REPRESENTATIONS OF THE ALTERNATING GROUP WHICH ARE IRREDUCIBLE OVER SUBGROUPS

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ABSTRACT. Let F be an algebraically closed field of characteristic $p \geq 0$ and A_n be the alternating group on n letters. The main goal of this paper is to describe the pairs (G, E), where E is an irreducible FA_n-module and $G < A_n$ is a proper subgroup of A_n such that the restriction $E{\downarrow}_G$ is irreducible. We will be able to give a list of all such pairs, provided $p > 3$. The case $p = 0$ was treated by Saxl. The problem is important for the classification of maximal subgroups in finite classical groups.

INTRODUCTION

Let F be an algebraically closed field of characteristic $p \geq 0$ and A_n be the alternating group on n letters. The main goal of this paper is to describe the pairs (G, E), where E is an irreducible FA_n-module and $G < A_n$ is a proper subgroup of A_n such that the restriction $E{\downarrow}_G$ is irreducible. We will be able to give a list of all such pairs, provided $p > 3$. The case $p = 0$ was treated by Saxl [25]. We note that (in characteristic p) the problem is important for the classification of maximal subgroups in finite classical groups.

A similar problem for the symmetric group has been solved in [3] (also for $p > 3$). This yields a solution of our problem for all irreducible FA_n-modules E which lift to the symmetric group Σ_n. However, not all irreducible FA_n-modules lift, and so we need some further investigation to complete the problem.

To describe the main result we first explain how the irreducible FA_n-modules can be parametrised. From now on we will assume that $p > 2$ as the case $p = 2$ is quite exceptional for A_n and also because it is excluded in our main result anyway. We start from the symmetric group, referring the reader to [11] for the standard facts on its representation theory. In particular, to every p-regular partition λ of n one can associate the irreducible $F\Sigma_n$-module D^λ. Denote by $\mathbf{sgn} = \mathbf{sgn}_{\Sigma_n}$ the 1-dimensional sign representation of Σ_n. Then $D^\lambda \otimes \mathbf{sgn}$ is irreducible so there should exist a p-regular partition $\lambda^{\mathbf{M}}$ with

$$D^\lambda \otimes \mathbf{sgn} \cong D^{\lambda^{\mathbf{M}}}.$$

The bijection $\lambda \mapsto \lambda^{\mathbf{M}}$ on the set of p-regular partitions is called the *Mullineux bijection*. This bijection can be described explicitly using a combinatorial algorithm suggested by Mullineux, see [24, 21, 10, 1].

It follows easily from the Clifford theory (see e.g. [9]) that the restriction $D^\lambda{\downarrow}_{A_n}$ is irreducible if and only if $D^\lambda \otimes \mathbf{sgn} \not\cong D^\lambda$. If this is the case, denote this irreducible restriction by E^λ. Of course, $E^\lambda \cong E^{\lambda^{\mathbf{M}}}$. On the other hand, if $D^\lambda \otimes \mathbf{sgn} \cong D^\lambda$ then the restriction $D^\lambda{\downarrow}_{A_n}$ splits as a direct sum $E^\lambda_+ \oplus E^\lambda_-$ of two irreducible FA_n-modules. We have $\lambda = \lambda^{\mathbf{M}}$, in which case we say that λ is *Mullineux-fixed*. Finally,

$$\{E^\lambda \mid \lambda \neq \lambda^{\mathbf{M}}\} \cup \{E^\lambda_+, E^\lambda_- \mid \lambda = \lambda^{\mathbf{M}}\}$$

is a complete set of irreducible FA_n-modules, and distinct modules L and M from this set are isomorphic if and only if $L \cong E^\lambda$, $M \cong E^{\lambda^{\mathbf{M}}}$ for some p-regular partition λ with $\lambda \neq \lambda^{\mathbf{M}}$.

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We introduce some elementary combinatorial notions used in the statement of the main result. Let λ be a p-regular partition of n. Represent it in the form

$$\lambda = (l_1^{a_1}, l_2^{a_2}, \dots, l_k^{a_k}) \quad \text{for } l_1 > l_2 > \dots > l_k > 0,\ a_i > 0.$$

Then λ is called a *Jantzen-Seitz partition* (or *JS-partition* for short) if

$$l_i - l_{i+1} + a_i + a_{i+1} \equiv 0 \pmod p \quad \text{for all } i \text{ with } 1 \le i \le k-1.$$

We will not distinguish between a partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \dots)$ and its *Young diagram*

$$\lambda = \{(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid j \le \lambda_i\}.$$

Elements $(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ are called *nodes*. The *residue* of a node $A = (i,j)$, written $\operatorname{res} A$, is defined to be the residue class $(j-i) \pmod p$. A node $(i, \lambda_i) \in \lambda$ is called *removable* if $\lambda_i > \lambda_{i+1}$. A node $(i, \lambda_i + 1)$ is called *addable* if $i = 1$ or $i > 1$ and $\lambda_i < \lambda_{i-1}$. We say a node (i,j) is above (resp. below) (i',j') if $i < i'$ (resp. $i > i'$). A removable node A of λ is called *normal* if for every addable node B above A with $\operatorname{res} B = \operatorname{res} A$ there exists a removable node $C(B)$ above A and below B with $\operatorname{res} C(B) = \operatorname{res} A$, and such that $B \neq B'$ implies $C(B) \neq C(B')$.

Now we can state our main result.

Main Theorem. *Let $p > 3$, $n \ge 5$, E be an irreducible FA_n-module of dimension greater than one, and $G < A_n$ be a proper subgroup. Then $E{\downarrow}_G$ is irreducible if and only if one of the following happens:*

- (i) *$E = E^\lambda$ for $\lambda \neq \lambda^{\mathbf{M}}$, and (G, D^λ) is as in* [3], *Main Theorem.*
- (ii) *$E = E^\lambda_\pm$ for $\lambda = \lambda^{\mathbf{M}}$, and one of the following happens:*
  - (a) *λ is a JS-partition, and $G \ge A_{n-2}$, the stabilizer in A_n of two points;*
  - (b) *λ has exactly two normal nodes A and B, $\operatorname{res} A \neq \operatorname{res} B$, and $G = A_{n-1}$, the stabilizer in A_n of one point;*
  - (c) *$\lambda = (3^3)$, $p > 5$, $n = 9$, and G is $\mathrm{P\Gamma L}(2,8)$ embedded via its permutation representation of degree 9 or $\mathrm{AGL}(3,2)$ embedded into $A_8 < A_9$ via its permutation representation of degree 8;*
  - (d) *$\lambda = (3^2, 2)$, $p > 5$, $n = 8$, and G is $\mathrm{AGL}(3,2)$ embedded via its permutation representation of degree 8.*

1. Preliminaries

Let G be a group, and L, M be FG-modules. We denote by M^G the space of G-invariants in M. The space $\operatorname{Hom}_F(L,M)$ has a natural structure of an FG-module, with $\operatorname{Hom}_{FG}(L,M) = \operatorname{Hom}_F(L,M)^G$. If $D_1, \dots, D_k$ are irreducible FG-modules then the notation $M = D_1|\dots|D_k$ means that M is a *uniserial* FG-module with composition factors $D_1, \dots, D_k$ counted from bottom to top. The trivial FG-module is denoted by $\mathbf{1}_G$ or simply $\mathbf{1}$.

We record two well-known general facts (for the first one see the explanations in [2, 5.1]).

Lemma 1.1. *Let G be a group, M be an FG-module and $H \lhd G$ be a normal subgroup with the index $[G:H]$ prime to p. Then the socle series for M and $M{\downarrow}_H$ coincide.*

Lemma 1.2. *Let X, Y, Z be FG-modules. Assume that $Z \subseteq X \oplus Y$ and that Z has a simple socle. Then either X or Y (or both) contains an isomorphic copy of Z as a submodule.*

Proof. Let $f : Z \to X \oplus Y$ be an embedding, and π_X, π_Y be projections of $X \oplus Y$ to X, Y, respectively. Then $\pi_X \circ f$ or $\pi_Y \circ f$ must be an injection, since otherwise both maps annihilate the simple socle of Z, which is impossible as $\pi_X \circ f + \pi_Y \circ f = f$ is injective. $\square$

Let λ be a partition of n. As in [11], we denote by Σ_λ, $M^\lambda = (\mathbf{1}_{\Sigma_\lambda}){\uparrow}^{\Sigma_n}$ and S^λ the corresponding Young subgroup, permutation module and Specht module, respectively. We will often use the description of the blocks of Σ_n, known as the *'Nakayama Conjecture'*, see [12, 2.7.41, 6.1.21].

