\wedge^r S^{(n-1,1)}$ are isomorphic, the element $\Delta(x_{i_0}, x_{i_1}, \ldots, x_{i_r})$ in Peel's notation corresponding to our wedge product $v(i_0, i_1, \ldots, i_r)$. We give a proof, similar to that of Peel, using our notation.

Lemma 1.4. *There exists a* $K\Sigma_n$*-monomorphism*

$$\eta_r : \wedge^r S^{(n-1,1)} \to M^{(n-r,r)}$$

with

$$\eta_r(v(i_0, i_1, \ldots, i_r)) = d(i_0, i_1, \ldots, i_r).$$

Proof. It follows from equation (4) that

$$\{v(1, i_1, \ldots, i_r) : 1 < i_1 < \ldots < i_r \le n\}$$

is a basis of $\wedge^r S^{(n-1,1)}$. Using Corollary 1.3, we see that we may define a vector space isomorphism $\eta_r : \wedge^r S^{(n-1,1)} \to L(n, r)$ with

$$\eta_r(v(1, i_1, \ldots, i_r)) = d(1, i_1, \ldots, i_r).$$

Equations (2) and (4) show that

$$\eta_r(v(i_0, i_1, \ldots, i_r)) = d(i_0, i_1, \ldots, i_r)$$

and equations (1) and (3) show that η_r is a $K\Sigma_n$- homomorphism, as required. □

Let ψ denote the restriction to $S^{(n-1,1)} \times S^{(n-1,1)}$ of the symmetric bilinear form f_1 defined on $M^{(n-1,1)} \times M^{(n-1,1)}$. Let v_1, w_1, ... , v_r, w_r be vectors in $S^{(n-1,1)}$. We define a symmetric bilinear form $\psi_r : \wedge^r S^{(n-1,1)} \times \wedge^r S^{(n-1,1)} \to K$ by the formula

$$\psi_r(v_1 \wedge v_2 \wedge \ldots \wedge v_r, w_1 \wedge w_2 \wedge \ldots \wedge w_r) = \det(\psi(v_i, w_j))_{1\le i,j\le r}.$$

Lemma 1.5. *Suppose that $1 \le r < n$. Then*

$$\begin{aligned}&\psi_r(v(1,i_1,\ldots,i_r), v(1,j_1,\ldots,j_r))\\ =&f_r(d(1,i_1,\ldots,i_r), d(1,j_1,\ldots,j_r))\end{aligned} \tag{5}$$

for all i_1, j_1, ... , i_r, j_r with $1 < i_1 < \ldots < i_r \le n$, $1 < j_1 < \ldots < j_r \le n$.

Proof. Let $I = \{i_1, i_2, \ldots, i_r\}$ and $J = \{j_1, j_2, \ldots, j_r\}$. It is straightforward to check that the right hand side of equation (5) is equal to

$$\begin{cases} r+1, & \text{if } I = J;\\ 1, & \text{if } |I\cap J| = r-1;\\ 0, & \text{otherwise.}\end{cases}$$

On the other hand, the left hand side of equation (5) equals

$$\det(\delta_{i_s j_t} + 1)_{1\le s,t\le r},$$

where δ_{ij} denotes the usual Kronecker delta. Set

$$A = (\delta_{i_s j_t} + 1)_{1\le s,t\le r}.$$

If $I = J$, then A is the $r\times r$ matrix with zeros on the main diagonal and ones everywhere else. One can easily check that $\det A = (-1)^{r-1}(r-1)$, which is the same as $r+1$ in characteristic 2.

If $|I \cap J| = r-1$, then A is the matrix with zeros at all but one position on the main diagonal and ones everywhere else. Then we calculate that $\det A = 1$, since K has characteristic 2. Finally, if $|I\cap J| < r-1$, then A has two identical rows (consisting entirely of ones), and thus $\det A = 0$, as required. □

Thus η_r identifies $L(n,r)$ with $\wedge^r S^{(n-1,1)}$. Moreover, η_r is an isometry between the natural forms defined on these spaces. For our next result, which shows that $L(n,r)$ contains $S^{(n-r,r)}$, we quote part of the proof of [25, Theorem 4, p.148].

Lemma 1.6. *Suppose that $2r \le n$. Let a_1, b_1, ... , a_r, b_r be different integers that satisfy $1 \le a_1, b_1, \ldots, a_r, b_r \le n$. Then*

$$z(a_1, b_1, \ldots, a_r, b_r) = \sum d(1, c_1, \ldots, c_r),$$

where the summation runs over all sequences $(c_1, \ldots, c_r)$ in which, for each i, either $c_i = a_i$ or $c_i = b_i$.

Corollary 1.7. *Suppose that $2r \le n$. Let T_r be the subspace of $\wedge^r S^{(n-1,1)}$ spanned by all vectors of the form*

$$(x_{a_1} + x_{b_1}) \wedge \ldots \wedge (x_{a_r} + x_{b_r})$$

where a_1, b_1, ... , a_r, b_r are different integers that satisfy $1 \le a_1, b_1, \ldots, a_r, b_r \le n$. Then

(i) *T_r is a $K\Sigma_n$-submodule of $\wedge^r S^{(n-1,1)}$ and $\eta_r \downarrow_{T_r}$ is a $K\Sigma_n$-isomorphism of T_r onto $S^{(n-r,r)}$. Moreover, $\psi_r(v,w) = f_r(\eta_r(v), \eta_r(w))$ for any v and w in $\wedge^r S^{(n-1,1)}$.*

(ii) *If $\operatorname{rad} T_r$ denotes the radical of T_r with respect to ψ_r, $\operatorname{rad} T_r$ is a $K\Sigma_n$-submodule of T_r and*

$$T_r/\operatorname{rad} T_r \cong D^{(n-r,r)}.$$

Proof. By Lemma 1.6, the preimage $\eta_r^{-1}(S^{(n-r,r)})$ is spanned by the vectors of the form

$$\sum v(1, c_1, \ldots, c_r),$$

where the summation extends over all sequences $(c_1, \ldots, c_r)$ with $c_i = a_i$ or b_i. However,

$$\begin{aligned} & \sum v(1, c_1, \ldots, c_r) = \sum (x_{c_1} + x_1) \wedge \ldots \wedge (x_{c_r} + x_1) \\ = \quad & (x_{a_1} + x_1 + x_{b_1} + x_1) \wedge \ldots \wedge (x_{a_r} + x_1 + x_{b_r} + x_1) \\ = \quad & (x_{a_1} + x_{b_1}) \wedge \ldots \wedge (x_{a_r} + x_{b_r}). \end{aligned}$$

This proves the first part of (i). The second part of (i) follows from Lemma 1.5. Finally, (ii) follows from (i) and Lemma 1.1. □

We note that the first part of Corollary1.7 (i) is proved in [25, Theorem 4].

We wish now to bring the symplectic group into consideration. Let l be a positive integer and let E be a vector space of dimension $2l$ over K. Let $b : E \times E \to K$ be a non-degenerate alternating bilinear form. Let $\{e_1, \ldots, e_l, e_{-1}, \ldots, e_{-l}\}$ be a symplectic basis of E. Thus we have

$$b(e_i, e_j) = b(e_{-i}, e_{-j}) = 0, \quad b(e_i, e_{-j}) = \delta_{ij}. \qquad (1 \leq i, j \leq l)$$

The group of all isometries of b is the symplectic group $\mathrm{Sp}_{2l}(K)$. $\mathrm{Sp}_{2l}(K)$ acts on the exterior powers $\wedge^r E$ and it is the composition factors of this action that we wish to investigate. Suppose that r satisfies $1 \leq r \leq l$. Elements $v_1 \wedge \ldots \wedge v_r$ of $\wedge^r E$ such that $v_1, \ldots, v_r$ are basis vectors of a totally isotropic r-dimensional subspace of E are called isotropic r-vectors. Let $F_r(K)$ denote the subspace of $\wedge^r E$ spanned by all the isotropic r-vectors. $F_r(K)$ is clearly a $K\mathrm{Sp}_{2l}(K)$-submodule of $\wedge^r E$. Witt's theorem shows that $\mathrm{Sp}_{2l}(K)$ permutes the totally isotropic r-dimensional subspaces of E transitively. It follows that if M is a $K\mathrm{Sp}_{2l}(K)$- submodule of $\wedge^r E$ that contains an isotropic r-vector, then $F_r(K) \leq M$. (This is true for any field K.) We shall now prove the inequality

$$\dim F_r(K) \leq \tbinom{2l}{r} - \tbinom{2l}{r-2}.$$

We do this by using the representation theory of the symplectic group over fields of characteristic 0.

By a theorem of MacLane [24, Theorem 2] (see also [17, Theorem 1.5 on p. 142]), there is a complete discrete valuation ring R of characteristic 0 with unique maximal ideal $2R$ such that $R/2R$ is isomorphic to K. Let L be the quotient field of R. Let V be a vector space of dimension $2l$ over L and let $\beta : V \times V \to L$ be a non-degenerate alternating bilinear form. Let $\{u_1, \ldots, u_l, u_{-1}, \ldots, u_{-l}\}$ be a symplectic basis of V. Let V_R denote the R-lattice in V spanned by the R-linear combinations of this symplectic basis.

