CARTAN DETERMINANTS AND SHAPOVALOV FORMS

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ABSTRACT. We compute the determinant of the Gram matrix of the Shapovalov form on weight spaces of the basic representation of an affine Kac-Moody algebra of ADE type (possibly twisted). As a consequence, we obtain explicit formulae for the determinants of the Cartan matrices of p-blocks of the symmetric group and its double cover, and of the associated Hecke algebras at roots of unity.

1. INTRODUCTION

Let $\mathfrak{g}$ be an affine Kac-Moody algebra of type $X_N^{(r)}$ as in the table:

$X_N^{(r)}$	$A_\ell^{(1)}$	$D_\ell^{(1)}$	$E_\ell^{(1)}$	$A_{2\ell-1}^{(2)}$	$A_{2\ell}^{(2)}$	$D_{\ell+1}^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
ℓ	≥ 1	≥ 4	$6, 7$ or 8	≥ 3	≥ 1	≥ 2	4	2
k	0	0	0	$\ell - 1$	ℓ	1	2	1
α	$\ell + 1$	4	$9 - \ell$	2	1	2	1	1
β	1	1	1	ℓ	$2\ell + 1$	2	3	2

We are interested here in the basic representation $V = V(\Lambda_0)$ of $\mathfrak{g}$, see [11]. Let $|0\rangle$ be a vacuum vector and define the lattice $V_\mathbb{Z} := U_\mathbb{Z}|0\rangle$ in V, where $U_\mathbb{Z}$ is the $\mathbb{Z}$-subalgebra of the universal enveloping algebra of $\mathfrak{g}$ generated by the divided powers

$$e_i^n/n!, \quad f_i^n/n! \qquad (i = 0, 1, \dots, \ell,\ n \geq 1)$$

in the Chevalley generators. Let $(.\,,.)_S$ denote the Shapovalov form, the unique Hermitian form on V satisfying $(|0\rangle, |0\rangle)_S = 1$ and $(e_i v, v')_S = (v, f_i v')_S$ for $i = 0, \dots, \ell$ and all $v, v' \in V$. Its restriction to $V_\mathbb{Z}$ gives a symmetric bilinear form

$$(.\,,.)_S : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{Z}.$$

Our Main Theorem gives an explicit formula for the determinant of the Gram matrix of this form on each weight space of $V_\mathbb{Z}$.

To state the result precisely, recall the description of the weights of V [11, §12.6]: every weight is of the form $w\Lambda_0 - d\delta$ for some w in the Weyl group W associated to $\mathfrak{g}$ and some integer $d \geq 0$. Also let $\mathscr{P}(d)$ denote the set of all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ of d. Given $\lambda \in \mathscr{P}(d)$, we can gather together its equal parts to represent it as $\lambda = (1^{r_1} 2^{r_2} \dots)$. Also recall the number $r \in \{1, 2, 3\}$ which comes from the type $X_N^{(r)}$. Then:

Main Theorem. *The determinant of the Gram matrix of the Shapovalov form on the $(w\Lambda_0 - d\delta)$-weight space of $V_\mathbb{Z}$ is $\alpha^{a(d)}\beta^{b(d)}$ where $a(d) = \sum_{\lambda \in \mathscr{P}(d)} a_\lambda$, $b(d) = \sum_{\lambda \in \mathscr{P}(d)} b_\lambda$*

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and for $\lambda = (1^{r_1}2^{r_2}\dots)$,

$$a_\lambda = \prod_{i \text{ with } r|i} \binom{\ell + r_i - 1}{r_i} \cdot \prod_{i \text{ with } r\nmid i} \binom{k + r_i - 1}{r_i} \cdot \sum_{i \text{ with } r|i} \frac{r_i}{\ell},$$

$$b_\lambda = \prod_{i \text{ with } r|i} \binom{\ell + r_i - 1}{r_i} \cdot \prod_{i \text{ with } r\nmid i} \binom{k + r_i - 1}{r_i} \cdot \sum_{i \text{ with } r\nmid i} \frac{r_i}{k},$$

ℓ, k, α, β being as in the above table. The generating functions $a(q) = \sum_{d\geq 0} a(d)q^d$ and $b(q) = \sum_{d \geq 0} b(d)q^d$ are given by the formulae

$$a(q) = T(q^r)P(q)^k P(q^r)^{\ell - k},$$
$$b(q) = (T(q) - T(q^r))P(q)^k P(q^r)^{\ell-k}$$

where $P(q) = \prod_{i\geq 1} \frac{1}{1-q^i}$ is the generating function for the number of partitions of d and $T(q) = \sum_{i \geq 1} \frac{q^i}{1-q^i}$ is the generating function for the number of divisors of d.

In [5], De Concini, Kac and Kazhdan constructed the basic representation over $\mathbb{Z}$ (at least in the untwisted cases) using an integral version of the vertex operator construction of [8]. They showed in particular that the basic representation remains irreducible on reduction modulo p if and only if $p \nmid \det X_N$, where X_N is the Cartan matrix of the underlying finite root system; this also follows immediately from our Main Theorem on noting that $\det X_N = \alpha\beta^{r-1}$.

Our interest in the theorem comes instead from modular representation theory. Suppose now that $\mathfrak{g}$ is of type $A_\ell^{(1)}$ and set $p = (\ell+1)$. Let FS_n denote the group algebra of the symmetric group over a field F of characteristic p (assuming in this case that p is prime), and let H_n denote the Iwahori-Hecke algebra associated to S_n over an arbitrary field but at a primitive pth root of 1 (this case making sense for arbitrary $p \geq 2$). By [1, 9], there is an isomorphism between the basic representation $V_\mathbb{Z}$ of $\mathfrak{g}$ and the direct sum $K = \bigoplus_{n\geq 0} K_n$ of the Grothendieck groups K_n of finitely generated projective FS_n- (resp. H_n-) modules for all n. Under the isomorphism, the weight spaces of $V_\mathbb{Z}$ are in 1–1 correspondence with the block components of K, a weight space of the form $w\Lambda - d\delta$ corresponding to a block of p-weight d (see e.g. [13, §5.3] for the definition of the p-weight of a block). Moreover, according to [9, Theorem 14.2], the Shapovalov form corresponds to the usual Cartan pairing $([P],[Q]) = \dim \operatorname{Hom}(P,Q)$ between projective modules P, Q. Thus the theorem has the following immediate corollary:

Corollary 1. *Let B be a block of p-weight d of either the group algebra FS_n of the symmetric group over a field of prime characteristic p, or the Hecke algebra H_n over an arbitrary field but at a primitive pth root of unity, in which case $p \geq 2$ is an arbitrary integer. Then the determinant of the Cartan matrix of B is $p^{N(d)}$ where*

$$N(d) = \sum_{\lambda = (1^{r_1}2^{r_2}\dots) \in \mathscr{P}(d)} \frac{r_1 + r_2 + \dots}{p-1} \binom{p-2+r_1}{r_1}\binom{p-2+r_2}{r_2}\cdots.$$

The generating function $N(q) = \sum_{d\geq 0} N(d)q^d$ equals $T(q)P(q)^{p-1}$.

It is a classical result of Brauer that the determinant of the Cartan matrix of a block of FS_n is a power of p (see [6, 84.17]). Donkin [7] has proved similarly that the determinant of the Cartan matrix of a block of H_n *divides* a power of p. The corollary shows in particular that the determinant is exactly a power of p, even in those cases where p is not prime, as had been conjectured by Mathas. We remark that in the case of blocks of FS_n, but not of H_n, the explicit generating function given in the corollary has also recently been obtained by Bessenrodt and Olsson [2] using methods from block theory.