Let $\sigma \in \Sigma_n$ be a transposition. Then $A_n \to A_n, g \mapsto \sigma g \sigma^{-1}$ is an (outer) automorphism of A_n. If M is an FA_n-module we can use the automorphism to twist the action of A_n on M:

$$g \cdot m = \sigma g \sigma^{-1} m,\ g \in A_n,\ m \in M.$$

This defines a new FA_n-module denoted by ${}^{\sigma}M$. One can easily check for the dual module: $({}^{\sigma}M)^* \cong {}^{\sigma}(M^*)$. The following lemma and corollary are well known and follow immediately from Clifford theory and definitions:

Lemma 1.3. *Let λ be a p-regular partition of n with $\lambda = \lambda^{\mathbf{M}}$. Then ${}^{\sigma}E^{\lambda}_{\pm} \cong E^{\lambda}_{\mp}$.*

Corollary 1.4. *Let λ be a p-regular partition of n with $\lambda = \lambda^{\mathbf{M}}$. Then we have isomorphisms of FA_n-modules:*

$$\begin{aligned} {}^{\sigma}\operatorname{Hom}_F(E^{\lambda}_{\pm}, E^{\lambda}_{\pm}) &\cong \operatorname{Hom}_F(E^{\lambda}_{\mp}, E^{\lambda}_{\mp}), \\ {}^{\sigma}\operatorname{Hom}_F(E^{\lambda}_{\pm}, E^{\lambda}_{\mp}) &\cong \operatorname{Hom}_F(E^{\lambda}_{\mp}, E^{\lambda}_{\pm}). \end{aligned}$$

The following known fact often helps to prove that some module is reducible:

Lemma 1.5. *Let G be a finite group. Then $\dim D \leq \sqrt{|G|}$ for any irreducible FG-module D.*

Another trivial tool which we occasionally employ, is the following

Lemma 1.6. *Suppose that E is an irreducible FA_n-module which is a reduction modulo p of some $\mathbb{C}A_n$-module $E_{\mathbb{C}}$. Then $E{\downarrow_G}$ is irreducible if and only if the restriction $E_{\mathbb{C}}{\downarrow_G}$ is an irreducible $\mathbb{C}G$-module whose reduction modulo p is also irreducible.*

Proof. It suffices to note that restriction to G and reduction modulo p commute. $\square$

A typical situation where Lemma 1.6 is useful is as follows. Recall that a partition is called a *p-core* if none of its hook lengths is divisible by p, see [12, § 2.7]. For p-cores we have $\lambda^{\mathbf{M}} = \lambda^t$, the *transpose* (or conjugate) partition, see for example [3, 2.1]. Now let $\lambda = \lambda^t$ be a Mullineux-fixed p-core partition. Consider the Specht module $S^{\lambda}_{\mathbb{C}}$ over $\mathbb{C}\Sigma_n$. This is an irreducible $\mathbb{C}\Sigma_n$-module, which splits into a direct sum of two irreducible modules under the restriction to A_n: $S^{\lambda}_{\mathbb{C}}{\downarrow_{A_n}} \cong T^{\lambda}_{\mathbb{C},+} \oplus T^{\lambda}_{\mathbb{C},-}$. By 'Nakayama's Conjecture', reduction modulo p of $S^{\lambda}_{\mathbb{C}}$ is the irreducible module D^{λ}, which splits into a direct sum of E^{λ}_{+} and E^{λ}_{-}. It follows that (for the proper choice of $+$ and $-$) reduction modulo p of the $\mathbb{C}A_n$-module $T^{\lambda}_{\mathbb{C},\pm}$ is the irreducible module $E^{\lambda}_{\pm}$. So the following result follows readily from Lemma 1.6.

Lemma 1.7. *Let $G < A_n$, and $\lambda = \lambda^t$ be a p-core. In the notation of the previous paragraph, $E^{\lambda}_{\pm}{\downarrow_G}$ is irreducible if and only if the restriction $T^{\lambda}_{\mathbb{C},\pm}{\downarrow_G}$ is an irreducible $\mathbb{C}G$-module whose reduction modulo p is also irreducible.*

The next result shows that partitions with few rows are usually not Mullineux-fixed. The proof uses the Mullineux algorithm a detailed description of which can be found e.g. in [10]. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ is a partition with $\lambda_h > 0$ then we write $h(\lambda)$ for h and call it the *height* of λ.

Lemma 1.8. *Let $n \geq 5$ and λ be a partition of n.*

(i) *If $p > 2$ and $h(\lambda) \leq 2$ then $h(\lambda^{\mathbf{M}}) \neq h(\lambda)$. In particular, $\lambda \neq \lambda^{\mathbf{M}}$.*

(ii) *If $p > 3$ and $h(\lambda) = 3$ then $\lambda^{\mathbf{M}} \neq \lambda$, except for the cases where $p > 5$ and λ is one of the following: $(3, 1^2)$, $(3, 2, 1)$, $(3^2, 2)$, (3^3).*

(iii) *If $p > 3$ and $h(\lambda) = 4$ then $\lambda^{\mathbf{M}} \neq \lambda$, except for the cases where $p > 7$ and λ is one of the following: $(4, 1^3)$, $(4, 2, 1^2)$, $(4, 3, 2, 1)$, $(4, 3^2, 1)$, $(4^2, 2^2)$, $(4^2, 3, 2)$, $(4^3, 3)$, (4^4) or $p = 5$, and λ is one of the following: $(4, 1^3)$, $(4, 2, 1^2)$, $(5, 3, 1^2)$, $(6, 3, 1^2)$.*

(iv) *Let $p > 5$ and $h(\lambda) = 5$. Then $\lambda^{\mathbf{M}} \neq \lambda$ unless $n \leq 25$. Moreover, the only Mullineux-fixed partition for $n = 16$ is $(5^2, 2^3)$ for $p > 7$.*

Proof. Let a be the number of nodes in the p-rim of λ. Set $\varepsilon = 0$ if $p|a$ and $\varepsilon = 1$ otherwise.

(i) If $h(\lambda) = 1$, use [11, 24.5(iii)]. Let $h(\lambda) = 2$. Then $h(\lambda^{\mathbf{M}}) = a + \varepsilon - 2$, see [3, 2.4]. Assume $h(\lambda^{\mathbf{M}}) = 2$. If $\varepsilon = 0$ then $a = 4$, which is a contradiction since $p > 2$ does not divide 4. Otherwise $a = 3$ and $p > 3$. But in this case the p-rim of λ coincides with its rim, which contradicts the assumption $n \geq 5$.

(ii) The height of $\lambda^{\mathbf{M}}$ is $a + \varepsilon - 3$. Assume this height is 3. If $\varepsilon = 0$ then $a = 6$, which is a contradiction since $p > 3$. So $a = 5$ and $p > 5$. In this case the p-rim of λ coincides with its rim and λ is a p-core, which gives the required list of exceptions.

(iii) As above we show that $a = 7$ and $p \neq 7$. Now the case $p > 7$ is similar to the case $p > 5$ in (ii). Let $p = 5$. Then either the Mullineux symbol of λ has just one column, which implies $\lambda = (4, 1^3)$, or the second column of the Mullineux symbol is $\binom{a'}{r'}$ with $r' < 4$. This shows that $\lambda^{(1)} := \lambda \setminus \{5\text{-rim}\}$ is a Mullineux-fixed partition with at most three rows. By (i) and (ii), $\lambda^{(1)}$ is a partition of $n' \leq 4$. Going through all possibilities for $\lambda^{(1)}$ we get the desired list.

(iv) As above, $a = 9$, and $\lambda \setminus \{p\text{-rim}\}$ is a Mullineux-invariant partition with at most four rows. The result now follows from (i)-(iii). $\square$

Now we describe the structure of some modules associated with two row partitions. Recall that the Specht module S^λ is by definition a submodule of the permutation module M^λ. Denote by $Y^{(n-k,k)}$ the block component of $M^{(n-k,k)}$ containing $S^{(n-k,k)}$.

Lemma 1.9. [3, 3.3] *Let $p > k \geq 0$ and $n \geq 2k$. If there exists l with $0 \leq l < k$ and $n \equiv k + l - 1 \pmod p$, then*

$$\begin{aligned} S^{(n-k,k)} &= D^{(n-l,l)}|D^{(n-k,k)}, \\ Y^{(n-k,k)} &= D^{(n-l,l)}|D^{(n-k,k)}|D^{(n-l,l)}, \text{ and} \\ Y^{(n-k,k)}/D^{(n-k,k)} &\cong (S^{(n-k,k)})^*. \end{aligned}$$

Otherwise $Y^{(n-k,k)} = S^{(n-k,k)} = D^{(n-k,k)}$.

Lemma 1.10. *Let $k \geq 0$, $p > \max(2, k)$, and $n \geq \max(2k, 5)$. If there exists l with $0 \leq l < k$ and $n \equiv k + l - 1 \pmod p$, then*

$$\begin{aligned} S^{(n-k,k)}{\downarrow_{A_n}} &= E^{(n-l,l)}|E^{(n-k,k)}, \\ Y^{(n-k,k)}{\downarrow_{A_n}} &= E^{(n-l,l)}|E^{(n-k,k)}|E^{(n-l,l)}, \text{ and} \\ (Y^{(n-k,k)}{\downarrow_{A_n}})/E^{(n-l,l)} &\cong (S^{(n-k,k)})^*{\downarrow_{A_n}}. \end{aligned}$$

Otherwise $Y^{(n-k,k)}{\downarrow_{A_n}} = S^{(n-k,k)}{\downarrow_{A_n}} = E^{(n-k,k)}$.

Proof. This follows from Lemmas 1.9, 1.1 and 1.8(i). $\square$

Now, we state some results about dimensions of irreducible FA_n-modules.

