REPRESENTATIONS OF THE SYMMETRIC GROUP ARE REDUCIBLE OVER SINGLY TRANSITIVE SUBGROUPS

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Introduction

Let F be an algebraically closed field of characteristic $p \geq 0$ and $\Sigma_n = \mathrm{Sym}(\Omega)$ be the symmetric group on an n-element set Ω. We are interested in the following

Problem. For any n describe all pairs (G, D) where $G \leq \Sigma_n$ is a subgroup and D is a simple $F\Sigma_n$-module of dimension greater than 1 such that the restriction D_G is irreducible.

If the characteristic of F is zero, this problem has been solved by Saxl [16]. An important feature of Saxl's result is that a group G as in the problem above is either 2-transitive or fixes a point, i.e. is contained in some Σ_{n-1}. The case $G \leq \Sigma_{n-1}$ can then be settled using the branching rule and induction. On the other hand, an explicit list of k-transitive groups (for $k \geq 2$) is available, which can be used to complete the proof in characteristic zero, see [16] for more details.

From now on we assume that $p > 0$. In this case the problem is important for determining maximal subgroups of finite classical groups [1],[13]. However, the situation is now more complicated. For example, to determine the pairs (G, D) as above with $G = A_n$ one needs the Mullineux conjecture [7], [2].

If G is intransitive then, up to a conjugation, it is contained in a standard Young subgroup of the form $\Sigma_{n-k} \times \Sigma_k$. The irreducible restrictions from Σ_n to Σ_{n-1} have been described in [14], see also [11], [6]. In [11] it is also shown for $p > 2$ that the Young subgroups $\Sigma_{n-k} \times \Sigma_k$ with $1 < k < n-1$ never act irreducibly on a simple $F\Sigma_n$-module of dimension > 1 (this fact is also proved in this paper, see Theorem 3.10). This reduces the intransitive case to the subgroups of Σ_{n-1}. The main result here is

Main Theorem. Let $p > 2, n \geq 4$, $G \leq \Sigma_n$, and D be a simple $F\Sigma_n$-module with $\dim D > 1$. If the restriction D_G is irreducible then either $G \leq \Sigma_{n-1}$ or G is 2-transitive on Ω.

We refer the reader to Theorem 3.10 for a partial result in the case $p = 2$.

For $p > 2$ the problem above is thus reduced to 2-transitive groups. This turns out to be a very important step in a complete solution of the problem, which is obtained in [3] (for the case $p > 3$).

1. Preliminaries

Let Ω be a finite set with $|\Omega| = n$, and $k \leq n/2$. A group $G \leq \mathrm{Sym}(\Omega)$ is called ***k*-homogeneous** (resp. ***k*-transitive**) on Ω if G acts transitively on the unordered (resp. ordered) k-element subsets of Ω. Moreover, G is called ***k*-homogeneous*** on Ω if G is k-homogeneous but not k-transitive on Ω.

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Let F be an algebraically closed field of characteristic $p > 0$, G be a group, and V be an FG-module. Denote by V^G the space of G-fixed points in V. We write $V \cong V_1|\cdots|V_k$ if V admits a *unique* filtration with sections isomorphic to $V_j, 1 \leq j \leq k$, counted from bottom to top. In particular, if the sections are irreducible this means that V is uniserial. If $H \leq G$ is a subgroup, then V_H or $V{\downarrow_H}$ denotes the restriction of V to H. If W is an FH-module then $W{\uparrow^G}$ denotes the induced module. If $V \cong V^*$ as FG-modules then $V_H \cong (V_H)^*$ as FH-modules. We let $\mathbf{1}{=}\mathbf{1}_G$ denote the trivial representation of G. If V and W are FG-modules we write $\mathrm{Hom}_G(V, W)$ (resp. $\mathrm{Hom}(V, W)$) for the space of all FG-homomorphisms (resp. F-linear maps) from V to W. Note that $\mathrm{Hom}(V, W)$ is an FG-module with $\mathrm{Hom}_G(V, W) \cong \mathrm{Hom}(V, W)^G$.

