PARABOLIC PRESENTATIONS OF THE YANGIAN $Y(\mathfrak{gl}_n)$

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1. Introduction

Let $Y_n = Y(\mathfrak{gl}_n)$ denote the Yangian associated to the Lie algebra $\mathfrak{gl}_n$ over the ground field $\mathbb{C}$; see e.g. [D1], [CP, ch.12] and [MNO]. In this article, we record some new presentations of Y_n that are adapted to standard parabolic subalgebras. To formulate our main result precisely, let $\mathfrak{gl}_\nu = \mathfrak{gl}_{\nu_1} \oplus \cdots \oplus \mathfrak{gl}_{\nu_m}$ be a standard Levi subalgebra of $\mathfrak{gl}_n$, so $\nu = (\nu_1, \dots, \nu_m)$ is a tuple of positive integers summing to n.

Theorem A. *The algebra Y_n is generated by elements $\{D_{a;i,j}^{(r)}, \widetilde{D}_{a;i,j}^{(r)}\}_{1\le a\le m, 1\le i,j\le \nu_a, r\ge 0}$, $\{E_{a;i,j}^{(r)}\}_{1\le a<m, 1\le i\le\nu_a, 1\le j\le \nu_{a+1}, r\ge 1}$ and $\{F_{a;i,j}^{(r)}\}_{1\le a<m,1\le i\le \nu_{a+1},1\le j\le\nu_a,r\ge 1}$ subject only to the relations*

$$D_{a;i,j}^{(0)} = \delta_{i,j}, \tag{1.1}$$

$$\sum_{t=0}^{r} D_{a;i,p}^{(t)} \widetilde{D}_{a;p,j}^{(r-t)} = -\delta_{r,0}\delta_{i,j}, \tag{1.2}$$

$$[D_{a;i,j}^{(r)}, D_{b;h,k}^{(s)}] = \delta_{a,b} \sum_{t=0}^{\min(r,s)-1} \left(D_{a;i,k}^{(r+s-1-t)} D_{a;h,j}^{(t)} - D_{a;i,k}^{(t)} D_{a;h,j}^{(r+s-1-t)} \right), \tag{1.3}$$

$$[E_{a;i,j}^{(r)}, F_{b;h,k}^{(s)}] = \delta_{a,b} \sum_{t=0}^{r+s-1} \widetilde{D}_{a;i,k}^{(t)} D_{a+1;h,j}^{(r+s-1-t)}, \tag{1.4}$$

$$[D_{a;i,j}^{(r)}, E_{b;h,k}^{(s)}] = \delta_{a,b} \sum_{t=0}^{r-1} D_{a;i,p}^{(t)} E_{a;p,k}^{(r+s-1-t)} \delta_{h,j} - \delta_{a,b+1} \sum_{t=0}^{r-1} D_{b+1;i,k}^{(t)} E_{b;h,j}^{(r+s-1-t)}, \tag{1.5}$$

$$[D_{a;i,j}^{(r)}, F_{b;h,k}^{(s)}] = \delta_{a,b+1} \sum_{t=0}^{r-1} F_{b;i,k}^{(r+s-1-t)} D_{b+1;h,j}^{(t)} - \delta_{a,b}\delta_{i,k} \sum_{t=0}^{r-1} F_{a;h,p}^{(r+s-1-t)} D_{a;p,j}^{(t)}, \tag{1.6}$$

$$[E_{a;i,j}^{(r)}, E_{a;h,k}^{(s)}] = \sum_{t=1}^{s-1} E_{a;i,k}^{(t)} E_{a;h,j}^{(r+s-1-t)} - \sum_{t=1}^{r-1} E_{a;i,k}^{(t)} E_{a;h,j}^{(r+s-1-t)}, \tag{1.7}$$

$$[F_{a;i,j}^{(r)}, F_{a;h,k}^{(s)}] = \sum_{t=1}^{r-1} F_{a;i,k}^{(r+s-1-t)} F_{a;h,j}^{(t)} - \sum_{t=1}^{s-1} F_{a;i,k}^{(r+s-1-t)} F_{a;h,j}^{(t)}, \tag{1.8}$$

$$[E_{a;i,j}^{(r)}, E_{a+1;h,k}^{(s+1)}] - [E_{a;i,j}^{(r+1)}, E_{a+1;h,k}^{(s)}] = -E_{a;i,q}^{(r)} E_{a+1;q,k}^{(s)} \delta_{h,j}, \tag{1.9}$$

$$[F_{a;i,j}^{(r+1)}, F_{a+1;h,k}^{(s)}] - [F_{a;i,j}^{(r)}, F_{a+1;h,k}^{(s+1)}] = -\delta_{i,k} F_{a+1;h,q}^{(s)} F_{a;q,j}^{(r)}, \tag{1.10}$$

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$$[E_{a;i,j}^{(r)}, E_{b;h,k}^{(s)}] = 0 \qquad \textit{if } b > a+1 \textit{ or if } b = a+1 \textit{ and } h \neq j, \tag{1.11}$$

$$[F_{a;i,j}^{(r)}, F_{b;h,k}^{(s)}] = 0 \qquad \textit{if } b > a+1 \textit{ or if } b = a+1 \textit{ and } i \neq k, \tag{1.12}$$

$$[E_{a;i,j}^{(r)}, [E_{a;h,k}^{(s)}, E_{b;f,g}^{(t)}]] + [E_{a;i,j}^{(s)}, [E_{a;h,k}^{(r)}, E_{b;f,g}^{(t)}]] = 0 \qquad \textit{if } |a-b| = 1, \tag{1.13}$$

$$[F_{a;i,j}^{(r)}, [F_{a;h,k}^{(s)}, F_{b;f,g}^{(t)}]] + [F_{a;i,j}^{(s)}, [F_{a;h,k}^{(r)}, F_{b;f,g}^{(t)}]] = 0 \qquad \textit{if } |a-b| = 1, \tag{1.14}$$

for all admissible $a, b, f, g, h, i, j, k, r, s, t$. (By convention the index p resp. q appearing here should be summed over $1, \dots, \nu_a$ resp. $1, \dots, \nu_{a+1}$.)

In the special case $\nu = (n)$, this presentation is the RTT presentation of Y_n originating in the work of Faddeev, Reshetikhin andTakhtadzhyan [FRT], while if $\nu = (1^n)$ the presentation is a variation on Drinfeld's presentation from [D2] (see Remark 5.12 for the precise relationship). One reason that Drinfeld's presentation is important is because it allows one to define subalgebras of Y_n which play the role of the Cartan subalgebra and Borel subalgebra in classical Lie theory. Our presentations allow us to define standard Levi and parabolic subalgebras. In the notation of Theorem A, let Y_ν, Y_ν^+ and Y_ν^- denote the subalgebras of Y_n generated by all the $D_{a;i,j}^{(r)}$'s, the $E_{a;i,j}^{(r)}$'s or the $F_{a;i,j}^{(r)}$'s, respectively. Then, $Y_\nu = Y(\mathfrak{gl}_\nu)$ is the *standard Levi subalgebra* of Y_n, isomorphic to $Y_{\nu_1} \otimes \cdots \otimes Y_{\nu_m}$. The *standard parabolic subalgebras* $Y_\nu^\sharp$ and $Y_\nu^\flat$ of Y_n are the subalgebras generated by Y_ν, Y_ν^+ and by Y_ν, Y_ν^- respectively. We also construct a PBW basis for each of these algebras. To write this down, define elements $E_{a,b;i,j}^{(r)}$ and $F_{a,b;j,i}^{(r)}$ for $1 \leq a < b \leq m$ and $1 \leq i \leq \nu_a, 1 \leq j \leq \nu_b$ by setting $E_{a,a+1;i,j}^{(r)} := E_{a;i,j}^{(r)}, F_{a,a+1;j,i}^{(r)} := F_{a;j,i}^{(r)}$ and then inductively defining

$$E_{a,b;i,j}^{(r)} := [E_{a,b-1;i,k}^{(r)}, E_{b-1;k,j}^{(1)}], \qquad F_{a,b;j,i}^{(r)} := [F_{b-1;j,k}^{(1)}, F_{a,b-1;k,i}^{(r)}],$$

if $b > a+1$, where $1 \leq k \leq \nu_{b-1}$. The relations imply that these definitions are independent of the particular choice of k; see (6.9).

Theorem B. (i) *The set of all monomials in $\{D_{a;i,j}^{(r)}\}_{a=1,\dots,m,1\leq i,j\leq \nu_a,r\geq 1}$ taken in some fixed order forms a basis for Y_ν.*

(ii) *The set of all monomials in $\{E_{a,b;i,j}^{(r)}\}_{1\leq a<b\leq m,1\leq i\leq \nu_a,1\leq j\leq \nu_b,r\geq 1}$ taken in some fixed order forms a basis for Y_ν^+.*

(iii) *The set of all monomials in $\{F_{a,b;i,j}^{(r)}\}_{1\leq a<b\leq m,1\leq i\leq \nu_b,1\leq j\leq \nu_a,r\geq 1}$ taken in some fixed order forms a basis for Y_ν^-.*

(iv) *The set of all monomials in the union of the elements listed in (i),(ii) and (iii) taken in some fixed order forms a basis for Y_n.*

Theorem B implies that the natural multiplication maps $Y_\nu^- \otimes Y_\nu \otimes Y_\nu^+ \to Y_n$, $Y_\nu \otimes Y_\nu^+ \to Y_\nu^\sharp$ and $Y_\nu^- \otimes Y_\nu \to Y_\nu^\flat$ are vector space isomorphisms. Moreover, there are natural projections $Y_\nu^\sharp \twoheadrightarrow Y_\nu$ and $Y_\nu^\flat \twoheadrightarrow Y_\nu$ with kernels generated by all $E_{a;i,j}^{(r)}$ and by all $F_{a;i,j}^{(r)}$ respectively.

The rest of the article is organized as follows. To start with, §§2-3 are expository, giving the necessary definitions and a proof of the PBW theorem for Y_n. In §4, we define Levi subalgebras. Then in §5 we give a complete proof of the equivalence of the RTT presentation of Y_n with Drinfeld's presentation, filling a gap in the literature. Note though that a proof of the analogous but harder result in the quantum affine case can be found in work of Ding and Frenkel [DF], and the basic trick of considering certain Gauss factorizations is the same here. The main theorems are proved in §6. The argument involves *partial* Gauss factorizations, an idea already exploited by Ding [D] to study the embedding of $U_q(\widehat{\mathfrak{gl}}_{n-1})$ in $U_q(\widehat{\mathfrak{gl}}_n)$ in the quantum affine setting. The remaining two sections are again expository in nature: in §7, we record proofs of some known results about centers and centralizers, and in §8 we explain the relationship between our approach and the *quantum determinants* which are used to define the Drinfeld generators elsewhere in the literature.

The results of this article play a central role in [BK], where we derive generators and relations for the finite W-algebras associated to nilpotent matrices in the general linear Lie algebras.

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Notation. Throughout the article, we work over the ground field $\mathbb{C}$. We write M_n for the associative algebra of all $n \times n$ matrices over $\mathbb{C}$, and $\mathfrak{gl}_n$ for the corresponding Lie algebra. The ij-matrix unit is denoted $e_{i,j}$.

2. RTT presentation

To define the *Yangian* $Y_n = Y(\mathfrak{gl}_n)$ we use the RTT formalism; see [ES, ch. 11] or [FRT]. Our basic reference for this material in the case of the Yangian is [MNO, §1]. Let

$$R(u) = u - \sum_{i,j=1}^{n} e_{i,j} \otimes e_{j,i} \in M_n \otimes M_n[u] \tag{2.1}$$

denote Yang's R-matrix with parameter u. This satisfies the QYBE with spectral parameters:

$$R^{[1,2]}(u-v)R^{[1,3]}(u)R^{[2,3]}(v) = R^{[2,3]}(v)R^{[1,3]}(u)R^{[1,2]}(u-v), \tag{2.2}$$

equality in $M_n \otimes M_n \otimes M_n[u,v]$. The superscripts in square brackets here and later on indicate the embedding of a smaller tensor into a bigger tensor, inserting the identity into all other tensor positions. Now, Y_n is defined to be the associative algebra on generators $\{T_{i,j}^{(r)}\}_{1 \leq i,j \leq n, r \geq 1}$ subject to certain relations. To write down these relations, let

$$T_{i,j}(u) := \sum_{r \geq 0} T_{i,j}^{(r)} u^{-r} \in Y_n[[u^{-1}]]$$

where $T_{i,j}^{(0)} := \delta_{i,j}$, and

$$T(u) := \sum_{i,j=1}^{n} e_{i,j} \otimes T_{i,j}(u) \in M_n \otimes Y_n[[u^{-1}]]. \tag{2.3}$$

We often think of $T(u)$ as an $n \times n$ matrix with ij-entry $T_{i,j}(u)$. Now the relations are given by the equation

$$R^{[1,2]}(u-v)T^{[1,3]}(u)T^{[2,3]}(v) = T^{[2,3]}(v)T^{[1,3]}(u)R^{[1,2]}(u-v), \tag{2.4}$$

equality in $M_n \otimes M_n \otimes Y_n((u^{-1}, v^{-1}))$ (the localization of $M_n \otimes M_n \otimes Y_n[[u^{-1}, v^{-1}]]$ at the multiplicative set consisting of the non-zero elements of $\mathbb{C}[[u^{-1}, v^{-1}]]$). Equating $e_{i,j} \otimes e_{h,k} \otimes ?$-coefficients on either side of (2.4), the relations are equivalent to

$$(u-v)[T_{i,j}(u), T_{h,k}(v)] = T_{h,j}(u)T_{i,k}(v) - T_{h,j}(v)T_{i,k}(u). \tag{2.5}$$

Swapping i with h, j with k and u with v, we get equivalently that

$$(u-v)[T_{i,j}(u), T_{h,k}(v)] = T_{i,k}(v)T_{h,j}(u) - T_{i,k}(u)T_{h,j}(v). \tag{2.6}$$

Yet another formulation of the relations is given by

$$[T_{i,j}^{(r)}, T_{h,k}^{(s)}] = \sum_{t=0}^{\min(r,s)-1} \left(T_{i,k}^{(r+s-1-t)} T_{h,j}^{(t)} - T_{i,k}^{(t)} T_{h,j}^{(r+s-1-t)} \right) \tag{2.7}$$

for every $1 \le h, i, j, k \le n$ and $r, s \ge 0$; see [MNO, Proposition 1.2].

Using (2.4), one checks that the following are (anti)automorphisms of Y_n; see [MNO, Proposition 1.12].

(A1) (*Translation*) For $a \in \mathbb{C}$, let $\eta_a : Y_n \to Y_n$ be the automorphism defined from $\eta_a^{[2]}(T(u)) = T(u+a)$, i.e. $\eta_a(T_{i,j}^{(r)}) = \sum_{s=0}^{r-1} \binom{r-1}{s} (-a)^s T_{i,j}^{(r-s)}$.

(A2) (*Multiplication by a power series*) For $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$, let $\mu_f : Y_n \to Y_n$ be the automorphism defined from $\mu_f^{[2]}(T(u)) = f(u)T(u)$, i.e. $\mu_f(T_{i,j}^{(r)}) = \sum_{s=0}^{r} a_s T_{i,j}^{(r-s)}$ if $f(u) = \sum_{s \ge 0} a_s u^{-s}$.

(A3) (*Sign change*) Let $\sigma : Y_n \to Y_n$ be the antiautomorphism of order 2 defined from $\sigma^{[2]}(T(u)) = T(-u)$, i.e. $\sigma(T_{i,j}^{(r)}) = (-1)^r T_{i,j}^{(r)}$.