There are contraction maps

$$\partial_r : \wedge^r V \to \wedge^{r-2} V$$

given by

$$\partial_r(v_1 \wedge \cdots \wedge v_r) = \sum_{i<j} f(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_r,$$

where $v_1, \ldots, v_r$ are any vectors in V. See, for example, [12, p.260]. We take ∂_0 and ∂_1 to be the zero maps of $\wedge^0 V = L$ and $\wedge^1 V = V$.

We now define a contraction operator ∂ on the whole exterior algebra $\wedge V$, whose restriction to $\wedge^r V$ equals ∂_r. It is not hard to see that ∂ commutes with the action of $\mathrm{Sp}_{2l}(L)$ on $\wedge V$ and thus it follows that $\ker \partial \cap \wedge^r V$ is an $L\mathrm{Sp}_{2l}(L)$-submodule of $\wedge^r V$. It is known that if $1 \leq r \leq l$, $\ker \partial \cap \wedge^r V$ is the irreducible $L\mathrm{Sp}_{2l}(L)$-module with fundamental highest weight ω_r, [12, 17.5]. Since $\ker \partial \cap \wedge^r V$ contains, for example, the isotropic r-vector $u_1 \wedge \ldots \wedge u_r$, it follows from the irreducibility of $\ker \partial \cap \wedge^r V$, together with our remarks on the transitive action of $\mathrm{Sp}_{2l}(L)$ on isotropic r-vectors, that $\ker \partial \cap \wedge^r V = F_r(L)$, where $F_r(L)$ denotes the subspace of $\wedge^r V$ spanned by all the isotropic r-vectors. It is also true that ∂ maps $\wedge^r V$ onto $\wedge^{r-2} V$ in this case, and thus it follows that

$$\dim F_r(L) = \binom{2l}{r} - \binom{2l}{r-2}.$$

Now we set $F_r(R) = F_r(L) \cap \wedge^r V_R$. Then $F_r(R)$ is an R-lattice in $F_r(L)$ of rank $\dim F_r(L)$, see for example [8, ch. VII, \$4, n°1].

Given a non-zero element u of V and a non-zero element λ of L, the isometry $t_{\lambda,u}$ of β given by

$$t_{\lambda,u}(v) = v + \lambda\beta(v,u)u$$

is called a symplectic transvection. Let $\mathrm{Sp}_{2l}(R)$ denote the subgroup of $\mathrm{Sp}_{2l}(L)$ generated by all symplectic transvections as above, where $\lambda \in R$ and $u \in V_R$. It is clear that V_R is invariant under $\mathrm{Sp}_{2l}(R)$ and it follows that $F_r(R)$ is also invariant under $\mathrm{Sp}_{2l}(R)$.

Let α denote the natural homomorphism $R \to R/2R \cong K$. We also let α denote the epimorphism $V_R \to E$ given by

$$\alpha\Big(\sum_{i=-l}^{l} \lambda_i u_i\Big) = \sum_{i=-l}^{l} \alpha(\lambda_i) e_i.$$

It is clear then that $\alpha(\beta(u,w)) = b(\alpha(u), \alpha(w))$ for all u and w in V_R. There is also a group homomorphism $\mathrm{Sp}_{2l}(R) \to \mathrm{Sp}_{2l}(K)$, which we will denote by θ, induced by reducing coefficients modulo 2. We readily check that

$$\theta(\sigma)\alpha(v) = \alpha(\sigma v) \tag{6}$$

for all $\sigma \in \mathrm{Sp}_{2l}(R)$ and all $v \in V_R$. The image of the symplectic transvection $t_{\lambda,u}$ under θ is the symplectic transvection $t_{\alpha(\lambda),\alpha(u)}$. Since we can clearly obtain all symplectic transvections of $\mathrm{Sp}_{2l}(K)$ in this way, and $\mathrm{Sp}_{2l}(K)$ is known to be generated by its transvections, it follows that θ is surjective.

We note that $\wedge^r V_R$ is an R-lattice in $\wedge^r V$ invariant under $\mathrm{Sp}_{2l}(R)$ and $F_r(R)$ is an R-sublattice in $\wedge^r V_R$ also invariant under $\mathrm{Sp}_{2l}(R)$. Moreover, α induces an epimorphism $\alpha_r : \wedge^r V_R \to \wedge^r E$, defined by

$$\alpha_r(v_1 \wedge \ldots \wedge v_r) = \alpha(v_1) \wedge \ldots \wedge \alpha(v_r) \tag{7}$$

for any $v_1, \ldots, v_r$ in V_R. Finally, for σ in $\mathrm{Sp}_{2l}(R)$ we have

$$\theta(\sigma)\alpha_r(v_1 \wedge \ldots \wedge v_r) = \alpha_r\sigma(v_1 \wedge \ldots \wedge v_r), \tag{8}$$

a result that follows easily from equations (6) and (7). With these preliminaries described, we can proceed to the proof of our inequality.

Lemma 1.8. *Let K be a field of characteristic 2 and let l be a positive integer. Let E be a vector space of dimension $2l$ over K and let $b : E \times E \to K$ be a non-degenerate alternating bilinear form. Given an integer r with $1 \le r \le l$, let $F_r(K)$ denote the subspace of $\wedge^r E$ spanned by the isotropic r-vectors. Then we have*

$$\dim F_r(K) \le \binom{2l}{r} - \binom{2l}{r-2}.$$

Proof. We identify $\mathrm{Sp}_{2l}(K)$ with the image of $\mathrm{Sp}_{2l}(R)$ under θ. The subspace $\alpha_r F_r(R)$ of $\wedge^r E$ has dimension equal to the rank of $F_r(R)$, which is $\binom{2l}{r} - \binom{2l}{r-2}$. Equation (8) shows that it is also $\mathrm{Sp}_{2l}(K)$-invariant. Since $\alpha_r F_r(R)$ contains the isotropic r-vector $e_1 \wedge \ldots \wedge e_r$, our earlier remarks show that $\alpha_r F_r(R)$ contains $F_r(K)$ and the desired inequality follows. □

Returning to Σ_n, we suppose that $n = 2l+1$ is odd, where l is a positive integer. The Specht module $S^{(2l,1)}$ has dimension $2l$ and the form ψ defined on $S^{(2l,1)} \times S^{(2l,1)}$ is alternating and non-degenerate. For notational convenience, we will identify $S^{(2l,1)}$ with the $2l$-dimensional vector space E previously studied and the form ψ with b. We thus have the inclusion $\Sigma_{2l+1} \le \mathrm{Sp}_{2l}(K)$.

Lemma 1.9. *Let K be a field of characteristic 2 and let l be a positive integer. Let r be an integer with $1 \le r \le l$ and let $F_r(K)$ denote the subspace of $\wedge^r S^{(2l,1)}$ spanned by the isotropic r-vectors. Let T_r be the $K\Sigma_{2l+1}$ submodule of $\wedge^r S^{(2l,1)}$ described in Corollary 1.7. Then we have $T_r = F_r(K)$ and thus*

$$\dim F_r(K) = \tbinom{2l}{r} - \tbinom{2l}{r-2}.$$

Proof. We know that T_r is spanned by r-vectors of the form $(x_{a_1} + x_{b_1}) \wedge \ldots \wedge (x_{a_r} + x_{b_r})$ where $a_1, b_1, \ldots, a_r, b_r$ are different integers lying between 1 and $2l+1$. Now if a_i, b_i, a_j, b_j are different integers as above, $x_{a_i} + x_{b_i}$ and $x_{a_j} + x_{b_j}$ are orthogonal with respect to ψ. Thus, T_r is spanned by isotropic r-vectors and it follows that $T_r \le F_r(K)$. Corollary 1.7 shows that T_r is isomorphic to the Specht module $S^{(2l+1-r,r)}$, and thus our remarks after Lemma 1.1 imply that

$$\dim T_r = \tbinom{2l+1}{r} - \tbinom{2l+1}{r-1}.$$

But

$$\tbinom{2l}{r} - \tbinom{2l}{r-2} = \tbinom{2l+1}{r} - \tbinom{2l+1}{r-1}$$

and we deduce that

$$\dim F_r(K) \ge \tbinom{2l}{r} - \tbinom{2l}{r-2}.$$

This inequality and the inequality in Lemma 1.8 imply that we have

$$\dim F_r(K) = \tbinom{2l}{r} - \tbinom{2l}{r-2}$$

and thus $T_r = F_r(K)$, as required. □

We note that Premet and Suprunenko obtained the same formula for the dimension of $F_r(K)$ when K has odd prime characteristic [26, Remark, p.1337]. Their dimension formula also depends on the characteristic 0 theory, as ours does.

Let $\mathrm{rad}\, F_r(K)$ denote the radical of $F_r(K)$ with respect to the bilinear form ψ_r. Then $F_r(K)/\mathrm{rad}\, F_r(K)$ is a $K\mathrm{Sp}_{2l}(K)$-module and we now show that it is irreducible.