Finally suppose that $\mathfrak{g}$ is of type $A_{2\ell}^{(2)}$ and set $p = (2\ell + 1)$. In this case, the Main Theorem can be reinterpreted as a computation of Cartan determinants of the p-blocks of the double covers $\widehat{S}_n$ of the symmetric group. Following [3, §9-c] for notation, let $S(n)$ be the twisted group algebra of S_n over an algebraically closed field F of characteristic p (assuming p is an odd prime in this case), and let $W(n)$ be the Hecke-Clifford superalgebra over an algebraically closed field of characteristic different from 2 at a primitive pth root of unity (for arbitrary odd $p \geq 3$). By [3, 7.16, 8.13, 9.9], there is an isomorphism between the basic representation $V_{\mathbb{Z}}$ and the direct sum $K = \bigoplus_{n\geq 0} K_n$ of the Grothendieck groups of finitely generated projective $S(n)$- (resp. $W(n)$-) supermodules, under which a weight space of the form $w\Lambda - d\delta$ maps to a superblock of p-bar weight d (see [3, §9-a] for the definition of p-bar weight of a superblock), and the Shapovalov form corresponds to the Cartan pairing on projective supermodules (see [3, §7-c]). So:

Corollary 2. *Let B be a superblock of p-bar weight d of either $S(n)$ in odd characteristic p, or $W(n)$ at a primitive pth root of unity, in which case $p \geq 3$ is an arbitrary odd integer. Then the determinant of the Cartan matrix of B is $p^{N(d)}$ where*

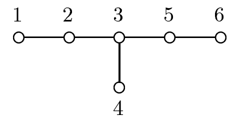
$$N(d) = \sum_{\lambda=(1^{r_1}2^{r_2}\dots)\in\mathscr{P}(d)} \frac{2r_1 + 2r_3 + 2r_5 + \dots}{p-1} \binom{\frac{p-3}{2} + r_1}{r_1}\binom{\frac{p-3}{2} + r_2}{r_2}\binom{\frac{p-3}{2} + r_3}{r_3}\cdots.$$

The generating function $N(q) = \sum_{d\geq 0} N(d)q^d$ equals $(T(q) - T(q^2))P(q)^{(p-1)/2}$.

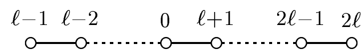
It is more natural from the point of view of finite group theory to ask for the Cartan determinant of a block B of the twisted group algebra $S(n)$ in the usual ungraded sense. According to Humphreys' classification [10], see also [3, 9.16], we can associated to B its p-bar weight d and a type $\varepsilon \in \{\mathtt{M}, \mathtt{Q}\}$. In case $\varepsilon = \mathtt{M}$, B coincides with a superblock of p-bar weight d and it is immediate that its Cartan determinant is as in Corollary 2. But in the cases when $\varepsilon = \mathtt{Q}$ and $d > 0$, the Cartan matrix of B has twice as many rows and columns as the Cartan matrix of the corresponding superblock. Nevertheless, we believe the Cartan determinant is the same, based on explicit computations for small d. In other words, we conjecture that Cartan determinants of p-blocks of $S(n)$ depend only on the p-bar weight d, not on the type ε, of the block.

2. The affine algebras

We begin by recalling the construction of the affine Lie algebras from [11, Chapter 8]. Let $X_N^{(r)}$ be an affine Dynkin diagram of ADE type as in the introduction, and let X_N be the underlying finite Dynkin diagram. We use the same numbering of Dynkin diagrams as [11, §4.8] with two exceptions: in the case $X_N^{(r)} = E_6^{(2)}$ we will number the vertices of the finite Dynkin diagram $X_N = E_6$ by



and in the case $X_N^{(r)} = A_{2\ell}^{(2)}$ we will number the vertices of the finite Dynkin diagram $X_N = A_{2\ell}$ by



Let Q' denote the root lattice of type X_N, with simple roots α_i' and invariant bilinear form $(.|.)'$ normalized so that each $(\alpha_i'|\alpha_i')' = 2$. Let $\mu : Q' \to Q'$ be a graph automorphism of order r, as in e.g. [11, §7.9]. Let

$$\varepsilon : Q' \times Q' \to \{\pm 1\}$$

be an asymmetry function as in [11, §7.8] chosen so that $\varepsilon(\mu(\alpha'), \mu(\beta')) = \varepsilon(\alpha', \beta')$. In case $X_N^{(r)} = A_{2\ell}^{(2)}$ this is not possible so we instead require here that $\varepsilon(\mu(\alpha'), \mu(\beta')) = \varepsilon(\beta', \alpha')$. Let $\mathfrak{h}' = \mathbb{C} \otimes_{\mathbb{Z}} Q'$ viewed as an abelian Lie algebra, and extend μ and $(.|.)'$ linearly to $\mathfrak{h}'$. Then we can construct the finite dimensional simple Lie algebra $\mathfrak{g}'$ of type X_N as the vector space

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{\alpha' \text{ a root}} \mathbb{C} E_{\alpha'}$$

viewed as a Lie algebra so that $\mathfrak{h}'$ is abelian and

$$[\alpha', E_{\beta'}] = (\alpha'|\beta')' E_{\beta'}, \qquad [E_{\alpha'}, E_{-\alpha'}] = -\alpha',$$
$$[E_{\alpha'}, E_{\beta'}] = \begin{cases} \varepsilon(\alpha', \beta') E_{\alpha'+\beta'} & \text{if } \alpha' + \beta' \text{ is a root,} \\ 0 & \text{otherwise.} \end{cases}$$

The invariant form on $\mathfrak{h}'$ extends to $\mathfrak{g}'$ by $(\mathfrak{h}'|E_{\alpha'})' = 0$ and $(E_{\alpha'}|E_{\beta'})' = -\delta_{\alpha', -\beta'}$ for all roots α', β'.

Let $a_i, a_i^\vee\, (i = 0, \dots, \ell)$ be the numerical labels on the Dynkin diagram $X_N^{(r)}$ and its dual as in [11, §4.8]. We note especially that $a_0 = 1$ if $X_N^{(r)} \neq A_{2\ell}^{(2)}$ and $a_0 = 2$ if $X_N^{(r)} = A_{2\ell}^{(2)}$. It will also be convenient to define

$$c_i = \begin{cases} 2 & \text{if } X_N^{(r)} = A_{2\ell}^{(2)} \text{ and } i = 0, \\ 1 & \text{otherwise;} \end{cases}$$
$$d_i = c_i a_i^\vee a_i^{-1} \in \{1, r\}$$

for $i = 0, 1, \dots, \ell$. Let $m = a_0 r$ and fix a primitive mth root of unity $\omega \in \mathbb{C}$. In all types other than $A_{2\ell}^{(2)}$, let $\eta : Q' \to \mathbb{C}^\times$ denote the constant function with $\eta(\alpha') = 1$ for all $\alpha' \in Q'$; in type $A_{2\ell}^{(2)}$, define η instead by the rules

$$\eta(0) = 1, \quad \eta(\alpha' + \beta') = \eta(\alpha')\eta(\beta')(-1)^{(\alpha'|\beta')'}, \qquad \eta(\alpha_j') = \begin{cases} 1 & j \neq 0, \ell+1, \\ \omega & j = 0, \ell+1. \end{cases}$$

Now extend μ from $\mathfrak{h}'$ to $\mathfrak{g}'$ by declaring that $\mu(E_{\alpha'}) = \eta(\alpha') E_{\mu(\alpha')}$ for all roots $\alpha' \in Q'$. The order of the resulting automorphism μ of $\mathfrak{g}'$ is equal to m in all cases.