(A4) (*Transposition*) Let $\tau : Y_n \to Y_n$ be the antiautomorphism of order 2 defined from $\tau^{[2]}(T(u)) = (T(u))^t$ (transpose matrix), i.e. $\tau(T_{i,j}^{(r)}) = T_{j,i}^{(r)}$.

(A5) (*Inversion*) Let $\omega : Y_n \to Y_n$ be the automorphism of order 2 defined from the equation $\omega^{[2]}(T(u)) = T(-u)^{-1}$.

The Yangian Y_n is a Hopf algebra with comultiplication $\Delta : Y_n \to Y_n \otimes Y_n$, counit $\varepsilon : Y_n \to \mathbb{C}$ and antipode $S : Y_n \to Y_n$ defined from

$$\Delta^{[2]}(T(u)) = T^{[1,2]}(u)T^{[1,3]}(u), \quad \varepsilon^{[2]}(T(u)) = I, \quad S^{[2]}(T(u)) = T(u)^{-1},$$

equalities written in $M_n \otimes Y_n \otimes Y_n[[u^{-1}]], M_n[[u^{-1}]]$ and $M_n \otimes Y_n[[u^{-1}]]$ respectively. Note that $S = \omega \circ \sigma$. The involutions ω and σ do *not* commute, so S is not of order 2; a precise description of S^2 is given in [MNO, Theorem 5.11] or Corollary 8.4 below. Since it arises quite often, we let $\widetilde{T}_{i,j}(u) = \sum_{r \ge 0} \widetilde{T}_{i,j}^{(r)} u^{-r} := -S(T_{i,j}(u))$, so

$$\widetilde{T}(u) := \sum_{i,j=1}^{n} e_{i,j} \otimes \widetilde{T}_{i,j}(u) = -T(u)^{-1}. \tag{2.8}$$

To work out commutation relations between $T_{i,j}(u)$ and $\widetilde{T}_{h,k}(v)$, it is useful to rewrite the RTT presentation in the form

$$\widetilde{T}^{[2,3]}(v)R^{[1,2]}(u-v)T^{[1,3]}(u) = T^{[1,3]}(u)R^{[1,2]}(u-v)\widetilde{T}^{[2,3]}(v). \tag{2.9}$$

We record [NT, Lemma 1.1]:

Lemma 2.1. *Given $i \neq k$ and $h \neq j$, $T_{i,j}^{(r)}$ and $\widetilde{T}_{h,k}^{(s)}$ commute for all $r, s \ge 1$.*

Proof. Compute the $e_{i,j} \otimes e_{h,k}$-coefficients on each side of (2.9). $\square$

We often work with the *canonical filtration*

$$\mathrm{F}_0 Y_n \subseteq \mathrm{F}_1 Y_n \subseteq \mathrm{F}_2 Y_n \subseteq \cdots \tag{2.10}$$

on Y_n defined by declaring that the generator $T_{i,j}^{(r)}$ is of degree r for each $r \geq 1$, i.e. $\mathrm{F}_d Y_n$ is the span of all monomials of the form $T_{i_1,j_1}^{(r_1)} \cdots T_{i_s,j_s}^{(r_s)}$ with total degree $r_1 + \cdots + r_s$ at most d. It is easy to see using (2.7) that the associated graded algebra $\operatorname{gr} Y_n$ is commutative. From this one deduces by induction on d that $\mathrm{F}_d Y_n$ is already spanned by the set of all monomials of total degree at most d in the elements $\{T_{i,j}^{(r)}\}_{1 \leq i,j \leq n, r \geq 1}$ *taken in some fixed order*. In fact it is known that such ordered monomials are linearly independent, hence the set of all monomials in the elements $\{T_{i,j}^{(r)}\}_{1 \leq i,j \leq n, r \geq 1}$ taken in some fixed order gives a basis for Y_n; see [MNO, Corollary 1.23] or Corollary 3.2 below. In other words, the associated graded algebra $\operatorname{gr} Y_n$ is the free commutative algebra on generators $\{\operatorname{gr}_r T_{i,j}^{(r)}\}_{1 \leq i,j \leq n, r \geq 1}$.

There is a second important filtration which we call the *loop filtration*

$$\mathrm{L}_0 Y_n \subseteq \mathrm{L}_1 Y_n \subseteq \mathrm{L}_2 Y_n \subseteq \cdots \tag{2.11}$$

defined by declaring that the generator $T_{i,j}^{(r)}$ is of degree $(r-1)$ for each $r \geq 1$. We denote the associated graded algebra by $\operatorname{gr}^{\mathrm{L}} Y_n$. Let $\mathfrak{gl}_n[t]$ denote the Lie algebra $\mathfrak{gl}_n \otimes \mathbb{C}[t]$ with basis $\{e_{i,j} t^r\}_{1 \leq i,j \leq n, r \geq 0}$. By the relations (2.7), there is a surjective homomorphism $U(\mathfrak{gl}_n[t]) \twoheadrightarrow \operatorname{gr}^{\mathrm{L}} Y_n$ mapping $e_{i,j} t^r$ to $\operatorname{gr}_r^{\mathrm{L}} T_{i,j}^{(r+1)}$ for each $1 \leq i,j \leq n, r \geq 0$. By the PBW theorem for Y_n described in the previous paragraph, this map is actually an isomorphism, hence $\operatorname{gr}^{\mathrm{L}} Y_n \cong U(\mathfrak{gl}_n[t])$; see also [MNO, Theorem 1.26] where this argument is explained in more detail.

3. PBW theorem

In this section, we give a proof of the PBW theorem for Y_n different from the one in [MNO]. It was inspired by the realization of the Yangian found in [BR]. Let $U(\mathfrak{gl}_n)$ denote the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n$. We have the *evaluation homomorphism*

$$\kappa_1 : Y_n \to U(\mathfrak{gl}_n), \quad T_{i,j}^{(1)} \mapsto \begin{cases} e_{i,j} & \text{if } r = 1, \\ 0 & \text{if } r > 1. \end{cases} \tag{3.1}$$

More generally for $l \geq 1$, consider the homomorphism

$$\kappa_l := \kappa_1 \otimes \cdots \otimes \kappa_1 \circ \Delta^{(l)} : Y_n \to U(\mathfrak{gl}_n)^{\otimes l}, \tag{3.2}$$

where $\Delta^{(l)} : Y_n \to Y_n^{\otimes l}$ denotes the lth iterated comultiplication. We define the algebra $Y_{n,l}$ to be the image $\kappa_l(Y_n)$ of Y_n under this homomorphism. Writing $e_{i,j}^{[s]}$ for the element $1^{\otimes(s-1)} \otimes e_{i,j} \otimes 1^{\otimes(l-s)} \in U(\mathfrak{gl}_n)^{\otimes l}$, we have by the definition of Δ that

$$\kappa_l(T_{i,j}^{(r)}) = \sum_{1 \leq s_1 < \cdots < s_r \leq l} \sum_{\substack{1 \leq i_0, \cdots, i_r \leq n \\ i_0 = i, i_r = j}} e_{i_0,i_1}^{[s_1]} e_{i_1,i_2}^{[s_2]} \cdots e_{i_{r-1},i_r}^{[s_r]}. \tag{3.3}$$

In particular, we see from this that $\kappa_l(T_{i,j}^{(r)}) = 0$ for all $r > l$.

Theorem 3.1. *The set of all monomials in the elements* $\{\kappa_l(T_{i,j}^{(r)})\}_{1 \leq i,j \leq n, r=1,\dots,l}$ *taken in some fixed order forms a basis for* $Y_{n,l}$.

Proof. It is obvious that such monomials span $Y_{n,l}$, so we just have to show that they are linearly independent. Consider the standard filtration

$$\mathrm{F}_0 U(\mathfrak{gl}_n)^{\otimes l} \subseteq \mathrm{F}_1 U(\mathfrak{gl}_n)^{\otimes l} \subseteq \mathrm{F}_2 U(\mathfrak{gl}_n)^{\otimes l} \subseteq \cdots$$

on $U(\mathfrak{gl}_n)^{\otimes l}$, so each generator $e_{i,j}^{[r]}$ is of degree 1. The associated graded algebra $\operatorname{gr} U(\mathfrak{gl}_n)^{\otimes l}$ is the free polynomial algebra on generators $x_{i,j}^{[r]} := \operatorname{gr}_1 e_{i,j}^{[r]}$. Let $y_{i,j}^{(r)} := \operatorname{gr}_r \kappa_l(T_{i,j}^{(r)})$, i.e.

$$y_{i,j}^{(r)} = \sum_{1 \leq s_1 < \cdots < s_r \leq l} \sum_{\substack{1 \leq i_0, \cdots, i_r \leq n \\ i_0 = i, i_r = j}} x_{i_0,i_1}^{[s_1]} x_{i_1,i_2}^{[s_2]} \cdots x_{i_{r-1},i_r}^{[s_r]}.$$

To complete the proof of the theorem, we show that the elements $\{y_{i,j}^{(r)}\}_{1 \leq i,j \leq n, r=1,\dots,l}$ are algebraically independent.

Let us identify $\operatorname{gr} U(\mathfrak{gl}_n)^{\otimes l}$ with the coordinate algebra $\mathbb{C}[M_n^{\times l}]$ of the affine variety $M_n^{\times l}$ of l-tuples $(A_1, \dots, A_l)$ of $n \times n$ matrices, so that $x_{i,j}^{[r]}$ is the function picking out the ij-entry of the rth matrix A_r. Let $\theta : M_n^{\times l} \to M_n^{\times l}$ be the morphism defined by $(A_1, \dots, A_l) \mapsto (B_1, \dots, B_l)$, where B_r is the rth elementary symmetric function

$$e_r(A_1, \dots, A_l) := \sum_{1 \leq s_1 < \cdots < s_r \leq l} A_{s_1} \cdots A_{s_r}$$

in the matrices $A_1, \dots, A_l$. The comorphism θ^* maps $x_{i,j}^{[r]}$ to $y_{i,j}^{(r)}$. So to show that the $y_{i,j}^{(r)}$ are algebraically independent, we need to show that θ^* is injective, i.e. that θ is a dominant morphism of affine varieties. For this it suffices to show that the differential of θ is surjective at some point $x \in M_n^{\times l}$.

Pick pairwise distinct scalars $c_1, \dots, c_l \in \mathbb{C}$ and consider $x := (c_1 I_n, \dots, c_l I_n)$. Identifying the tangent space $T_x(M_n^{\times l})$ with the vector space $M_n^{\oplus l}$, a calculation shows that the differential $d\theta_x$ maps $(A_1, \dots, A_l)$ to $(B_1, \dots, B_l)$ where

$$B_r = \sum_{s=1}^{l} e_{r-1}(c_1, \dots, \widehat{c_s}, \dots, c_l) A_s.$$

Here, $e_{r-1}(c_1, \dots, \widehat{c_s}, \dots, c_l)$ denotes the $(r-1)$th elementary symmetric function in the scalars $c_1, \dots, c_l$ excluding c_s. We just need to show this linear map is surjective, for which it clearly suffices to consider the case $n = 1$. But in that case its determinant is the Vandermonde determinant $\prod_{1 \leq r < s \leq l} (c_s - c_r)$, so it is non-zero by the choice of the scalars $c_1, \dots, c_l$. $\square$

Corollary 3.2. *The set of all monomials in the elements* $\{T_{i,j}^{(r)}\}_{1 \leq i,j \leq n, r \geq 1}$ *taken in some fixed order forms a basis for* Y_n.

Proof. We have already observed that such monomials span Y_n. The fact that they are linearly independent follows from Theorem 3.1 by taking sufficiently large l. $\square$

Corollary 3.3. *The kernel of* $\kappa_l : Y_n \twoheadrightarrow Y_{n,l}$ *is the two-sided ideal of* Y_n *generated by the elements* $\{T_{i,j}^{(r)}\}_{1 \leq i,j \leq n, r > l}$.

Proof. Let I denote the two-sided ideal of Y_n generated by $\{T_{i,j}^{(r)}\}_{1 \le i,j \le n, r > l}$. It is obvious that κ_l induces a map $\bar{\kappa}_l : Y_n/I \twoheadrightarrow Y_{n,l}$. Since Y_n/I is spanned by the set of all monomials in the elements $\{T_{i,j}^{(r)} + I\}_{1 \le i,j \le n, r=1,\dots,l}$ taken in some fixed order, Theorem 3.1 now implies that $\bar{\kappa}_l$ is an isomorphism. $\square$

The first of these corollaries proves the PBW theorem for Y_n. The second corollary shows that the algebra $Y_{n,l}$ is the *Yangian of level* l defined by Cherednik [C]. Moreover, by Corollary 3.3, the maps κ_l induce an inverse system

$$Y_{n,1} \twoheadleftarrow Y_{n,2} \twoheadleftarrow \cdots$$

of filtered algebras, where each $Y_{n,l}$ is filtered by the canonical filtration defined by declaring that the generators $\kappa_l(T_{i,j}^{(r)})$ are of degree r. It is easy to see using Theorem 3.1 and Corollary 3.2 that the Yangian Y_n is the inverse limit $\varprojlim Y_{n,l}$ of this system taken in the category of filtered algebras. This gives a concrete realization of the Yangian.

4. Levi subalgebras

Our exposition is biased towards the *standard embedding* $Y_n \hookrightarrow Y_{n+1}$ under which $T_{i,j}^{(r)} \in Y_n$ maps to the element with the same name in Y_{n+1}. We warn the reader that the element $\widetilde{T}_{i,j}^{(r)} \in Y_n$ does *not* map to the element with the same name in Y_{n+1}! The standard embeddings define a tower of algebras

$$Y_1 \subset Y_2 \subset Y_3 \subset \cdots \tag{4.1}$$

which will be implicit in our work from now on. Since most of the automorphisms of the Yangian defined in §2 do *not* commute with the standard embeddings, we sometimes add a subscript to clarify notation; for example we write $\omega_n : Y_n \to Y_n$ for the automorphism ω if confusion seems likely.

For $m \ge 0$, we let $\varphi_m : Y_n \hookrightarrow Y_{m+n}$ denote the obvious injective algebra homomorphism mapping $T_{i,j}^{(r)} \in Y_n$ to $T_{m+i,m+j}^{(r)} \in Y_{m+n}$ for each $1 \le i,j \le n, r \ge 1$. Following [NT], we define another injective algebra homomorphism $\psi_m : Y_n \hookrightarrow Y_{m+n}$ by

$$\psi_m := \omega_{m+n} \circ \varphi_m \circ \omega_n. \tag{4.2}$$

Observe that ψ_m maps $\widetilde{T}_{i,j}^{(r)} \in Y_n$ to $\widetilde{T}_{m+i,m+j}^{(r)} \in Y_{m+n}$. So the subalgebra $\psi_m(Y_n)$ of Y_{m+n} is generated by the elements $\{\widetilde{T}_{m+i,m+j}^{(r)}\}_{1 \le i,j \le n, r \ge 1}$. Given this, the following lemma is an immediate consequence of Lemma 2.1.