Corollary 1.10. *Let K be a field of characteristic 2 and let l be a positive integer. Let r be an integer with $1 \le r \le l$ and let $F_r(K)$ denote the subspace of $\wedge^r S^{(2l,1)}$ spanned by the isotropic r-vectors. Then $F_r(K)/\mathit{rad}\, F_r(K)$ is an irreducible $K\mathrm{Sp}_{2l}(K)$-module whose restriction to Σ_{2l+1} is isomorphic to the irreducible module $D^{(2l+1-r,r)}$.*

Proof. The result follows from Corollary 1.7, since we have seen in Lemma 1.9 that $F_r(K) = T_r$. □

Suppose now that K is algebraically closed and E is our vector space of dimension $2l$ over K. We wish to identify the irreducible $K\mathrm{Sp}_{2l}(K)$-module $F_r(K)/\mathrm{rad}\, F_r(K)$ and the composition factors of $\wedge^r E$ in this case. With the usual notation, let $B = TU$ be a Borel subgroup of $\mathrm{Sp}_{2l}(K)$ containing the maximal torus T. We may choose a symplectic basis $\{e_1, \ldots, e_l, e_{-1}, \ldots, e_{-l}\}$ of E so that these vectors are weight vectors with respect to T with corresponding weights $\{\varepsilon_1, \ldots, \varepsilon_l, -\varepsilon_1, \ldots, -\varepsilon_l\}$ in the notation of [7, Table III]. Moreover, the l subspaces $\langle e_1\rangle$, $\langle e_1, e_2\rangle$, $\ldots$, $\langle e_1, e_2, \ldots, e_l\rangle$ are all B-invariant. Again with the usual notation, the fundamental dominant weights are

$$\omega_1 = \varepsilon_1, \quad \omega_2 = \varepsilon_1 + \varepsilon_2, \quad \ldots, \quad \omega_l = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_l.$$

Let $L(\omega_r)$ denote the fundamental irreducible module with highest weight ω_r.

Theorem 1.11. *Let K be an algebraically closed field of characteristic 2 and let l be a positive integer. Let E be a vector space of dimension $2l$ over K and let b be a non-degenerate alternating bilinear form on $E \times E$. Let r be an integer with $1 \leq r \leq l$ and let $F_r(K)$ denote the subspace of $\wedge^r E$ spanned by the isotropic r-vectors. Then the irreducible $K\mathrm{Sp}_{2l}(K)$-module $F_r(K)/\mathrm{rad}\, F_r(K)$ is isomorphic to $L(\omega_r)$. Thus $L(\omega_r)\downarrow_{\Sigma_{2l+1}} \cong D^{(2l+1-r,r)}$.*

Proof. The one-dimensional subspace spanned by the isotropic r-vector $e_1 \wedge \ldots \wedge e_r$ is contained in $F_r(K)$ and is B-invariant. Furthermore, since

$$\psi_r(e_1 \wedge \ldots \wedge e_r, e_{-1} \wedge \ldots \wedge e_{-r}) = 1,$$

$e_1 \wedge \ldots \wedge e_r + F_r(K)$ is a (non-zero) maximal vector in $F_r(K)/\mathrm{rad}\, F_r(K)$ for which the associated weight is $\varepsilon_1 + \cdots + \varepsilon_r = \omega_r$. The identification of $F_r(K)/\mathrm{rad}\, F_r(K)$ with $L(\omega_r)$ now follows from the isomorphism theorem for irreducible rational modules of semisimple algebraic groups (see, for example, [13, 31.3]). □

We note that there is a subgroup Σ_{2l+2} in $\mathrm{Sp}_{2l}(K)$ with $\Sigma_{2l+1} < \Sigma_{2l+2} \leq \mathrm{Sp}_{2l}(K)$ when $l \geq 2$. Indeed, let $D^{(2l+1,1)}$ be the $2l$-dimensional irreducible quotient of the Specht module $S^{(2l+1,1)}$. This is a faithful module provided $l \geq 2$. Then the bilinear form f_1 on $M^{(2l+1,1)} > S^{(2l+1,1)}$ gives rise to a non-degenerate Σ_{2l+2}-invariant alternating form on $D^{(2l+1,1)}$. Thus $\Sigma_{2l+2} \leq \mathrm{Sp}_{2l}(K)$ for $l \geq 2$. This agrees with the natural embedding $\Sigma_{2l+1} < \Sigma_{2l+2}$ and our previous embedding $\Sigma_{2l+1} < \mathrm{Sp}_{2l}(K)$ since the restriction of $D^{(2l+1,1)}$ to Σ_{2l+1} is naturally isomorphic to $S^{(2l,1)}$.

Corollary 1.12. *Assume the hypotheses and notation of Theorem 1.11 and also that $l \geq 2$. Then*

$$L(\omega_r)\downarrow_{\Sigma_{2l+2}} \cong D^{(2l+2-r,r)}$$

for $1 \leq r \leq l$.

Proof. It follows from Theorem 1.11 that $L(\omega_r)\downarrow_{\Sigma_{2l+2}}$ is irreducible. Put $L(\omega_r)\downarrow_{\Sigma_{2l+2}} \cong D^\lambda$, where λ is a 2-regular partition of $2l+2$. Then $D^\lambda \downarrow_{\Sigma_{2l+1}} \cong D^{(2l+1-r,r)}$. It follows from [20, Theorem 0.6] that $\lambda = (2l+2-r, r)$. □

We turn to a consideration of the $\mathrm{Sp}_{2l}(K)$ composition factors of $\wedge^r E$. In what follows, $L(\omega_0)$ will denote the trivial one-dimensional irreducible module.

Lemma 1.13. *Assume the notation of Theorem 1.11. Then the composition factors of $\wedge^r E$ have the form $L(\omega_{r-2i})$, where i is an integer satisfying $0 \leq 2i \leq r$.*

Proof. Following our previous notation, the weight vectors of $\wedge^r E$ with respect to T are the r-vectors $e_{i_1} \wedge \ldots \wedge e_{i_r}$, where the i_j are integers satisfying $i_1 < \ldots < i_r$ and $-l \leq i_j \leq l$. The corresponding weights have the form

$$\pm\varepsilon_{j_1} \pm \ldots \pm \varepsilon_{j_s},$$

where $s = r - 2k$, k is an integer satisfying $0 \leq 2k \leq r$, and $j_1, \ldots, j_s$ is a strictly increasing sequence of integers, each lying between 1 and l.

Since the action of $\mathrm{Sp}_{2l}(K)$ on $\wedge^r E$ is rational, the classification of finite dimensional irreducible rational modules for semisimple algebraic groups shows that any composition factor for $\mathrm{Sp}_{2l}(K)$ in $\wedge^r E$ is determined up to isomorphism by a dominant weight. We thus seek the dominant weights among the weights $\pm\varepsilon_{j_1} \pm \ldots \pm \varepsilon_{j_s}$ that occur in $\wedge^r E$. Now it is known that any weight is conjugate under the Weyl group to a unique dominant weight. Given the action of the Weyl group W of $\mathrm{Sp}_{2l}(K)$, we easily see that $\pm\varepsilon_{j_1} \pm \ldots \pm \varepsilon_{j_s}$ is W-conjugate to $\varepsilon_1 + \ldots + \varepsilon_s = \omega_s$, which is a dominant weight. Thus, the only dominant weights occurring among the weights of $\wedge^r E$ are the ω_s, with s as above, and the isomorphism theorem [13, 31.3] now tells us that the composition factors of $\wedge^r E$ have the form $L(\omega_s)$, as required. □

We proceed next to find the multiplicity of each composition factor $L(\omega_j)$ in $\wedge^r E$.

Theorem 1.14. *Assume the notation of Theorem 1.11. Let* $[\wedge^r E : L(\omega_j)]$ *denote the multiplicity of* $L(\omega_j)$ *as a composition factor of* $\wedge^r E$*. Then we have*

$$[\wedge^r E : L(\omega_j)] = \sum_{i=0}^{[(r-j)/2]} \binom{2l+2-2j}{r-j-2i}',$$

where $[x]$ *denotes the greatest integer not exceeding* x *and* $\binom{a}{b}' = 0$ *if* $\binom{a}{b}$ *is even, and* $\binom{a}{b}' = 1$ *otherwise.*

Proof. Since

$$E \downarrow^{Sp_{2l}(K)}_{\Sigma_{2l+1}} \cong S^{(2l,1)},$$

we conclude that

$$\wedge^r E \downarrow^{Sp_{2l}(K)}_{\Sigma_{2l+1}} \cong \wedge^r S^{(2l,1)}.$$

Now we remarked before the proof of Lemma 1.4 that $\wedge^r S^{(n-1,1)}$ and $S^{(n-r,1^r)}$ are isomorphic as $K\Sigma_n$-modules. Since, by Lemma 1.13, all composition factors of $\wedge^r E$ are of the form $L(\omega_j)$, it follows from Theorem 1.11 that all composition factors of $S^{(2l+1-r,1^r)}$ are of the form $D^{(2l+1-j,j)}$, and

$$[\wedge^r E : L(\omega_j)] = [S^{(2l+1-r,1^r)} : D^{(2l+1-j,j)}].$$

The calculation on [14, p.93] shows that

$$[S^{(2l+1-r,1^r)} : D^{(2l+1-j,j)}] = \sum_{i\geq 0} [S^{(2l+1-r+2i,r-2i)} : D^{(2l+1-j,j)}]$$

and the multiplicity $[S^{(2l+1-r+2i,r-2i)} : D^{(2l+1-j,j)}]$ is given by $\binom{2l+2-2j}{r-j-2i}'$ in accordance with [14, 24.15] . ∎

While the multiplicities above depend on the parity of binomial coefficients, the multiplicity $[\wedge E : L(\omega_j)]$ of $L(\omega_j)$ in the entire exterior algebra is more easily expressed, as it proves to be a power of 2. We begin by finding the multiplicity of the trivial module $L(\omega_0)$ in $\wedge E$.