Basic facts on the representation theory of the symmetric group can be found in [8]. For $\lambda = (\lambda_1, \ldots, \lambda_r)$, a composition of n with non-zero parts, we let $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_r}$ denote the corresponding Young subgroup of Σ_n. We may identify Σ_{n-1} with $\Sigma_{(n-1,1)}$. We let Y^λ, S^λ and $M^\lambda = (\mathbf{1}_{\Sigma_\lambda}){\uparrow^{\Sigma_n}}$ denote the Young, Specht, and permutation modules labelled by λ, respectively. If λ is p-regular we write D^λ for the unique irreducible quotient of S^λ. It is well known that M^λ, Y^λ, and D^λ are all self-dual $F\Sigma_n$-modules. Let **sgn** denote the sign representation of Σ_n.

We will need the structure of the permutation modules $M^{(n-1,1)}$ and $M^{(n-2,2)}$. The proof of the next three lemmas is obtained by applying [8, 17.17, 24.15] and the 'Nakayama Conjecture' [10, 6.1.21, 2.7.41].

Lemma 1.1. *The module $M^{(n-1,1)}$ is isomorphic to $D^{(n-1,1)} \oplus \mathbf{1}$ if $n \not\equiv 0 \pmod p$, and $\mathbf{1}|D^{(n-1,1)}|\mathbf{1} \cong S^{(n-1,1)}|\mathbf{1} \cong \mathbf{1}|(S^{(n-1,1)})^*$ otherwise.*

Lemma 1.2. *Let $p > 2$ and $n \geq 4$.*

(i) *If $n \not\equiv 1, 2 \pmod p$ then $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus M^{(n-1,1)}$ where $Y^{(n-2,2)} \cong S^{(n-2,2)} \cong D^{(n-2,2)}$.*

(ii) *If $n \equiv 1 \pmod p$ then $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus D^{(n-1,1)}$ where $Y^{(n-2,2)} \cong \mathbf{1}|D^{(n-2,2)}|\mathbf{1} \cong \mathbf{1}|(S^{(n-2,2)})^*$.*

(iii) *If $n \equiv 2 \pmod p$ then $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus \mathbf{1}$ where*

$$Y^{(n-2,2)} \cong D^{(n-1,1)}|D^{(n-2,2)}|D^{(n-1,1)} \cong D^{(n-1,1)}|(S^{(n-2,2)})^*.$$

We will only need the case $p = 2$ when n is odd:

Lemma 1.3. *Let $p = 2$ and $n \geq 4$ be odd.*

(i) *If $n \equiv 1 \pmod 4$ then $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus D^{(n-1,1)}$ where $Y^{(n-2,2)} \cong \mathbf{1}|D^{(n-2,2)}|\mathbf{1} \cong \mathbf{1}|(S^{(n-2,2)})^*$.*

(ii) *If $n \equiv 3 \pmod 4$ then $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus M^{(n-1,1)}$ where $Y^{(n-2,2)} \cong D^{(n-2,2)}$.*

The following easy result is well known.

Lemma 1.4. *Let X be a G-set, and M be the corresponding permutation FG-module. Then $\dim M^G$ equals the number of G-orbits on X.*

Proof. It suffices to consider the case where G is transitive on Ω. Then $M = (1_H){\uparrow^G}$ where H is a point stabilizer. But $\dim \mathrm{Hom}_G(\mathbf{1}_G, (\mathbf{1}_H){\uparrow^G}) \cong \dim \mathrm{Hom}_H(\mathbf{1}_H, \mathbf{1}_H) = 1$ by Frobenius reciprocity. □

For $1 \leq k \leq n$ denote by d_k the number of G-orbits on the unordered k-element subsets of Ω. The following lemma characterizes d_k in terms of the corresponding permutation module.