Lemma 4.1. *The subalgebras Y_m and $\psi_m(Y_n)$ of Y_{m+n} centralize each other.*

Let us give another description of the map ψ_m in terms of the quasi-determinants of Gelfand and Retakh; see e.g. [GKLLRT, §2.2]. Suppose that A, B, C and D are $m \times m$, $m \times n$, $n \times m$ and $n \times n$ matrices respectively with entries in some ring R. Assuming that the matrix A is invertible, we define

$$\begin{vmatrix} A & B \\ C & \boxed{D} \end{vmatrix} := D - CA^{-1}B. \tag{4.3}$$

Then:

Lemma 4.2. *For any* $1 \leq i,j \leq n$,

$$\psi_m(T_{i,j}(u)) = \begin{vmatrix} T_{1,1}(u) & \cdots & T_{1,m}(u) & T_{1,m+j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ T_{m,1}(u) & \cdots & T_{m,m}(u) & T_{m,m+j}(u) \\ T_{m+i,1}(u) & \cdots & T_{m+i,m}(u) & \boxed{T_{m+i,m+j}(u)} \end{vmatrix}.$$

Proof. Let $T(u)$ denote the matrix $(T_{i,j}(u))_{1\leq i,j\leq n}$ with entries in $Y_n[[u^{-1}]]$ as usual and let $\widetilde{T}(u) := -T(u)^{-1}$. Also define the matrices

$$A(u) = (T_{i,j}(u))_{1\leq i,j\leq m}\,, \qquad B(u) = (T_{i,j}(u))_{1\leq i\leq m, m+1\leq j\leq m+n}\,,$$
$$C(u) = (T_{i,j}(u))_{m+1\leq i\leq m+n, 1\leq j\leq m}\,, \qquad D(u) = (T_{i,j}(u))_{m+1\leq i,j\leq m+n}$$

with entries in $Y_{m+n}[[u^{-1}]]$ and let

$$\begin{pmatrix} \widetilde{A}(u) & \widetilde{B}(u) \\ \widetilde{C}(u) & \widetilde{D}(u) \end{pmatrix} := -\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}^{-1}.$$

By block multiplication, one checks the classical identities

$$\begin{aligned} \widetilde{A}(u) &= -\left(A(u) - B(u)D(u)^{-1}C(u)\right)^{-1}, \\ \widetilde{B}(u) &= A(u)^{-1}B(u)\left(D(u) - C(u)A(u)^{-1}B(u)\right)^{-1}, \\ \widetilde{C}(u) &= D(u)^{-1}C(u)\left(A(u) - B(u)D(u)^{-1}C(u)\right)^{-1}, \\ \widetilde{D}(u) &= -\left(D(u) - C(u)A(u)^{-1}B(u)\right)^{-1}. \end{aligned}$$

Now, by its definition, the homomorphism ψ_m maps $\widetilde{T}(u)$ to $\widetilde{D}(u)$. Hence, it maps $T(u)$ to $-\widetilde{D}(u)^{-1} = D(u) - C(u)A(u)^{-1}B(u)$. The lemma follows from this on computing ij-entries. $\square$

The description of $\psi_m(T_{i,j}(u))$ given by Lemma 4.2 does not depend on n. This means that the maps ψ_m are compatible with the standard embeddings, in the sense that the following diagram commutes

$$\begin{array}{ccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \cdots \\ {\scriptstyle\psi_m}\big\downarrow & & {\scriptstyle\psi_m}\big\downarrow & & {\scriptstyle\psi_m}\big\downarrow & & \\ Y_{m+1} & \longrightarrow & Y_{m+2} & \longrightarrow & Y_{m+3} & \longrightarrow & \cdots \end{array} \tag{4.4}$$

where the horizontal maps are standard embeddings. So our notation for the maps ψ_m is unambiguous as n varies. We also note that

$$\psi_m \circ \psi_{m'} = \psi_{m+m'} \tag{4.5}$$

for any $m, m' \geq 0$, which is an obvious consequence of our original definition.

Now we can define the *standard Levi subalgebras* of Y_n. Given a tuple $\nu = (\nu_1, \dots, \nu_m)$ of positive integers summing to n, define Y_ν to be the subalgebra

$$Y_\nu := Y_{\nu_1}\psi_{\nu_1}(Y_{\nu_2})\psi_{\nu_1+\nu_2}(Y_{\nu_3})\cdots\psi_{\nu_1+\cdots+\nu_{m-1}}(Y_{\nu_m}) \tag{4.6}$$

of Y_n. For $a = 1, \dots, m$ and $1 \leq i,j \leq \nu_a$, we let

$$D_{a;i,j}(u) = \sum_{r\geq 0} D_{a;i,j}^{(r)}u^{-r} := \psi_{\nu_1+\cdots+\nu_{a-1}}(T_{i,j}(u)). \tag{4.7}$$

By Lemma 4.1 and induction on m, the various "blocks" of Y_ν centralize each other, hence the map

$$\psi_0 \bar{\otimes} \psi_{\nu_1} \bar{\otimes} \cdots \bar{\otimes} \psi_{\nu_1+\cdots+\nu_{m-1}} : Y_{\nu_1} \otimes Y_{\nu_2} \otimes \cdots \otimes Y_{\nu_m} \to Y_\nu$$

is an algebra isomorphism. This means that the elements $\{D_{a;i,j}^{(r)}\}_{1\leq a\leq m, 1\leq i,j\leq \nu_a, r\geq 1}$ generate Y_ν subject only to the following relations:

$$[D_{a;i,j}^{(r)}, D_{b;h,k}^{(s)}] = \delta_{a,b} \sum_{t=0}^{\min(r,s)-1} \left(D_{a;i,k}^{(r+s-1-t)} D_{a;h,j}^{(t)} - D_{a;i,k}^{(t)} D_{a;h,j}^{(r+s-1-t)} \right) \tag{4.8}$$

where $D_{a;i,j}^{(0)} := \delta_{i,j}$. The special case $\nu = (1^n)$ is particularly important: $Y_{(1^n)} \cong Y_1 \otimes \cdots \otimes Y_1$ is a commutative subalgebra of Y_n which plays the role of Cartan subalgebra.

5. Drinfeld presentation

Since the leading minors of the matrix $T(u)$ are invertible, it possesses a Gauss factorization

$$T(u) = F(u)D(u)E(u) \tag{5.1}$$

for unique matrices

$$D(u) = \begin{pmatrix} D_1(u) & 0 & \cdots & 0 \\ 0 & D_2(u) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_n(u) \end{pmatrix},$$

$$E(u) = \begin{pmatrix} 1 & E_{1,2}(u) & \cdots & E_{1,n}(u) \\ 0 & 1 & \cdots & E_{2,n}(u) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \; F(u) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ F_{1,2}(u) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{1,n}(u) & F_{2,n}(u) & \cdots & 1 \end{pmatrix}.$$

This defines power series $D_i(u) = \sum_{r\geq 0} D_i^{(r)} u^{-r}$, $E_{i,j}(u) = \sum_{r\geq 1} E_{i,j}^{(r)} u^{-r}$ and $F_{i,j}(u) = \sum_{r\geq 1} F_{i,j}^{(r)} u^{-r}$. Let $E_i(u) = \sum_{r\geq 1} E_i^{(r)} u^{-r} := E_{i,i+1}(u)$ and $F_i(u) = \sum_{r\geq 1} F_i^{(r)} u^{-r} := F_{i,i+1}(u)$ for short. Also let $\widetilde{D}_i(u) = \sum_{r\geq 0} \widetilde{D}_i^{(r)} u^{-r} := -D_i(u)^{-1}$.

In terms of quasi-determinants, we have the following more explicit descriptions; see [GR1, Theorem 4.4] or [GR2, Theorem 2.2.6].

$$D_i(u) = \begin{vmatrix} T_{1,1}(u) & \cdots & T_{1,i-1}(u) & T_{1,i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ T_{i-1,1}(u) & \cdots & T_{i-1,i-1}(u) & T_{i-1,i}(u) \\ T_{i,1}(u) & \cdots & T_{i,i-1}(u) & \boxed{T_{i,i}(u)} \end{vmatrix}, \tag{5.2}$$

$$E_{i,j}(u) = D_i(u)^{-1} \begin{vmatrix} T_{1,1}(u) & \cdots & T_{1,i-1}(u) & T_{1,j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ T_{i-1,1}(u) & \cdots & T_{i-1,i-1}(u) & T_{i-1,j}(u) \\ T_{i,1}(u) & \cdots & T_{i,i-1}(u) & \boxed{T_{i,j}(u)} \end{vmatrix}, \tag{5.3}$$

$$F_{i,j}(u) = \begin{vmatrix} T_{1,1}(u) & \cdots & T_{1,i-1}(u) & T_{1,i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ T_{i-1,1}(u) & \cdots & T_{i-1,i-1}(u) & T_{i-1,i}(u) \\ T_{j,1}(u) & \cdots & T_{j,i-1}(u) & \boxed{T_{j,i}(u)} \end{vmatrix} D_i(u)^{-1}. \tag{5.4}$$

Since $E_{j-1}^{(1)} = T_{j-1,j}^{(1)}$ and $F_{j-1}^{(1)} = T_{j,j-1}^{(1)}$, it follows easily that

$$E_{i,j}^{(r)} = [E_{i,j-1}^{(r)}, E_{j-1}^{(1)}], \qquad F_{i,j}^{(r)} = [F_{j-1}^{(1)}, F_{i,j-1}^{(r)}]. \tag{5.5}$$

for $i+1 < j \leq n$. Comparing (5.2)–(5.4) with Lemma 4.2 we deduce:

Lemma 5.1. *For all admissible i, we have that*

(i) $D_i(u) = \psi_{i-1}(D_1(u)) = \psi_{i-1}(T_{1,1}(u))$;
(ii) $E_i(u) = \psi_{i-1}(E_1(u)) = \psi_{i-1}(T_{1,1}(u)^{-1}T_{1,2}(u))$;
(iii) $F_i(u) = \psi_{i-1}(F_1(u)) = \psi_{i-1}(T_{2,1}(u)T_{1,1}(u)^{-1})$.

In particular Lemma 5.1(i) shows that the elements $D_i^{(r)}$ here are the same as the elements denoted $D_{i;1,1}^{(r)}$ from §4, so they generate the Cartan subalgebra $Y_{(1^n)}$. Also let $Y_{(1^n)}^+$ resp. $Y_{(1^n)}^-$ denote the subalgebra of Y_n generated by the elements $\{E_i^{(r)}\}_{i=1,\dots,n-1,r\geq 1}$ resp. $\{F_i^{(r)}\}_{i=1,\dots,n-1,r\geq 1}$. In view of (5.5), all the elements $E_{i,j}^{(r)}$ belong to $Y_{(1^n)}^+$ and all the elements $F_{i,j}^{(r)}$ belong to $Y_{(1^n)}^-$. By applying the antiautomorphism τ to the factorization (5.1), one checks:

$$\tau(E_{i,j}(u)) = F_{i,j}(u), \qquad \tau(F_{i,j}(u)) = E_{i,j}(u), \qquad \tau(D_i(u)) = D_i(u). \tag{5.6}$$

Hence, τ fixes $Y_{(1^n)}$ elementwise and interchanges the subalgebras $Y_{(1^n)}^+$ and $Y_{(1^n)}^-$.

Now we state the main theorem of the section. This is essentially due to Drinfeld [D2]; see the remark at the end of the section for the precise relationship.

Theorem 5.2. *The algebra Y_n is generated by the elements $\{D_i^{(r)}, \widetilde{D}_i^{(r)}\}_{1\leq i\leq n, r\geq 0}$ and $\{E_i^{(r)}, F_i^{(r)}\}_{1\leq i<n, r\geq 1}$ subject only to the following relations:*

$$D_i^{(0)} = 1, \tag{5.7}$$

$$\sum_{t=0}^{r} D_i^{(t)}\widetilde{D}_i^{(r-t)} = -\delta_{r,0}, \tag{5.8}$$

$$[D_i^{(r)}, D_j^{(s)}] = 0, \tag{5.9}$$

$$[E_i^{(r)}, F_j^{(s)}] = \delta_{i,j} \sum_{t=0}^{r+s-1} \widetilde{D}_i^{(t)} D_{i+1}^{(r+s-1-t)}, \tag{5.10}$$

$$[D_i^{(r)}, E_j^{(s)}] = (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} D_i^{(t)} E_j^{(r+s-1-t)}, \tag{5.11}$$

$$[D_i^{(r)}, F_j^{(s)}] = (\delta_{i,j+1} - \delta_{i,j}) \sum_{t=0}^{r-1} F_j^{(r+s-1-t)} D_i^{(t)}, \tag{5.12}$$

$$[E_i^{(r)}, E_i^{(s)}] = \sum_{t=1}^{s-1} E_i^{(t)} E_i^{(r+s-1-t)} - \sum_{t=1}^{r-1} E_i^{(t)} E_i^{(r+s-1-t)}, \tag{5.13}$$

$$[F_i^{(r)}, F_i^{(s)}] = \sum_{t=1}^{r-1} F_i^{(r+s-1-t)} F_i^{(t)} - \sum_{t=1}^{s-1} F_i^{(r+s-1-t)} F_i^{(t)}, \tag{5.14}$$

$$[E_i^{(r)}, E_{i+1}^{(s+1)}] - [E_i^{(r+1)}, E_{i+1}^{(s)}] = -E_i^{(r)} E_{i+1}^{(s)}, \tag{5.15}$$

$$[F_i^{(r+1)}, F_{i+1}^{(s)}] - [F_i^{(r)}, F_{i+1}^{(s+1)}] = -F_{i+1}^{(s)} F_i^{(r)}, \tag{5.16}$$

$$[E_i^{(r)}, E_j^{(s)}] = 0 \qquad \text{if } |i-j| > 1, \tag{5.17}$$

$$[F_i^{(r)}, F_j^{(s)}] = 0 \qquad \text{if } |i-j| > 1, \tag{5.18}$$

$$[E_i^{(r)}, [E_i^{(s)}, E_j^{(t)}]] + [E_i^{(s)}, [E_i^{(r)}, E_j^{(t)}]] = 0 \qquad \text{if } |i-j| = 1, \tag{5.19}$$

$$[F_i^{(r)}, [F_i^{(s)}, F_j^{(t)}]] + [F_i^{(s)}, [F_i^{(r)}, F_j^{(t)}]] = 0 \qquad \text{if } |i-j| = 1, \tag{5.20}$$

for all admissible i, j, r, s, t.

Remark 5.3. The relations (5.13) and (5.14) are equivalent to the relations

$$[E_i^{(r)}, E_i^{(s+1)}] - [E_i^{(r+1)}, E_i^{(s)}] = E_i^{(r)} E_i^{(s)} + E_i^{(s)} E_i^{(r)}, \tag{5.21}$$

$$[F_i^{(r+1)}, F_i^{(s)}] - [F_i^{(r)}, F_i^{(s+1)}] = F_i^{(r)} F_i^{(s)} + F_i^{(s)} F_i^{(r)}, \tag{5.22}$$

respectively.

In the remainder of the section, we are going to write down a proof, since we could not find one in the literature. There are two parts to the proof: first we must show that all these relations are satisfied in Y_n; second we must show that we have found enough relations.

Let us begin with some reductions to the first part of the proof. We have already noted that the elements $\{D_i^{(r)}\}_{i=1,\dots,n, r\geq 1}$ commute, hence the relations (5.7)–(5.9) hold. Also by Lemma 5.1, $D_i^{(r)} \in \psi_{i-1}(Y_1)$ and $E_j^{(s)} \in \psi_{j-1}(Y_2)$, so Lemma 4.1 implies that (5.11) holds if either $i < j$ or $i > j+1$. Similar reasoning shows that (5.10) holds if $|i-j| > 1$ and (5.17)–(5.18) hold always. Having made these remarks, Lemma 5.1 and (5.6) reduces the verification of all the remaining relations to checking the following special cases: (5.10) with $i = 1, j = 1$ or $i = 2, j = 1$; (5.11) with $i = 1, j = 1$ or $i = 2, j = 1$; (5.13) with $i = 1$; (5.15) with $i = 1$; (5.19) with $i = 2, j = 1$ or $i = 1, j = 2$.