Decompose

$$\mathfrak{g}' = \bigoplus_{n \in \mathbb{Z}/m} \mathfrak{g}'_n \qquad \text{where} \qquad \mathfrak{g}'_n = \{X \in \mathfrak{g}' \mid \mu(X) = \omega^n X\}.$$

Also write $\mathfrak{h}'_n = \mathfrak{h}' \cap \mathfrak{g}'_n$. Introduce the infinite dimensional Lie algebras

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}'_n \otimes t^n \oplus \mathbb{C}c \oplus \mathbb{C}d \subseteq \mathfrak{g}' \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

$$\mathfrak{h} = \mathfrak{h}'_0 \otimes 1 \oplus \mathbb{C}c \oplus \mathbb{C}d \subset \mathfrak{g},$$

$$\mathfrak{t} = \mathfrak{t}^+ \oplus \mathbb{C}c \oplus \mathfrak{t}^- \subset \mathfrak{g} \quad \text{where} \quad \mathfrak{t}^{\pm} = \bigoplus_{\pm n > 0} \mathfrak{h}'_n \otimes t^n.$$

Multiplication is defined by the rules

$$[d, X \otimes t^n] = nX \otimes t^n, \qquad [c, \mathfrak{g}] = 0,$$

$$[X \otimes t^n, Y \otimes t^k] = [X, Y] \otimes t^{n+k} + \delta_{n,-k} n \frac{(X|Y)'}{m} c.$$

Then $\mathfrak{g}$ is the affine Lie algebra of type $X_N^{(r)}$ with canonical central element c and scaling element d, and $\mathfrak{h}$ is a Cartan subalgebra. As a matter of notation, we will write

$$X(n) := \sum_{j=0}^{m-1} \omega^{-nj} \mu^j(X) \otimes t^n \in \mathfrak{g}'_n \otimes t^n$$

for $X \in \mathfrak{g}'$ and $n \in \mathbb{Z}$. The normalized invariant form on $\mathfrak{g}$ will be denoted $(.|.)$, and is defined by

$$(X \otimes t^n | Y \otimes t^k) = \delta_{n,-k} (X|Y)'/r$$

for all $X \in \mathfrak{g}_n, Y \in \mathfrak{g}_k$.

In order to write down a choice of Chevalley generators for $\mathfrak{g}$, let ℓ denote the number of μ-orbits on the simple roots in Q'. Let

$$\varepsilon = \begin{cases} 0 & \text{if } X_N^{(r)} \neq A_{2\ell}^{(2)}, \\ \ell & \text{if } X_N^{(r)} = A_{2\ell}^{(2)}, \end{cases}$$

and set

$$I = \{0, 1, \dots, \ell\} - \{\varepsilon\}.$$

Then, the α'_i for $i \in I$ give a set of representatives for the μ-orbits on the simple roots. Define

$$-\alpha'_\varepsilon = \begin{cases} \text{the longest root in } Q' & \text{if } r = 1 \text{ or } X_N^{(r)} = A_{2\ell}^{(2)}, \\ \alpha'_1 + \cdots + \alpha'_{2\ell-2} & \text{if } X_N^{(r)} = A_{2\ell-1}^{(2)}, \\ \alpha'_1 + \cdots + \alpha'_\ell & \text{if } X_N^{(r)} = D_{\ell+1}^{(2)}, \\ \alpha'_2 + \alpha'_3 + \alpha'_4 & \text{if } X_N^{(r)} = D_4^{(3)}, \\ \alpha'_1 + 2\alpha'_2 + 2\alpha'_3 + \alpha'_4 + \alpha'_5 + \alpha'_6 & \text{if } X_N^{(r)} = E_6^{(2)}. \end{cases}$$

For $i = 0, 1, \dots, \ell$, write

$$e_i(n) = \frac{\sqrt{c_i}}{a_0 d_i} E_{\alpha'_i}(n) \quad \text{and} \quad f_i(n) = -\frac{\sqrt{c_i}}{a_0 d_i} E_{-\alpha'_i}(n).$$

The Chevalley generators of $\mathfrak{g}$ are $e_0 = e_0(1), e_i = e_i(0)$ and $f_0 = f_0(-1), f_i = f_i(0)$ for $i = 1, \dots, \ell$, as is proved in [11, §8.7] (taking $s_0 = 1, s_1 = \dots = s_\ell = 0$). We also define

$$h_i = [e_i, f_i] = \delta_{i,0} c + \frac{c_i}{a_0 d_i} \alpha'_i(0).$$

Next let $Q \subset \mathfrak{h}^*$ denote the root lattice associated to $\mathfrak{g}$. So following [11, §6.2],

$$Q = \bigoplus_{i=0}^{\ell} \mathbb{Z}\alpha_i \oplus \mathbb{Z}\Lambda_0$$

where $\alpha_0, \dots, \alpha_\ell$ are the simple roots corresponding to $h_0, \dots, h_\ell$ and Λ_0 is the zeroth fundamental dominant weight, i.e.

$$\begin{aligned}
\langle h_i, \alpha_j \rangle &= \text{the } ij\text{-entry of the Cartan matrix of type } X_N^{(r)}, \\
\langle h_i, \Lambda_0 \rangle &= \langle d, \alpha_i \rangle = \delta_{i,0}, \\
\langle d, \Lambda_0 \rangle &= 0,
\end{aligned}$$

for $i, j = 0, \dots, \ell$. Also as in [11, §6.2], we have the normalized invariant form $(.|.)$ on $\mathfrak{h}^*$ and the element $\delta = \sum_{i=0}^{\ell} a_i \alpha_i \in Q$.

To conclude, we explain the relationship between the form $(.|.)'$ on Q' and the form $(.|.)$ on Q. Introduce the new symmetric bilinear form $(.|.)_\mu$ on Q' defined by

$$(\alpha'|\beta')_\mu = (\alpha'|\sum_{j=0}^{r-1} \mu^j(\beta'))'$$

for all $\alpha', \beta' \in Q'$. There is an orthogonal decomposition

$$\mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0)$$

where $\mathring{\mathfrak{h}}^* = \bigoplus_{i=1}^{\ell} \mathbb{C}\alpha_i$, see [11, §6.2]. As in *loc. cit.* we write $- : \mathfrak{h}^* \to \mathring{\mathfrak{h}}^*$ for the orthogonal projection, in particular $\overline{Q}$ denotes the orthogonal projection of Q onto $\mathring{\mathfrak{h}}^*$. Define a $\mathbb{Z}$-linear map

$$\iota : Q' \to \overline{Q} \tag{2.1}$$

by $\iota(\mu^j(\alpha'_i)) = \overline{\alpha}_i$ for each $i \in I$ and $j \geq 0$. The kernel of ι is the space

$$M' = \{\alpha' - \mu(\alpha') \mid \alpha' \in Q'\} \tag{2.2}$$

which is precisely the radical of the bilinear form $(.|.)_\mu$. Moreover, ι induces an isometry between Q'/M' and $\overline{Q}$ with respect to the forms induced by $(.|.)_\mu$ and $(.|.)$ respectively.