Lemma 1.11. *Let $p > 3$, $n \geq 5$, and $\lambda = \lambda^{\mathbf{M}}$. Then* $\dim E^\lambda_\pm \geq (n^3 - 9n^2 + 14n)/12$. *Moreover,* $\dim E^\lambda_\pm \geq 35, 21, 21, 10, 8$ *and* 3 *for* $n = 10, 9, 8, 7, 6$ *and* 5, *respectively.*

Proof. If $n \geq 13$ then, by [3, 1.20], either λ has at most 2 nodes outside of the first row or $\dim D^\lambda > (n^3 - 9n^2 + 14n)/6$. As $\dim E^\lambda_\pm = (\dim D^\lambda)/2$, the result for $n \geq 13$ now follows from Lemma 1.8(i)(ii). For $n \leq 12$ use e.g. [6, 14]. $\square$

Lemma 1.12. *Let $p > 2$, and λ be a p-regular partition of $n = 22, 23$ or 24 with $\lambda = \lambda^{\mathbf{M}}$. Then,* $\dim E^\lambda_\pm > 10395$.

Proof. This is a computer calculation involving a GAP program written by J. Brundan, see [3, 1.23] for more details. $\square$

2. Main technical results

Throughout this section G is a (fixed) subgroup of Σ_n, and D^λ is a (fixed) irreducible $F\Sigma_n$-module. For $k \geq 0$ we set

$$\begin{aligned} r_k(G) &:= \dim(M^{(n-k,k)})^G \quad \text{and} \\ d_k(\lambda) &:= \dim \operatorname{Hom}_{F\Sigma_n}(M^{(n-k,k)}, \operatorname{End}_F(D^\lambda)). \end{aligned}$$

By agreement, $r_{-1}(G) := 0$ and $d_{-1}(\lambda) := 0$. Note that we always have $r_0(G) = 1$ and $d_0(\lambda) = 1$. In fact,

$$r_k(G) = \sharp\{G\text{-orbits on unordered } k\text{-element subsets of } \{1, 2, \dots, n\}\}, \tag{1}$$

see e.g. [23, 1.5]. Also note that by Frobenius reciprocity,

$$d_k(\lambda) = \dim \operatorname{End}_{F[\Sigma_{n-k}\times\Sigma_k]}(D^\lambda{\downarrow_{\Sigma_{n-k}\times\Sigma_k}}). \tag{2}$$

The following lemma concerns with G-fixed points in $Y^{(n-k,k)}$ and $(S^{(n-k,k)})^*$.

Lemma 2.1. *Let $k \geq 0$, $p > k$, $n \geq 2k$, and $r_{k-1}(G) < r_k(G)$. Then*

$$(Y^{(n-k,k)})^G \neq 0 \quad \textit{and} \quad ((S^{(n-k,k)})^*)^G \neq 0.$$

If, moreover, $0 \leq l < k$ for some l with $n \equiv k + l - 1 \pmod p$ and $r_{l-1}(G) < r_l(G)$ then

$$\dim(Y^{(n-k,k)})^G \geq 2.$$

Proof. For $k = 0$ we have $Y^{(n-k,k)} = S^{(n-k,k)} \cong \mathbf{1}$ and the result follows. Let $k > 0$. By Lemmas 3.1 and 3.2 in [3], we have the exact sequence

$$0 \longrightarrow S^{(n-k,k)} \longrightarrow M^{(n-k,k)} \xrightarrow{f} M^{(n-k+1,k-1)} \longrightarrow 0. \tag{3}$$

Now the assumption $r_{k-1}(G) < r_k(G)$ implies $(S^{(n-k,k)})^G \neq 0$. By definition, $S^{(n-k,k)}$ is contained in $Y^{(n-k,k)}$, so $(S^{(n-k,k)})^G$ is contained in $(Y^{(n-k,k)})^G$, hence $(Y^{(n-k,k)})^G \neq 0$.

By dualizing the exact sequence above we get another exact sequence:

$$0 \longrightarrow (M^{(n-k+1,k-1)})^* \longrightarrow (M^{(n-k,k)})^* \longrightarrow (S^{(n-k,k)})^* \longrightarrow 0.$$

As $(M^\mu)^* \cong M^\mu$ for any μ, the assumption $r_{k-1}(G) < r_k(G)$ yields $((S^{(n-k,k)})^*)^G \neq 0$.

Finally, let $0 \leq l < k$ and $n \equiv k + l - 1 \pmod p$. Then, by Lemma 1.9, we have

$$Y^{(n-k,k)} = D^{(n-l,l)}|D^{(n-k,k)}|D^{(n-l,l)} = S^{(n-k,k)}|D^{(n-l,l)} \quad \text{and} \quad Y^{(n-l,l)} = D^{(n-l,l)}.$$

Now, from (3), $M^{(n-k,k)} = Y^{(n-k,k)} \oplus N$, where $N \cong M^{(n-k+1,k-1)}/D^{(n-l,l)}$. Therefore

$$\dim(Y^{(n-k,k)})^G = \dim(M^{(n-k,k)})^G - \dim(M^{(n-k+1,k-1)}/D^{(n-l,l)})^G. \tag{4}$$

Using Lemma 1.9 and the 'Nakayama conjecture' we deduce that $D^{(n-l,l)} = Y^{(n-l,l)}$ is a block component and hence a direct summand of $M^{(n-k+1,k-1)}$. By the first part of the lemma applied to $k = l$, we see that $r_{l-1}(G) < r_l(G)$ implies $(D^{(n-l,l)})^G \neq 0$. Therefore

$$\dim(M^{(n-k+1,k-1)}/D^{(n-l,l)})^G = r_{k-1}(G) - \dim(D^{(n-l,l)})^G \leq r_{k-1}(G) - 1 \leq r_k(G) - 2.$$

By (4), this implies $\dim(Y^{(n-k,k)})^G \geq 2$. $\square$

Recall the definition of JS-partitions from the introduction. The following result was first proved in [18] (cf. also [15, 8, 19]).

Theorem 2.2. [18, Theorem D] *The restriction $D^\lambda{\downarrow_{\Sigma_{n-1}}}$ is irreducible if and only if λ is a JS-partition.*

We need some relations between various $d_k(\lambda)$.

Proposition 2.3. *Let $0 \le l < k < p$, $n \ge 2k$, and D^λ be an irreducible $F\Sigma_n$-module with $h(\lambda) \ge k$ and $h(\lambda^{\mathbf{M}}) \ge k$. Then $d_l(\lambda) < d_k(\lambda)$ unless $(l,k) = (0,1)$ and λ is a JS-partition, in which case $d_0(\lambda) = d_1(\lambda) = 1$.*

Proof. By [3, 3.5], we have $d_l(\lambda) \le d_{l+1}(\lambda) \le \cdots \le d_k(\lambda)$. Moreover, by [3, Corollary 3.9], $d_k(\lambda) > d_{k-1}(\lambda)$, provided $k > 1$. Finally, $d_1(\lambda) = 1$ if and only if $D^\lambda{\downarrow}_{\Sigma_{n-1}}$ is irreducible, which by (2), Theorem 2.2 and self-duality of irreducible $F\Sigma_n$-modules, is equivalent to λ being a JS-partition. $\square$

Lemma 2.4. *Let $1 \le k < p$, $n \ge 2k$ and V be an $F\Sigma_n$-module. If*

$$\dim \mathrm{Hom}_{F\Sigma_n}(M^{(n-k,k)}, V) > \dim \mathrm{Hom}_{F\Sigma_n}(M^{(n-k+1,k-1)}, V).$$

then V contains either $Y^{(n-k,k)}$ or $(S^{(n-k,k)})^*$ (or both) as a submodule.

Proof. Consider the exact sequence (3). By assumption, there exists an $F\Sigma_n$-homomorphism $\theta : M^{(n-k,k)} \to V$ which does not factor through f. If the restriction $\theta|Y^{(n-k,k)}$ is injective then $Y^{(n-k,k)}$ is a submodule of V. Otherwise, in view of Lemma 1.9, the kernel of $\theta|Y^{(n-k,k)}$ is certain $D^{(n-l,l)}$ and $Y^{(n-k,k)}/D^{(n-l,l)} \cong (S^{(n-k,k)})^*$ embeds into V. $\square$

The next result is our main tool to test the restrictions of FA_n-modules for reducibility.

Theorem 2.5. *Let $1 \le l < k < p$, $n \ge \max(5, 2k)$, $G \le A_n$, and a p-regular partition $\lambda \vdash n$ satisfy $\lambda^{\mathbf{M}} = \lambda$ and $h(\lambda) \ge k$. Assume that*

$$r_{l-1}(G) < r_l(G),\ r_{k-1}(G) < r_k(G), \quad \text{and} \quad d_{l-1}(\lambda) < d_l(\lambda),\ d_{k-1}(\lambda) < d_k(\lambda).$$

Then $E^\lambda_+{\downarrow}_G$ and $E^\lambda_-{\downarrow}_G$ are reducible.