Let n be a positive integer and let

$$n = 2^{c_1} + \cdots + 2^{c_t}$$

be the 2-adic decomposition of n, where the c_i are integers satisfying $0 \leq c_1 < \ldots < c_t$. Let r be an integer satisfying $1 \leq r \leq n$ and let

$$r = 2^{b_1} + \cdots + 2^{b_u}$$

be the 2-adic decomposition of r, with $0 \leq b_1 < \ldots < b_u$. Then $\binom{n}{r}$ is odd if and only if $\{b_1, \ldots, b_u\}$ is a subset of $\{c_1, \ldots, c_t\}$. The case that $r = 0$ will be covered by taking the empty subset of $\{c_1, \ldots, c_t\}$. Thus, since $\{c_1, \ldots, c_t\}$ has exactly 2^t subsets, there are 2^t different odd binomial coefficients $\binom{n}{r}$.

Consider now the exterior power $\wedge^k E$. It is well known that as a $K\mathrm{GL}_{2l}(K)$-module, $\wedge^{2l-k}E$ is isomorphic to the dual of $\wedge^k E$. But, as an $K\mathrm{Sp}_{2l}(K)$-module, $\wedge^k E$ is isomorphic to its dual, the isomorphism being provided by an invariant non-degenerate symmetric bilinear form constructed from the underlying alternating form b in the manner described before Lemma 1.5. Thus it follows that $\wedge^{2l-k}E$ is isomorphic to $\wedge^k E$ as a $K\mathrm{Sp}_{2l}(K)$-module.

Having made these observations, we proceed to the proof of the following multiplicity theorem.

Theorem 1.15. *Assume the notation of Theorem 1.11. Let*

$$l+1=2^{a_1}+\cdots+2^{a_s}$$

be the 2-adic decomposition of $l+1$, where the a_i are integers satisfying $0\le a_1<\ldots<a_s$. Then the multiplicity $[\wedge E:L(\omega_0)]$ of the trivial $K\mathrm{Sp}_{2l}(K)$-module in the exterior algebra is 2^{s+a_s-1}.

Proof. It follows from Theorem 1.14 that the trivial module is a composition factor of $\wedge^k E$, where $0\le k\le l$, if and only if k is even, say $k=2i$. By Theorem 1.14, the multiplicity of the trivial module in $\wedge^{2i}E$ equals the number of odd binomial coefficients among

$$\binom{2l+2}{0},\quad \binom{2l+2}{2},\ldots,\quad \binom{2l+2}{2i}.$$

Since $\binom{2a}{2b}\equiv\binom{a}{b}$ mod 2, this multiplicity is the number of odd numbers among

$$\binom{l+1}{0},\quad \binom{l+1}{1},\ldots,\quad \binom{l+1}{i}.$$

Suppose first that l is odd, with $l=2m+1$, where m is a non-negative integer. Then by our remarks above,

$$[\wedge E:L(\omega_0)]=2\sum_{i=0}^{m}[\wedge^{2i}E:L(\omega_0)].$$

Let the different integers j such that $0\le j\le m$ and $\binom{l+1}{j}=\binom{2m+2}{j}$ is odd be $j_1,\ldots,j_u$. Then it follows from Lemma 1.13 that

$$\sum_{i=0}^{m}[\wedge^{2i}E:L(\omega_0)]=\sum_{i=1}^{u}(m-j_i+1)=u(m+1)-\sum_{i=1}^{u}j_i. \tag{9}$$

We proceed to evaluate u and the sum of the j_i.

Let j be a positive integer with $j\le m$ and $\binom{2m+2}{j}$ odd. Let

$$j=2^{b_1}+\cdots+2^{b_t}$$

be the 2-adic decomposition of j, with $b_1<\ldots<b_t$. Then it is easy to see that $b_t\ne a_s$, since $j\le m$, and thus $t\le s-1$. In particular, $\{b_1,\ldots,b_t\}$ must be a subset of $\{a_1,\ldots,a_{s-1}\}$, but it is easy to see that any subset of $\{a_1,\ldots,a_{s-1}\}$ is admissible as $\{b_1,\ldots,b_t\}$. Taking into account the additional number $j=0$ not included above, we see that there are 2^{s-1} possibilities for j and hence $u=2^{s-1}$.

If we consider all those numbers j above for which the 2-adic decomposition contains exactly t summands, a straightforward count shows that the sum of these numbers is

$$(2^{a_1}+\cdots+2^{a_{s-1}})\binom{s-2}{t-1}.$$

Thus we have

$$\textstyle\sum_{i=1}^{u}j_i=(2^{a_1}+\cdots+2^{a_{s-1}})\sum_{t=1}^{s-1}\binom{s-2}{t-1}=(2m+2-2^{a_s})2^{s-2}.$$

We therefore obtain

$$\begin{aligned}\sum_{i=0}^{m}[\wedge^{2i}E:L(\omega_0)] &= 2^{s-1}(m+1)-(2m+2-2^{a_s})2^{s-2}\\ &= 2^{s+a_s-2}\end{aligned}$$

and it follows that

$$[\wedge E:L(\omega_0)] = 2\times 2^{s+a_s-2}=2^{s+a_s-1},$$

as required.

To finish the proof, we assume that $l = 2m$ is even. Then

$$[\wedge E : L(\omega_0)] = [\wedge^{2m} E : L(\omega_0)] + 2\sum_{i=0}^{m-1} [\wedge^{2i} E : L(\omega_0)].$$

The multiplicity $[\wedge^{2m} E : L(\omega_0)]$ is the number of odd binomial coefficients among

$$\binom{2m+1}{0}, \quad \binom{2m+1}{1}, \dots, \quad \binom{2m+1}{m},$$

which is half the number of odd binomial coefficients among $\binom{2m+1}{i}$, $0 \le i \le 2m+1$. Since

$$l + 1 = 2m + 1 = 2^{a_1} + \cdots + 2^{a_s},$$

we see that

$$[\wedge^{2m} E : L(\omega_0)] = 2^{s-1}.$$

Now let the different integers t such that $0 \le t \le m-1$ and $\binom{2m+1}{t}$ is odd be $t_1, \dots, t_v$. Then

$$\sum_{i=0}^{m-1} [\wedge^{2i} E : L(\omega_0)] = \sum_{i=1}^{v} (m - t_i) = vm - \sum_{i=1}^{v} t_i.$$

Let t be one of the t_i above. Let

$$t = 2^{c_1} + \cdots + 2^{c_w}$$

be the 2-adic decomposition of t, with $0 \le c_1 < \ldots < c_w$. As before, we have $c_w \ne a_s$, and thus $w \le s-1$. Therefore, $\{c_1, \dots, c_w\}$ is again a subset of $\{a_1, \dots, a_{s-1}\}$. Furthermore, any subset of $\{a_1, \dots, a_{s-1}\}$ is admissible as $\{c_1, \dots, c_w\}$, except if

$$a_1 = 0, \quad a_2 = 1, \dots, \quad a_{s-1} = s-2, \quad a_s = s-1,$$

in which case the subset $\{0, 1, \dots, s-2\}$ is inadmissible. Let us omit this exceptional case $2m+1 = 2^s - 1$ for the time being. Then, following the earlier argument, we have $v = 2^{s-1}$ and

$$\sum_{i=1}^{v} t_i = (2m + 1 - 2^{a_s})2^{s-2}.$$

Thus, we deduce that

$$\begin{aligned}
[\wedge E : L(\omega_0)] &= 2^{s-1} + 2vm - 2\sum t_i \\
&= 2^{s-1} + 2^s m - 2^{s-1}(2m + 1 - 2^{a_s}) \\
&= 2^{s+a_s-1},
\end{aligned}$$

as required.

Finally, suppose that $2m+1 = 2^s - 1$. Then all the binomial coefficients involved in the sum giving $[\wedge^{2r} E : L(\omega_0)]$ are odd. Thus we get

$$\begin{aligned}
[\wedge E : L(\omega_0)] &= (m+1) + 2 \times \frac{m(m+1)}{2} = (m+1)^2 \\
&= 2^{2s-2} = 2^{s+a_s-1},
\end{aligned}$$

in accordance with our theorem. □

The multiplicity $[\wedge E : L(\omega_r)]$ for $1 \le r \le l$ is also a power of 2, for it is straightforward to see from Theorem 1.14 that this number equals the multiplicity of the trivial $K\mathrm{Sp}_{2l-2r}(K)$-module in the exterior algebra $\wedge E(2l-2r)$, where $E(2l-2r)$ is a symplectic space of dimension $2l - 2r$ over K, which may be found from Theorem 1.15.

2. Investigation of further irreducible modules for $\mathrm{Sp}_{2l}(K)$ and Σ_{2l+2}

The previous section has shown that there are certain irreducible $K\mathrm{Sp}_{2l}(K)$ modules that remain irreducible on restriction to both Σ_{2l+1} and Σ_{2l+2}. An obvious question to ask is whether there are additional irreducible $K\mathrm{Sp}_{2l}(K)$ modules that remain irreducible on restriction to Σ_{2l+1}, or failing that, on restriction to Σ_{2l+2}. We have not found any further such modules that remain irreducible on restriction to Σ_{2l+1}, but the purpose of this section is to construct, using the fundamental modules for $\mathrm{Sp}_{2l}(K)$, additional modules that do remain irreducible on restriction to Σ_{2l+2}. It is unlikely that the modules that we consider are the only ones with the sought-for property.