Lemma 1.5. *Let $1 \leq k \leq n$ and $G \leq \Sigma_n$ be a subgroup. Then $\dim(M^{(n-k,k)})^G = d_k$.*

Proof. Note that $M_G^{(n-k,k)}$ is the permutation FG-module on the k-element subsets. Now we can use Lemma 1.4. $\square$

Lemma 1.6. *If $G \leq \Sigma_n$ and $n \not\equiv 0 \pmod p$ then $\dim(D^{(n-1,1)})^G = d_1 - 1$.*

Proof. This follows from Lemmas 1.1 and 1.5. $\square$

Lemma 1.7. *Let $n \geq 4, G \leq \Sigma_n$ and n be odd if $p = 2$. Suppose $d_1 < d_2$. Then $(Y^{(n-2,2)})^G$ and $((S^{(n-2,2)})^*)^G$ are non-zero. Moreover if $p > 2$ and $n \equiv 1 \pmod p$, or $p = 2$ and $n \equiv 1 \pmod 4$, then $\dim(Y^{(n-2,2)})^G \geq 2$.*

Proof. By Lemmas 1.2 and 1.3 (or [8, 17.17]), there is an exact sequence

$$0 \to M^{(n-1,1)} \to M^{(n-2,2)} \to (S^{(n-2,2)})^* \to 0,$$

which implies that $((S^{(n-2,2)})^*)^G \neq 0$, by assumption and Lemma 1.5. By 1.2(i) and 1.3(ii), $Y^{(n-2,2)} = (S^{(n-2,2)})^*$, unless $p > 2$, $n \equiv 1, 2 \pmod p$ or $p = 2$, $n \equiv 1 \pmod 4$. If $p > 2$ and $n \equiv 2 \pmod p$ then by Lemma 1.2(iii), $\dim(Y^{(n-2,2)})^G = d_2 - 1 \geq d_1 > 0$. In the remaining cases, using Lemma 1.6, we get $\dim(Y^{(n-2,2)})^G = d_2 - (d_1 - 1) \geq 2$. $\square$

Lemma 1.8. *Let $G \leq \Sigma_n$ be transitive and $n \not\equiv 0 \pmod p$. Then G is 2-transitive if and only if $\dim \mathrm{End}_G(D_G^{(n-1,1)}) = 1$.*

Proof. Note that G is 2-transitive on Ω if and only if G has exactly two orbits on $\Omega \times \Omega$. However, the corresponding permutation module is isomorphic to $M_G^{(n-1,1)} \otimes M_G^{(n-1,1)}$. By Lemma 1.4, we now have that G is 2-transitive if and only if $\dim(M_G^{(n-1,1)} \otimes M_G^{(n-1,1)})^G = 2$. By Lemmas 1.6 and 1.1, this is equivalent to $\dim(D^{(n-1,1)} \otimes D^{(n-1,1)})^G = 1$, which by self-duality of irreducible modules is equivalent to our claim. $\square$

2. 2^*-homogeneous groups

The main result of this section is Theorem 2.5 which shows that the restriction of a simple $F\Sigma_n$-module D of dimension > 1 to a 2^*-homogeneous subgroup is reducible. We will use a theorem of Kantor [12] (see also [4, 9.4B]) which describes 2^*-homogeneous groups. Let $\Omega = \mathbb{F}_q$ be a finite field of order $q = r^e$ for a prime r. Let $\mathrm{Aut}_{\mathbb{F}_r}(\mathbb{F}_q)$ denote the Galois group, which is a cyclic group C_e of order e. Let $\mathrm{A\Sigma L}_1(q)$ (resp. $\mathrm{ASL}_1(q)$) be the group of all transformations of $\mathbb{F}_q$ of the form $x \mapsto a^2x^\sigma + b$ (resp. $x \mapsto a^2x + b$) with $a \in \mathbb{F}_q^*, b \in \mathbb{F}_q$ and $\sigma \in \mathrm{Aut}_{\mathbb{F}_r}(\mathbb{F}_q)$. These are permutation groups on the elements of $\mathbb{F}_q$.

Proposition 2.1. [12] *Let G be a 2^*-homogeneous group. Then, up to permutation isomorphism, $\mathrm{ASL}_1(q) \leq G \leq \mathrm{A\Sigma L}_1(q)$ with $q \equiv 3 \pmod 4$, $q > 3$.*

Let $R_n(m)$ denote the class of simple $F\Sigma_n$-modules D such that for some p-regular partition $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq n - m$, we have $D \cong D^\lambda$ or $D \cong D^\lambda \otimes \mathbf{sgn}$. We need the following results from [9] and the main results in [17],[18].