Lemma 5.4. *The following identities hold in* $Y_2((u^{-1}, v^{-1}))$*:*

(i) $(u-v)[D_1(u), E_1(v)] = D_1(u)(E_1(v) - E_1(u))$;

(ii) $(u-v)[E_1(u), \widetilde{D}_2(v)] = (E_1(u) - E_1(v))\widetilde{D}_2(v)$;

(iii) $(u-v)[E_1(u), F_1(v)] = \widetilde{D}_1(v)D_2(v) - \widetilde{D}_1(u)D_2(u)$;

(iv) $(u-v)[E_1(u), E_1(v)] = (E_1(v) - E_1(u))^2$.

Proof. Compute the $e_{1,1} \otimes e_{1,2}$-, $e_{1,2} \otimes e_{2,2}$-, $e_{1,2} \otimes e_{2,1}$- and $e_{1,2} \otimes e_{1,2}$-coefficients on each side of (2.9) and rearrange the resulting four equations to obtain the following identities:

(i)′ $(u-v)[T_{1,1}(u), \widetilde{T}_{1,2}(v)] = T_{1,1}(u)\widetilde{T}_{1,2}(v) + T_{1,2}(u)\widetilde{T}_{2,2}(v)$;

(ii)′ $(u-v)[T_{1,2}(u),\widetilde{T}_{2,2}(v)] = T_{1,1}(u)\widetilde{T}_{1,2}(v) + T_{1,2}(u)\widetilde{T}_{2,2}(v)$;
(iii)′ $(u-v)[T_{1,2}(u),\widetilde{T}_{2,1}(v)] = T_{1,1}(u)\widetilde{T}_{1,1}(v) + T_{1,2}(u)\widetilde{T}_{2,1}(v) - \widetilde{T}_{2,1}(v)T_{1,2}(u) - \widetilde{T}_{2,2}(v)T_{2,2}(u)$;
(iv)′ $[T_{1,2}(u),\widetilde{T}_{1,2}(v)] = 0$.

We also note that

$$\begin{pmatrix} T_{1,1}(u) & T_{1,2}(u) \\ T_{2,1}(u) & T_{2,2}(u) \end{pmatrix} = \begin{pmatrix} D_1(u) & D_1(u)E_1(u) \\ F_1(u)D_1(u) & D_2(u)+F_1(u)D_1(u)E_1(u) \end{pmatrix}$$

and that

$$\begin{pmatrix} \widetilde{T}_{1,1}(v) & \widetilde{T}_{1,2}(v) \\ \widetilde{T}_{2,1}(v) & \widetilde{T}_{2,2}(v) \end{pmatrix} = \begin{pmatrix} \widetilde{D}_1(v)+E_1(v)\widetilde{D}_2(v)F_1(v) & -E_1(v)\widetilde{D}_2(v) \\ -\widetilde{D}_2(v)F_1(v) & \widetilde{D}_2(v) \end{pmatrix}.$$

Substituting from these into (i)′ and using the known fact that $D_1(u)$ commutes with $\widetilde{D}_2(v)$ gives the identity

$$(u-v)[D_1(u),E_1(v)]\widetilde{D}_2(v) = D_1(u)(E_1(v)-E_1(u))\widetilde{D}_2(v).$$

Multiplying on the right by $D_2(v)$ gives (i). The deduction of (ii) from (ii)′ is entirely similar.

Next we deduce (iii) from (iii)′. Rewriting (i) and (ii) using τ gives that

$$E_1(v)\widetilde{D}_2(v) + (u-v-1)E_1(u)\widetilde{D}_2(v) = (u-v)\widetilde{D}_2(v)E_1(u)$$
$$F_1(u)D_1(u) + (u-v-1)F_1(v)D_1(u) = (u-v)D_1(u)F_1(v).$$

Also rearranging (iii)′ gives

$$D_1(u)\widetilde{D}_1(v) + D_1(u)(E_1(v)\widetilde{D}_2(v) + (u-v-1)E_1(u)\widetilde{D}_2(v))F_1(v) = \\ D_2(u)\widetilde{D}_2(v) + \widetilde{D}_2(v)(F_1(u)D_1(u) + (u-v-1)F_1(v)D_1(u))E_1(u).$$

Now substituting the first two of these identities into the third and multiplying on the left by $\widetilde{D}_1(u)D_2(v)$ gives (iii).

Finally we must deduce (iv). By (iv)′, we have that

$$D_1(u)E_1(u)E_1(v)\widetilde{D}_2(v) = E_1(v)D_1(u)\widetilde{D}_2(v)E_1(u).$$

Multiply both sides by $(u-v)^2$ and use (i) and (ii) to move $D_1(u)$ to the left and $\widetilde{D}_2(v)$ to the right, then cancel the leading $D_1(u)$'s and trailing $\widetilde{D}_2(v)$'s to get

$$(u-v)^2E_1(u)E_1(v) = ((u-v)E_1(v) - E_1(v) + E_1(u))\,((u-v)E_1(u) + E_1(v) - E_1(u))\,.$$

Hence,

(iv)″ $(u-v)^2[E_1(u),E_1(v)] = (E_1(v)-E_1(u))(E_1(u)-E_1(v)) + (u-v)E_1(v)(E_1(v)-E_1(u)) + (u-v)(E_1(u)-E_1(v))E_1(u)$.

Now subtract $(u-v)[E_1(u),E_1(v)]$ from both sides of (iv)″ to deduce that

$$(u-v)(u-v-1)[E_1(u),E_1(v)] = (u-v-1)(E_1(v)-E_1(u))^2.$$

Hence (iv) follows on dividing both sides by $(u-v-1)$. □

Lemma 5.5. *The following identities hold in* $Y_3((u^{-1},v^{-1}))$*:*

(i) $[E_1(u),F_2(v)] = 0$;
(ii) $(u-v)[E_1(u),E_2(v)] = E_1(u)E_2(v) - E_1(v)E_2(v) - E_{1,3}(u) + E_{1,3}(v)$;
(iii) $[E_{1,3}(u),E_2(v)] = E_2(v)[E_1(u),E_2(v)]$;

(iv) $[E_1(u), E_{1,3}(v) - E_1(v)E_2(v)] = -[E_1(u), E_2(v)]E_1(u)$.

Proof. Arguing as in the proof of the previous lemma, we compute the $e_{1,2} \otimes e_{3,2}$-, $e_{1,2} \otimes e_{2,3}$-, $e_{1,3} \otimes e_{2,3}$- and $e_{1,2} \otimes e_{1,3}$-coefficients of (2.9) respectively to obtain the identities

(i)′ $[D_1(u)E_1(u), \widetilde{D}_3(v)F_2(v)] = 0$

(ii)′ $(u-v)[D_1(u)E_1(u), E_2(v)\widetilde{D}_3(v)] = D_1(u)(E_1(u)E_2(v) - E_1(v)E_2(v) + E_{1,3}(v) - E_{1,3}(u))\widetilde{D}_3(v)$;

(iii)′ $[D_1(u)E_{1,3}(u), E_2(v)\widetilde{D}_3(v)] = 0$;

(iv)′ $[D_1(u)E_1(u), (E_{1,3}(v) - E_1(v)E_2(v))\widetilde{D}_3(v)] = 0$.

Using commutation relations already derived, (i) and (ii) follow easily from (i)′ and (ii)′. To prove (iii), we need one more identity. By Lemma 5.4(ii), we know that

$$(u-v)[\widetilde{D}_3(v), E_2(u)] = (E_2(v) - E_2(u))\widetilde{D}_3(v).$$

Considering u^0-coefficients gives $[\widetilde{D}_3(v), E_2^{(1)}] = E_2(v)\widetilde{D}_3(v)$. Hence, recalling (5.5),

$$\begin{aligned}[E_{1,3}(u), \widetilde{D}_3(v)] &= [[E_1(u), E_2^{(1)}], \widetilde{D}_3(v)] = [E_1(u), [E_2^{(1)}, \widetilde{D}_3(v)]] \\ &= -[E_1(u), E_2(v)\widetilde{D}_3(v)] = -[E_1(u), E_2(v)]\widetilde{D}_3(v).\end{aligned}$$

Now take (iii)′, cancel the leading $D_1(u)$ and then simplify to get

$$\begin{aligned}[E_{1,3}(u), E_2(v)\widetilde{D}_3(v)] &= [E_{1,3}(u), E_2(v)]\widetilde{D}_3(v) + E_2(v)[E_{1,3}(u), \widetilde{D}_3(v)] \\ &= ([E_{1,3}(u), E_2(v)] - E_2(v)[E_1(u), E_2(v)])\,\widetilde{D}_3(v) = 0.\end{aligned}$$

This proves (iii). For (iv), note to start with by considering the u^0-coefficients of (ii) that $[E_1^{(1)}, E_2(v)] = E_{1,3}(v) - E_1(v)E_2(v)$. Lemma 5.4(i) implies that $[D_1(u), E_1^{(1)}] = D_1(u)E_1(u)$. Now compute:

$$\begin{aligned}[D_1(u), E_{1,3}(v) - E_1(v)E_2(v)] &= [D_1(u), [E_1^{(1)}, E_2(v)]] = [[D_1(u), E_1^{(1)}], E_2(v)] \\ &= [D_1(u)E_1(u), E_2(v)] = D_1(u)[E_1(u), E_2(v)].\end{aligned}$$

Using this identity to rewrite (iv)′ we get

$$\begin{aligned}[D_1(u)E_1(u), E_{1,3}(v) - E_1(v)E_2(v)] &= D_1(u)[E_1(u), E_2(v)]E_1(u) \\ &\quad + D_1(u)[E_1(u), E_{1,3}(v) - E_1(v)E_2(v)] = 0.\end{aligned}$$

Now (iv) follows on cancelling $D_1(u)$. □

Lemma 5.6. *The following relations hold:*

(i) $[[E_1(u), E_2(v)], E_2(v)] = 0$;

(ii) $[E_1(u), [E_1(u), E_2(v)]] = 0$.

Proof. (i) Compute using Lemma 5.5(ii) and (iii):

$$\begin{aligned}(u-v)[[E_1(u), E_2(v)], E_2(v)] &= [E_1(u)E_2(v) - E_1(v)E_2(v) - E_{1,3}(u) + E_{1,3}(v), E_2(v)] \\ &= [E_1(u), E_2(v)]E_2(v) - [E_1(v), E_2(v)]E_2(v) \\ &\qquad + E_2(v)[E_1(v), E_2(v)] - E_2(v)[E_1(u), E_2(v)] \\ &= [[E_1(u), E_2(v)], E_2(v)] - [[E_1(v), E_2(v)], E_2(v)].\end{aligned}$$

We conclude that $(u-v-1)[[E_1(u),E_2(v)],E_2(v)] = -[[E_1(v),E_2(v)],E_2(v)]$. Now let $u = v+1$ to deduce that the right hand side equals zero, then divide by $(u-v-1)$ to proof the lemma.

(ii) Similar calculation using Lemma 5.5(iv) instead of (iii). □

Lemma 5.7. *The following relations hold:*

(i) $[[E_1(u),E_2(v)],E_2(w)] + [[E_1(u),E_2(w)],E_2(v)] = 0$;

(ii) $[E_1(u),[E_1(v),E_2(w)]] + [E_1(v),[E_1(u),E_2(w)]] = 0$.

Proof. (i) We show that the expression $(u-v)(u-w)(v-w)[[E_1(u),E_2(v)],E_2(w)]$ is symmetric in v and w. By Lemma 5.5(ii) it equals

$$(u-w)(v-w)[E_1(u)E_2(v) - E_1(v)E_2(v) + E_{1,3}(v) - E_{1,3}(u), E_2(w)]$$

Using Lemmas 5.5(iii) and 5.6(i) this equals

$$\begin{aligned}&(u-w)(v-w)[E_1(u),E_2(w)]E_2(v) + (u-w)(v-w)E_1(u)[E_2(v),E_2(w)]\\&-(u-w)(v-w)[E_1(v),E_2(w)]E_2(v) - (u-w)(v-w)E_1(v)[E_2(v),E_2(w)]\\&+(u-w)(v-w)[E_1(v),E_2(w)]E_2(w) - (u-w)(v-w)[E_1(u),E_2(w)]E_2(w)\end{aligned}$$

Now use Lemmas 5.5(ii) and 5.4(iv) to expand the commutators once more to get

$$\begin{aligned}&(v-w)(E_1(u)E_2(w)E_2(v) - E_1(w)E_2(w)E_2(v) + E_{1,3}(w)E_2(v) - E_{1,3}(u)E_2(v))\\&+(u-w)(E_1(u)E_2(v)^2 - E_1(u)E_2(v)E_2(w) - E_1(u)E_2(w)E_2(v) + E_1(u)E_2(w)^2)\\&-(u-w)(E_1(v)E_2(w)E_2(v) - E_1(w)E_2(w)E_2(v) + E_{1,3}(w)E_2(v) - E_{1,3}(v)E_2(v))\\&-(u-w)(E_1(v)E_2(v)^2 - E_1(v)E_2(v)E_2(w) - E_1(v)E_2(w)E_2(v) + E_1(v)E_2(w)^2)\\&+(u-w)(E_1(v)E_2(w)E_2(w) - E_1(w)E_2(w)E_2(w) + E_{1,3}(w)E_2(w) - E_{1,3}(v)E_2(w))\\&-(v-w)(E_1(u)E_2(w)E_2(w) - E_1(w)E_2(w)E_2(w) + E_{1,3}(w)E_2(w) - E_{1,3}(u)E_2(w)).\end{aligned}$$

Now open the parentheses and check that the resulting expression is symmetric in v and w to complete the proof.

(ii) A similar calculation using Lemma 5.5(iv) instead of (iii) and Lemma 5.6(ii) instead of (i) shows that the expression $(u-v)(u-w)(v-w)[E_1(u),[E_1(v),E_2(w)]]$ is symmetric in u and v. □

Now we can verify the remaining relations needed for the first part of the proof. Note that

$$(E_1(v) - E_1(u))/(u-v) = \sum_{r,s\geq 1} E_1^{(r+s-1)} u^{-r} v^{-s}. \tag{5.23}$$

Using this, divide both sides of the identity from Lemma 5.4(i) by $(u-v)$ and equate $u^{-r}v^{-s}$-coefficients on both sides to prove (5.11) with $i=1, j=1$. Next, multiplying Lemma 5.4(ii) on the left and right by $D_2(v)$ then swapping u and v gives the identity

$$(u-v)[D_2(u),E_1(v)] = -D_2(u)(E_1(v) - E_1(u)). \tag{5.24}$$

Now argue using (5.23) again to deduce (5.11) with $i=2, j=1$ from this. Similarly one gets (5.10) with $i=1, j=1$ from Lemma 5.4(iii), (5.10) with $i=2, j=1$ from Lemma 5.5(i), (5.13) with $i=1$ from Lemma 5.4(iv), (5.15) with $i=1$ from Lemma 5.5(ii), (5.19) with $i=2, j=1$ from Lemma 5.7(i) and (5.19) with $i=1, j=2$ from Lemma 5.7(ii).