Notation 2.1. *Let G be a finite group and let V be a finite-dimensional KG-module. Let $V_1, \ldots, V_r$ be the non-isomorphic irreducible KG-modules that occur as composition factors of V and let $a_i = [V : V_i]$ denote the multiplicity with which V_i occurs as a composition factor of V, $1 \le i \le r$. Then we write*

$$V \sim a_1V_1 + \cdots + a_rV_r.$$

If V is a completely reducible KG-module, we write

$$V = a_1V_1 \oplus \cdots \oplus a_rV_r.$$

Sheth [28] described the branching rules for two-row partitions. The following result is a special case we need.

Theorem 2.2. *Let n be a positive integer with $n \equiv 3 \bmod 4$ and let m be an odd positive integer with $1 \le m < n/2$. Then, over a field of characteristic 2,*

$$D^{(n-m,m)} \downarrow_{\Sigma_{n-1}} \sim D^{(n-m-1,m)} + 2D^{(n-m,m-1)}.$$

Furthermore,

$$D^{(\frac{n+1}{2},\frac{n-1}{2})} \downarrow_{\Sigma_{n-1}} \sim 2D^{(\frac{n+1}{2},\frac{n-3}{2})}.$$

Corollary 2.3. *Assume the hypotheses of Theorem 2.2. Then $D^{(n-m,m)} \downarrow_{\Sigma_{n-1}}$ is an indecomposable uniserial module, with socle and head both isomorphic to $D^{(n-m,m-1)}$.*

Proof. The result follows from [20, Theorem 0.5] which shows that the socle of the restriction is $D^{(n-m,m-1)}$. □

Our next result is a straightforward consequence of the Mackey subgroup theorem, and we therefore omit its proof.

Lemma 2.4. *Let K be an arbitrary field and let W be a $K\Sigma_n$-module. Let Π be the natural permutation module for $K\Sigma_n$ of degree n. Then we have*

$$(W \uparrow^{\Sigma_{n+1}}) \downarrow_{\Sigma_n} = W \oplus (W \otimes \Pi).$$

We proceed to investigate the tensor products of certain irreducible $K\Sigma_n$-modules.

Theorem 2.5. *Let K be a field of characteristic 2 and let $n = 4l + 2$, where l is a positive integer. Let j be an integer with $1 \le j \le l - 1$. Then we have*

$$D^{(4l+1,1)} \otimes D^{(4l-2j+2,2j)} = D^{(4l-2j+1,2j,1)} \oplus D^{(4l-2j+1,2j+1)}$$

and

$$D^{(4l+1,1)} \otimes D^{(2l+2,2l)} = D^{(2l+1,2l,1)}.$$

Proof. We assume first that $1 \le j \le l-1$. Since $4l-2j+2$ and $2j$ are both even, we have

$$D^{(4l-2j+2,2j)} \uparrow^{\Sigma_{4l+3}} = D^{(4l-2j+2,2j,1)} \oplus D^{(4l-2j+2,2j+1)},$$

by [22, Corollary D]. Furthermore,

$$D^{(4l-2j+2,2j,1)}\downarrow_{\Sigma_{4l+2}}= D^{(4l-2j+1,2j,1)}\oplus D^{(4l-2j+2,2j)}$$

by [20, Theorem 0.6]. Since $4l+3\equiv 3 \bmod 4$, Theorem 2.2 implies that

$$D^{(4l-2j+2,2j+1)}\downarrow_{\Sigma_{4l+2}}\sim D^{(4l-2j+1,2j+1)}+2D^{(4l-2j+2,2j)}.$$

Since n is even, we have $\Pi\sim D^{(4l+1,1)}+2D^{(4l+2)}$, where Π is the permutation module described in Lemma 2.4. Lemma 2.4 now implies that

$$\begin{aligned} & D^{(4l-2j+1,2j,1)}+D^{(4l-2j+1,2j+1)}+3D^{(4l-2j+2,2j)} \\ \sim\ & D^{(4l-2j+2,2j)}\otimes(D^{(4l+1,1)}+2D^{(4l+2)})+D^{(4l-2j+2,2j)}. \end{aligned}$$

Multiplying out, we obtain

$$D^{(4l+1,1)}\otimes D^{(4l-2j+2,2j)}\sim D^{(4l-2j+1,2j,1)}+D^{(4l-2j+1,2j+1)}.$$

As $D^{(4l+1)}\otimes D^{(4l-2j+2,2j)}$ is certainly self-dual, it must be the direct sum of the two non-isomorphic irreducible submodules above, as required.

Finally, when $j=l$, the only difference in the argument is that

$$D^{(2l+2,2l+1)}\downarrow_{\Sigma_{4l+2}}\sim 2D^{(2l+2,2l)}$$

and the amended result in this case is thus clear. □

Remark. The assumption that $n\equiv 2 \bmod 4$ is essential to the truth of Theorem 2.5. We have been able to prove the following more involved tensor product theorems when $n\equiv 0 \bmod 4$:

$$\begin{aligned} D^{(n-1,1)}\otimes D^{(n-2,2)} &\sim D^{(n-3,2,1)}+D^{(n-3,3)}+2D^{(n-1,1)}, \quad n\equiv 0 \bmod 4 \\ D^{(n-1,1)}\otimes D^{(n-4,4)} &\sim D^{(n-5,4,1)}+D^{(n-5,5)}+2D^{(n-3,3)}, \quad n\equiv 4 \bmod 8 \\ D^{(n-1,1)}\otimes D^{(n-4,4)} &\sim D^{(n-5,4,1)}+D^{(n-5,5)}+2D^{(n-3,3)}+2D^{(n-1,1)}, \\ & \qquad n\equiv 0 \bmod 8. \end{aligned}$$

We proceed to give an interpretation of Theorem 2.5 in terms of the representations of $\mathrm{Sp}_{4l}(K)$.

Theorem 2.6. *Let K be an algebraically closed field of characteristic 2. Then for the group $\mathrm{Sp}_{4l}(K)$ we have*

$$\begin{aligned} L(\omega_1)\otimes L(\omega_0) &= L(\omega_1), \\ L(\omega_1)\otimes L(\omega_{2j}) &= L(\omega_{2j+1})\oplus L(\omega_1+\omega_{2j}), \quad 1\le j\le l-1, \\ L(\omega_1)\otimes L(\omega_{2l}) &= L(\omega_1+\omega_{2l}). \end{aligned}$$

Proof. We proceed by induction on j, the result being obvious when $j=0$. We note that if λ and μ are dominant weights and $L(\lambda)$ and $L(\mu)$ are the corresponding irreducible $K\mathrm{Sp}_{4l}(K)$-modules, $L(\lambda)\otimes L(\mu)$ contains $L(\lambda+\mu)$ as a composition factor (see, for example, [13, 31.4]). Thus $L(\omega_1)\otimes L(\omega_{2j})$ certainly contains $L(\omega_1+\omega_{2j})$ as a composition factor for $1\le j\le l$.

We know from Lemma 1.13 that, provided $1\le j\le l$, the composition factors of $\wedge^{2j}E$ consist of $L(\omega_{2j})$, with multiplicity 1, together with various $L(\omega_{2j-2s})$, where $1\le s\le j$. (Here we take E to be the natural vector space of dimension $4l$ over K on which $\mathrm{Sp}_{4l}(K)$ acts.) Thus the composition factors of $E\otimes\wedge^{2j}E$ are the composition factors of $L(\omega_1)\otimes L(\omega_{2j})$ and the various $L(\omega_1)\otimes L(\omega_{2j-2s})$, where $1\le s\le l$. Since we have an $K\mathrm{Sp}_{4l}(K)$-epimorphism from $E\otimes\wedge^{2j}E$ onto $\wedge^{2j+1}E$ given by

$$v\otimes(v_1\wedge\ldots\wedge v_{2j})\mapsto v\wedge v_1\wedge\ldots\wedge v_{2j},$$

and since $\wedge^{2j+1}E$ contains $L(\omega_{2j+1})$ provided $j \leq l-1$, it follows that for $1 \leq j \leq l-1$, there must be an index $2k$ such that $L(\omega_1) \otimes L(\omega_{2k})$ contains $L(\omega_{2j+1})$ as a composition factor. By induction, this index must be $2j$. Thus by our earlier observation, $L(\omega_1) \otimes L(\omega_{2j})$ contains both $L(\omega_{2j+1})$ and $L(\omega_1 + \omega_{2j})$ as composition factors, for $1 \leq j \leq l-1$. Now by Theorem 1.11 we have

$$L(\omega_1) \downarrow_{\Sigma_{4l+2}} = D^{(4l+1,1)}, \quad L(\omega_{2j}) \downarrow_{\Sigma_{4l+2}} = D^{(4l-2j+2,2j)}$$

for $1 \leq j \leq l$ and by Theorem 2.5

$$D^{(4l+1,1)} \otimes D^{(4l-2j+2,2j)} = D^{(4l-2j+1,2j+1)} \oplus D^{(4l-2j+1,2j,1)}$$

for $1 \leq j \leq l-1$. It therefore follows that $L(\omega_1) \otimes L(\omega_{2j})$ contains at most two composition factors for $1 \leq j \leq l-1$. Thus our theorem is proved for $1 \leq j \leq l-1$. Our theorem also follows for $j = l$, since we have already seen that $D^{(4l+1,1)} \otimes D^{(2l+2,2l)} = D^{(2l+1,2l,1)}$ is irreducible. □

Corollary 2.7. *Assume the notation of Theorem 2.6 above. Then*

$$L(\omega_1 + \omega_{2j}) \downarrow_{\Sigma_{4l+2}} = D^{(4l+1-2j,2j,1)}$$

is an irreducible $K\Sigma_{4l+2}$-module for $1 \leq j \leq l$.