3. The basic representation

Next we recall the construction of the basic representation $V = V(\Lambda_0)$ of $\mathfrak{g}$, following Lepowsky [12]. Let $Z = \langle -1, \omega \rangle \subset \mathbb{C}^\times$ be the multiplicative group generated by -1 and ω. Form the central extension

$$1 \longrightarrow Z \longrightarrow \widehat{Q} \xrightarrow{\pi} Q' \longrightarrow 1,$$

namely, $\widehat{Q}=\{e_x^{\alpha'} \mid \alpha'\in Q', x\in Z\}$ with multiplication

$$e_x^{\alpha'}e_y^{\beta'}=\begin{cases} e^{\alpha'+\beta'}_{xy\varepsilon(\alpha',\beta')} & \text{if } X_N^{(r)}\neq A_{2\ell}^{(2)}, D_4^{(3)},\\ e^{\alpha'+\beta'}_{xy\varepsilon(\alpha',\beta')(-\omega)^{-(\alpha'|\mu(\beta'))'}} & \text{if } X_N^{(r)}=A_{2\ell}^{(2)} \text{ or } D_4^{(3)},\end{cases}$$

for $\alpha',\beta'\in Q',\ x,y\in Z$. The map $\pi:\widehat{Q}\to Q'$ here is defined by $\pi(e_x^{\alpha'})=\alpha'$. Let $\widehat{M}=\pi^{-1}(M')$, where M' is as in (2.2). There is a well-defined multiplicative character $\tau:\widehat{M}\to\mathbb{C}^\times$ defined in [12, Proposition 6.1] by

$$\tau(e_x^{\alpha'-\mu(\alpha')})=(-1)^{(\alpha'|\alpha')'/2}x\eta(\alpha')\varepsilon(\alpha',\mu(\alpha'))\omega^{-a_0^2(\alpha'|\alpha')_\mu/2}.$$

So we can form the induced $\widehat{Q}$-module $\mathbb{C}[\widehat{Q}]\otimes_{\mathbb{C}[\widehat{M}]}\tau$. We note the useful formula

$$e_1^{\alpha'}\otimes\tau=\eta(\alpha')\omega^{a_0(\alpha'|\alpha')_\mu/2}e_1^{\mu(\alpha')}\otimes\tau\qquad(\alpha'\in Q').$$

View the symmetric algebra $S(\mathfrak{t}^-)$ as a $\mathfrak{t}$-module in the unique way so that c acts as 1, elements of $\mathfrak{t}^-$ act by multiplication, and elements of $\mathfrak{t}^+$ annihilate 1. It is $\mathbb{Z}$-graded by declaring that

$$\deg(h\otimes t^{-n})=\frac{n}{a_0}$$

for each $h\in\mathfrak{h}'_{-n}, n\geq 1$. Let

$$V=S(\mathfrak{t}^-)\otimes\mathbb{C}[\widehat{Q}]\otimes_{\mathbb{C}[\widehat{M}]}\tau.$$

Let $\mathfrak{t}$ act on $S(\mathfrak{t}^-)$ as given and trivially on $\mathbb{C}[\widehat{Q}]\otimes_{\mathbb{C}[\widehat{M}]}\tau$, let $h\otimes t^0$ for $h\in\mathfrak{h}'_0$ act by

$$(h\otimes t^0)(f\otimes e_x^{\alpha'}\otimes\tau)=(h|\alpha')'f\otimes e_x^{\alpha'}\otimes\tau,$$

and let d act by

$$d(f\otimes e_x^{\alpha'}\otimes\tau)=-a_0\left(\deg(f)+(\alpha'|\alpha')_\mu/2\right)f\otimes e_x^{\alpha'}\otimes\tau.$$

We have now defined the action of $\mathfrak{h}+\mathfrak{t}$ on V. To extend the action to all of $\mathfrak{g}$, let $\alpha'\in Q'$ be a root. As in [12, (4.8)], let

$$\sigma(\alpha')=\begin{cases}1 & r=1,\\ \sqrt{2}^{(\alpha'|\mu(\alpha'))'} & \text{if } X_N^{(r)}=A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)} \text{ or } E_6^{(2)},\\ (1-\omega^{-1})^{(\alpha'|\mu(\alpha'))'} & \text{if } X_N^{(r)}=D_4^{(3)},\\ 2(1+\omega)^{(\alpha'|\mu(\alpha'))'} & \text{if } X_N^{(r)}=A_{2\ell}^{(2)}.\end{cases}$$

Also define

$$P_{\alpha'}(z)=\exp\left(\sum_{n\geq1}\frac{\alpha'(-n)z^n}{n}\right),\qquad Q_{\alpha'}(z)=\exp\left(-\sum_{n\geq1}\frac{\alpha'(n)z^n}{n}\right),$$

viewed as elements of $\mathrm{End}(V)[[z^{\pm1}]]$. Let

$$E_{\alpha'}(z)=\sigma(\alpha')P_{\alpha'}(z)Q_{\alpha'}(z^{-1})e_1^{\alpha'}z^{a_0\alpha'}z^{a_0(\alpha'|\alpha')_\mu/2-1}.$$

Here, $z^{a_0\alpha'}$ denotes the operator with

$$z^{a_0\alpha'}(f\otimes e_x^{\beta'}\otimes\tau)=z^{(a_0\alpha'|\beta')_\mu}f\otimes e_x^{\beta'}\otimes\tau$$

for each $f \in S(\mathfrak{t}^-)$ and $\beta \in Q'$, and

$$e_1^{\alpha'}(f \otimes e_x^{\beta'} \otimes \tau) = f \otimes (e_1^{\alpha'} e_x^{\beta'}) \otimes \tau.$$

Expanding $E_{\alpha'}(z)$ in powers of z we get the required action of $E_{\alpha'}(n) \in \mathfrak{g}$ on V for each root $\alpha' \in Q'$ and each $n \in \mathbb{Z}$:

$$E_{\alpha'}(z) = \sum_{n \in \mathbb{Z}} E_{\alpha'}(n) z^{-n-1}.$$

For a proof that this is a well-defined irreducible representation of $\mathfrak{g}$ in case $r = 1$ see [11, §14.8]; the general case is due to Lepowsky [12].

Let $\mathbb{C}[\overline{Q}]$ denote the group algebra of $\overline{Q}$, with natural basis e^α for $\alpha \in \overline{Q}$ and multiplication $e^\alpha e^\beta = e^{\alpha+\beta}$. Note $\mathbb{C}[\widehat{Q}] \otimes_{\mathbb{C}[\widehat{M}]} \tau$ has a basis given by the elements $e_1^{\alpha'} \otimes \tau$ for all $\alpha' \in \sum_{i \in I} \mathbb{Z}\alpha'_i$. For such an α', let

$$\iota(e_1^{\alpha'} \otimes \tau) = \begin{cases} e^{\iota(\alpha')} & \text{if } X_N^{(r)} \neq A_{2\ell}^{(2)},\, D_4^{(3)}, \\ (-\omega)^{(\alpha'|\mu(\alpha'))'/2} e^{\iota(\alpha')} & \text{if } X_N^{(r)} = D_4^{(3)}, \\ \left(\frac{1-\omega}{\sqrt{2}}\right)^{(\alpha'|\mu(\alpha'))'} e^{\iota(\alpha')} & \text{if } X_N^{(r)} = A_{2\ell}^{(2)}, \end{cases}$$

recalling the map $\iota : Q' \to \overline{Q}$ defined in (2.1). Extending linearly, we obtain a vector space isomorphism $\iota : \mathbb{C}[\widehat{Q}] \otimes_{\mathbb{C}[\widehat{M}]} \tau \to \mathbb{C}[\overline{Q}]$. For $i = 0, 1, \dots, \ell$, we define functions $\sigma_i^\pm : \overline{Q} \to \mathbb{C}^\times$ by the equation

$$\sigma_i^\pm(\alpha) e^{\alpha \pm \overline{\alpha}_i} = \pm \frac{\sqrt{c_i}}{a_0 d_i} \sigma(\alpha'_i) \iota(e_1^{\pm\alpha'_i} \iota^{-1}(e^\alpha))$$

for all $\alpha \in \overline{Q}$. The choice of the renormalization map ι above ensures:

Lemma 3.1. *For all $i = 0, 1, \dots, \ell$ and $\alpha \in \overline{Q}$, $\sigma_i^\pm(\alpha) \in \{\pm 1\}$. Moreover, for $i \in I$, we have that $\sigma_i^- = -\sigma_i^+$, and $\sigma_i^+ : \overline{Q} \to \{\pm 1\}$ is a group homomorphism such that $\sigma_i^+(\overline{\alpha}_j) = \varepsilon(\alpha'_i, \alpha'_j)$ for each $j \in I$.*