Proof. We establish the result for $E^\lambda_+{\downarrow}_G$, the proof for $E^\lambda_-{\downarrow}_G$ being similar. As $D^\lambda{\downarrow}_{A_n} = E^\lambda_+ \oplus E^\lambda_-$, we get an isomorphism of FA_n-modules

$$\mathrm{End}_F(D^\lambda){\downarrow}_{A_n} \cong \mathrm{End}_F(E^\lambda_+) \oplus \mathrm{End}_F(E^\lambda_-) \oplus \mathrm{Hom}_F(E^\lambda_+, E^\lambda_-) \oplus \mathrm{Hom}_F(E^\lambda_-, E^\lambda_+). \tag{5}$$

Let X denote either $Y^{(n-l,l)}{\downarrow}_{A_n}$ or $(S^{(n-l,l)})^*{\downarrow}_{A_n}$, and Z denote $Y^{(n-k,k)}{\downarrow}_{A_n}$ or $(S^{(n-k,k)})^*{\downarrow}_{A_n}$. By Lemma 2.4 and the assumption $d_l(\lambda) > d_{l-1}(\lambda)$, X is a submodule of $\mathrm{End}_F(D^\lambda){\downarrow}_{A_n}$. Moreover, X has a simple socle by Lemma 1.10, so Lemma 1.2 implies that X is a submodule of at least one of the four summands in the right hand side of (5). Moreover, ${}^\sigma(X) \cong X$ as X 'lifts' to Σ_n. Now, by Corollary 1.4, X is a submodule of both $\mathrm{End}_F(E^\lambda_+)$, $\mathrm{End}_F(E^\lambda_-)$ or X is a submodule of both $\mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)$, $\mathrm{Hom}_F(E^\lambda_-, E^\lambda_+)$. The same argument shows that a similar statement holds for Z. So we may assume that one of the following holds:

(a) $X \subseteq \mathrm{End}_F(E^\lambda_+)$, $\ Z \subseteq \mathrm{End}_F(E^\lambda_+)$;
(b) $X \subseteq \mathrm{End}_F(E^\lambda_+)$, $\ Z \subseteq \mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)$;
(c) $X \subseteq \mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)$, $\ Z \subseteq \mathrm{End}_F(E^\lambda_+)$;
(d) $X \subseteq \mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)$, $\ Z \subseteq \mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)$.

Moreover, $\mathbf{1}_{A_n} \subseteq \mathrm{End}_F(E^\lambda_+)$. Suppose that $E^\lambda_+{\downarrow}_G$ is irreducible. Then $\mathrm{End}_{FG}(E^\lambda_+{\downarrow}_G) \cong \mathrm{End}_F(E^\lambda_+{\downarrow}_G)^G$ is 1-dimensional by Schur's lemma. We consider the following two cases.

Case 1: (a), (b), or (c) holds. First assume $n \not\equiv l-1 \pmod p$ and $n \not\equiv k-1 \pmod p$. Then by Lemma 1.10, $\mathbf{1}_{A_n}$ is not in the socle of X and Z. So $\mathbf{1}_{A_n} \oplus X \subseteq \mathrm{End}_F(E^\lambda_+)$ or $\mathbf{1}_{A_n} \oplus Z \subseteq \mathrm{End}_F(E^\lambda_+)$. By Lemma 2.1, this implies $\dim \mathrm{End}_F(E^\lambda_+)^G \ge 2$, giving a contradiction.

Now assume that $n \equiv l-1 \pmod p$. Then $n \not\equiv k-1 \pmod p$ as $0 < k-l < p$ by assumption. By Lemma 1.10, $X = E^{(n-l,l)}|\mathbf{1}_{A_n}$ or $X = \mathbf{1}_{A_n}|E^{(n-l,l)}|\mathbf{1}_{A_n}$. In the former case, either $\mathbf{1}_{A_n} \oplus X \subseteq \mathrm{End}_F(E^\lambda_+)^G$ or $\mathbf{1}_{A_n} \oplus Z \subseteq \mathrm{End}_F(E^\lambda_+)^G$, giving a contradiction as above. In the latter case, $\dim X^G \ge 2$ by Lemma 2.1 (applied with $k = l$ and $l = 0$). If (a) or (b) holds, this shows $\dim \mathrm{End}_F(E^\lambda_+)^G \ge 2$, a contradiction. If (c) holds, we have $\dim \mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)^G \ge$

2, but this is impossible as $\mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)^G \cong \mathrm{Hom}_{FG}(E^\lambda_+{\downarrow_G}, E^\lambda_-{\downarrow_G})$, $E^\lambda_+{\downarrow_G}$ is irreducible by assumption, and $\dim E^\lambda_+ = \dim E^\lambda_-$. The case $n \equiv k-1 \pmod p$ is considered similarly.

Case 2: (d) holds. If $n \not\equiv k+l-1 \pmod p$ then X and Z are in different blocks by the 'Nakayama Conjecture'. So $X \oplus Z \subseteq \mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)$. By Lemma 2.1, X^G and Z^G are both non-trivial, hence $\dim \mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)^G \geq 2$, which is impossible as explained in the previous paragraph.

Finally, let $n \equiv k+l-1 \pmod p$. Then by Lemma 1.9, $X = E^{(n-l,l)}$, and either $Z = E^{(n-k,k)}|E^{(n-l,l)}$ or $Z = E^{(n-l,l)}|E^{(n-k,k)}|E^{(n-l,l)}$. In the former case we have $X \oplus Z \subseteq \mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)$, giving a contradiction as above. In the latter case, $\dim Z^G \geq 2$ by Lemma 2.1, whence $\dim \mathrm{Hom}_F(E^\lambda_+, E^\lambda_-)^G \geq 2$, giving a contradiction again. □

3. Reduction to 2-homogeneous subgroups

From now on we consider restrictions of an irreducible FA_n-module of the form $E^\lambda_\pm$ to subgroups $G < A_n$. In particular, it is always assumed that $\lambda^{\mathbf{M}} = \lambda$. Recall the constants $r_i(G)$ and their characterization (1).

We first consider the case where G is intransitive. Then, up to a conjugation, G is contained in a subgroup of the form

$$A_{(n-k,k)} := A_n \cap (\Sigma_{n-k} \times \Sigma_k)$$

for some $1 \leq k \leq n/2$. The following result is straightforward.

Lemma 3.1. *Let $n \geq 5$, $1 < k \leq n/2$, and $G = A_{(n-k,k)}$. Then $r_1(G) = 2$, $r_2(G) = 3$, $r_3(G) = 4$ for $k > 2$, and $r_3(G) = 3$ for $k = 2$.*

Proposition 3.2. *Let $p > 3$, $2 < k \leq n/2$, and $G \leq A_{(n-k,k)}$. Then $E^\lambda_+{\downarrow_G}$ and $E^\lambda_-{\downarrow_G}$ are reducible.*

Proof. We may assume that $G = A_{(n-k,k)}$. By Lemma 1.8(i), $h(\lambda) \geq 3$. So Proposition 2.3 yields $d_1(\lambda) < d_2(\lambda) < d_3(\lambda)$. Hence $E^\lambda_\pm{\downarrow_{A_{(n-k,k)}}}$ is reducible by Lemma 3.1 and Theorem 2.5. □

Now we turn to the case $k = 2$.

Proposition 3.3. *Let $p > 2$, $n \geq 5$, $G \leq A_{(n-2,2)}$, and assume that λ is not a JS-partition. Then $E^\lambda_+{\downarrow_G}$ and $E^\lambda_-{\downarrow_G}$ are reducible.*

Proof. We may assume that $G = A_{(n-2,2)}$. By Lemma 1.8(i), $h(\lambda) \geq 3$. So Proposition 2.3 gives $d_0(\lambda) < d_1(\lambda) < d_2(\lambda)$. By Lemma 3.1, we have $1 = r_0(G) < r_1(G) < r_2(G)$. Now apply Theorem 2.5. □

Proposition 3.3 shows in particular that for $A_{n-2} \leq G \leq A_{(n-2,2)}$, the restriction $E^\lambda_\pm{\downarrow_G}$ is irreducible only if λ is a JS-partition. We now work towards proving Theorem 3.7 below which establishes the converse of this fact. We will use results from [20]. Recall the combinatorial notions defined in the Introduction, in particular the notion of a normal node. A removable node is called *good* if it is the lowest among the normal nodes of a fixed residue. By definition there is at most one good node of each residue. If $A = (i, \lambda_i)$ is a removable node for $\lambda = (\lambda_1, \lambda_2, \dots)$ then

$$\lambda_A := \lambda \setminus \{A\} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots)$$

denotes the Young diagram of $n-1$ obtained from λ by removing A. We will use the following version of the modular branching rule.

Theorem 3.4. [20, Theorem 0.6] *Let D^λ be an arbitrary irreducible $F\Sigma_n$-module. The restriction $D^\lambda{\downarrow_{\Sigma_{n-1}}}$ is completely reducible if and only if every normal node of λ is good. In this case*

$$D^\lambda{\downarrow_{\Sigma_{n-1}}} \cong \oplus D^{\lambda_A},$$

where the sum is over all good nodes of λ.

We also need the following relation between the Mullineux bijection and good nodes:

Lemma 3.5. [21, 4.8] *Let λ be a p-regular partition, and A be a good node for λ of residue α. Then there exists a good node B for $\lambda^{\mathbf{M}}$ of residue $-\alpha$ such that $(\lambda_A)^{\mathbf{M}} = (\lambda^{\mathbf{M}})_B$.*

Now we apply the results above to prove the following

Lemma 3.6. *Let $p > 2$, $n \geq 5$, λ be a Mullineux-fixed JS-partition, and D^λ be the corresponding irreducible $F\Sigma_n$-module. Then*

$$D^\lambda \downarrow_{\Sigma_{n-2}\times\Sigma_2} = (D^\mu \otimes \mathbf{1}_{\Sigma_2}) \oplus (D^\nu \otimes \mathbf{sgn}_{\Sigma_2}),$$

where μ, ν are distinct p-regular partitions of $n-2$, obtained from λ by consecutive removal of two good nodes, and $\mu^{\mathbf{M}} = \nu$.