Proposition 2.2. [9, Theorem 7] *If $n \geq 15$ and D is a simple $F\Sigma_n$-module, then either $\dim D > \frac{1}{2}(n-1)(n-2)$ or $D \in R_n(2)$.*

Proposition 2.3. [9, p.420] *If $D \in R_n(2) \setminus R_n(1)$ then $\dim D \geq \frac{1}{2}n(n-5)$. If $D \in R_n(1)$ then $\dim D \geq n - 2$.*

Proposition 2.4. [17], [18], [9, Theorem 6(i)] *If $n \geq 7$ and D is a simple $F\Sigma_n$-module, then either $\dim D > n - 1$ or $D \in R_n(1)$.*

Proposition 2.5. *Let G be a 2*-homogeneous subgroup of Σ_n and D be a simple $F\Sigma_n$-module with* $\dim D > 1$. *Then the restriction D_G is reducible.*

Proof. In view of Proposition 2.1 we may assume $G = \mathrm{A\Sigma L}_1(q) < \Sigma_q$, $q \equiv 3 \pmod 4$, and $q > 3$. In particular, $q \geq 7$ and $r > 2$. Let D_G be irreducible. Then

$$\dim D \leq \sqrt{|G|} = \sqrt{q(q-1)e/2}. \tag{1}$$

Moreover, $e = \log_r q < q - 5$ as $q \geq 7$ and $r > 2$. So

$$\dim D < q\sqrt{(q-5)/2} \leq \frac{1}{2}q(q-5). \tag{2}$$

If $q = 7$ or $q = 11$ then $e = 1$ and we get a contradiction by (1) and Propositions 2.4, 2.3. Let $q \geq 15$. Then from (2) and Propositions 2.2, 2.3 we conclude that $D \in R_n(1)$, i.e. $D \cong D^{(n-1,1)}$ or $D \cong D^{(n-1,1)} \otimes \mathbf{sgn}$. If D_G is irreducible, we have by Schur's lemma

$$1 = \dim \mathrm{End}_G(D_G) = \dim \mathrm{End}_G(D_G^{(n-1,1)}).$$

Now, if p does not divide q then G is 2-transitive on $\mathbb{F}_q$ by Lemma 1.8, giving a contradiction. Otherwise the restriction of D to $\mathbb{F}_q \lhd G$ is semisimple by Clifford's theorem. But the trivial module is the only simple module over the group $\mathbb{F}_q$, as $\mathbb{F}_q$ is a p-group. Hence $\mathbb{F}_q$ acts trivially on D. This contradicts the fact that D is faithful over Σ_q. $\square$

3. Main Results

Denote by σ the transposition $(n-1, n) \in \Sigma_n$. We will write $\Sigma_{n-2,2}$ for the Young subgroup $\Sigma_{(n-2,2)} \cong \Sigma_{n-2} \times \Sigma_2$. For an $F\Sigma_n$-module V and a homomorphism $\theta \in \mathrm{End}_{\Sigma_{n-1}}(V{\downarrow_{\Sigma_{n-1}}})$ define a map $\hat{\theta} : V \to V$ by setting

$$\hat{\theta}(v) := \theta(v) + \sigma\theta(\sigma v) \quad (v \in V).$$

Lemma 3.1. *Let $p > 2$ and $n \geq 4$. Then the map $\theta \mapsto \hat{\theta}$ is an injective map from* $\mathrm{End}_{\Sigma_{n-1}}(V{\downarrow_{\Sigma_{n-1}}})$ *to* $\mathrm{End}_{\Sigma_{n-2,2}}(V{\downarrow_{\Sigma_{n-2,2}}})$.