Remark. (i) While Theorem 2.5 does not apply to Σ_{4l}, and not all the statements of Theorem 2.6 are true in general for $\mathrm{Sp}_{4l-2}(K)$, some parts of Corollary 2.7 are true when we replace $4l+2$ by $4l$. Thus, with a little more work, we can prove that

$$L(\omega_1 + \omega_2) \downarrow_{\Sigma_{4l}} = D^{(4l-3,2,1)}$$

is an irreducible $K\Sigma_{4l}$-module. On the other hand,

$$L(\omega_1 + \omega_{2l-1}) \downarrow_{\Sigma_{4l}} = D^{(2l,2l-1,1)} \oplus D^{(2l+1,2l-1)},$$

so that not all parts of Corollary 2.7 can be translated to Σ_{4l}.

(ii) None of the modules $L(\omega_1 + \omega_{2j})$ remains irreducible on restriction to Σ_{4l+1} by Theorem D of [19] (see also [11]).

3. Tensoring with the spin module

Let K be an algebraically closed field of characteristic 2. The irreducible module $L(\omega_l)$ for $\mathrm{Sp}_{2l}(K)$ is known as the spin module and it has dimension 2^l. Steinberg, [29, Theorem 11.1] has shown that, if λ is a dominant weight expressible in terms of the fundamental weights as

$$\lambda = \omega_{i_1} + \cdots + \omega_{i_r}$$

where $1 \leq i_1 < i_2 \ldots < i_r \leq l-1$, and $L(\lambda)$ is the corresponding irreducible $K\mathrm{Sp}_{2l}(K)$-module, then $L(\lambda) \otimes L(\omega_l)$ is irreducible and isomorphic to $L(\lambda + \omega_l)$. The final part of Theorem 2.6 is a very special case of this theorem. Steinberg also proved, [29, Theorem 1.3], that the irreducible modules $L(\lambda)$ and $L(\lambda + \omega_l)$ remain irreducible and non-isomorphic on restriction to $\mathrm{Sp}_{2l}(\mathbf{F}_2)$ and constitute a complete set of 2^l absolutely irreducible modules for $\mathrm{Sp}_{2l}(\mathbf{F}_2)$ over K (and they may be realized over $\mathbf{F}_2$). Since we have $\Sigma_{2l+2} \leq \mathrm{Sp}_{2l}(\mathbf{F}_2)$ for $l \geq 2$, it is the restriction of these 2^l irreducible modules to Σ_{2l+2} that we wish to check for irreducibility.

As we have seen, the restriction of spin module $L(\omega_l)$ to Σ_{2l+2} is the irreducible module $D^{(l+2,l)}$ and its restriction to Σ_{2l+1} is the irreducible module $D^{(l+1,l)}$. We shall refer to these modules as spin modules for the symmetric group over $\mathbf{F}_2$. We now make use of properties of the characteristic 0 spin modules for the proper double covers of the symmetric groups to investigate these characteristic 2 spin modules in more detail.

Let Γ_n denote either of the two non-isomorphic proper double covers of the symmetric group Σ_n. Γ_n has a faithful irreducible complex module Δ, say, of dimension $2^{[(n-1)/2]}$, known as the basic spin representation. By finding a suitable integral lattice in a basic spin module, we may reduce coefficients modulo 2 to form the module $\overline{\Delta}$, say, for Σ_n in characteristic 2. It is known, [4, 5.1], that $\overline{\Delta}$ is isomorphic to the spin module described in the paragraph above. This enables us to calculate the Brauer characters of the spin modules in characteristic 2.

Let $\varphi^{(l+2,l)}$ denote the Brauer character of $D^{(l+2,l)}$ and let α be an element of odd order in Σ_{2l+2}. Let $\sigma(\alpha)$ denote the number of cycles that appear in the cycle decomposition of α. Then by [27, VII, p.203], we have

$$\varphi^{(l+2,l)}(\alpha) = \pm 2^{[(\sigma(\alpha)-1)/2]}.$$

It follows that $\varphi^{(l+2,l)}(\alpha)$ is an odd integer if and only if $\sigma(\alpha) = 2$, in which case α is a product of two disjoint cycles of length $2r+1$ and $2l-2r+1$, where $0 \leq r \leq [l/2]$. Thus there are exactly $[l/2]+1$ conjugacy classes of elements of odd order on which $\varphi^{(l+2,l)}$ does not take even values. The restriction of $\varphi^{(l+2,l)}$ to Σ_{2l+1} is the Brauer character of the irreducible module $D^{(l+1,l)}$, which we denote by $\varphi^{(l+1,l)}$. Our discussion above shows that the only conjugacy class of elements of odd order in Σ_{2l+1} on which $\varphi^{(l+1,l)}$ does not take even values is that containing the cycles of length $2l+1$. We intend to show how this information bounds the number of non-isomorphic irreducible modules D in characteristic 2 for the symmetric group with the property that D tensored with the spin module is also irreducible. Before starting our investigation, we first note a property of Brauer characters. Let G be any finite group and let $\vartheta_1, \dots, \vartheta_s$ be the absolutely irreducible 2-modular Brauer characters of G. Let

$$\vartheta = a_1\vartheta_1 + \cdots + a_s\vartheta_s,$$

where the a_i are rational integers. Suppose that $\vartheta(g)/2$ is an algebraic integer for each element g of odd order. Then, since the ϑ_i are linearly independent modulo 2 (see, for example, [9, p.599]), it follows that each coefficient a_i is even.

Theorem 3.1. *Let K be a field of characteristic 2 and let D^λ be an irreducible $K\Sigma_n$-module, where λ is a 2-regular partition of n. Let S denote the irreducible spin $K\Sigma_n$-module.*

(a) *Suppose that n is odd. Then unless D^λ is the trivial module (so that $\lambda = (n)$), $D^\lambda \otimes S$ is not irreducible. More precisely, let φ^λ denote the Brauer character of D^λ and let π denote an n-cycle in Σ_n. Then if $\varphi^\lambda(\pi)$ is an even integer, each composition factor of $D^\lambda \otimes S$ occurs with even multiplicity. If $\varphi^\lambda(\pi)$ is an odd integer, S occurs with odd multiplicity as a composition factor of $D^\lambda \otimes S$ and there are other composition factors all occuring with even multiplicity.*

(b) *Suppose that n is even, with $n = 2m$. In this case, there are at most $[(m-1)/2]+1$ 2-regular partitions λ for which $D^\lambda \otimes S$ is irreducible.*

Proof. Let φ^λ be the Brauer character of D^λ and let φ be the Brauer character of S. We note that φ^λ takes rational integer values, since this is true for all complex characters of Σ_n. Suppose first that n is odd and $\varphi^\lambda(\pi)$ is an even integer. Then the Brauer character $\varphi^\lambda\varphi$ of $D^\lambda \otimes S$ takes even integer values on all elements of odd order. By our earlier remarks, each absolutely irreducible 2-modular Brauer character of Σ_n occurs as a constituent of $\varphi^\lambda\varphi$ with even multiplicity. Since the multiplicity of each composition factor of $D^\lambda \otimes S$ equals the multiplicity of its corresponding irreducible Brauer character as a constituent of $\varphi^\lambda\varphi$, we see that each composition factor of $D^\lambda \otimes S$ occurs with even multiplicity. Suppose next that n is odd and $\varphi^\lambda(\pi)$ is an odd integer. Then the Brauer character $\varphi^\lambda\varphi + \varphi$ also takes even integer values on all elements of odd order. It follows that φ is a constituent of $\varphi^\lambda\varphi$ with odd multiplicity and every other irreducible Brauer character is a constituent with even multiplicity. The corresponding properties of the composition factors of $D^\lambda \otimes S$ follow from this deduction. This proves part (a).

Suppose finally that $n = 2m$ is even and there are t different 2-regular partitions $\lambda_1, \ldots, \lambda_t$ such that $D^{\lambda_i} \otimes S$ is irreducible. Let $\vartheta_1, \ldots, \vartheta_t$ be the Brauer characters of these irreducible tensor products. Since the irreducible Brauer characters are linearly independent modulo 2, there must exist elements $g_1, \ldots, g_t$ of odd order in Σ_n such that the $t \times t$ integer matrix

$$(\vartheta_i(g_j)), \quad 1 \leq i,j \leq t$$

is non-singular modulo 2. Now, as each ϑ_i is a product of an irreducible Brauer character with φ, the g_j must be non-conjugate elements such that $\varphi(g_j)$ is an odd integer. By our earlier observations, each g_j is a product of two disjoint odd cycles and hence $t \leq [(m-1)/2]+1$. This completes the proof of part (b). □

Concerning the construction of irreducible tensor products using the spin module, as described in part (b) of Theorem 3.1, Theorem 2.5 has already provided an example of such a non-trivial irreducible tensor product when $n \equiv 2 \bmod 4$, $n \geq 6$. We take $\lambda = (n-1,1)$ in this case. On the basis of looking at examples for $n \leq 14$, we speculate that the following may be true. Suppose that $n \equiv 2 \bmod 4$ and write $n = 4m+2$. Let λ be the partition $(n-2r-1, 2r+1)$. Then for $0 \leq r \leq m-1$, $D^\lambda \otimes S$ is irreducible and isomorphic to $D^{d(\lambda)}$, where $d(\lambda)$ is the double of λ, as defined by Benson in [4, p.250]. Specifically, if $\lambda = (n-2r-1, 2r+1)$,

$$d(\lambda) = (n/2 - r, n/2 - r - 1, r+1, r).$$

If this is true, then taking into account the trivial case when $\lambda = (n)$, we would have $m+1$ different partitions λ for which $D^\lambda \otimes S$ is irreducible, in accordance with the upper bound given in part (b) of Theorem 3.1. On the other hand, we speculate that there are no non-trivial examples for which $D^\lambda \otimes S$ is irreducible when $n \equiv 0 \bmod 4$.