Proof. Let $n = k(p-1) + d$, with $0 \leq d < p-1$, and

$$\varepsilon_n := ((k+1)^d, k^{p-1-d}).$$

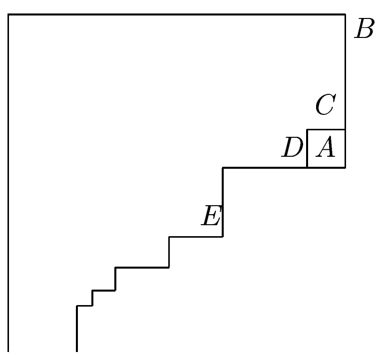
By [11, 24.5(iii)], $D^{\varepsilon_n} \cong \mathbf{sgn}$, so ε_n is not Mullineux-fixed. Hence $\lambda \neq \varepsilon_n$.

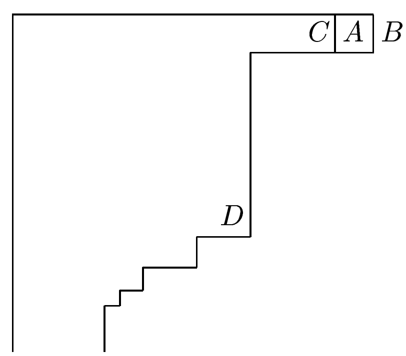
It follows from definitions and Theorem 3.4 that the top removable node A of λ is its only normal node and $D^\lambda{\downarrow_{\Sigma_{n-1}}} = D^{\lambda_A}$. Moreover,

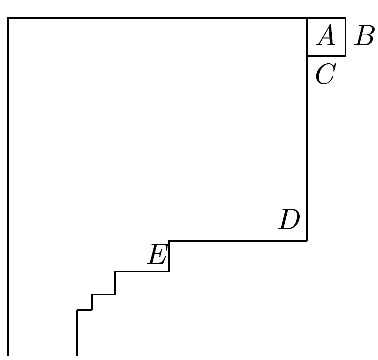
$$(D^\lambda \otimes \mathbf{sgn}){\downarrow_{\Sigma_{n-1}}} \cong (D^\lambda{\downarrow_{\Sigma_{n-1}}}) \otimes \mathbf{sgn} \tag{6}$$

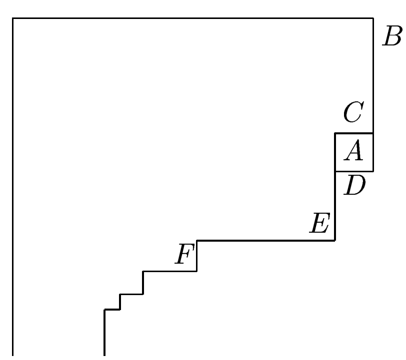
implies $\lambda_A = (\lambda_A)^{\mathbf{M}}$. So $\operatorname{res} A = 0$ in view of Lemma 3.5. Similarly, $(D^\lambda{\downarrow_{\Sigma_{n-2}}}) \otimes \mathbf{sgn} \cong D^\lambda{\downarrow_{\Sigma_{n-2}}}$. So if we prove that $D^\lambda{\downarrow_{\Sigma_{n-2}}} \cong D^\mu \oplus D^\nu$ and $\mu^{\mathbf{M}} \neq \mu$ then it will follow that $\mu^{\mathbf{M}} = \nu$. Moreover, the whole lemma will follow as by [3, 1.5], both $\mathbf{1}_{\Sigma_2}$ and $\mathbf{sgn}_{\Sigma_2}$ must appear in $D^\lambda{\downarrow_{\Sigma_2}}$.

We have the following four possibilities for λ :

(a) $\lambda =$ 

(b) $\lambda =$ 

(c) $\lambda =$ 

(d) $\lambda =$ 

Consider the case (a). Note that C is a normal node for λ_A. Moreover, D is a normal node for λ_A if and only if $\operatorname{res} D \neq \operatorname{res} B$. If this is the case, then E is not normal for λ_A as

λ is JS and so $\operatorname{res} E = \operatorname{res} B$. Similarly all lower removable nodes are not normal. Moreover, $\operatorname{res} C \neq \operatorname{res} D$ as $p > 2$. Thus C and D are the only normal nodes of λ_A and they have different residues. By Theorem 3.4 we now have $D^\lambda{\downarrow}_{\Sigma_{n-2}} \cong D^{(\lambda_A)_C} \oplus D^{(\lambda_A)_D}$. Next, observe that $\operatorname{res} C \neq \operatorname{res} A = 0$, so by Lemma 3.5 we have $(\lambda_A)_C \neq ((\lambda_A)_C)^{\mathbf{M}}$, as needed.

Now assume $\operatorname{res} D = \operatorname{res} B$. Then E must exists as $\lambda \neq \varepsilon_n$. Moreover, in this case $\operatorname{res} E = \operatorname{res} D = \operatorname{res} B$, and so E is normal. Moreover, $\operatorname{res} E \neq \operatorname{res} C$, and C, E are the only two normal nodes of λ_A. Now apply Theorem 3.4 and Lemma 3.5 as above.

Case (b). Here C and D are the only normal nodes for λ_A, and we use Theorem 3.4 and Lemma 3.5 as above. (Note that D exists as $\lambda \neq (n)$.)

Case (c). As λ is JS, we have $\operatorname{res} D = \operatorname{res} B$ and $\operatorname{res} E = \operatorname{res} C$ (note that E exists as $\lambda \neq \varepsilon_n$). It follows that λ_A is a p-regular partition whose only normal nodes are D and E, with $\operatorname{res} D \neq \operatorname{res} E$. Now apply Theorem 3.4 and Lemma 3.5 as above.

Case (d). As λ is JS, we have $\operatorname{res} E = \operatorname{res} B$ and $\operatorname{res} F = \operatorname{res} D$ (note that F exists since $\lambda \neq \varepsilon_n$). Now, C and F are the only normal nodes for λ_A and they have different residues since $p > 2$. It remains to use Theorem 3.4 and Lemma 3.5 as above. ☐

Theorem 3.7. *Let $p > 2$, $n \geq 5$ and $A_{n-2} \leq G \leq A_{(n-2,2)}$. Then $E^\lambda_\pm{\downarrow}_G$ is irreducible if and only if λ is a JS-partition.*

Proof. The 'only-if' part follows from Proposition 3.3, and the 'if' part from Lemma 3.6. ☐

Finally, we consider the subgroup A_{n-1}.

Proposition 3.8. *Let $p > 2$ and $n \geq 5$. The restriction $E^\lambda_\pm{\downarrow}_{A_{n-1}}$ is irreducible if and only if one of the following happens:*

- (i) *λ is a JS-partition, in which case $E^\lambda_\pm{\downarrow}_{A_{n-1}} \cong E^{\lambda_A}_\pm$ or $E^{\lambda_A}_\mp$, where A is the top removable node of λ;*
- (ii) *λ has exactly two normal nodes A, B, and $\operatorname{res} A \neq \operatorname{res} B$, in which case $E^\lambda_\pm{\downarrow}_{A_{n-1}} \cong E^{\lambda_A} \cong E^{\lambda_B}$.*

Proof. 'Only-if' part. By 1.3, $E^\lambda_+{\downarrow}A_{n-1}$ is irreducible if and only if $E^\lambda_-{\downarrow}A_{n-1}$ is irreducible. So $D^\lambda{\downarrow}_{A_{n-1}} \cong E^\lambda_+{\downarrow}_{A_{n-1}} \oplus E^\lambda_-{\downarrow}_{A_{n-1}}$ is semisimple. Now by Lemma 1.1, the restriction $D^\lambda{\downarrow}_{\Sigma_{n-1}}$ is semisimple. Moreover, if λ is not JS, then $D^\lambda{\downarrow}_{\Sigma_{n-1}}$ must have exactly two composition factors. Now use Theorem 3.4.

'If' part. If λ is JS then $E^\lambda_+{\downarrow}_{A_{n-1}}$ is irreducible by Theorem 3.7. Moreover, in this case $D^\lambda{\downarrow}_{\Sigma_{n-1}} \cong D^{\lambda_A}$ where A is the top removable node of λ, see Theorem 3.4. This proves that $E^\lambda_\pm{\downarrow}_{A_{n-1}} \cong E^{\lambda_A}_\pm$ or $E^{\lambda_A}_\mp$ as λ_A must be Mullineux-invariant by (6). Now, let λ be as in (ii). As $\lambda^{\mathbf{M}} = \lambda$, we get from Lemma 3.5 that $\operatorname{res} A = -\operatorname{res} B$, and so $\operatorname{res} A, \operatorname{res} B \neq 0$. By Lemma 3.5 again, λ_A and λ_B are not Mullineux-invariant. Hence $D^{\lambda_A}{\downarrow}_{A_{n-1}}$ and $D^{\lambda_B}{\downarrow}_{A_{n-1}}$ are irreducible, and one of them must be $E^\lambda_+{\downarrow}_{A_{n-1}}$, while the other must be $E^\lambda_-{\downarrow}_{A_{n-1}}$. ☐

Remark 3.9. Bessenrodt and Olsson [2] described the partitions satisfying (i) and (ii) in Proposition 3.8 in terms of Mullineux symbols.