Now we consider the second part of the proof. Let $\widehat{Y}_n$ denote the algebra with generators and relations as in the statement of Theorem 5.2. For $1 \leq i < j \leq n$, define elements $E_{i,j}^{(r)}, F_{i,j}^{(r)} \in \widehat{Y}_n$ by the equations (5.5). Let $\widehat{Y}_{(1^n)}$ resp. $\widehat{Y}_{(1^n)}^+$ resp. $\widehat{Y}_{(1^n)}^-$ denote the subalgebra of $\widehat{Y}_n$ generated by the elements $\{D_i^{(r)}\}_{i=1,\dots,n,r\geq 1}$ resp. $\{E_i^{(r)}\}_{i=1,\dots,n-1,r\geq 1}$ resp. $\{F_i^{(r)}\}_{i=1,\dots,n-1,r\geq 1}$. Define an ascending filtration

$$\mathrm{L}_0\widehat{Y}_{(1^n)}^+ \subseteq \mathrm{L}_1\widehat{Y}_{(1^n)}^+ \subseteq \cdots$$

on $\widehat{Y}_{(1^n)}^+$ by declaring that the generator $E_i^{(r)}$ is of degree $(r-1)$, i.e. $\mathrm{L}_d\widehat{Y}_{(1^n)}^+$ is the span of all monomials in these generators of total degree at most d. Let $\operatorname{gr}^{\mathrm{L}} \widehat{Y}_{(1^n)}^+$ denote the associated graded algebra, and let $e_{i,j;r} := \operatorname{gr}_r^{\mathrm{L}} E_{i,j}^{(r+1)} \in \operatorname{gr}^{\mathrm{L}} \widehat{Y}_{(1^n)}^+$ for each $1 \leq i < j \leq n$ and $r \geq 0$.

Lemma 5.8. *For $1 \leq i < j \leq n, 1 \leq h < k \leq n$ and $r, s \geq 0$, we have that*

$$[e_{i,j;r}, e_{h,k;s}] = e_{i,k;r+s}\delta_{h,j} - \delta_{i,k}e_{h,j;r+s}.$$

Proof. By the defining relations for $\widehat{Y}_n$, we have easily that

(i) $[e_{i,i+1;r}, e_{j,j+1;s}] = 0$ if $|i-j| \neq 1$;
(ii) $[e_{i,i+1;r+1}, e_{j,j+1;s}] = [e_{i,i+1;r}, e_{j,j+1;s+1}]$ if $|i-j| = 1$;
(iii) $[e_{i,i+1;r}, [e_{i,i+1;s}, e_{j,j+1;t}]] = -[e_{i,i+1;s}, [e_{i,i+1;r}, e_{j,j+1;t}]]$ if $|i-j| = 1$.

We also have by definition that

(iv) $e_{i,j;r} = [e_{i,j-1;r}, e_{j-1,j;0}]$ for $j > i+1$.

Now we consider seven cases.

(1) $j < h$. Obviously, $[e_{i,j;r}, e_{h,k;s}] = 0$.
(2) $j = h$. By (ii) and (iv), $[e_{j-1,j;r}, e_{j,j+1;s}] = e_{j-1,j+1;r+s}$. Now bracket with $e_{j+1,j+2;0}, \dots, e_{k-1,k;0}$ to deduce that $[e_{j-1,j;r}, e_{j,k;s}] = e_{j-1,k;r+s}$. Finally bracket with $e_{j-2,j-1;0}, \dots, e_{i,i+1;0}$.
(3) $i < h, j = k$. Let us just show that $[e_{1,3;r}, e_{2,3;s}] = 0$, since the general case is an easy consequence. Note that by (iii), $[e_{1,3;r}, e_{2,3;s}] = [[e_{1,2;r}, e_{2,3;0}], e_{2,3;s}] = -[[e_{1,2;r}, e_{2,3;s}], e_{2,3;0}]$. By (ii) this equals $-[[e_{1,2;r+s}, e_{2,3;0}], e_{2,3;0}]$ which is zero by (iii).
(4) $i = h, j < k$. Similar to (3).
(5) $i = h, j = k$. If $j = i+1$, we are done by (i); else,

$$\begin{aligned}[e_{i,j;r}, e_{i,j;s}] &= [[e_{i,j-1;r}, e_{j-1,j;0}], e_{i,j;s}] \\ &= [[e_{i,j-1;r}, e_{i,j;s}], e_{j-1,j;0}] + [e_{i,j-1;r}, [e_{j-1,j;0}, e_{i,j;s}]]\end{aligned}$$

which is zero by (3) and (4).

(6) $i < h < j < k$. We just show $[e_{1,3;r}, e_{2,4;s}] = 0$. It equals

$$\begin{aligned}[[e_{1,2;r}, e_{2,3;0}], [e_{2,3;0}, e_{3,4;s}]] &= [e_{2,3;0}, [[e_{1,2;r}, e_{2,3;0}], e_{3,4;s}]] \\ &= [e_{2,3;0}, [e_{1,2;r}, [e_{2,3;0}, e_{3,4;s}]]] \\ &= [[e_{2,3;0}, e_{1,2;r}], [e_{2,3;0}, e_{3,4;s}]] \\ &= -[[e_{1,2;r}, e_{2,3;0}], [e_{2,3;0}, e_{3,4;s}]].\end{aligned}$$

Hence it is zero.

(7) $i < h, k < j$. We just show $[e_{2,3;r}, e_{1,4;s}] = 0$. It equals $[e_{2,3;r}, [e_{1,3;s}, e_{3,4;0}]] = [e_{1,3;s}, e_{2,4;r}]$, which is zero by (6).

□

Lemma 5.9. *The algebra $\widehat{Y}_n$ is spanned by the set of monomials in $\{D_i^{(r)}\}_{i=1,\dots,n,r\geq 1} \cup \{E_{i,j}^{(r)}, F_{i,j}^{(r)}\}_{1\leq i<j\leq n,r\geq 1}$, taken in some fixed order so that F's come before D's and D's come before E's.*

Proof. Using Lemma 5.8, one shows easily that the associated graded algebra $\operatorname{gr}^{\mathrm{L}} \widehat{Y}^+_{(1^n)}$ is spanned by the set of all ordered monomials in the elements $\{e_{i,j;r}\}_{1\leq i<j\leq n,r\geq 0}$ taken in some fixed order. Hence $\widehat{Y}^+_{(1^n)}$ itself is spanned by the corresponding monomials in $\{E_{i,j}^{(r)}\}_{1\leq i<j\leq n,r\geq 1}$. By the defining relations for $\widehat{Y}_n$, there is an antiautomorphism τ of $\widehat{Y}_n$ fixing each $D_i^{(r)}$ and interchanging each $E_{i,j}^{(r)}$ with $F_{i,j}^{(r)}$. It follows on applying τ that the set of monomials in $\{F_{i,j}^{(r)}\}_{1\leq i<j\leq n}$ taken in some fixed order span $\widehat{Y}^-_{(1^n)}$, while obviously the ordered monomials in the elements $\{D_i^{(r)}\}_{i=1,\dots,n,r\geq 1}$ span $\widehat{Y}_{(1^n)}$. Since by the defining relations the natural multiplication map $\widehat{Y}^-_{(1^n)} \otimes \widehat{Y}_{(1^n)} \otimes \widehat{Y}^+_{(1^n)} \to \widehat{Y}_n$ is surjective, the lemma follows. □

Now, the first part of the proof of Theorem 5.2 above implies that there is a surjective algebra homomorphism $\theta : \widehat{Y}_n \to Y_n$ sending $D_i^{(r)}, E_{i,j}^{(r)}, F_{i,j}^{(r)} \in \widehat{Y}_n$ to the elements with the same name in Y_n. To complete the proof of Theorem 5.2 we need to show that θ is an isomorphism. This follows immediately from Lemma 5.10 below, since it shows that the images of the monomials that span $\widehat{Y}_n$ from Lemma 5.9 are linearly independent in Y_n itself.

Lemma 5.10. *The set of monomials in $\{D_i^{(r)}\}_{i=1,\dots,n,r\geq 1} \cup \{E_{i,j}^{(r)}, F_{i,j}^{(r)}\}_{1\leq i<j\leq n,r\geq 1}$ taken in some fixed order is linearly independent in Y_n.*

Proof. As explained at the end of §2, we can identify the associated graded algebra $\operatorname{gr}^{\mathrm{L}} Y_n$ with $U(\mathfrak{gl}_n[t])$, so that $\operatorname{gr}^{\mathrm{L}}_r T_{i,j}^{(r+1)}$ is identified with $e_{i,j}t^r$. It is easy to see from (5.2)–(5.4) that under this identification $\operatorname{gr}^{\mathrm{L}}_r D_i^{(r+1)}$ resp. $\operatorname{gr}^{\mathrm{L}}_r E_{i,j}^{(r+1)}$ resp. $\operatorname{gr}^{\mathrm{L}}_r F_{i,j}^{(r+1)}$ is identified with $e_{i,i}t^r$ resp. $e_{i,j}t^r$ resp. $e_{j,i}t^r$. Hence by the PBW theorem for $U(\mathfrak{gl}_n[t])$, the set of all monomials in

$$\{\operatorname{gr}^{\mathrm{L}}_r D_i^{(r+1)}\}_{i=1,\dots,n,r\geq 0} \cup \{\operatorname{gr}^{\mathrm{L}}_r E_{i,j}^{(r+1)}, \operatorname{gr}^{\mathrm{L}}_r F_{i,j}^{(r+1)}\}_{1\leq i<j\leq n,r\geq 0}$$

taken in some fixed order forms a basis for $\operatorname{gr}^{\mathrm{L}} Y_n$. The lemma follows easily. □

This completes the proof of Theorem 5.2. Let us also state the following theorem which was obtained in the course of the above proof; cf. [L].

Theorem 5.11. (i) *The set of all monomials in $\{D_i^{(r)}\}_{i=1,\dots,n,r\geq 1}$ taken in some fixed order form a basis for $Y_{(1^n)}$.*

(ii) *The set of all monomials in $\{E_{i,j}^{(r)}\}_{1\leq i<j\leq n,r\geq 1}$ taken in some fixed order form a basis for $Y^+_{(1^n)}$.*

(iii) *The set of all monomials in $\{F_{i,j}^{(r)}\}_{1\leq i<j\leq n,r\geq 1}$ taken in some fixed order form a basis for $Y^-_{(1^n)}$.*

(iv) *The set of all monomials in* $\{D_i^{(r)}\}_{i=1,\dots,n,r\geq 1} \cup \{E_{i,j}^{(r)}, F_{i,j}^{(r)}\}_{1\leq i<j\leq n, r\geq 1}$ *taken in some fixed order form a basis for* Y_n.

Remark 5.12. Let us explain the relationship between the presentation given in Theorem 5.2 and Drinfeld's presentation from [D2], since there are some additional shifts in u. Actually, the latter is a presentation for the subalgebra

$$Y(\mathfrak{sl}_n) = \{x \in Y_n \mid \mu_f(x) = x \text{ for all } f(u) \in 1 + \mathbb{C}[[u^{-1}]]\};$$

see [MNO, Definition 2.14]. Define $\kappa_{i,k}, \xi^{\pm}_{i,k}$ for $i = 1, \dots, n-1$ and $k \geq 0$ from the equations

$$\kappa_i(u) = \sum_{k\geq 0} \kappa_{i,k} u^{-k-1} := 1 + \widetilde{D}_i\left(u - \frac{i-1}{2}\right) D_{i+1}\left(u - \frac{i-1}{2}\right), \tag{5.25}$$

$$\xi_i^+(u) = \sum_{k\geq 0} \xi^+_{i,k} u^{-k-1} := E_i\left(u - \frac{i-1}{2}\right), \tag{5.26}$$

$$\xi_i^-(u) = \sum_{k\geq 0} \xi^-_{i,k} u^{-k-1} := F_i\left(u - \frac{i-1}{2}\right). \tag{5.27}$$

One can check by equating coefficients in the identities from Lemmas 5.4, 5.5 and 5.7 that these elements generate $Y(\mathfrak{sl}_n)$ subject to the Drinfeld relations, namely:

$$[\kappa_{i,k}, \kappa_{j,l}] = 0, \tag{5.28}$$

$$[\xi^+_{i,k}, \xi^-_{j,l}] = \delta_{i,j}\kappa_{i,k+l}, \tag{5.29}$$

$$[\kappa_{i,0}, \xi^{\pm}_{j,l}] = \pm a_{i,j}\xi^{\pm}_{j,l}, \tag{5.30}$$

$$[\kappa_{i,k}, \xi^{\pm}_{j,l+1}] - [\kappa_{i,k+1}, \xi^{\pm}_{j,l}] = \pm\frac{a_{i,j}}{2}(\kappa_{i,k}\xi^{\pm}_{j,l} + \xi^{\pm}_{j,l}\kappa_{i,k}), \tag{5.31}$$

$$[\xi^{\pm}_{i,k}, \xi^{\pm}_{j,l+1}] - [\xi^{\pm}_{i,k+1}, \xi^{\pm}_{j,l}] = \pm\frac{a_{i,j}}{2}(\xi^{\pm}_{i,k}\xi^{\pm}_{j,l} + \xi^{\pm}_{j,l}\xi^{\pm}_{i,k}), \tag{5.32}$$

$$i \neq j, N = 1 - a_{i,j} \Rightarrow \mathrm{Sym}[\xi^{\pm}_{i,k_1}, [\xi^{\pm}_{i,k_2}, \cdots [\xi^{\pm}_{i,k_N}, \xi^{\pm}_{j,l}]\cdots]] = 0 \tag{5.33}$$

where $(a_{i,j})_{1\leq i,j<n}$ denotes the Cartan matrix of type A_{n-1} indexed in the standard way and Sym denotes symmetrization with respect to $k_1, \dots, k_N$. For example, let us verify (5.32) in the case $j = i+1$ and the sign is $+$: applying Lemma 5.5(ii)

$$\begin{aligned}
&(u-v)[\xi_i^+(u), \xi_{i+1}^+(v)] \\
&= \left(\left(u - \frac{i-1}{2}\right) - \left(v - \frac{i}{2}\right) - \frac{1}{2}\right)\left[E_i\left(u - \frac{i-1}{2}\right), E_{i+1}\left(v - \frac{i}{2}\right)\right] \\
&= E_i\left(u - \frac{i-1}{2}\right)E_{i+1}\left(v - \frac{i}{2}\right) - E_i\left(v - \frac{i}{2}\right)E_{i+1}\left(v - \frac{i}{2}\right) \\
&\quad - E_{i,i+2}\left(u - \frac{i-1}{2}\right) + E_{i,i+2}\left(v - \frac{i}{2}\right) - \frac{1}{2}\left[E_i\left(u - \frac{i-1}{2}\right), E_{i+1}\left(v - \frac{i}{2}\right)\right] \\
&= \frac{1}{2}\left(\xi_i^+(u)\xi_{i+1}^+(v) + \xi_{i+1}^+(v)\xi_i^+(u)\right) - E_i\left(v - \frac{i}{2}\right)E_{i+1}\left(v - \frac{i}{2}\right) \\
&\quad - E_{i,i+2}\left(u - \frac{i-1}{2}\right) + E_{i,i+2}\left(v - \frac{i}{2}\right).
\end{aligned}$$

Now equate $u^{-k-1}v^{-l-1}$-coefficients on both sides to get

$$[\xi^+_{i,k+1}, \xi^+_{i+1,l}] - [\xi^+_{i,k}, \xi^+_{i+1,l+1}] = \frac{1}{2}\left(\xi^+_{i,k}\xi^+_{i+1,l} + \xi^+_{i+1,l}\xi^+_{i,k}\right)$$

as required. (*Beware*: the relations (5.28)–(5.33) are actually not exactly the same as the relations in [D2] — one needs to swap $\xi^+_{i,k}$ and $\xi^-_{i,k}$ and replace $\kappa_{i,k}$ with $-\kappa_{i,k}$ to get those. The reason for the difference is that we have chosen to work with the opposite presentation to Drinfeld throughout.)