Now we can rewrite the construction of the basic representation V in terms of the Chevalley generators. We will identify

$$V = S(\mathfrak{t}^-) \otimes \mathbb{C}[\widehat{Q}] \otimes_{\mathbb{C}[\widehat{M}]} \tau = S(\mathfrak{t}^-) \otimes \mathbb{C}[\overline{Q}]$$

via the map $\mathrm{id} \otimes \iota$. Then, the actions of h_i for $i = 0, \dots, \ell$ and of d are as

$$h_i(f \otimes e^\alpha) = (\delta_{i,0} + \langle h_i, \alpha \rangle) f \otimes e^\alpha,$$
$$d(f \otimes e^\alpha) = -a_0 \left(\deg(f) + (\alpha|\alpha)/2\right) f \otimes e^\alpha$$

for all $\alpha \in \overline{Q}$. In particular, we note from this that

$$\mathrm{wt}(f \otimes e^\alpha) = \Lambda_0 + \alpha - \left(\deg(f) + (\alpha|\alpha)/2\right) \delta \tag{3.2}$$

for each homogeneous $f \in S(\mathfrak{t}^-)$ and $\alpha \in \overline{Q}$. This shows that $1 \otimes e^0$ is a highest weight vector in V of highest weight Λ_0 (cf. [11, Lemma 12.6]), identifying V with the irreducible highest weight module $V(\Lambda_0)$. Finally, for $i = 0, \dots, \ell$,

$$e_i(z) = \sum_{n\in\mathbb{Z}} e_i(n) \otimes z^{-n-1} = P_{\alpha_i'}(z) Q_{\alpha_i'}(z^{-1}) e^{\overline{\alpha}_i} z^{a_0\alpha_i} z^{a_0(\alpha_i|\alpha_i)/2-1} s_i^+, \tag{3.3}$$

$$f_i(z) = \sum_{n\in\mathbb{Z}} f_i(n) \otimes z^{-n-1} = P_{-\alpha_i'}(z) Q_{-\alpha_i'}(z^{-1}) e^{-\overline{\alpha}_i} z^{-a_0\alpha_i} z^{a_0(\alpha_i|\alpha_i)/2-1} s_i^-, \tag{3.4}$$

where

$$z^{\pm a_0\alpha_i}(f \otimes e^\beta) = z^{(\pm a_0\alpha_i|\beta)} f \otimes e^\beta,$$
$$s_i^\pm(f \otimes e^\beta) = \sigma_i^\pm(\beta) f \otimes e^\beta,$$
$$e^{\pm\overline{\alpha}_i}(f \otimes e^\beta) = f \otimes e^{\beta\pm\overline{\alpha}_i}.$$

The following lemma will be needed later on:

Lemma 3.5. *For $i_1, \dots, i_s \in \{0, \dots, \ell\}$, roots $\beta_1', \dots, \beta_t' \in Q'$ and $\gamma \in \overline{Q}$, we have that*

$$\begin{aligned}
&e_{i_1}(z_1) e_{i_2}(z_2) \dots e_{i_s}(z_s) P_{\beta_1'}(w_1) \dots P_{\beta_t'}(w_t) \otimes e^\gamma = \\
&\quad \pm \prod_{1\le u\le s} z_u^{\frac{a_0}{2}(\alpha_{i_u}|\alpha_{i_u})-1+a_0(\alpha_{i_u}|\gamma)} \\
&\quad \times \prod_{1\le u<v\le s} z_u^{a_0(\alpha_{i_u}|\alpha_{i_v})} \prod_{k\in\mathbb{Z}/m} \left(1 - \omega^{-k}\frac{z_v}{z_u}\right)^{(\mu^k(\alpha_{i_u}')|\alpha_{i_v}')'} \\
&\quad \times \prod_{1\le u\le s} \prod_{1\le v\le t} \prod_{k\in\mathbb{Z}/m} \left(1 - \omega^{-k}\frac{w_v}{z_u}\right)^{(\mu^k(\alpha_{i_u}')|\beta_v')'} \\
&\quad \times P_{\alpha_{i_1}'}(z_1) \dots P_{\alpha_{i_s}'}(z_s) P_{\beta_1'}(w_1) \dots P_{\beta_t'}(w_t) \otimes e^{\gamma+\overline{\alpha}_{i_1}+\dots+\overline{\alpha}_{i_s}}.
\end{aligned}$$

A similar formula holds for $f_{i_1}(z_1) \dots f_{i_s}(z_s) P_{\beta_1'}(w_1) \dots P_{\beta_s'}(w_s) \otimes e^\gamma$, replacing α_{i_u} by $-\alpha_{i_u}$, α_{i_u}' by $-\alpha_{i_u}'$, and $\bar{\alpha}_{i_u}$ by $-\bar{\alpha}_{i_u}$ everywhere.

Proof. This follows from the following commutation relation obtained in [12, 3.4]: for $\alpha', \beta' \in Q'$,

$$Q_{\alpha'}(z^{-1}) P_{\beta'}(w) = P_{\beta'}(w) Q_{\alpha'}(z^{-1}) \prod_{k\in\mathbb{Z}/m} (1 - \omega^{-k}\frac{w}{z})^{(\mu^k(\alpha')|\beta')'},$$

which is a consequence of the Campbell-Hausdorff formula, cf. [11, (14.8.12)]. $\square$

4. The Integral form

As in the introduction, let $U_\mathbb{Z}$ denote the $\mathbb{Z}$-subalgebra of the universal enveloping algebra of $\mathfrak{g}$ generated by the elements $e_i^r/r!, f_i^r/r!$ for $i = 0, \dots, \ell$ and $r \ge 0$, and let

$$V_\mathbb{Z} := U_\mathbb{Z}(1 \otimes e^0) \subset V.$$

In this section, we will give an explicit description of $V_\mathbb{Z}$.

To start with, let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the antilinear Chevalley antiautomorphism, so

$$\tau(d) = d, \quad \tau(e_i(n)) = f_i(-n), \quad \tau(f_i(n)) = e_i(-n)$$

for each $i = 0, \dots, \ell$ and $n \in \mathbb{Z}$, cf. [11, §§7.6, 8.3]. The *Shapovalov form* $(.|.)_S$ on V is the unique Hermitian form such that

$$(1 \otimes e^0, 1 \otimes e^0)_S = 1 \quad \text{and} \quad (xv, w)_S = (v, \tau(x)w)_S$$

for all $v, w \in V$, $x \in \mathfrak{g}$. The restriction of τ to $\mathfrak{t}$ gives the antilinear Chevalley antiautomorphism of $\mathfrak{t}$, and we can also consider the Shapovalov form on $S(\mathfrak{t}^-)$, satisfying $(1,1)_S = 1$ and $(xf, g)_S = (f, \tau(x)g)_S$ for all $f, g \in S(\mathfrak{t}^-)$, $x \in \mathfrak{t}$.

Lemma 4.1. *For all $f, g \in S(\mathfrak{t}^-)$ and $\alpha, \beta \in \overline{Q}$, $(f \otimes e^\alpha, g \otimes e^\beta)_S = (f, g)_S$.*

Proof. Since different weight spaces are orthogonal and in view of (3.2), this reduces to checking that $(1 \otimes e^\alpha, 1 \otimes e^\alpha)_S = 1$ for all $\alpha \in \overline{Q}$. Proceeding by induction, we may assume that there is some $\beta \in \overline{Q}$ and $i \in I$ such that $(1 \otimes e^\beta, 1 \otimes e^\beta)_S = 1$ and either $\alpha = \beta + \overline{\alpha}_i$ or $\alpha = \beta - \overline{\alpha}_i$.