Now we consider the case where G is a transitive subgroup of A_n. We will show in Theorem 3.14 that $E^\lambda_+{\downarrow}_G$ and $E^\lambda_-{\downarrow}_G$ are reducible, unless G is 2-homogeneous. Recall that for $1 \leq k \leq n/2$ a group G is called *k-homogeneous* if $r_k(G) = 1$. The following relation between various $r_k(G)$ is well known, see, for example, [7, 9.4A(i)]:

Lemma 3.10. *Let $G \leq \Sigma_n$. If $0 \leq k \leq l$ and $k + l \leq n$, then $r_k(G) \leq r_l(G)$.*

The proof of Theorem 3.14 relies on the next result of Cameron, Neumann and Saxl.

Theorem 3.11. [5, §5, Corollary] *Let $n > 5$, $G < \Sigma_n$ be transitive, and $r_2(G) = r_3(G)$. Then one of the following holds:*

(i) *n is even and G has 2 blocks of imprimitivity of size $n/2$.*
(ii) *n is even and G has $n/2$ blocks of imprimitivity of size 2.*
(iii) *G is 3-homogeneous.*

If n is even, set $m = n/2$. In the cases (i) and (ii) of Theorem 3.11 the group G is contained in the wreath products $\Sigma_m \wr \Sigma_2$ and $\Sigma_2 \wr \Sigma_m$, respectively (see e.g. [7, 2.6]). So we need to study restrictions to such wreath products or, more precisely, their intersections with A_n. The next result if straightforward.

Lemma 3.12. *Let $n = 2m \geq 8$, and $G = (\Sigma_m \wr \Sigma_2) \cap A_n$ or $G = (\Sigma_2 \wr \Sigma_m) \cap A_n$. Then $r_1(G) = 1$, $r_2(G) = r_3(G) = 2$, and $r_4(G) \geq 3$.*

Now we show that restrictions to the wreath products above are reducible.

Proposition 3.13. *Let $p > 3$, $n = 2m \geq 8$, and $G = (\Sigma_m \wr \Sigma_2) \cap A_n$ or $G = (\Sigma_2 \wr \Sigma_m) \cap A_n$. Then the restrictions $E^\lambda_+{\downarrow}_G$ and $E^\lambda_-{\downarrow}_G$ are reducible.*

Proof. If $h(\lambda) \geq 4$ the result follows from Lemma 3.12, Proposition 2.3 and Theorem 2.5. So by Lemma 1.8, $\lambda = (3^2, 2)$, which is a p-core. Moreover the group G does not appear in the list of [25, Theorem 2] where all irreducible restriction in characteristic 0 are given. So for the corresponding module in characteristic p the result follows from Lemma 1.7. ☐

Now we are able to reduce the transitive case to the 2-homogeneous case:

Theorem 3.14. *Let $p > 3$, $n \geq 7$, and $G < A_n$ be transitive. If $E^\lambda_+{\downarrow}_G$ or $E^\lambda_-{\downarrow}_G$ is irreducible then G is 2-homogeneous.*

Proof. Assume G is not 2-homogeneous, i.e. $r_2(G) > 1 = r_1(G)$. By Theorem 3.11, either $r_3(G) > r_2(G)$ or n is even and G is contained in one of the subgroups $(\Sigma_m \wr \Sigma_2) \cap A_n$, $G = (\Sigma_2 \wr \Sigma_m) \cap A_n$. In the latter case $E^\lambda_\pm{\downarrow}_G$ is reducible by Proposition 3.13. In the former case, we have $r_1(G) < r_2(G) < r_3(G)$. Since $h(\lambda) \geq 3$ by Lemma 1.8(i), we can apply Proposition 2.3 and Theorem 2.5 to conclude that $E^\lambda_\pm{\downarrow}_G$ is reducible. ☐

4. 2-homogeneous subgroups

Now we consider 2-homogeneous subgroups $G < A_n$. Our goal is to prove the following

Theorem 4.1. *Let $p > 3$, $n \geq 7$, and $G < A_n$ be 2-homogeneous. Then $E^\lambda_\pm{\downarrow}_G$ is irreducible if and only if $p > 5$ and either $\lambda = (3^2, 2)$ and G is $\mathrm{AGL}(3,2)$ or $\lambda = (3^3)$ and G is $\mathrm{P\Gamma L}(2,8)$.*

The proof will occupy the rest of the section. We are going to use the classification of 2-homogeneous groups. The list of 2-transitive groups convenient for our purposes can be found in [16]. There are 'very few' 2-homogeneous groups which are not 2-transitive, and all of them are subgroups of certain $\mathrm{A\Gamma L}(1,q)$'s, see [17]. As in Theorem 4.1 we have $n \geq 7$, we will keep this assumption throughout the section.

On representations of $\mathbf{AGL}(m,q)$ and exceptional cases in small heights. Assume that $p = \operatorname{char} F$ does not divide a prime power q. We describe a parametrization of irreducible $F\,\mathrm{AGL}(m,q)$-modules, referring the reader to [27, 4, 3] for more details. Let V_m be the m-dimensional $\mathbb{F}_q$-vector space with basis $\{v_1, v_2, \dots, v_m\}$ and the natural action of $\mathrm{GL}(m,q)$. We have $\mathrm{AGL}(m,q) = V_m \rtimes \mathrm{GL}(m,q)$. Denote by $\operatorname{Irr} V_m$ the set of irreducible characters $V_m \to F^*$ and by $\bar{0} \in \operatorname{Irr} V_m$ the trivial character. Fix a non-trivial character $\chi_0 : (\mathbb{F}_q, +) \to F^\times$. For any element θ of the $\mathbb{F}_q$-linear dual V_m^*, let ε_θ denote the character sending $v \in V_m$ to $\chi_0(\theta(v)) \in F^*$. We let $\theta_1, \dots, \theta_m$ denote the basis of V_m^* dual to $v_1, \dots, v_m$, and set $\varepsilon_i := \varepsilon_{\theta_i} \in \operatorname{Irr} V_m$. The group $\mathrm{GL}(m,q)$ acts on $\operatorname{Irr} V_m$ by

$$(g\varepsilon)(v) = \varepsilon(g^{-1}v), \quad g \in \mathrm{GL}(m,q),\ v \in V_m,\ \varepsilon \in \operatorname{Irr} V_m.$$

Under this action, the trivial character $\bar{0} \in \operatorname{Irr} V_m$ is fixed by $\mathrm{GL}(m,q)$ and all other characters are permuted transitively. The centralizer $C_{\mathrm{GL}(m,q)}(\varepsilon_m)$ is isomorphic to $\mathrm{AGL}(m-1,q)$, embedded into $\mathrm{GL}(m,q)$ as the subgroup of matrices of the form

$$\left[\begin{array}{ccc|c} & * & & * \\ & & & \\ \hline 0 & \dots & 0 & 1 \end{array}\right].$$

We have a chain of subgroups

$$\mathrm{GL}(1,q) \subset \mathrm{AGL}(1,q) \subset \mathrm{GL}(2,q) \subset \mathrm{AGL}(2,q) \subset \dots \subset \mathrm{GL}(m,q) \subset \mathrm{AGL}(m,q).$$

Given an $F\,\mathrm{AGL}(m,q)$-module M and $\varepsilon \in \operatorname{Irr} V_m$, let

$$M_\varepsilon = \{m \in M \mid vm = \varepsilon(v)m \text{ for all } v \in V_m\}$$

be the corresponding 'weight space'. Since FV_m is a semisimple algebra, M decomposes as a direct sum of such weight spaces. Observe that the action of $\mathrm{GL}(m,q)$ on M induces an action on the weight spaces, so that $gM_\varepsilon = M_{g\varepsilon}$ for all $g \in \mathrm{GL}(m,q), \varepsilon \in \operatorname{Irr} V_m$. In particular, M_0 is stable under the action of $\mathrm{GL}(m,q)$, while M_{ε_n} is stable under the action of the subgroup $\mathrm{AGL}(m-1,q)$, since $\mathrm{AGL}(m-1,q)$ centralizes ε_m.

Now, let L be any irreducible $F\,\mathrm{AGL}(m-1,q)$-module. Consider the restriction $L{\downarrow}_{V_m}$. If V_m acts trivially on L, the action of $\mathrm{AGL}(m,q)$ factors through the surjection $\mathrm{AGL}(m,q) \to \mathrm{GL}(m,q)$ to give an irreducible $\mathrm{GL}(m,q)$-module. If this module is X, we denote L by $L(X;0)$. Next, assume that V_m does not act trivially on L. Then it not hard to see that L_{ε_m} is an irreducible $F\,\mathrm{AGL}(m-1,q)$-module in a natural way, and

$$L \cong (L_{\varepsilon_n}){\uparrow}_{V_m.\mathrm{AGL}(m-1,q)}^{\mathrm{AGL}(m,q)}$$

If V_{m-1} acts trivially on L_{ε_m}, this module factors through to give an irreducible $F\,\mathrm{AGL}(m-1,q)$-module, say X. In this case we denote our L by $L(X;1)$. Otherwise iterate by taking $(L_{\varepsilon_m})_{\varepsilon_{m-1}}$, etc. In this way the module L will be labelled by $L(X;j)$, where $0 \le j \le m-1$ and X is an irreducible $FGL(m-j,q)$-module. If we do the same construction over $\mathbb{C}$ we will get a similar parametrization. To distinguish from the modular case we use the notation $L_{\mathbb{C}}(X_{\mathbb{C}};j)$.