We now want to prove a result on the 'modular Littlewood-Richardson rule' to be used later. Let us write λ^t for a partition conjugate to λ, i.e. the partition whose Young diagram is the transpose of that of λ. Given partitions $\mu = (\mu_1, \mu_2, \ldots)$, $\nu = (\nu_1, \nu_2, \ldots)$ we denote the partition $(\mu_1+\nu_1, \mu_2+\nu_2, \ldots)$ by $\mu+\nu$. Moreover, we denote by $m(\mu,\nu)$ the partition $(\mu^t+\nu^t)^t$. If μ, ν, λ are partitions of $m, n, m+n$ respectively, let $c^\lambda_{\mu\nu}$ denote the corresponding Littlewood-Richardson coefficient, see [23, (5.2)]. It follows from the Littlewood-Richardson rule [23, (9.2)] that $c^{m(\mu,\nu)}_{\mu\nu} = 1$ and $c^\lambda_{\mu\nu} = 0$ unless $\lambda \trianglerighteq m(\mu,\nu)$ where $\triangleright$ is the dominance order on partitions, as defined in [14].

Lemma 3.2. *Let F be a field of characteristic p, and let μ and ν be p-regular partitions of m and n respectively. Then*

$$[(D^\mu \otimes D^\nu)\uparrow^{F\Sigma_{m+n}}_{F[\Sigma_m\times\Sigma_n]} : D^\lambda] = 0 \quad \textit{unless } \lambda \trianglerighteq m(\mu,\nu),$$

and if $m(\mu,\nu)$ is p-regular then

$$[(D^\mu \otimes D^\nu)\uparrow^{F\Sigma_{m+n}}_{F[\Sigma_m\times\Sigma_n]} : D^{m(\mu,\nu)}] = 1$$

Proof. By [16], $(S^\mu \otimes S^\nu)\uparrow^{F\Sigma_{m+n}}_{F[\Sigma_m\times\Sigma_n]}$ has a Specht filtration with module S^λ appearing $c^\lambda_{\mu\nu}$ times for any λ. Moreover D^γ is a composition factor of S^λ only if $\gamma \trianglerighteq \lambda$, and $[S^\lambda : D^\lambda] = 1$ if λ is p-regular. It follows from the remarks preceeding this theorem that

$$[(S^\mu \otimes S^\nu)\uparrow^{F\Sigma_{m+n}}_{F[\Sigma_m\times\Sigma_n]} : D^\lambda] = 0 \quad \text{unless } \lambda \trianglerighteq m(\mu,\nu),$$

and if $m(\mu,\nu)$ is p-regular then

$$[(S^\mu \otimes S^\nu)\uparrow_{F[\Sigma_m\times\Sigma_n]}^{F\Sigma_{m+n}} : D^{m(\mu,\nu)}] = 1$$

Let us pick a total order '<' on the pairs (μ,ν) as above such that $\mu' \trianglerighteq \mu, \nu' \trianglerighteq \nu$ implies $(\mu',\nu') \leq (\mu,\nu)$. We prove the lemma by induction on this total order. We start induction from the minimal pair $((m),(n))$. Since $D^{(m)} = S^{(m)}$, $D^{(n)} = S^{(n)}$, the result follows from the remarks above. For an arbitrary (μ,ν) the composition factors of the $F[\Sigma_m \times \Sigma_n]$-module $S^\mu \otimes S^\nu$ are $D^{\mu'} \otimes D^{\nu'}$ with $\mu' \trianglerighteq \mu, \nu' \trianglerighteq \nu$, and $D^\mu \otimes D^\nu$ appears exactly once. Note that $\mu' \trianglerighteq \mu, \nu' \trianglerighteq \nu$ and $(\mu',\nu') \neq (\mu,\nu)$ imply $m(\mu',\nu') \triangleright m(\mu,\nu)$. So the result now follows from the inductive hypothesis. □

If $\mu(i)$ is a partition of n_i, $i = 1,\ldots,a$, we define $m(\mu(1),\ldots,\mu(a))$ inductively using the formula $m(\mu(1),\ldots,\mu(i)) := m(m(\mu(1),\ldots,\mu(i-1)),\mu(i))$. The following corollary follows from Lemma 3.2 by induction.

Corollary 3.3. *Let F be a field of characteristic p, and let $\mu(i)$ be a p-regular partition of n_i, $i = 1,\ldots,a$. Then*

$$[(D^{\mu(1)} \otimes \cdots \otimes D^{\mu(i)})\uparrow_{F[\Sigma_{n_1}\times\cdots\times\Sigma_{n_a}]}^{F\Sigma_{n_1+\cdots+n_a}} : D^\lambda] = 0 \quad \textit{unless } \lambda \triangleright m(\mu(1),\ldots,\mu(a)),$$

and if $m(\mu(1),\ldots,\mu(a))$ is p-regular then

$$[(D^{\mu(1)} \otimes \cdots \otimes D^{\mu(i)})\uparrow_{F[\Sigma_{n_1}\times\cdots\times\Sigma_{n_a}]}^{F\Sigma_{n_1+\cdots+n_a}} : D^{m(\mu(1),\ldots,\mu(a))}] = 1.$$

We wish now to consider the connection between the tensor products $D^\lambda \otimes S$ and the modules $D^{d(\lambda)}$, where $d(\lambda)$ is the double of λ, as defined by Benson, [4]. Let $\lambda = (\lambda_1,\ldots,\lambda_r)$ be a 2-regular partition of n and let $\Sigma_\lambda = \Sigma_{\lambda_1}\times\cdots\times\Sigma_{\lambda_r}$ be the corresponding Young subgroup of Σ_n. Let $D^{d(n)}$ be the spin module for Σ_n over $\mathbf{F}_2$. The character formula of Schur, [27, VII, p.203], implies that

$$D^{d(n)}\downarrow_{\Sigma_\lambda} = 2^{k-1+[(l+1)/2]} D^{d(\lambda_1)} \otimes \cdots \otimes D^{d(\lambda_r)},$$

where k is the number of parts of λ of even size and l is the number of parts of odd size. Now, provided that λ is spin regular, as defined in [4, p.250], we have

$$d(\lambda) = m(d(\lambda_1),\ldots,d(\lambda_r)).$$

So Corollary 3.3 implies that

$$D^{d(\lambda_1)} \otimes \cdots \otimes D^{d(\lambda_r)}\uparrow^{\Sigma_n}$$

contains $D^{d(\lambda)}$ with multiplicity 1, together with other D^μ, where μ is a 2-regular partition with $\mu \triangleright d(\lambda)$. It follows that the composition factors of

$$(D^{d(n)}\downarrow_{\Sigma_\lambda})\uparrow^{\Sigma_n}$$

are $2^{k-1+[(l+1)/2]}$ copies of $D^{d(\lambda)}$ together with various D^μ, where $\mu \triangleright d(\lambda)$. But we also have

$$(D^{d(n)}\downarrow_{\Sigma_\lambda})\uparrow^{\Sigma_n} \cong M^\lambda \otimes D^{d(n)},$$

where M^λ is the permutation module for Σ_n obtained by permuting the cosets of Σ_λ, [9, 38.5]. By comparing these two decompositions, we can obtain some information about tensoring modules with the spin module. Before proving our main result, we briefly discuss some properties of partitions.

Let λ and μ be 2-regular partitions of n with $\mu \triangleright \lambda$. Suppose that λ and μ are both spin regular. Then it is straightforward to prove that $d(\mu) \triangleright d(\lambda)$. However, if we assume only that λ is spin regular, we cannot deduce that μ is also spin regular. For example, taking $n = 15$ and $\lambda = (9,5,1)$, λ is spin regular but there are several 2-regular partitions

μ that dominate λ and are not spin regular. This seems to introduce complications into the development of the theory. In the special case that λ and μ are both partitions of n into at most two parts, with $\mu \rhd \lambda$, it is easy to see that if λ is spin regular, then μ is also spin regular. We use this fact in our final result.

Theorem 3.4. *Let K be a field of characteristic 2 and let λ be a spin regular 2-row partition of n. Let $d(\lambda)$ denote the double of λ and $D^{d(n)}$ denote the spin module for Σ_n over K. Then the composition factors of $D^\lambda \otimes D^{d(n)}$ are $D^{d(\lambda)}$ with multiplicity $2^{[(k+1)/2]}$, where $k = 0$, 1 or 2 is the number of even parts of λ, and various D^ν, where $\nu \rhd d(\lambda)$.*

Proof. We shall prove by induction on the dominance order that for any spin regular partition λ into at most two parts the composition factors of $D^\lambda \otimes D^{d(n)}$ are $D^{d(\lambda)}$ with multiplicity $2^{k-1+[(l+1)/2]}$, where k is the number of even parts and l is the number of odd parts of λ, and various D^ν, where $\nu \rhd d(\lambda)$. The result will then follow as for 2-row partitions one has $k-1+[(l+1)/2] = [(k+1)/2]$.