Suppose that $\alpha = \beta + \overline{\alpha}_i$. Letting $n = -a_0(\alpha_i|\beta) - a_0(\alpha_i|\alpha_i)/2$, one checks easily using (3.3), (3.4) that

$$e_i(n)(1 \otimes e^\beta) = \sigma_i^+(\alpha)(1 \otimes e^\alpha), \qquad f_i(-n)(1 \otimes e^\alpha) = \sigma_i^-(\beta)(1 \otimes e^\beta). \tag{4.2}$$

Hence,

$$\begin{aligned}(1 \otimes e^\alpha, 1 \otimes e^\alpha)_S &= \sigma_i^+(\beta)(e_i(n)(1 \otimes e^\beta), 1 \otimes e^\alpha)_S \\ &= \sigma_i^+(\beta)(1 \otimes e^\beta, f_i(-n)(1 \otimes e^\alpha))_S \\ &= \sigma_i^-(\alpha)\sigma_i^+(\beta)(1 \otimes e^\beta, 1 \otimes e^\beta)_S = 1,\end{aligned}$$

since $\sigma_i^-(\alpha) = \sigma_i^+(\beta)$ by Lemma 3.1.

A similar argument in the case that $\alpha = \beta - \overline{\alpha}_i$ completes the proof. $\square$

Lemma 4.3. *For all $i = 0, 1, \dots, \ell$ and $n \in \mathbb{Z}$, the elements $e_i(n)$ and $f_i(n)$ belong to $U_\mathbb{Z}$.*

Proof. Suppose that $e_i(n) \neq 0$. Then, $\mathrm{wt}(e_i(n)) = \overline{\alpha}_i + \frac{n}{a_0}\delta$ is a real root, hence is conjugate under the Weyl group W associated to $\mathfrak{g}$ to some simple root α_j. So we can find simple reflections $s_{i_1}, \dots, s_{i_t} \in W$ such that $\overline{\alpha}_i + \frac{n}{a_0}\delta = s_{i_1} \dots s_{i_t}\alpha_j$. Let r_i^{ad} be the automorphism of $\mathfrak{g}$ defined by $r_i^{\mathrm{ad}} = \exp(\mathrm{ad} f_i)\exp(-\mathrm{ad} e_i)\exp(\mathrm{ad} f_i)$, for $i = 0, 1, \dots, \ell$. Since real root spaces of $\mathfrak{g}$ are one dimensional,

$$r_{i_1}^{\mathrm{ad}} \dots r_{i_t}^{\mathrm{ad}} e_j = c e_i(n)$$

for some non-zero scalar c. Now, $\tau(\exp(\mathrm{ad} y)(x)) = \exp(-\mathrm{ad}\tau(y))(\tau(x))$, whence by an SL_2-calculation we have $r_i^{\mathrm{ad}}(x) = \tau(r_i^{\mathrm{ad}}(\tau(x)))$ for all $x \in \mathfrak{g}$, we also get that

$$r_{i_1}^{\mathrm{ad}} \dots r_{i_t}^{\mathrm{ad}} f_j = c f_i(-n).$$

But the r_i^{ad} preserve the normalized invariant form on $\mathfrak{g}$, so

$$a_j(a_j^\vee)^{-1} = (e_j|f_j) = (ce_i(n)|cf_i(-n)) = c^2 a_i(a_i^\vee)^{-1}.$$

Clearly, α_i and α_j are roots of the same length, i.e. $a_j(a_j^\vee)^{-1} = a_i(a_i^\vee)^{-1}$, so this gives that $c = \pm 1$. Finally, the action of r_i^{ad} on $\mathfrak{g}$ leaves $U_\mathbb{Z} \cap \mathfrak{g}$ invariant, so

$$e_i(n) = \pm r_{i_1}^{\mathrm{ad}} \dots r_{i_s}^{\mathrm{ad}} e_j \in U_\mathbb{Z},$$

and similarly $f_i(n) \in U_\mathbb{Z}$ too. $\square$

For $n \geq 1$ and $i = 0, 1, \dots, \ell$, define

$$y_{nd_i}^{(i)} = \frac{\alpha_i'(-a_0 n d_i)}{a_0 n d_i}, \qquad x_{nd_i}^{(i)} = \sum_{k_1 + 2k_2 + \cdots = n} \frac{{y_{d_i}^{(i)}}^{k_1}}{k_1!} \frac{{y_{2d_i}^{(i)}}^{k_2}}{k_2!} \frac{{y_{3d_i}^{(i)}}^{k_3}}{k_3!} \cdots .$$

Observe that

$$P_{\alpha_i'}(z) = \exp\left(\sum_{n\geq 1} y_{nd_i}^{(i)} z^{a_0 n d_i}\right) = 1 + \sum_{n \geq 1} x_{nd_i}^{(i)} z^{a_0 n d_i}. \tag{4.4}$$

The $y_{nd_i}^{(i)}$ for $n \geq 1$ and $i \in I$ give a basis for $\mathfrak{t}^-$. So $S(\mathfrak{t}^-)$ is equal to the free polynomial algebra

$$B := \mathbb{C}[y_{nd_i}^{(i)} \mid n \geq 1, i \in I].$$

Since the $x_{nd_i}^{(i)}$ are related to the $y_{nd_i}^{(i)}$ in a unitriangular way, we obtain a $\mathbb{Z}$-form

$$B_{\mathbb{Z}} := \mathbb{Z}[x_{nd_i}^{(i)} \mid n \geq 1, i \in I] \subset B$$

for B. As α_ε' is an integral linear combination of the α_i' with $i \in I$, it follows from (4.4) that the elements $x_{nd_\varepsilon}^{(\varepsilon)}$ also belong to the lattice $B_{\mathbb{Z}}$. The $\mathbb{Z}$-grading on $B_{\mathbb{Z}}$ induced by the grading on $S(\mathfrak{t}^-)$ is determined by $\deg(y_n^{(i)}) = \deg(x_n^{(i)}) = n$.

The following theorem (or rather its q-analogue) for the non-twisted case has been proved in [4]. Our argument for the general case is similar.

Theorem 4.5. $V_{\mathbb{Z}} = B_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\overline{Q}]$.

Proof. Let us first show that $B_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\overline{Q}] \subseteq V_{\mathbb{Z}}$. Fix $i_1, \dots, i_s \in I$, and let

$$M(i_1, \dots, i_s) = \{(n_1, \dots, n_s) \mid n_1 \geq \cdots \geq n_s \geq 0 \text{ and } d_{i_j} | n_j \text{ for all } j = 1, \dots, s\}.$$

Denote by $>$ the dominance ordering on partitions belonging to $M(i_1, \dots, i_s)$. We will show that $x_{n_1}^{(i_1)} \dots x_{n_s}^{(i_s)} \otimes e^\beta \in V_{\mathbb{Z}}$ for all $(n_1, \dots, n_s) \in M(i_1, \dots, i_s)$ and each $\beta \in \overline{Q}$. Clearly every monomial in $B_{\mathbb{Z}}$ is of the form $x_{n_1}^{(i_1)} \dots x_{n_s}^{(i_s)}$ for some choice of $i_1, \dots, i_s$ and $(n_1, \dots, n_s) \in M(i_1, \dots, i_s)$, so this is good enough.