6. Parabolic subalgebras

Now we are ready to prove Theorems A and B stated in the introduction. Fix throughout the section a tuple $\nu = (\nu_1, \dots, \nu_m)$ of non-negative integers summing to n. Note there is going to be some overlap between the notation here and that of the previous section, which is the special case $\nu = (1^n)$ of the present definitions. When necessary, we will add an additional superscript ν to our notation to avoid any ambiguity as ν varies. Factor the $n \times n$ matrix $T(u)$ as

$$T(u) = F(u)D(u)E(u) \tag{6.1}$$

for unique *block matrices*

$$D(u) = \begin{pmatrix} D_1(u) & 0 & \cdots & 0 \\ 0 & D_2(u) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_m(u) \end{pmatrix},$$

$$E(u) = \begin{pmatrix} I_{\nu_1} & E_{1,2}(u) & \cdots & E_{1,m}(u) \\ 0 & I_{\nu_2} & \cdots & E_{2,m}(u) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{\nu_m} \end{pmatrix}, F(u) = \begin{pmatrix} I_{\nu_1} & 0 & \cdots & 0 \\ F_{1,2}(u) & I_{\nu_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{1,m}(u) & F_{2,m}(u) & \cdots & I_{\nu_m} \end{pmatrix},$$

where $D_a(u) = (D_{a;i,j}(u))_{1 \le i,j \le \nu_a}$, $E_{a,b}(u) = (E_{a,b;i,j}(u))_{1 \le i \le \nu_a, 1 \le j \le \nu_b}$ and $F_{a,b}(u) = (F_{a,b;i,j}(u))_{1 \le i \le \nu_b, 1 \le j \le \nu_a}$ are $\nu_a \times \nu_a$, $\nu_a \times \nu_b$ and $\nu_b \times \nu_a$ matrices, respectively. Also define the $\nu_a \times \nu_a$ matrix $\widetilde{D}_a(u) = (\widetilde{D}_{a;i,j}(u))_{1 \le i,j \le \nu_a}$ by $\widetilde{D}_a(u) := -D_a(u)^{-1}$. The entries of these matrices define power series $D_{a;i,j}(u) = \sum_{r \ge 0} D^{(r)}_{a;i,j} u^{-r}$, $\widetilde{D}_{a;i,j}(u) = \sum_{r \ge 0} \widetilde{D}^{(r)}_{a;i,j} u^{-r}$, $E_{a,b;i,j}(u) = \sum_{r \ge 1} E^{(r)}_{a,b;i,j} u^{-r}$ and $F_{a,b;i,j}(u) = \sum_{r \ge 1} F^{(r)}_{a,b;i,j} u^{-r}$. We let $E_{a;i,j}(u) = \sum_{r \ge 1} E^{(r)}_{a;i,j} u^{-r} := E_{a,a+1;i,j}(u)$ and $F_{a;i,j}(u) = \sum_{r \ge 1} F^{(r)}_{a;i,j} u^{-r} := F_{a,a+1;i,j}(u)$ for short.

Like before, there are explicit descriptions of all these elements in terms of quasi-determinants. To write them down, write the matrix $T(u)$ in block form as

$$T(u) = \begin{pmatrix} {}^{\nu}T_{1,1}(u) & \cdots & {}^{\nu}T_{1,m}(u) \\ \vdots & \ddots & \dots \\ {}^{\nu}T_{m,1}(u) & \cdots & {}^{\nu}T_{m,m}(u) \end{pmatrix}$$

where ${}^\nu T_{a,b}(u)$ is a $\nu_a \times \nu_b$ matrix. Then, recalling the notation (4.3),

$$D_a(u) = \begin{vmatrix} {}^\nu T_{1,1}(u) & \cdots & {}^\nu T_{1,a-1}(u) & {}^\nu T_{1,a}(u) \\ \vdots & \ddots & \vdots & \vdots \\ {}^\nu T_{a-1,1}(u) & \cdots & {}^\nu T_{a-1,a-1}(u) & {}^\nu T_{a-1,a}(u) \\ {}^\nu T_{a,1}(u) & \cdots & {}^\nu T_{a,a-1}(u) & \boxed{{}^\nu T_{a,a}(u)} \end{vmatrix}, \tag{6.2}$$

$$E_{a,b}(u) = D_a(u)^{-1} \begin{vmatrix} {}^\nu T_{1,1}(u) & \cdots & {}^\nu T_{1,a-1}(u) & {}^\nu T_{1,b}(u) \\ \vdots & \ddots & \vdots & \vdots \\ {}^\nu T_{a-1,1}(u) & \cdots & {}^\nu T_{a-1,a-1}(u) & {}^\nu T_{a-1,b}(u) \\ {}^\nu T_{a,1}(u) & \cdots & {}^\nu T_{a,a-1}(u) & \boxed{{}^\nu T_{a,b}(u)} \end{vmatrix}, \tag{6.3}$$

$$F_{a,b}(u) = \begin{vmatrix} {}^\nu T_{1,1}(u) & \cdots & {}^\nu T_{1,a-1}(u) & {}^\nu T_{1,a}(u) \\ \vdots & \ddots & \vdots & \vdots \\ {}^\nu T_{a-1,1}(u) & \cdots & {}^\nu T_{a-1,a-1}(u) & {}^\nu T_{a-1,a}(u) \\ {}^\nu T_{b,1}(u) & \cdots & {}^\nu T_{b,a-1}(u) & \boxed{{}^\nu T_{b,a}(u)} \end{vmatrix} D_a(u)^{-1}. \tag{6.4}$$

It follows in particular from these descriptions that for $b > a+1$ and $1 \leq i \leq \nu_a, 1 \leq j \leq \nu_b$,

$$E^{(r)}_{a,b;i,j} = [E^{(r)}_{a,b-1;i,k}, E^{(1)}_{b-1;k,j}], \qquad F^{(r)}_{a,b;j,i} = [F^{(1)}_{b-1;j,k}, F^{(r)}_{a,b-1;k,i}], \tag{6.5}$$

for any $1 \leq k \leq \nu_{b-1}$. We also get the analogue of Lemma 5.1:

Lemma 6.1. *Fix $a \geq 1$ and let $\bar\nu := (\nu_a, \nu_{a+1}, \dots, \nu_m)$. Then, for all admissible i, j,*
(i) ${}^\nu D_{a;i,j}(u) = \psi_{\nu_1+\cdots+\nu_{a-1}}({}^{\bar\nu} D_{1;i,j}(u))$;
(ii) ${}^\nu E_{a;i,j}(u) = \psi_{\nu_1+\cdots+\nu_{a-1}}({}^{\bar\nu} E_{1;i,j}(u))$;
(iii) ${}^\nu F_{a;i,j}(u) = \psi_{\nu_1+\cdots+\nu_{a-1}}({}^{\bar\nu} F_{1;i,j}(u))$.

In particular Lemma 6.1(i) shows that the elements $D^{(r)}_{a;i,j}$ here are the same as the generators of the standard Levi subalgebra Y_ν introduced at the end of §4, so they satisfy the relations (4.8). We also let Y_ν^+ resp. Y_ν^- denote the subalgebra generated by $\{E^{(r)}_{a;i,j}\}_{1\leq a<m, 1\leq i\leq \nu_a, 1\leq j\leq\nu_{a+1}, r\geq 1}$ resp. $\{F^{(r)}_{a;i,j}\}_{1\leq a<m, 1\leq i\leq \nu_{a+1}, 1\leq j\leq\nu_{a}, r\geq 1}$. The antiautomorphism τ has the properties

$$\tau(D_{a;i,j}(u)) = D_{a;j,i}(u), \tag{6.6}$$

$$\tau(E_{a,b;i,j}(u)) = F_{a,b;j,i}(u), \tag{6.7}$$

$$\tau(F_{a,b;i,j}(u)) = E_{a,b;j,i}(u), \tag{6.8}$$

so it leaves Y_ν invariant and interchanges Y_ν^+ and Y_ν^-.

We have now defined the elements $D^{(r)}_{a;i,j}, E^{(r)}_{a;i,j}$ and $F^{(r)}_{a;i,j}$ appearing in Theorems A and B stated in the introduction. We are ready to explain the proofs of these theorems. Actually, the argument runs almost exactly parallel to the proofs of Theorems 5.2 and 5.11 given in the previous section. As before, there are two parts: first, to show all the relations (1.1)–(1.14) from Theorem A hold; second, to show we have enough relations by constructing the PBW bases described in Theorem B.

For the first part, one uses Lemma 6.1, (4.8) and (6.6)–(6.8) to reduce the problem to checking the following special cases: (1.4) with $a = 1, b = 1$ or $a = 2, b = 1$; (1.5) with $a = 1, b = 1$ or $a = 2, b = 1$; (1.7) with $a = 1$; (1.9) with $a = 1$; (1.11) with $a = 1, b = 2$; (1.13) with $a = 2, b = 1$ or $a = 1, b = 2$. These special cases may be deduced from the following four lemmas by equating coefficients. Note these lemmas are the exact analogues of Lemmas 5.4–5.7.

Lemma 6.2. *Suppose $m = 2$, i.e. $\nu = (\nu_1, \nu_2)$. The following identities hold for all admissible h, i, j, k:*

(i) $(u-v)[D_{1;i,j}(u), E_{1;h,k}(v)] = D_{1;i,p}(u)(E_{1;p,k}(v) - E_{1;p,k}(u))\delta_{h,j}$;

(ii) $(u-v)[E_{1;i,j}(u), \widetilde{D}_{2;h,k}(v)] = (E_{1;i,q}(u) - E_{1;i,q}(v))\widetilde{D}_{2;q,k}(v)\delta_{h,j}$;

(iii) $(u-v)[E_{1;i,j}(u), F_{1;h,k}(v)] = \widetilde{D}_{1;i,k}(v)D_{2;h,j}(v) - \widetilde{D}_{1;i,k}(u)D_{2;h,j}(u)$;

(iv) $(u-v)[E_{1;i,j}(u), E_{1;h,k}(v)] = (E_{1;i,k}(v) - E_{1;i,k}(u))(E_{1;h,j}(v) - E_{1;h,j}(u))$.

(Here, p resp. q should be summed over $1, \dots, \nu_1$ resp. $1, \dots, \nu_2$.)

Proof. Compute the $e_{i,j} \otimes e_{h,\nu_1+k}$-, $e_{i,\nu_1+j} \otimes e_{\nu_1+h,\nu_1+k}$-, $e_{i,\nu_1+j} \otimes e_{\nu_1+h,k}$- and $e_{i,\nu_1+j} \otimes e_{h,\nu_1+k}$-coefficients on each side of (2.9) and then rearrange the resulting identities like we did in the proof of Lemma 5.4 above to obtain:

(i)′ $(u-v)[D_{1;i,j}(u), E_{1;h,q}(v)\widetilde{D}_{2;q,k}(v)] =$
$D_{1;i,p}(u)(E_{1;p,q}(v) - E_{1;p,q}(u))\widetilde{D}_{2;q,k}(v)\delta_{h,j}$;

(ii)′ $(u-v)[D_{1;i,p}(u)E_{1;p,j}(u), \widetilde{D}_{2;h,k}(v)] =$
$D_{1;i,p}(u)(E_{1;p,q}(u) - E_{1;p,q}(v))\widetilde{D}_{2;q,k}(v)\delta_{h,j}$;

(iii)′ $(u-v)[D_{1;i,p}(u)E_{1;p,j}(u), \widetilde{D}_{2;h,q}(v)F_{1;q,k}(v)] =$
$\delta_{i,k}\widetilde{D}_{2;h,q}(v)\big\{D_{2;q,j}(u) + (F_{1;q,p}(u) - F_{1;q,p}(v))D_{1;p,p'}(u)E_{1;p',j}(u)\big\}$
$- D_{1;i,p}(u)\big\{\widetilde{D}_{1;p,k}(v) + (E_{1;p,q}(v) - E_{1;p,q}(u))\widetilde{D}_{2;q,q'}(v)F_{1;q',k}(v)\big\}\delta_{h,j}$;

(iv)′ $[D_{1;i,p}(u)E_{1;p,j}(u), E_{1;h,q}(v)\widetilde{D}_{2;q,k}(v)] = 0$.

(Here, p, p' resp. q, q' should also be summed over $1, \dots, \nu_1$ resp. $1, \dots, \nu_2$.) Now (i), (ii) and (iii) are deduced from (i)′, (ii)′ and (iii)′ by simplifying exactly like we did in the proof of Lemma 5.4. It turns out to be more difficult than before to deduce (iv) from (iv)′ so we explain this part of the argument more carefully. As before, one rewrites (iv)′ using (i) and (ii) to obtain:

(iv)″ $(u-v)^2[E_{1;i,j}(u), E_{1;h,k}(v)] = (E_{1;i,j}(v) - E_{1;i,j}(u))(E_{1;h,k}(u) - E_{1;h,k}(v)) +$
$(u-v)E_{1;h,j}(v)(E_{1;i,k}(v) - E_{1;i,k}(u)) + (u-v)(E_{1;i,k}(u) - E_{1;i,k}(v))E_{1;h,j}(u)$.

Now we deduce (iv) from this. For a power series X in $Y_n[[u^{-1}, v^{-1}]]$, let us write $\{X\}_d$ for the homogeneous component of X of total degree d in the variables u^{-1} and v^{-1}. We show by induction on $d = 1, 2, \dots$ that

$$(u-v)\{[E_{1;i,j}(u), E_{1;h,k}(v)]\}_{d+1} = \{(E_{1;i,k}(v) - E_{1;i,k}(u))(E_{1;h,j}(v) - E_{1;h,j}(u))\}_d.$$

For the base case $d = 1$, applying $\{.\}_0$ to (iv)″ shows $(u-v)^2\{[E_{1;i,j}(u), E_{1;h,k}(v)]\}_2 = 0$, hence $(u-v)\{[E_{1;i,j}(u), E_{1;h,k}(v)]\}_2 = 0$ as required. For the induction step, assume the statement is true for $d > 1$. Apply $\{.\}_d$ to (iv)″ to get that

$$\begin{aligned}(u-v)^2\{[E_{1;i,j}(u), E_{1;h,k}(v)]\}_{d+2} &= (u-v)\{E_{1;h,j}(v)(E_{1;i,k}(v) - E_{1;i,k}(u))\}_{d+1}\\ &\quad - (u-v)\{(E_{1;i,k}(v) - E_{1;i,k}(u))E_{1;h,j}(u)\}_{d+1}\\ &\quad - \{(E_{1;i,j}(v) - E_{1;i,j}(u))(E_{1;h,k}(v) - E_{1;h,k}(u))\}_d.\end{aligned}$$

Now use the induction hypothesis, together with the identity $\{[E_{1;h,j}(v), E_{1;i,k}(v)]\}_{d+1} = 0$ which follows by dividing both sides of the induction hypothesis by $(u-v)$ then setting $u = v$, to rewrite the right hand side to deduce that

$$(u-v)^2\{[E_{1;i,j}(u), E_{1;h,k}(v)]\}_{d+2} = (u-v)\{(E_{1;i,k}(v) - E_{1;i,k}(u))E_{1;h,j}(v)\}_{d+1} \\ - (u-v)\{(E_{1;i,k}(v) - E_{1;i,k}(u))E_{1;h,j}(u)\}_{d+1}.$$

Dividing both sides by $(u-v)$ completes the proof of the induction step. □

Lemma 6.3. *Suppose $m = 3$, i.e. $\nu = (\nu_1, \nu_2, \nu_3)$. The following identities hold for all admissible g, h, i, j, k:*

(i) $[E_{1;i,j}(u), F_{2;h,k}(v)] = 0$;
(ii) $(u-v)[E_{1;i,j}(u), E_{2;h,k}(v)] = (E_{1;i,q}(u)E_{2;q,k}(v) - E_{1;i,q}(v)E_{2;q,k}(v) - E_{1,3;i,k}(u) + E_{1,3;i,k}(v))\delta_{h,j}$;
(iii) $[E_{1,3;i,j}(u), E_{2;h,k}(v)] = E_{2;h,j}(v)[E_{1;i,g}(u), E_{2;g,k}(v)]$;
(iv) $[E_{1;i,j}(u), E_{1,3;h,k}(v) - E_{1;h,q}(v)E_{2;q,k}(v)] = -[E_{1;i,g}(u), E_{2;g,k}(v)]E_{1;h,j}(u)$.