The claim is clear when $\lambda = (n)$. We assume that the theorem is proved for all partitions μ of n into at most two parts with $\mu \rhd \lambda$. As we noted above, any such μ is spin regular. Our earlier discussion has shown $M^\lambda \otimes D^{d(n)}$ contains $D^{d(\lambda)}$ as a composition factor with multiplicity $2^{k-1+[(l+1)/2]}$, where k is defined as above, together with various D^ν, where $\nu \rhd d(\lambda)$. By [14, 12.1], the composition factors of M^λ are D^λ, with multiplicity 1, and various D^μ, where μ is a partition of n into at most two parts with $\mu \rhd \lambda$. By induction, $D^\mu \otimes D^{d(n)}$ cannot contain $D^{d(\lambda)}$, since $d(\mu) \rhd d(\lambda)$. It follows that $D^\lambda \otimes D^{d(n)}$ contains $D^{d(\lambda)}$ as a composition factor with the required multiplicity. $\square$

When n is even and λ is a spin regular 2-row partition of n into parts of odd size, we see that $D^\lambda \otimes D^{d(n)}$ contains $D^{d(\lambda)}$ as a composition factor with multiplicity 1. As we noted previously, examples suggest that $D^\lambda \otimes D^{d(n)} = D^{d(\lambda)}$ when $n \equiv 2 \bmod 4$, but that this is not true when $n \equiv 0 \bmod 4$. Thus, the decomposition matrix of Σ_{10} given in [14, p.139] shows that $\dim D^{(7,3)} = 48$ and $\dim D^{(4,3,2,1)} = 768$. Theorem 3.4 then implies that $D^{(7,3)} \otimes D^{(6,4)} = D^{(4,3,2,1)}$. Similarly, the decomposition matrix of Σ_{14} given in [5] enables us to deduce that $D^\lambda \otimes D^{(8,6)} = D^{d(\lambda)}$ when λ is any of $(13,1)$, $(11,3)$ and $(9,5)$.

While we have concentrated on looking for examples of irreducible modules D^λ for Σ_n with $D^\lambda \otimes D^{d(n)}$ irreducible, there may be examples for which $D^\lambda \otimes D^{d(n)}$ is homogeneous (i.e. a multiple of an irreducible module). We describe here one such non-trivial example which may be part of a more general scheme.

Example 3.5. *Let K be a field of characteristic 2 and let $\lambda = (9,5,1)$ be a partition of 15. Let $D^{(8,7)}$ denote the spin module for Σ_{15}. Then*

$$D^{(9,5,1)} \otimes D^{(8,7)} = D^{(5,4,3,2,1)} \oplus D^{(5,4,3,2,1)},$$

where $D^{(5,4,3,2,1)}$ is the unique irreducible projective module for Σ_{15}.

Proof. The partition λ is spin regular with $d(\lambda) = (5,4,3,2,1)$. Our discussion before the proof of Theorem 3.4 shows that $M^\lambda \otimes D^{(8,7)}$ contains $D^{d(\lambda)} = D^{(5,4,3,2,1)}$ as a composition factor with multiplicity 2. Using Young's rule and Benson's decomposition matrix for Σ_{15} in [5], we find that each composition factor of M^λ has multiplicity at least 2, with the exception of $D^\lambda = D^{(9,5,1)}$, which has multiplicity 1. We also know that for any 2-regular partition μ, $D^\mu \otimes D^{(8,7)}$ contains $D^{(5,4,3,2,1)}$ with even multiplicity by Theorem 3.1(a). Thus if there is a composition factor D^μ of M^λ, with $\mu \neq \lambda$, such that $D^{(5,4,3,2,1)}$ is a composition factor of $D^\mu \otimes D^{(8,7)}$, it follows that $D^{(5,4,3,2,1)}$ is a composition factor of $M^\lambda \otimes D^{(8,7)}$ with multiplicity at least 4, contrary to what we already know. Thus $D^{(5,4,3,2,1)}$ is a composition factor of $D^\lambda \otimes D^{(8,7)}$ with multiplicity 2. Benson's table shows that

$$\dim D^\lambda \dim D^{(8,7)} = 2 \dim D^{(5,4,3,2,1)}$$

and thus we deduce that $D^\lambda \otimes D^{(8,7)}$ has a composition series consisting of two copies of $D^{(5,4,3,2,1)}$. Finally, since $D^{(5,4,3,2,1)}$ is projective, $D^\lambda \otimes D^{(8,7)}$ is the direct sum of two copies of $D^{(5,4,3,2,1)}$. □

This example suggests the possibility of the following phenomenon. Suppose that $n = m+m-1+\cdots+1 = m(m+1)/2$ is a triangular number. Then if we set $\lambda = (2m-1, 2m-5, 2m-9, \ldots)$, we have $d(\lambda) = (m, m-1, \ldots, 1)$ and $D^{d(\lambda)}$ is an irreducible projective module obtained from the corresponding Specht lattice in characteristic 0 by reduction modulo 2. We speculate that the decomposition

$$D^\lambda \otimes D^{d(n)} = 2^{[(m-1)/4]} D^{d(\lambda)}$$

may hold, where the right hand side of the equality is the direct sum of copies of $D^{d(\lambda)}$. This decomposition holds for $m \leq 5$.

References

[1] A. M. Adamovich, Analogue of space of primitive forms over a field of positive characteristic, *Mosc. Univ. Math. Bull.* **39** (1984), No.1, 53-56.

[2] A. M. Adamovich, The submodule lattice of Weyl modules for symplectic groups with fundamental highest weights, *Mosc. Univ. Math. Bull.* **41** (1986), No.2, 6-9.

[3] M. Aschbacher, On the maximal subgroups of the finite classical groups, *Invent. Math.* **76**(1984), 469-514.

[4] D. Benson, Spin modules for symmetric groups, *J. London Math. Soc.* **38** (1988), 250-262.

[5] D. Benson, Some remarks on the decomposition numbers of the symmetric groups, *Proceedings of Symposia in Pure Mathematics* **47-1** (1987), 381-394. American Mathematical Society, Providence, Rhode Island, USA.

[6] C. Bessenrodt and A. Kleshchev, On Kronecker products of complex representations of the symmetric and alternating groups, *Pacif. J. Math.*, to appear.

[7] N. Bourbaki. *Groupes et Algèbres de Lie*, Hermann, Paris, 1968.

[8] N. Bourbaki, *Algèbre Commutative*, ch.7, Hermann, Paris, 1965.

[9] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras.* John Wiley, New York, 1962.

[10] K. Erdmann, Tensor products and dimensions of simple modules for symmetric groups, *Manuscripta Math.* **88** (1995), 357-386.

[11] B. Ford, Irreducible restrictions of representations of the symmetric groups, *Bull. Lond. Math. Soc.* **27**(1995), 453-459.

[12] W. Fulton and J. Harris, *Representation Theory.* Graduate Texts in Mathematics, 129. Springer-Verlag, New York, Berlin, Heidelberg, 1991.

[13] J.E. Humphreys, *Linear Algebraic Groups.* Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[14] G.D. James, *The Representation Theory of the Symmetric Groups.* Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[15] G.D. James, Representations of the symmetric group over the field of order 2, *J. Algebra* **38** (1976), 280-308.

[16] G.D. James and M.H. Peel, Specht series for skew representations of symmetric groups, *J. Algebra* **56** (1979), 343-364.

[17] G. Karpilovsky, *Group Representations*, Vol. 4, North-Holland, Amsterdam, 1995.

[18] P. Kleidman and M. Liebeck, *The Subgroup Structure of the Finite Classical Groups.* London Mathematical Society Lecture Note Series, 129. Cambridge University Press, Cambridge, 1990.

[19] A. Kleshchev, On restrictions of irreducible modular representations of semisimple algebraic groups and symmetric groups to some natural subgroups, I, *Proc. London Math. Soc.*(3) **69** (1994), 515-540.

[20] A. Kleshchev, Branching rules for modular representations of symmetric groups, II, *J. reine angew. Math.* **459** (1995), 163- 212.

[21] A. Kleshchev, Branching rules for modular representations of symmetric groups, III: some corollaries and a problem of Mullineux, *J. London Math. Soc.* **54** (1996), 25- 38.

[22] A. Kleshchev, Branching rules for modular representations of symmetric groups, IV, *J. Algebra* **201** (1998), 547-572.

[23] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.

[24] S. MacLane, Subfields and automorphism groups of p-adic fields, *Ann. of Math.* **40** (1939), 423-442.

[25] M. H. Peel, Hook representations of symmetric groups, *Glasgow Math. J.* **12** (1971), 136-149.
[26] A. A. Premet and I. D. Suprunenko, The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights, *Commun. Algebra* **11** (1983), 1309-1342.
[27] I. Schur, Über die Darstellung der symmetrischen und alternierenden Gruppe durch gebrochene lineare Substitutionen, *J. reine angew Math.* **139** (1911), 155-250.
[28] J. Sheth, Branching rules for two-part partitions and inductive systems, *Commun. Algebra* (to appear).
[29] R. Steinberg, Representations of algebraic groups, *Nagoya Math. J.* **22** (1963), 33-56.
[30] I. Zisser, Irreducible products of characters in A_n, *Isr. J. Math.* **84**(1993), 147-151.

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