To start with, each $e_i(n), f_i(-n) \in U_{\mathbb{Z}}$ by Lemma 4.3. So an obvious inductive argument using (4.2) gives that $1 \otimes e^\gamma \in V_{\mathbb{Z}}$ for each $\gamma \in \overline{Q}$. Hence, letting $\gamma = \beta - \bar{\alpha}_{i_1} - \cdots - \bar{\alpha}_{i_s}$, Lemma 4.3 implies that all coefficients of $e_{i_1}(z_1) \dots e_{i_s}(z_s) \otimes e^\gamma$ belong to $V_{\mathbb{Z}}$. Applying Lemma 3.5, we deduce that all the coefficients of

$$X := \left(\prod_{1 \leq u < v \leq s} \prod_{k \in \mathbb{Z}/m} \left(1 - \omega^{-k} \frac{z_v}{z_u}\right)^{(\mu^k(\alpha_{i_u}') | \alpha_{i_v}')'}\right) P_{\alpha_{i_1}'}(z_1) \dots P_{\alpha_{i_s}'}(z_s) \otimes e^\beta$$

belong to $V_{\mathbb{Z}}$. One checks that in all cases,

$$\prod_{k \in \mathbb{Z}/m} \left(1 - \omega^{-k} \frac{z_v}{z_u}\right)^{(\mu^k(\alpha_{i_u}') | \alpha_{i_v}')'} = 1 + (*),$$

where $(*)$ is a $\mathbb{Z}$-linear combination of $\left(\frac{z_v}{z_u}\right)^p$ for $p \geq 1$. It follows that the $z_1^{a_0 n_1} \dots z_s^{a_0 n_s}$-coefficient of X equals

$$x_{n_1}^{(i_1)} \dots x_{n_s}^{(i_s)} \otimes e^\beta + (**),$$

where $(**)$ is a $\mathbb{Z}$-linear combination of $x_{n'_1}^{(i_1)} \dots x_{n'_s}^{(i_s)} \otimes e^\beta$ for $(n'_1, \dots, n'_s) > (n_1, \dots, n_s)$. Using downward induction on this ordering, we deduce that $x_{n_1}^{(i_1)} \dots x_{n_s}^{(i_s)} \in V_\mathbb{Z}$.

Finally, we prove that $B_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}[\overline{Q}] \supseteq V_\mathbb{Z}$. As the high weight vector $1 \otimes e^0$ belongs to $B_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}[\overline{Q}] \supseteq V_\mathbb{Z}$, it suffices to show that $B_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}[\overline{Q}]$ is invariant under each of the operators $f_i(n)^s/s!$ for $n \in \mathbb{Z}, s \geq 1$ and $i = 0, 1, \dots, \ell$. Fix $i \in \{0, 1, \dots, \ell\}$ and consider

$$Y := f_i(z_1) \dots f_i(z_s) P_{\alpha'_{i_1}}(w_1) \dots P_{\alpha'_{i_t}}(w_t) \otimes e^{\gamma + s\overline{\alpha}_i}$$

for $i_1, \dots, i_t \in I$ and $\gamma \in \overline{Q}$. Applying Lemma 3.5 and simplifying,

$$\begin{aligned} Y \quad = \quad & \pm(z_1 \dots z_s)^{\frac{a_0}{2}(\alpha_i|\alpha_i) - 1 - a_0(\alpha_i|\gamma + s\overline{\alpha}_i)} \\ & \times \prod_{1 \leq u < v \leq s} \prod_{k \in \mathbb{Z}/m} \left(z_u - \omega^{-k} z_v\right)^{(\mu^k(\alpha'_i)|\alpha'_i)'} \\ & \times \prod_{\substack{1 \leq u \leq s \\ 1 \leq v \leq t}} \prod_{k \in \mathbb{Z}/m} \left(1 - \omega^{-k} \frac{w_v}{z_u}\right)^{-(\mu^k(\alpha'_i)|\alpha'_{i_v})'} \\ & \times P_{-\alpha'_i}(z_1) \dots P_{-\alpha'_i}(z_s) P_{\alpha'_{i_1}}(w_1) \dots P_{\alpha'_{i_t}}(w_t) \otimes e^\gamma. \end{aligned}$$

Certainly, each coefficient of $P_{-\alpha'_i}(z_1) \dots P_{-\alpha'_i}(z_s) P_{\alpha'_{i_1}}(w_1) \dots P_{\alpha'_{i_t}}(w_t) \otimes e^\gamma$ belongs to $V_\mathbb{Z}$. One checks that

$$\prod_{k \in \mathbb{Z}/m} \left(z_u - \omega^{-k} z_v\right)^{(\mu^k(\alpha'_i)|\alpha'_i)'} = \begin{cases} (z_u - z_v)^2 \frac{(z_u + z_v)^2}{z_u^2 + z_v^2} & \text{if } X_N^{(r)} = A_{2\ell}^{(2)} \text{ and } i = 0, \\ (z_u^{a_0 d_i} - z_v^{a_0 d_i})^2 & \text{otherwise.} \end{cases}$$

Hence in all cases, Y looks like $\prod_{1 \leq u < v \leq s} (z_u - z_v)^2$ times an expression that is symmetric in $z_1, \dots, z_s$. Hence, by [4, Lemma 2.5(ii)], the coefficient of $(z_1 \dots z_s)^{-n-1}$ in Y is divisible by $s!$. Hence, all coefficients of $(f_i(n)^s/s!) P_{\alpha'_{i_1}}(w_1) \dots P_{\alpha'_{i_s}}(w_s) \otimes e^{\gamma - s\overline{\alpha}_i}$ belong to $V_\mathbb{Z}$, which completes the proof. $\square$

5. The determinant

Fix now some $d \geq 0$. Lemma 4.1 and Theorem 4.5 reduce the problem of computing the determinant of the Shapovalov form on the $(w\Lambda_0 - d\delta)$ weight space of $V_\mathbb{Z}$ for any $w \in W$ to the problem of computing the determinant of the Shapovalov form on the degree d component of $B_\mathbb{Z}$. To tackle the latter question, observe

$$B = \bigotimes_{i \in I} \mathbb{C}[y_{d_i}^{(i)}, y_{2d_i}^{(i)}, \dots], \qquad B_\mathbb{Z} = \bigotimes_{i \in I} \mathbb{Z}[x_{d_i}^{(i)}, x_{2d_i}^{(i)}, \dots].$$

So according to [5, Corollary 2.1], there is a well-defined Hermitian form on B determined by the rules

$$(1,1)_K = 1, \qquad (ny^{(i)}_{nd_i} f, g)_K = (f, \frac{\partial}{\partial y^{(i)}_{nd_i}} g)_K$$

for all $i \in I, n \geq 1$ and $f, g \in B$. Moreover, there is a homogeneous basis for the lattice $B_{\mathbb{Z}}$ (given by Schur polynomials) that is orthonormal with respect to the form $(.,.)_K$. In particular, the determinant of the form $(.,.)_K$ on the degree d component of $B_{\mathbb{Z}}$ is equal to 1. Our strategy will therefore be to relate the Shapovalov form $(.,.)_S$ to the form $(.,.)_K$.

Define $I(n) = \{i \in I \mid d_i | n\}$. Introduce the matrices $A^{(n)} = (a^{(n)}_{i,j})_{i,j \in I(n)}$ with

$$a^{(n)}_{i,j} = \frac{1}{d_i}(\alpha'_i | \sum_{k=0}^{r-1} \omega^{a_0 n k} \mu^k(\alpha'_j))'.$$

Recall α, β from the introduction. One verifies:

Lemma 5.1. *For any $n \geq 0$, we have* $\det A^{(n)} = \begin{cases} \alpha & \text{if } r \mid n, \\ \beta & \text{if } r \nmid n, \end{cases}$

The significance of the matrices $A^{(n)}$ is that for $i \in I(n)$, the element $\tau(ny^{(i)}_n / d_i) = \alpha'_i(a_0 n)/a_0 d_i$ acts on B as the operator $\sum_{j \in I(n)} a^{(n)}_{i,j} \frac{\partial}{\partial y^{(j)}_n}$. We set

$$z^{(i)}_n = \sum_{j \in I(n)} a^{(n)}_{i,j} y^{(j)}_n.$$

For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_h > 0)$ we let $I(\lambda) = I(\lambda_1) \times \cdots \times I(\lambda_h)$. Given $\underline{i} = (i_1, \ldots, i_h) \in I(\lambda)$, let

$$x^{(\underline{i})}_\lambda = x^{(i_1)}_{\lambda_1} \ldots x^{(i_h)}_{\lambda_h}, \qquad y^{(\underline{i})}_\lambda = y^{(i_1)}_{\lambda_1} \ldots x^{(i_h)}_{\lambda_h}, \qquad z^{(\underline{i})}_\lambda = z^{(i_1)}_{\lambda_1} \ldots x^{(i_h)}_{\lambda_h},$$

all elements of B of degree $|\lambda|$.