(Here, q should be summed over $1, \dots, \nu_2$.)

Proof. One computes the $e_{i,\nu_1+j} \otimes e_{\nu_1+\nu_2+h,\nu_1+k}$-, $e_{i,\nu_1+j} \otimes e_{\nu_1+h,\nu_1+\nu_2+k}$-, $e_{i,\nu_1+\nu_2+j} \otimes e_{\nu_1+h,\nu_1+\nu_2+k}$- and $e_{i,\nu_1+j} \otimes e_{h,\nu_1+\nu_2+k}$-coefficients of (2.9) respectively like in the proof of Lemma 5.5 to obtain the identities

(i)′ $[D_{1;i,p}(u)E_{1;p,j}(u), \widetilde{D}_{3;h,r}(v)F_{2;r,k}(v)] = 0$;
(ii)′ $(u-v)[D_{1;i,p}(u)E_{1;p,j}(u), E_{2;h,r}(v)\widetilde{D}_{3;r,k}(v)] = D_{1;i,p}(u)(E_{1;p,q}(u)E_{2;q,r}(v) - E_{1;p,q}(v)E_{2;q,r}(v) + E_{1,3;p,r}(v) - E_{1,3;p,r}(u))\widetilde{D}_{3;r,k}(v)\delta_{h,j}$;
(iii)′ $[D_{1;i,p}(u)E_{1,3;p,j}(u), E_{2;h,r}(v)\widetilde{D}_{3;r,k}(v)] = 0$;
(iv)′ $[D_{1;i,p}(u)E_{1;p,j}(u), (E_{1,3;h,r}(v) - E_{1;h,q}(v)E_{2;q,r}(v))\widetilde{D}_{3;r,k}(v)] = 0$.

(Here, p, q and r sum over $1, \dots, \nu_1, 1, \dots, \nu_2$ and $1, \dots, \nu_3$ respectively.) Now (i)–(iv) are deduced from (i)′–(iv)′ by copying the arguments from the proof of Lemma 5.5. □

Lemma 6.4. *Suppose $m = 3$, i.e. $\nu = (\nu_1, \nu_2, \nu_3)$. The following identities hold for all admissible f, g, h, i, j, k:*

(i) $[[E_{1;i,j}(u), E_{2;h,k}(v)], E_{2;f,g}(v)] = 0$;
(ii) $[E_{1;i,j}(u), [E_{1;h,k}(u), E_{2;f,g}(v)]] = 0$.

Proof. Dividing both sides of Lemma 6.2(iv) by $(u-v)$ then setting $v = u$ shows that $[E_{a;i,j}(u), E_{a;h,k}(u)] = 0$. Given this and Lemma 6.3(ii), (i) is obvious unless $f = h = j$ and (ii) is obvious unless $f = k = j$. Now the proof in these cases is completed exactly like the proof of Lemma 5.6. □

Lemma 6.5. *Suppose $m = 3$, i.e. $\nu = (\nu_1, \nu_2, \nu_3)$. The following identities hold for all admissible f, g, h, i, j, k:*

(i) $[[E_{1;i,j}(u), E_{2;h,k}(v)], E_{2;f,g}(w)] + [[E_{1;i,j}(u), E_{2;h,k}(w)], E_{2;f,g}(v)] = 0$;
(ii) $[E_{1;i,j}(u), [E_{1;h,k}(v), E_{2;f,g}(w)]] + [E_{1;i,j}(v), [E_{1;h,k}(u), E_{2;f,g}(w)]] = 0$.

Proof. Show that $(u-v)(u-w)(v-w)[[E_{1;i,j}(u), E_{2;j,k}(v)], E_{2;f,g}(w)]$ is symmetric in v and w and that $(u-v)(u-w)(v-w)[E_{1;i,j}(u), [E_{1;h,k}(v), E_{2;k,g}(w)]]$ is symmetric in u and v, following the argument of Lemma 5.7 exactly. □

Now we consider the second part of the proof. Let $\widehat{Y}_n$ denote the algebra with generators and relations as in the statement of Theorem A. Define elements $E_{a,b;i,j}^{(r)}, F_{a,b;j,i}^{(r)} \in \widehat{Y}_n$ by the equations (6.5). We need to check that these definitions are independent of the particular choice of k. Well, given $1 \leq k, k' \leq \nu_{b-1}$ with $k \neq k'$, we have that $[E_{a,b-1;i,k}^{(r)}, E_{b-1;k',j}^{(s)}] = 0$ by (1.11). Bracketing with $D_{b-1;k,k'}^{(1)}$ and using (1.5), one deduces that

$$[E_{a,b-1;i,k}^{(r)}, E_{b-1;k,j}^{(s)}] = [E_{a,b-1;i,k'}^{(r)}, E_{b-1;k',j}^{(s)}] \tag{6.9}$$

as required to verify that the definition of the elements $E_{a,b;i,j}^{(r)}$ is independent of the choice of k. A similar argument shows that the definition of the elements $F_{a,b;j,i}^{(r)}$ is independent of k too.

Let $\widehat{Y}_\nu$, $\widehat{Y}_\nu^+$ and $\widehat{Y}_\nu^-$ denote the subalgebras of $\widehat{Y}_n$ generated by the D's, E's and F's respectively. By the first part of the proof, there is a surjective homomorphism $\theta : \widehat{Y}_n \to Y_n$ sending $\widehat{Y}_\nu$ onto Y_ν and $\widehat{Y}_\nu^\pm$ onto $Y_\nu^\pm$. We just need to show that θ is an isomorphism. This is done just like in the previous section by exhibiting a set of monomials that span $\widehat{Y}_n$ whose image in Y_n is linearly independent. We just explain the key step, namely, the analogue of Lemma 5.8 allowing one to construct the spanning set for $\widehat{Y}_\nu^+$. Given this, the rest of our earlier argument extends without further complication to complete the proof. Define a filtration

$$\mathrm{L}_0\widehat{Y}_\nu^+ \subseteq \mathrm{L}_1\widehat{Y}_\nu^+ \subseteq \cdots$$

of $\widehat{Y}_\nu^+$ by declaring that the generators $E_{a;i,j}^{(r)}$ are of degree $(r-1)$. Let $\mathrm{gr}^{\mathrm{L}}\, \widehat{Y}_\nu^+$ denote the associated graded algebra. Letting $n_a := \nu_1 + \cdots + \nu_{a-1}$ for short, define

$$e_{n_a+i,n_b+j;r} := \mathrm{gr}_r^{\mathrm{L}}\, E_{a,b;i,j}^{(r+1)} \in \mathrm{gr}^{\mathrm{L}}\, \widehat{Y}_\nu^+$$

for each $1 \leq a < b \leq m, 1 \leq i \leq \nu_a, 1 \leq j \leq \nu_b$ and $r \geq 0$. Then:

Lemma 6.6. *For $1 \leq a < b \leq m, 1 \leq c < d \leq m$, $r, s \geq 0$ and all admissible h, i, j, k, we have that*

$$[e_{n_a+i,n_b+j;r}, e_{n_c+h,n_d+k;s}] = e_{n_a+i,n_d+k;r+s}\delta_{n_c+h,n_b+j} - \delta_{n_a+i,n_d+k}e_{n_c+h,n_b+j;r+s}.$$

Proof. Like in the proof of Lemma 5.8, we split into seven cases: (1) $b < c$; (2) $b = c$; (3) $a < c, b = d$; (4) $a = c, b < d$; (5) $a = c, b = d$; (6) $a < c < b < d$; (7) $a < c; d < b$. Since the analysis of each of the cases is very similar to Lemma 5.8, we just illustrate the idea with the two hardest situations, both of which require the Serre relations.

First we check for case (3) that $[e_{n_1+i,n_3+j;r}, e_{n_2+h,n_3+k;s}] = 0$. For any $1 \leq g \leq \nu_2$, we have by (6.5) and the images of the relations (1.9) and (1.13) in $\mathrm{gr}^{\mathrm{L}}\, \widehat{Y}_\nu$ that

$$\begin{aligned}
[e_{n_1+i,n_3+j;r}, e_{n_2+h,n_3+k;s}] &= [[e_{n_1+i,n_2+g;r}, e_{n_2+g,n_3+j;0}], e_{n_2+h,n_3+k;s}] \\
&= -[[e_{n_1+i,n_2+g;r}, e_{n_2+g,n_3+j;s}], e_{n_2+h,n_3+k;0}] \\
&= -[[e_{n_1+i,n_2+g;r+s}, e_{n_2+g,n_3+j;0}], e_{n_2+h,n_3+k;0}] = 0.
\end{aligned}$$

Second we check for case (6) that $[e_{n_1+i,n_3+j;r}, e_{n_2+h,n_4+k;s}] = 0$. By the case (1), (6.5), (1.9) and (1.13), we have that

$$\begin{aligned}
[e_{n_1+i,n_3+j;r}, e_{n_2+h,n_4+k;s}] &= [[e_{n_1+i,n_2+h;r}, e_{n_2+h,n_3+j;0}], [e_{n_2+h,n_3+j;0}, e_{n_3+j,n_4+k;s}]] \\
&= [e_{n_2+h,n_3+j;0}, [[e_{n_1+i,n_2+h;r}, e_{n_2+h,n_3+j;0}], e_{n_3+j,n_4+k;s}]] \\
&= [e_{n_2+h,n_3+j;0}, [e_{n_1+i,n_2+h;r}, [e_{n_2+h,n_3+j;0}, e_{n_3+j,n_4+k;s}]]] \\
&= [[e_{n_2+h,n_3+j;0}, e_{n_1+i,n_2+h;r}], [e_{n_2+h,n_3+j;0}, e_{n_3+j,n_4+k;s}]] \\
&= -[[e_{n_1+i,n_2+h;r}, e_{n_2+h,n_3+j;0}], [e_{n_2+h,n_3+j;0}, e_{n_3+j,n_4+k;s}]] \\
&= -[e_{n_1+i,n_3+j;r}, e_{n_2+h,n_4+k;s}]
\end{aligned}$$

Hence it is zero. □

This completes the proof of Theorems A and B.

7. Centers and centralizers

In this section, we compute the centralizer in Y_n of the standard Levi subalgebra Y_ν. The argument depends on the following auxiliary lemma, which is a generalization of [MNO, Proposition 2.12]; the proof given here is based on the argument in *loc. cit.*.

Lemma 7.1. *Let $\mathfrak{h}$ be a reductive subalgebra of a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Let $\mathfrak{c}$ be the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Then, the centralizer of $U(\mathfrak{h}[t])$ in $U(\mathfrak{g}[t])$ is equal to $U(\mathfrak{c}[t])$.*

(We believe that the word "reductive" is unnecessary here, but we did not find a proof without it.)

Proof. The symmetrization map $S(\mathfrak{g}[t]) \to U(\mathfrak{g}[t])$ is an isomorphism of $\mathfrak{h}[t]$-modules. Using this, it suffices to show that the space of invariants of $\mathfrak{h}[t]$ acting on $S(\mathfrak{g}[t])$ is $S(\mathfrak{c}[t])$. Since $\mathfrak{h}$ is reductive, we can pick an $\operatorname{ad}\mathfrak{h}$-stable complement $\mathfrak{c}'$ to $\mathfrak{c}$ in $\mathfrak{g}$. Let $x_1, \dots, x_m$ be a basis for $\mathfrak{c}'$ and let $x_{m+1}, \dots, x_n$ be a basis for $\mathfrak{c}$. Let z be an $\mathfrak{h}[t]$-invariant in $S(\mathfrak{g}[t])$. Define $h \geq 0$ to be minimal such that z has the form

$$z = \sum_d z_d (x_1 t^h)^{d_1} \cdots (x_m t^h)^{d_m}$$

summing over $d = (d_1, \dots, d_m)$ with $d_1, \dots, d_m \geq 0$, where the coefficients z_d are polynomials in the variables $x_i t^k$ for $i = 1, \dots, m$ and $0 \leq k < h$ together with the variables $x_i t^k$ for $i = m+1, \dots, n$ and $k \geq 0$. Pick a basis $y_1, \dots, y_r$ for $\mathfrak{h}$ and let $[y_i, x_j] = \sum_{k=1}^m c_{i,j,k} x_k$ for each $j = 1, \dots, m$. Acting on z with $y_i t \in \mathfrak{h}[t]$ and taking the coefficient of $x_k t^{h+1}$ gives the equation

$$\sum_d z_d \sum_{j=1}^m c_{i,j,k} d_j (x_1 t^h)^{d_1} \cdots (x_j t^h)^{d_j - 1} \cdots (x_m t^h)^{d_m} = 0,$$

for each $i = 1, \dots, r$ and $k = 1, \dots, m$. Now fix $d = (d_1, \dots, d_m)$ with $d_1, \dots, d_m \geq 0$. Taking the coefficient of $(x_1 t^h)^{d_1} \cdots (x_m t^h)^{d_m}$ in our equation gives

$$\sum_{j=1}^m c_{i,j,k} (d_j + 1) z_{d+\delta_j} = 0 \qquad (i = 1, \dots, r, k = 1, \dots, m)$$

where $d + \delta_j$ denotes the tuple $(d_1, \dots, d_j + 1, \dots, d_m)$. Since $\mathfrak{h}$ has no non-trivial invariants in $\mathfrak{c}'$, the system of linear equations $[y_i, \sum_{j=1}^m \lambda_j x_j] = 0$ $(i = 1, \dots, r)$ has only the trivial solution $\lambda_1 = \dots = \lambda_m = 0$. Equivalently, the system of equations

$$\sum_{j=1}^m c_{i,j,k} \lambda_j = 0 \qquad (i = 1, \dots, r, k = 1, \dots, m)$$

has only the trivial solution $\lambda_1 = \dots = \lambda_m = 0$ too. We deduce that $(d_j + 1) z_{d+\delta_j} = 0$ for each $j = 1, \dots, m$. Hence $z_d = 0$ for all non-zero d, which implies by the minimality of the choice of h that $h = 0$ hence that $z \in S(\mathfrak{c}[t])$. $\square$

Now, working once more in terms of the usual Drinfeld generators from §5, define

$$C_n(u) = \sum_{r \geq 0} C_n^{(r)} u^{-r} := D_1(u) D_2(u-1) \cdots D_n(u-n+1). \tag{7.1}$$

The importance of the elements $C_n^{(r)}$ is due to the following theorem; cf. [MNO, Theorem 2.13]. We remark that this theorem implies in particular that the commutative subalgebra $Y_{(1^n)}$ of Y_n is generated by the centers $Z(Y_1), Z(Y_2), \dots, Z(Y_n)$ of the nested subalgebras $Y_1 \subset Y_2 \subset \dots \subset Y_n$ from (4.1).