Proof. It is routine that $\hat{\theta} \in \mathrm{End}_{\Sigma_{n-2,2}}(V{\downarrow_{\Sigma_{n-2,2}}})$. Assume $\hat{\theta} = 0$. Then $\theta(\sigma v) = -\sigma\theta(v)$ for all $v \in V$. But θ is also an $F\Sigma_{n-1}$-homomorphism. So

$$\theta(v) = \theta\big(\big(\sigma(1, n-1)\big)^3 v\big) = -(\sigma(1, n-1))^3\theta(v) = -\theta(v),$$

whence $\theta(v) = 0$ for any $v \in V$. $\square$

Lemma 3.2. *Let $p > 2$, and W be a $F\Sigma_4$-module with $((1,4) + (2,3) - (1,3) - (2,4))W = 0$. Then A_4 acts trivially on W.*

Proof. By assumption we have $((1,3) + (2,4))v = ((1,4) + (2,3))v$ for all $v \in W$. Conjugating by $(2,3)$ we get

$$((1,2) + (3,4))v = ((1,3) + (2,4))v = ((1,4) + (2,3))v. \tag{3}$$

The group algebra $F[\Sigma_2 \times \Sigma_2]$ is semisimple and has four irreducible modules: $\mathbf{1}_{\Sigma_2} \otimes \mathbf{1}_{\Sigma_2}$, $\mathbf{1}_{\Sigma_2} \otimes \mathbf{sgn}_{\Sigma_2}$, $\mathbf{sgn}_{\Sigma_2} \otimes \mathbf{1}_{\Sigma_2}$, and $\mathbf{sgn}_{\Sigma_2} \otimes \mathbf{sgn}_{\Sigma_2}$. We claim that only $\mathbf{1}_{\Sigma_2} \otimes \mathbf{1}_{\Sigma_2}$ and $\mathbf{sgn}_{\Sigma_2} \otimes \mathbf{sgn}_{\Sigma_2}$ may appear in the restriction $W{\downarrow_{\Sigma_2 \times \Sigma_2}}$. Indeed, assume for example that $\mathbf{1}_{\Sigma_2} \otimes \mathbf{sgn}_{\Sigma_2}$ appears in this restriction. Pick a non-zero vector v in the corresponding isotypic component. Then $(1,2)v = v$ and $(3,4)v = -v$. So $(1,2)(3,4)v = -v$ and $((1,2) + (3,4))v = 0$. By (3), we have $(1,3)v = -(2,4)v$ and $(1,4)v = -(2,3)v$. Hence

$$(1,3)(2,4)v = -v \quad \text{and} \quad (1,4)(2,3)v = -v. \tag{4}$$

But $(1,2)(3,4) = (1,3)(2,4)(1,4)(2,3)$. So (4) implies $(1,2)(3,4)v = v$, and we have arrived to a contradiction. The case of $\mathbf{sgn}_{\Sigma_2} \otimes \mathbf{1}_{\Sigma_2}$ is considered similarly.

As the element $(1,2)(3,4)$ acts trvially on both $\mathbf{1}_{\Sigma_2} \otimes \mathbf{1}_{\Sigma_2}$ and $\mathbf{sgn}_{\Sigma_2} \otimes \mathbf{sgn}_{\Sigma_2}$, it must act trivially on W. By conjugation, we conclude that the whole Klein group

$$H := \{1,\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\}$$

acts trivially on W. As $(1,2) \equiv (3,4) \pmod H$ and $(2,3) \equiv (1,4) \pmod H$, the equation (3) implies $(1,2)v = (2,3)v$. Hence $(1,2,3)v = v$, and so A_4 acts trivially on W. $\square$

Theorem 3.3. *Let $p > 2$ and $n \geq 4$. Assume that V is an $F\Sigma_n$-module such that the alternating group $A_n < \Sigma_n$ does not act trivially on V. Then*

$$\dim \mathrm{End}_{\Sigma_{n-1}}(V{\downarrow}_{\Sigma_{n-1}}) < \dim \mathrm{End}_{\Sigma_{n-2,2}}(V{\downarrow}_{\Sigma_{n-2,2}}).$$

Proof. By Lemma 3.1 it is enough to demonstrate an endomorphism $\psi \in \mathrm{End}_{\Sigma_{n-2,2}}(V{\downarrow}_{\Sigma_{n-2,2}})$ which is not in the image of $\theta \mapsto \hat{\theta}$. Set $\psi(v) := \sigma v$, $v \in V$. Assume that $\psi = \hat{\theta}$ for some $\theta \in \mathrm{End}_{\Sigma_{n-1}}(V{\downarrow}_{\Sigma_{n-1}})$. Then

$$\theta(\sigma v) = v - \sigma\theta(v) \quad (v \in V). \tag{5}$$

For $i = 1, 2, \dots, n-2$ denote $\pi_i := (i, n-1)$ and $\tau_i := (i, n)$. Then, using (5), we get

$$\theta(\pi_i\sigma\pi_i v) = \pi_i\theta(\sigma\pi_i v) = \pi_i(\pi_i v - \sigma\theta(\pi_i v)) = v - \pi_i\sigma\pi_i\theta(v).$$