Lemma 5.2. *For $\underline{i} \in I(\lambda)$ and any $f \in B$, we have $(y^{(\underline{i})}_\lambda, f)_S = (z^{(\underline{i})}_\lambda, f)_K$.*

Proof. Proceed by induction on the number of non-zero parts of λ, starting induction from the obvious fact that $(1, f)_S = (1, f)_K$. For the induction step, note that for $i \in I(n)$,

$$\begin{aligned}(y^{(i)}_n y^{(\underline{i})}_\lambda, f)_S &= (\frac{d_i}{n} y^{(\underline{i})}_\lambda, \sum_{j \in I(n)} a^{(n)}_{i,j} \frac{\partial}{\partial y^{(j)}_n} f)_S \\ &= (\frac{d_i}{n} z^{(\underline{i})}_\lambda, \sum_{j \in I(n)} a^{(n)}_{i,j} \frac{\partial}{\partial y^{(j)}_n} f)_K = (z^{(i)}_n z^{(\underline{i})}_\lambda, f)_K.\end{aligned}$$

□

Let

$$\Omega(\lambda) = \{\underline{i} \in I(\lambda) \mid i_j \leq i_{j+1} \text{ whenever } \lambda_j = \lambda_{j+1}\}.$$

Then $\{x_\lambda^{(\underline{i})} \,|\, \lambda \in \mathscr{P}(d), \underline{i} \in \Omega(\lambda)\}$, $\{y_\lambda^{(\underline{i})} \,|\, \lambda \in \mathscr{P}(d), \underline{i} \in \Omega(\lambda)\}$ and $\{z_\lambda^{(\underline{i})} \,|\, \lambda \in \mathscr{P}(d), \underline{i} \in \Omega(\lambda)\}$ give three different bases for the degree d component of B. Consider the transition matrices $P = (p_{\lambda,\mu}^{\underline{i},\underline{j}})$ and $Q = (q_{\lambda,\mu}^{\underline{i},\underline{j}})$ where $\lambda, \mu \in \mathscr{P}(d)$, $\underline{i} \in \Omega(\lambda), \underline{j} \in \Omega(\mu)$ defined from

$$x_\lambda^{(\underline{i})} = \sum_{\mu \in \mathscr{P}(d),\ \underline{j} \in \Omega(\mu)} p_{\lambda,\mu}^{\underline{i},\underline{j}} y_\mu^{(\underline{j})}, \qquad z_\lambda^{(\underline{i})} = \sum_{\mu \in \mathscr{P}(d),\ \underline{j} \in \Omega(\mu)} q_{\lambda,\mu}^{\underline{i},\underline{j}} y_\mu^{(\underline{j})}.$$

Lemma 5.3. *The matrix Q is block diagonal, i.e. $q_{\lambda,\mu}^{\underline{i},\underline{j}} = 0$ for $\lambda \neq \mu$. Moreover, the determinant of the λ-block $Q_\lambda = (q_{\lambda,\lambda}^{\underline{i},\underline{j}})_{\underline{i},\underline{j} \in \Omega(\lambda)}$ of Q is $\alpha^{a_\lambda} \beta^{b_\lambda}$, notation as in the introduction.*

Proof. It is immediate from the definition that $q_{\lambda,\mu}^{\underline{i},\underline{j}} = 0$ for $\lambda \neq \mu$. So consider the λ-block Q_λ of Q. Represent $\lambda = (\lambda_1 \geq \cdots \geq \lambda_h > 0)$ instead as $(1^{r_1} 2^{r_2} \dots s^{r_s})$. By definition,

$$z_\lambda^{(\underline{i})} = \sum_{\underline{j} \in I(\lambda)} a_{i_1,j_1}^{(\lambda_1)} \dots a_{i_h,j_h}^{(\lambda_h)} y_\lambda^{(\underline{j})}.$$

Thus Q_λ is the matrix $S^{r_1}(A^{(1)}) \otimes \cdots \otimes S^{r_s}(A^{(s)})$, a tensor product of symmetric powers of the matrices $A^{(n)}$. Now, note that for an $n \times n$ matrix A,

$$\det S^m(A) = (\det A)^{\binom{n+m-1}{n}},$$

while for an $n \times n$ matrix B and an $m \times m$ matrix C,

$$\det(B \otimes C) = (\det B)^m (\det C)^n.$$

These are both proved by reducing to the case that the matrices are diagonal. Combining the formulae with Lemma 5.1, one computes $\det Q_\lambda = \alpha^{a_\lambda} \beta^{b_\lambda}$. $\square$

Now we can prove the main theorem:

Theorem 5.4. *The determinant of the restriction of the Shapovalov form to the degree d part of $B_{\mathbb{Z}}$ is $\prod_{\lambda \in \mathscr{P}(d)} \alpha^{a_\lambda} \beta^{b_\lambda}$.*

Proof. Consider the matrices $M = (m_{\lambda,\mu}^{\underline{i},\underline{j}})$ and $N = (n_{\lambda,\mu}^{\underline{i},\underline{j}})$ for $\lambda, \mu \in \mathscr{P}(d)$, $\underline{i} \in \Omega(\lambda)$, $\underline{j} \in \Omega(\mu)$ defined from

$$m_{\lambda,\mu}^{\underline{i},\underline{j}} = (x_\lambda^{(\underline{i})}, x_\mu^{(\underline{j})})_S, \qquad n_{\lambda,\mu}^{\underline{i},\underline{j}} = (x_\lambda^{(\underline{i})}, x_\mu^{(\underline{j})})_K.$$

Recalling the transition matrices P and Q introduced above, Lemma 5.2 gives at once that $M = PQP^{-1}N$. On the other hand, N has determinant 1, since as we observed above the degree d component of $B_{\mathbb{Z}}$ admits an orthonormal basis with respect to the contravariant form $(.,.)_K$ (see [5, Corollary 2.1]). So we can compute $\det M$ at once using Lemma 5.3. $\square$

We finally indicate how to deduce the generating functions $a(q)$ and $b(q)$ stated in the introduction. By definition, a_λ is

$$\frac{h}{\ell} \times \left(\begin{array}{l} \text{the number of ways of coloring} \\ \text{the parts } \lambda_i \equiv 0\,(r) \text{ with } \ell \text{ colors} \end{array} \right) \times \left(\begin{array}{l} \text{the number of ways of coloring} \\ \text{the parts } \lambda_i \not\equiv 0\,(r) \text{ with } k \text{ colors} \end{array} \right)$$

where h is the number of $\lambda_i \equiv 0\,(r)$. Consider

$$G(q,t,u) = \left(\prod_{n\geq 1} \frac{1}{1-q^{nr}t}\right)^{\ell} \left(\prod_{n\geq 1} \frac{1-q^{nr}u}{1-q^{n}u}\right)^{k}.$$

The coefficient of $q^d t^h u^i$ is equal to the number of partitions of d with h parts divisible by r colored by ℓ different colors and with i parts not divisible by r colored by k different colors. Hence the generating function $a(q)$ for $a(d) = \sum_{\lambda \in \mathscr{P}(d)} a_\lambda$ is equal to $\frac{1}{\ell}\frac{d}{dt}G(q,t,u)|_{t=u=1}$. Similarly, $b(q) = \frac{1}{k}\frac{d}{du}G(q,t,u)|_{t=u=1}$.

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