Theorem 7.2. *The elements $C_n^{(1)}, C_n^{(2)}, \dots$ are algebraically independent and generate the center $Z(Y_n)$.*

Proof. First we check that the $C_n^{(r)}$ are central. For this, it suffices to show that $[D_i(u)D_{i+1}(u-1), E_i(v)] = 0 = [D_i(u)D_{i+1}(u-1), F_i(v)]$ for each $i = 1, \dots, n-1$. Actually we just need to check the first equality, since the second then follows on applying τ. By Lemma 5.4(i),

$$(u - v)E_i(v)D_i(u) = (u - v - 1)D_i(u)E_i(v) + D_i(u)E_i(u)$$

By (5.24),

$$(u - v - 1)E_i(v)D_{i+1}(u - 1) = (u - v)D_{i+1}(u - 1)E_i(v) - D_{i+1}(u - 1)E_i(u - 1).$$

Hence, setting $v = u$,

$$E_i(u)D_{i+1}(u - 1) = D_{i+1}(u - 1)E_i(u - 1).$$

Now calculate $(u - v)E_i(v)D_i(u)D_{i+1}(u - 1)$ using these identities to show that it equals $(u - v)D_i(u)D_{i+1}(u - 1)E_i(v)$. Hence $[D_i(u)D_{i+1}(u - 1), E_i(v)] = 0$.

Now we complete the proof by following the argument of [MNO, Theorem 2.13]. Recall the filtration (2.11) of Y_n, with associated graded algebra $\operatorname{gr}^{\mathrm{L}} Y_n = U(\mathfrak{gl}_n[t])$. Let $z = e_{1,1} + \dots + e_{n,n} \in \mathfrak{gl}_n$. One checks from the definition (7.1) that

$$\operatorname{gr}^{\mathrm{L}}_{r-1} C_n^{(r)} = z t^{r-1}.$$

By Lemma 7.1 (taking $\mathfrak{h} = \mathfrak{g} = \mathfrak{gl}_n$) the center of $U(\mathfrak{gl}_n[t])$ is freely generated by the elements $\{zt^r \mid r \geq 0\}$. The theorem now follows on combining these two observations with the fact already proved that each $C_n^{(r)}$ belongs to $Z(Y_n)$. $\square$

We now use essentially the same argument to prove the following theorem, which is a variation on a result of Olshanskii [O, §2.1].

Theorem 7.3. *The centralizer of Y_m in Y_{m+n} is equal to $Z(Y_m)\psi_m(Y_n)$.*

Proof. Lemma 4.1 shows that $Z(Y_m)\psi_m(Y_n)$ centralizes Y_m, so we just need to show that the centralizer is no larger. Consider the associated graded algebra $\mathrm{gr}^{\mathrm{L}} Y_{m+n} = U(\mathfrak{gl}_{m+n}[t])$. Since

$$\mathrm{gr}^{\mathrm{L}}_{r-1} \psi_m(T_{i,j}^{(r)}) = e_{m+i,m+j}t^{r-1},$$

we have that $\mathrm{gr}^{\mathrm{L}} Y_m = U(\mathfrak{gl}_m[t])$ (where $\mathfrak{gl}_m$ is embedded into the top left corner of $\mathfrak{gl}_{m+n}$) and $\mathrm{gr}^{\mathrm{L}} \psi_m(Y_n) = U(\mathfrak{gl}_n[t])$ (where $\mathfrak{gl}_n$ is embedded into the bottom right corner of $\mathfrak{gl}_{m+n}$). By Lemma 7.1, the centralizer of $U(\mathfrak{gl}_m[t])$ in $U(\mathfrak{gl}_{m+n}[t])$ is equal to $Z(U(\mathfrak{gl}_m[t]))U(\mathfrak{gl}_n[t])$. The theorem follows. □

Corollary 7.4. *Let $\nu = (\nu_1, \dots, \nu_m)$ be a tuple of non-negative integers summing to n. The centralizer of the Levi subalgebra Y_ν in Y_n is equal to $Z(Y_\nu)$.*

Proof. Proceed by induction on m, the case $m = 1$ being vacuous. By (4.5) $Y_\nu = Y_{\nu_1}\psi_{\nu_1}(Y_{\bar\nu})$ where $\bar\nu = (\nu_2, \dots, \nu_m)$. By the theorem, the centralizer in Y_n of Y_{ν_1} is $Z(Y_{\nu_1})\psi_{\nu_1}(Y_{\nu_2+\dots+\nu_m})$. By induction, the centralizer in $Y_{\nu_2+\dots+\nu_m}$ of $Y_{\bar\nu}$ is $Z(Y_{\bar\nu})$. Hence, the centralizer of Y_ν in Y_n is $Z(Y_{\nu_1})\psi_{\nu_1}(Z(Y_{\bar\nu})) = Z(Y_\nu)$. (Alternatively one can prove the corollary directly using Lemma 7.1 once more.) □

Corollary 7.5. *$Y_{(1^n)}$ is a maximal commutative subalgebra of Y_n.*

Proof. By the previous corollary, $Y_{(1^n)}$ is its own centralizer. □

8. Quantum determinants

In the literature, Drinfeld generators are usually expressed in terms quantum determinants, rather than the quasi-determinants used up to now. In this section we complete the picture by relating quasi-determinants to quantum determinants. We begin by introducing quantum determinants following [MNO, §2]. Fix $d \geq 1$ and let $A_d \in M_n^{\otimes d}$ denote the antisymmetrization operator, i.e. the endomorphism

$$v_1 \otimes \cdots \otimes v_d \mapsto \sum_{\pi \in S_d} \mathrm{sgn}(\pi) v_{\pi 1} \otimes \cdots \otimes v_{\pi d}$$

of the natural space $(\mathbb{C}^n)^{\otimes d}$ that $M_n^{\otimes d}$ acts on. Note that $A_d^2 = (d!)A_d$. We have the following fundamental identity

$$\begin{aligned} A_d^{[1,\dots,d]} T^{[1,d+1]}(u) T^{[2,d+1]}(u-1) \cdots T^{[d,d+1]}(u-d+1) = \\ T^{[d,d+1]}(u-d+1) \cdots T^{[2,d+1]}(u-1) T^{[1,d+1]}(u) A_d^{[1,\dots,d]} \end{aligned} \tag{8.1}$$

equality written in $M_n^{\otimes d} \otimes Y_n[[u^{-1}]]$; see [MNO, Proposition 2.4]. For tuples $\boldsymbol{i} = (i_1, \dots, i_d)$ and $\boldsymbol{j} = (j_1, \dots, j_d)$ of integers from $\{1, \dots, n\}$, the *quantum determinant* $T_{\boldsymbol{i},\boldsymbol{j}}(u) \in Y_n[[u^{-1}]]$ is defined to be the coefficient of $e_{\boldsymbol{i},\boldsymbol{j}} = e_{i_1,j_1} \otimes \cdots \otimes e_{i_d,j_d} \in M_n^{\otimes d}$ on either side of the equation (8.1). Explicit computation using the left and the right hand sides of (8.1) respectively gives that

$$T_{\boldsymbol{i},\boldsymbol{j}}(u) = \sum_{\pi \in S_d} \mathrm{sgn}(\pi) T_{i_{\pi 1},j_1}(u) T_{i_{\pi 2},j_2}(u-1) \cdots T_{i_{\pi d},j_d}(u-d+1) \tag{8.2}$$

$$= \sum_{\pi \in S_d} \mathrm{sgn}(\pi) T_{i_d,j_{\pi d}}(u-d+1) \cdots T_{i_2,j_{\pi 2}}(u-1) T_{i_1,j_{\pi 1}}(u), \tag{8.3}$$

where S_d is the symmetric group. It is obvious from these formulae that

$$T_{\boldsymbol{i}\cdot\pi,\boldsymbol{j}}(u) = \operatorname{sgn}(\pi) T_{\boldsymbol{i},\boldsymbol{j}}(u) = T_{\boldsymbol{i},\boldsymbol{j}\cdot\pi}(u) \tag{8.4}$$

for any permutation $\pi \in S_d$ (acting naturally on the tuples $\boldsymbol{i}, \boldsymbol{j}$ by place permutation). Using (8.4) one obtains further variations on the formulae (8.2)–(8.3) as in [MNO, Remark 2.8], for instance:

$$T_{\boldsymbol{i},\boldsymbol{j}}(u) = \sum_{\pi \in S_d} \operatorname{sgn}(\pi) T_{i_1,j_{\pi 1}}(u-d+1) T_{i_2,j_{\pi 2}}(u-d+2) \cdots T_{i_d,j_{\pi d}}(u). \tag{8.5}$$

The following properties of quantum determinants are easily derived from (8.2)–(8.4).

$$\tau(T_{\boldsymbol{i},\boldsymbol{j}}(u)) = T_{\boldsymbol{j},\boldsymbol{i}}(u), \tag{8.6}$$

$$\sigma(T_{\boldsymbol{i},\boldsymbol{j}}(u)) = T_{\boldsymbol{i},\boldsymbol{j}}(-u+d-1). \tag{8.7}$$

In the special case $\boldsymbol{i} = \boldsymbol{j} = (1, \dots, n)$, we denote the quantum determinant $T_{\boldsymbol{i},\boldsymbol{j}}(u)$ instead by $C_n(u)$, i.e.

$$C_n(u) := T_{(1,\dots,n),(1,\dots,n)}(u). \tag{8.8}$$

We will show in Theorem 8.6 below that this agrees with the definition (7.1), hence the coefficients of the series $C_n(u)$ generate the center of Y_n, but we do not know this yet.

The next few results taken from [NT] describe the effect of the maps Δ and S on quantum determinants. Actually we do not need the first of these here, but include it for the sake of completeness.

Lemma 8.1. *Let $\boldsymbol{i}, \boldsymbol{j}$ be d-tuples of distinct integers from $\{1, \dots, n\}$. Then,*

$$\Delta(T_{\boldsymbol{i},\boldsymbol{j}}(u)) = \sum_{\boldsymbol{k}} T_{\boldsymbol{i},\boldsymbol{k}}(u) \otimes T_{\boldsymbol{k},\boldsymbol{j}}(u)$$

where the sum is over all $\boldsymbol{k} = (k_1, \dots, k_d)$ with $1 \leq k_1 < \dots < k_d \leq n$.

Proof. See [NT, Proposition 1.11]. □

Lemma 8.2. *Let $\boldsymbol{i}, \boldsymbol{j}$ be d-tuples of distinct integers from $\{1, \dots, n\}$. Choose $\boldsymbol{i}' = (i_{d+1}, \dots, i_n)$ and $\boldsymbol{j}' = (j_{d+1}, \dots, j_n)$ so that $\{i_1, \dots, i_n\} = \{j_1, \dots, j_n\} = \{1, \dots, n\}$, and let ε denote the sign of the permutation $(i_1, \dots, i_n) \mapsto (j_1, \dots, j_n)$. Then,*

$$\omega(T_{\boldsymbol{i},\boldsymbol{j}}(u)) = \varepsilon C_n(-u+n-1)^{-1} T_{\boldsymbol{j}',\boldsymbol{i}'}(-u+n-1).$$

Proof. This is proved in [NT, Lemma 1.5] but for the opposite algebra, so we repeat the argument once more. By the identity (8.1) and the definition (8.8), we have that

$$A_n^{[1,\dots,n]} T^{[1,n+1]}(u) T^{[2,n+1]}(u-1) \cdots T^{[n,n+1]}(u-n+1) = C_n(u)^{[n+1]} A_n^{[1,\dots,n]};$$

see also [MNO, Proposition 2.5]. Hence,

$$A_n^{[1,\dots,n]} T^{[1,n+1]}(u) \cdots T^{[d,n+1]}(u-d+1) = \\ (-1)^{n-d} C_n(u)^{[n+1]} A_n^{[1,\dots,n]} \widetilde{T}^{[n,n+1]}(u-n+1) \cdots \widetilde{T}^{[d+1,n+1]}(u-d).$$

Now equate the $e_{(i_1,\dots,i_d,i_n,\dots,i_{d+1}),(j_1,\dots,j_d,i_n,\dots,i_{d+1})}$-coefficients on each side and use (8.2) to deduce that

$$T_{\boldsymbol{i},\boldsymbol{j}}(u) = \varepsilon C_n(u) \omega(T_{\boldsymbol{j}',\boldsymbol{i}'}(-u+n-1)).$$

The lemma follows on making some obvious substitutions. □

Corollary 8.3. *In the notation of Lemma 8.2,* $S(T_{\boldsymbol{i},\boldsymbol{j}}(u)) = \varepsilon C_n(u+n-d)^{-1} T_{\boldsymbol{j}',\boldsymbol{i}'}(u+n-d)$.

Proof. Recalling that $S = \omega \circ \sigma$, this is a consequence of (8.7) and Lemma 8.2. □

Corollary 8.4. $S^2(T_{i,j}(u)) = C_n(u+n)^{-1} T_{i,j}(u+n) C_n(u+n-1)$.

Proof. Apply Corollary 8.3 twice. □

Now we describe the embedding $\psi_m : Y_n \hookrightarrow Y_{m+n}$ from §4 in terms of quantum determinants.

Lemma 8.5. *Let* $\boldsymbol{i}, \boldsymbol{j}$ *be* d*-tuples of distinct integers from* $\{1, \dots, n\}$*. Then,*

$$\psi_m(T_{\boldsymbol{i},\boldsymbol{j}}(u)) = C_m(u+m)^{-1} T_{m\#\boldsymbol{i}, m\#\boldsymbol{j}}(u+m)$$

where $m\#\boldsymbol{i}$ *denotes the* $(m+d)$*-tuple* $(1, \dots, m, m+i_1, \dots, m+i_d)$ *and* $m\#\boldsymbol{j}$ *is defined similarly.*

Proof. Calculate using (4.2) and Lemma 8.2 twice. □

Now we can verify that $C_n(u)$ as defined in this section is the same as the earlier definition (7.1). Note this theorem is the Yangian analogue of [GKLLRT, Theorem 7.24].

Theorem 8.6. $C_n(u) = D_1(u) D_2(u-1) \cdots D_n(u-n+1)$.

Proof. Recalling that $D_i(u) = \psi_{i-1}(T_{1,1}(u))$, Lemma 8.5 implies that $D_i(u) = C_{i-1}(u+i-1)^{-1} C_i(u+i-1)$. The lemma follows easily from this by induction. □

Finally, we apply Lemma 8.5 once more to express the Drinfeld generators from §5 in terms of quantum determinants.

Theorem 8.7. *For* $i \geq 1$*,*

(i) $D_i(u) = T_{(1,\dots,i-1),(1,\dots,i-1)}(u+i-1)^{-1} T_{(1,\dots,i),(1,\dots,i)}(u+i-1)$;

(ii) $E_i(u) = T_{(1,\dots,i),(1,\dots,i)}(u+i-1)^{-1} T_{(1,\dots,i),(1,\dots,i-1,i+1)}(u+i-1)$;

(iii) $F_i(u) = T_{(1,\dots,i-1,i+1),(1,\dots,i)}(u+i-1) T_{(1,\dots,i),(1,\dots,i)}(u+i-1)^{-1}$.

Proof. Calculate using Lemmas 5.1 and 8.5 and the formulae (5.6) and (8.6). □

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