Note that $\pi_i\sigma\pi_i = \tau_i$ so we have proved that

$$\theta(\tau_i v) = v - \tau_i\theta(v), \quad i = 1, 2, \dots, n-2 \quad (v \in V). \tag{6}$$

Using (5), (6) and the equality $\pi_i = \tau_i\sigma\tau_i$, we get for any $v \in V$ and $1 \leq i \leq n-2$

$$\begin{aligned}
\pi_i\theta(v) &= \theta(\pi_i v) = \theta(\tau_i\sigma\tau_i v) = \sigma\tau_i v - \tau_i\theta(\sigma\tau_i v)\\
&= \sigma\tau_i v - \tau_i(\tau_i v - \sigma\theta(\tau_i v)) = \sigma\tau_i v - v + \tau_i\sigma(v - \tau_i\theta(v))\\
&= \sigma\tau_i v - v + \tau_i\sigma v - \tau_i\sigma\tau_i\theta(v) = \sigma\tau_i v - v + \tau_i\sigma v - \pi_i\theta(v).
\end{aligned}$$

Solving for $\pi_i\theta(v)$ and multiplying by π_i, we find

$$\theta(v) = (1/2)(\sigma v - \tau_i\sigma\tau_i v + \tau_i v), \quad i = 1, 2, \dots, n-2 \quad (v \in V). \tag{7}$$

Taking $i = 1$ and $i = 2$, we have

$$(1/2)(\sigma v - \tau_1\sigma\tau_1 v + \tau_1 v) = (1/2)(\sigma v - \tau_2\sigma\tau_2 v + \tau_2 v) \quad (v \in V).$$

Conjugating by $(3, n-1)(4, n)$, we get

$$((1,4) + (2,3) - (1,3) - (2,4))V = 0.$$

By Lemma 3.2, the natural subgroup $A_4 < \Sigma_n$ acts trivially on V, which implies by conjugation that A_n acts trivially on V, giving a contradiction. $\square$

Now we study the case $p = 2$. If $n = 2l$ is even we write S for the irreducible module $D^{(l+1,l-1)}$ and if $n = 2l+1$ is odd we write S for $D^{(l+1,l)}$. We call S the *spinor* representation of Σ_n. The following result is proved in [15], see also [5] for a generalization.

Lemma 3.4. *Let $p = 2$, $n \geq 3$, and D be an irreducible $F\Sigma_n$-module different from the spinor representation S. Then for any 3-cycle $\gamma \in \Sigma_n$ there exists a non-zero vector $v \in D$ such that $\gamma v = v$.*

Proof. It is proved in [15] that the only irreducible representations of the alternating group A_n for which 3 cycles act fixed-point-freely come from the restriction of the spinor module to A_n. This implies the result as these irreducible A_n-modules do not appear in the restriction of D^λ to A_n, unless $D^\lambda \cong S$. $\square$

Lemma 3.5. *Let $p = 2$, $n \geq 3$, $b, c \in F$, and D be an irreducible $F\Sigma_n$-module. If $b \neq 0$ and*

$$(b(1,2) + c(1,2,3) + c(1,3,2) + c)v = 0$$

for all $v \in D$ then D is the trivial module $\mathbf{1}_{\Sigma_n}$.

Proof. By conjugating with $(2,3)$ we also get

$$(b(1,3) + c(1,2,3) + c(1,3,2) + c)D = 0,$$

so $(1,2)v = (1,3)v$, hence $(1,2,3)v = v$ for all $v \in D$. By conjugating we also get $(1,3,2)v = v$ for any v. Now it follows from the assumption that $(1,2)$ acts trivially on D, whence any transposition acts trivially. □

Theorem 3.6. *Let $p = 2$, $n \geq 4$, and D be a non-trivial irreducible $F\Sigma_n$-module. Then*

$$\dim \mathrm{End}_{\Sigma_{n-1}}(D{\downarrow_{\Sigma_{n-1}}}) < \dim \mathrm{End}_{\Sigma_{n-2,2}}(D{\downarrow_{\Sigma_{n-2,2}}}),$$

unless n is odd and D is the spinor module S.