Now we can deal with the exceptional cases in Lemma 1.8(ii).

Lemma 4.2. *Let λ and p be as in one of the exceptional cases in Lemma 1.8(ii), and G be a transitive subgroup of A_n. Then $E^\lambda_\pm{\downarrow}_G$ is irreducible if and only $\lambda = (3^2,2)$ and G is* $\mathrm{AGL}(3,2)$ *or $\lambda = (3^3)$ and G is* $\mathrm{P\Gamma L}(2,8)$.

Proof. Note that λ is a p-core, so by Lemma 1.7 we only have to deal with the subgroups G for which the corresponding modules restrict irreducibly in characteristic 0. Checking the list in [25, Theorem 2], we see that $\lambda = (3^2,2)$, $G = \mathrm{AGL}(3,2)$ and $\lambda = (3^3)$, $G = \mathrm{P\Gamma L}(2,8)$ are the only two cases. By the hook formula, in both cases we have $\dim E^\lambda_\pm = 21$. Now one checks using [6, p. 6] and [14, p. 6] that the 21-dimensional $\mathbb{C}\mathrm{P\Gamma L}(2,8)$-module is irreducible modulo $p > 5$. By Lemma 1.7, the restriction to $\mathrm{P\Gamma L}(2,8)$ is indeed irreducible in characteristic p. Finally, a 21-dimensional irreducible $\mathbb{C}\,\mathrm{AGL}(3,2)$-module must be of the form $L_{\mathbb{C}}(X_{\mathbb{C}};j)$, where either $j=1$ and $\dim X_{\mathbb{C}} = 3$ or $j=2$ and $\dim X_{\mathbb{C}} = 1$. Let X be a reduction modulo p of $X_{\mathbb{C}}$. Then X is irreducible as the order of the corresponding group is not divisible by p. It follows that reduction modulo p of $L_{\mathbb{C}}(X_{\mathbb{C}};j)$ is the irreducible module $L(X;j)$. So, by Lemma 1.7, the restriction to $\mathrm{AGL}(3,2)$ is indeed irreducible. ∎

It is now convenient to establish a special case of the 'if' part of the Main Theorem:

Lemma 4.3. *Let $p > 5$, $\lambda = (3^3)$, and $G = \mathrm{AGL}(3,2)$ be embedded into $A_8 < A_9$ via its permutation representation of degree 8. Then $E^\lambda_\pm{\downarrow}_G$ is irreducible.*

Proof. It follows from Theorem 3.4 that $E^\lambda_\pm{\downarrow}_{A_8} \cong E^{(3^2,2)}_\pm$ or $E^{(3^2,2)}_\mp$. Now use Lemma 4.2. □

Lemma 4.2 treats the cases where $\lambda = \lambda^{\mathbf{M}}$ and $h(\lambda) = 3$. Moreover $\lambda \neq \lambda^{\mathbf{M}}$ for $h(\lambda) \leq 2$ by Lemma 1.8(i). So from now on we may assume that

$$h(\lambda) \geq 4. \tag{7}$$

Groups with a regular normal subgroup. A 2-homogeneous group G in this class is always a subgroup of the group $\mathrm{A\Gamma L}(m,q)$ of all non-degenerate semilinear affine transformations of the affine space $V = \mathbb{F}_q^m$ acting naturally on the q^m points of V. We have $n = q^m$, and $|\,\mathrm{A\Gamma L}(m,q)| = fq^{m(m+1)/2}(q^m-1)(q^{m-1}-1)\dots(q-1)$, if $q = \ell^f$ for a prime ℓ.

Assume first that $m = 1$. Then $q = n \geq 7$. We have $|G| \leq |\,\mathrm{A\Gamma L}(1,q)| = fq(q-1)$. By Lemma 1.5, if $E^\lambda_\pm{\downarrow}_G$ is irreducible then $\dim E^\lambda_\pm \leq \sqrt{fq(q-1)}$, which contradicts Lemma 1.11. So from now on we may assume that $m \geq 2$.

If, moreover, $q > 2$ and $(m,q) \neq (2,3)$, then for $G = \mathrm{A\Gamma L}(m,q)$ we have $r_4(G) > r_3(G) > r_2(G)$, see [3, §5]. Hence $E^\lambda_\pm{\downarrow}_G$ is reducible by (7), Theorem 2.5 and Proposition 2.3. Now, let $(m,q) = (2,3)$. Then $|G| \leq |\,\mathrm{A\Gamma L}(2,3)| = 432$, and so $E^\lambda_\pm{\downarrow}_G$ is reducible by Lemmas 1.5 and 1.11.

It remains to prove that $E^\lambda_\pm{\downarrow}_G$ is reducible for $G = \mathrm{AGL}(m,2)$. If $m = 3$ then $n = 8$, and $\lambda = \lambda^{\mathbf{M}}$ implies that either $\lambda = (3^2,2)$ and $p \neq 5$ or $\lambda = (4,2,1^2)$ and $p \neq 7$. The former case is treated in Lemma 4.2. In the latter one, λ is a p-core and we find using the hook formula that $\dim E^\lambda_\pm = 45$, so $E^\lambda_\pm{\downarrow}_G$ is reducible by virtue of Lemma 1.5.

So let $m \geq 4$. Then it is shown for example in [5, §4, p.166] that $r_4(G) > r_3(G)$ and $r_6(G) > r_5(G)$. Now, if $p > 5$ and $h(\lambda) \geq 6$, we may apply Proposition 2.3 and Theorem 2.5. Otherwise we may assume that $m = 4$, $p > 7$, and $\lambda = (4^4)$ or $(5^2,2^3)$, thanks to Lemma 1.8. Now, λ is a p-core, and $E^\lambda_\pm{\downarrow}_G$ is reducible by dimensions, using the hook formula and Lemma 1.5.

The case $p = 5$ for $\mathrm{AGL}(m,2)$ turns out to be harder so we consider it separately in the next subsection.

The group $\mathbf{AGL(m,2)}$ in characteristic 5. Throughout the subsection, $p = 5$, $G = \mathrm{AGL}(m,2)$, $n = 2^m$, $k = n/2$. Until Corollary 4.7 we do not assume that $\lambda = \lambda^{\mathbf{M}}$.

For $0 \leq j \leq 2$, set

$$H_j(n) := \{D^{(n-i,1^i)}, D^{(n-i,1^i)} \otimes \mathbf{sgn} \mid 0 \leq i \leq j\}.$$

Also set $H_2'(n) := H_2(n) \setminus H_0(n)$.

Lemma 4.4. *Let $n \geq 6$, and D^λ be an irreducible $F\Sigma_n$-module. If every composition factor of the restriction $D^\lambda{\downarrow}_{\Sigma_{n-1}}$ belongs to $H_2(n-1)$ then $D^\lambda \in H_2(n)$.*

Proof. By tensoring D^λ with $\mathbf{sgn}$ if necessary, we may assume that one of the modules $D^{(n-1)}, D^{(n-2,1)}, D^{(n-3,1^2)}$ is in the socle of $D^\lambda{\downarrow}_{\Sigma_{n-1}}$. Now repeat the proof of [3, 5.2]. □

Lemma 4.5. *Let $p = 5$, $m \geq 3$, $n = 2^m$, $k = n/2$, and λ be a p-regular partition of n. Assume that every composition factor of $D^\lambda{\downarrow}_{\Sigma_k\times\Sigma_k}$ has one of the following forms:*

(1) $D^\mu \otimes D^\mu$ for $D^\mu \in H_1(k)$.

(2) $D^\mu \otimes D^\nu$ or $D^\nu \otimes D^\mu$ with $D^\mu \in H_2'(k) \cup \{D^{(5,3)}, D^{(5,3)} \otimes \mathbf{sgn}\}$ and $D^\nu \in H_0(k) \cup \{D^{(2,2)}\}$. Then $D^\lambda \in H_2(n) \cup \{D^{(5,3)}, D^{(5,3,3)} \otimes \mathbf{sgn}\}$.

Proof. If $m > 4$, it follows from the assumption that any composition factor of $D^\lambda{\downarrow}_{\Sigma_k}$ belongs to $H_2(k)$. So $D^\lambda \in H_2(n)$ by Lemma 4.4.

The cases $m = 4$ and $m = 3$ are similar to those in Lemma 5.3 of [3]. □

Recall that $\mathrm{AGL}(m,2) = V_m \rtimes \mathrm{GL}(m,2)$. So, in particular, $V_m < \Sigma_n$.

Proposition 4.6. *Let $m \geq 2$, and D^λ be an irreducible $F\Sigma_n$-module. Then the fixed point space $(D^\lambda)^{V_m} = 0$ if and only if $D^\lambda \in H_2'(n)$ or $n = 8$ and $D^\lambda \cong D^{(5,3)}$ or $D^{(5,3)} \otimes \mathbf{sgn}$.*

Proof. The proof is similar to that of Lemma 5.6 of [3] but uses Lemma 4.5 instead of Lemma 5.3 there. ☐

Corollary 4.7. *Let $\lambda = \lambda^{\mathbf{M}}$. Then $E^\lambda_\pm{\downarrow}_{\mathrm{AGL}(m,2)}$ is reducible.*

Proof. It follows from Proposition 4.6 and Lemma 1.8(i)(ii) that $(D^\lambda)^{V_m} \neq 0$. So $(E^\lambda_+)^{V_m} \neq 0$ or $(E^\lambda_-)^{V_m} \neq 0$. We claim that actually both invariant spaces are non-trivial. To prove this, it suffices to verify that $E^\lambda_+{\downarrow}_{V_m} \cong E^\lambda_-{\downarrow}_{V_m}$. Note that all non-trivial elements of V_m belong to the same A_n-conjugacy class, the class corresponding to the cycle-shape (2^k). Now it follows from Lemma 1.3 that the Brauer characters of the two restrictions coincide. Since the modules are completely reducible, this implies the required isomorphism.

Now, note that $(E^\lambda_\pm)^{V_m} \neq 0$ is $\mathrm{GL}(m,2)$-invariant. So if $E^\lambda_\pm{\downarrow}_{\mathrm{AGL}(m,2)}$ is irreducible then $(E^\lambda_\pm)^{V_m} = E^\lambda_\pm$, which contradicts the faithfulness of the representation. ☐

Groups G with $\mathbf{PSL}(m,q) \leq G \leq \mathbf{P\Gamma L}(m,q)$, $m \geq 2$. The group $\mathrm{P\Gamma L}(m,q)$ acts naturally on the points of $\mathbb{P}^{m-1}(\mathbb{F}_q)$, yielding a permutation representation of degree $n = (q^m-1)/(q-1)$. If we twist this permutation representation with an outer automorphism of G, we get another permutation representation, non-equivalent to the first one. However, in both cases we get the same subgroup of A_n, and for the problem we are considering we may assume without loss of generality that the action is natural.

We consider the case $m = 2$ first. We have $n = q+1$ and $|G| \leq |\,\mathrm{P\Gamma L}(2,q)| = fq(q^2-1)$, where $q = \ell^f$ for a prime ℓ. Using Lemmas 1.5 and 1.11, we may assume that $q = 9$ or $q = 8$.

If $q = 9$ we have $\mathrm{P\Gamma L}(2,9) \cong \mathrm{PSL}(2,9)\,.2^2$, see [6]. By [6, 14], the dimensions of irreducible $F\,\mathrm{PSL}(2,9)$-modules are among 1, 5, 8, 9, 10. It follows that the dimensions of irreducible $F\,\mathrm{P\Gamma L}(2,9)$-modules, exceeding 34, can only be $4 \cdot 10 = 40$ and $4 \cdot 9 = 36$. Now we can apply Lemma 1.11 and [6, 14].

If $q = 8$, then $n = 9$ and so $\lambda = \lambda^{\mathbf{M}}$ implies that either $\lambda = (5,1^4)$ or $\lambda = (3^3)$ and $p > 5$. The latter case has been treated in Lemma 4.2. For the former case, note that $(5,1^4)$ is a p-core for any $p > 5$, so using the hook formula we get $\dim E^\lambda_\pm = 35$. On the other hand, by [6, 14], the group $\mathrm{P\Gamma L}(2,8)$ does not have irreducible modules of dimension 35.

Let $m = 3$ and $q = 2$. By [6, 14], the irreducible modules over $\mathrm{P\Gamma L}(3,2) \cong \mathrm{GL}_3(2)$ have dimension at most 8, and we can apply Lemma 1.11.

If $m \geq 3$ and $(m,q) \neq (3,2)$, then for the group $G = \mathrm{P\Gamma L}(m,q)$ we have $r_4(G) > r_3(G) > r_2(G)$, see [3, §5]. Hence $E^\lambda_\pm{\downarrow}_G$ is reducible by (7), Theorem 2.5 and Proposition 2.3.

The group $G = \mathbf{Sp}(2m,2)$, $m \geq 3$. In this case G has two 2-transitive representations Ω^0 and Ω^1 of degrees $2^{m-1}(2^m+1)$ and $2^{m-1}(2^m-1)$, respectively. It is proved in [3, 5.10,5.11] that $r_4(G) > r_3(G) > r_2(G)$. So we can apply Theorem 2.5 and Proposition 2.3 as above.

The groups $\mathbf{M_{23}}$ and $\mathbf{M_{24}}$. The result follows by dimensions using Lemma 1.12 and [6, 14].

The remaining 2-homogeneous groups. In the remaining cases one can easily check the dimensions by using Lemmas 1.11, 1.5 and [6, 14]. (One may consult [3, §5] for a similar problem for symmetric groups).

5. Proof of the Main Theorem

If $\lambda \neq \lambda^{\mathbf{M}}$ then $E^\lambda = D^\lambda{\downarrow}_{A_n}$, and the result follows from [3, Main Theorem]. So we may assume that $\lambda = \lambda^{\mathbf{M}}$. Now the 'if' part follows from Theorem 3.7, Proposition 3.8 and Lemmas 4.2, 4.3. For the 'only-if' part, suppose that $E^\lambda_+{\downarrow}_G$ is irreducible (the argument for E^λ_- being similar). We may assume that $n \geq 8$. Indeed, we only have to deal with the cases where $p \leq n$, as otherwise one can use the characteristic 0 result [25]. For $5 \leq n \leq 7$ and $p \leq n$, the only Mullineux-fixed partition is $(4,1^3)$ for $p = 5$, see Lemma 1.8. But this partition is a 5-core and so this case can be dealt with using Lemma 1.7 and [25] or just using [6, 14]. Now, by Theorem 3.14 and Proposition 3.2, either G is 2-homogeneous or $G \le A_{(n-k,k)}$ for $k \le 2$. If G is 2-homogeneous we apply Theorem 4.1. So let $G \le A_{(n-k,k)}$ for $k \le 2$.

Assume first that $k = 2$. Recall that $A_{(n-2,2)} = A_n \cap (\Sigma_{n-2} \times \Sigma_2)$. Let $\sigma \in \Sigma_n$ be the transposition which generates the subgroup Σ_2 in $\Sigma_{n-2} \times \Sigma_2$. For $g \in \Sigma_{n-2}$ let $\varepsilon(g)$ be 1 if g is odd and 0 otherwise. Then

$$\varphi : \Sigma_{n-2} \to A_{(n-2,2)},\ g \mapsto (g, \sigma^{\varepsilon(g)})$$

is an isomorphism. As $G < A_{(n-2,2)}$, the restriction $E^\lambda_\pm \downarrow_{A_{(n-2,2)}}$ is irreducible and via the isomorphism φ it can be considered as an irreducible $F\Sigma_{n-2}$-module, D^γ say, where γ is a p-regular partition of $n-2$. By Theorem 3.7, λ is a JS-partition. It follows from Lemma 3.6 that γ is obtained from λ by consecutively removing two good nodes. Applying [3, Main Theorem] to the group Σ_{n-2} we see that one of the following happens: (a) G contains A_{n-2}, (b) γ or $\gamma^{\mathbf{M}}$ belongs to the set of partitions $X := \{(n-3,1), (n-4,2), (n-4,1^2), (21,2,1), (21,1^3), (5,3)\}$ or (c) $G < A_{(n-3,3)}$. But (c) is impossible by Proposition 3.2. If (b) happens then using Lemma 3.5 we may assume that λ is obtained from a partition γ in X by adding two nodes. Using Lemma 1.8, we conclude that there are no such Mullineux-fixed Jantzen-Seitz partitions λ. So (b) is also impossible.

Now let $k = 1$, i.e. $G \le A_{n-1}$. We may assume that $G \not\le A_{(n-2,2)}$. Then we have to prove that $G = A_{n-1}$, unless $G = \mathrm{AGL}(3,2)$ embedded as in Main Theorem (ii)(c) and $\lambda = (3^3)$. Since the restriction $E^\lambda_+\downarrow_{A_{n-1}}$ is irreducible, Proposition 3.8 implies that either (1) λ is a JS-partition, in which case $E^\lambda_+\downarrow_{A_{n-1}} \cong E^{\lambda_A}_\pm$ or $E^{\lambda_A}_\mp$, where A is the top removable node of λ, or (2) λ has exactly two normal nodes A,B, and $\operatorname{res} A \ne \operatorname{res} B$, in which case $E^\lambda_+\downarrow_{A_{n-1}} \cong E^{\lambda_A} \cong E^{\lambda_B}$. As $G \not\le A_{(n-2,2)}$, Proposition 3.2 implies that G is transitive in A_{n-1}. Consider the case (1). By Theorems 3.14 and 4.1, either $\lambda_A = (3^2,2)$ and G is $\mathrm{AGL}(3,2)$ or $\lambda_A = (3^3)$ and G is $\mathrm{P\Gamma L}(2,8)$. By Theorem 3.4, only the former case is possible, in which case we get $\lambda = (3^3)$ and $G = \mathrm{AGL}(3,2)$, as required. Finally, in case (2) it follows from [3, Main Theorem] that λ_A or λ_B belongs to the set X defined in the previous paragraph. Using Lemma 1.8, we conclude that there are no Mullineux-fixed partitions λ which satisfy (2) and have λ_A in X.

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