Proof. We first note that the kernel of the linear map $\theta \mapsto \hat{\theta}$ defined in the beginning of this section is 1-dimensional and is spanned by the identity map id_D. Indeed, $\hat{\theta}(v) = 0$ is equivalent to $\theta(\sigma v) = \sigma\theta(v)$. But θ is also an $F\Sigma_{n-1}$-homomorphism. Hence θ is an $F\Sigma_n$-endomorphism of D so it must be proportional to id_D, by Schur's lemma.

Let $\mathrm{id}_D, \theta_1, \dots, \theta_k$ be a basis of the vector space $\mathrm{End}_{\Sigma_{n-1}}(D{\downarrow_{\Sigma_{n-1}}})$. Define a map $\psi : D \to D$ by setting $\psi(v) = \sigma v$, $v \in D$. Then ψ is an element of $\mathrm{End}_{\Sigma_{n-2,2}}(D{\downarrow_{\Sigma_{n-2,2}}})$. We claim that

$$\hat{\theta}_1,\ \hat{\theta}_2,\ \dots,\ \hat{\theta}_k,\ \mathrm{id}_D,\ \psi$$

are linearly independent elements of $\mathrm{End}_{\Sigma_{n-2,2}}(D{\downarrow_{\Sigma_{n-2,2}}})$. This of course implies the theorem.

A linear dependence between our endomorphisms looks like

$$a_1\hat{\theta}_1 + a_2\hat{\theta}_2 + \cdots + a_k\hat{\theta}_k + b\,\mathrm{id}_D + c\psi \equiv 0 \tag{8}$$

with $a_i, b, c \in F$. We may assume that $a_i \neq 0$ for some i, since otherwise we would get a linear dependence between id_D and ψ, which is only possible if D is trivial. Let $\theta := a_1\theta_1 + \cdots + a_k\theta_k$. Then $a_1\hat{\theta}_1 + \cdots + a_k\hat{\theta}_k = \hat{\theta}$, and we may rewrite (8) as

$$\hat{\theta} + b\,\mathrm{id}_D + c\psi \equiv 0.$$

By the first paragraph of the proof, we have $(b, c) \neq (0, 0)$. The last equality is equivalent to

$$\theta(\sigma v) = \sigma\theta(v) + b\sigma v + cv. \qquad (v \in D)$$

Let $\pi_i = (i, n-1)$, $\tau_i = (i, n) \in \Sigma_n$. Then $\tau_i = \pi_i\sigma\pi_i$. So for any $v \in D$ we have

$$\begin{aligned} \theta(\tau_i v) &= \theta(\pi_i\sigma\pi_i v) = \pi_i\theta(\sigma\pi_i v) = \pi_i(\sigma\theta(\pi_i v) + b\sigma\pi_i v + c\pi_i v) \\ &= \pi_i\sigma\theta(\pi_i v) + b\pi_i\sigma\pi_i v + cv = \tau_i\theta(v) + b\tau_i v + cv. \qquad (v \in D) \end{aligned}$$

As $\pi_i = \tau_i\sigma\tau_i$ we also have for any $v \in D$

$$\begin{aligned} \pi_i\theta(v) &= \theta(\pi_i v) = \theta(\tau_i\sigma\tau_i v) = \tau_i\theta(\sigma\tau_i v) + b\tau_i\sigma\tau_i v + c\sigma\tau_i v \\ &= \tau_i(\sigma\theta(\tau_i v) + b\sigma\tau_i v + c\tau_i v) + b\tau_i\sigma\tau_i v + c\sigma\tau_i v \\ &= \tau_i\sigma\theta(\tau_i v) + b\tau_i\sigma\tau_i v + cv + b\tau_i\sigma\tau_i v + c\sigma\tau_i v \\ &= \tau_i\sigma(\tau_i\theta(v) + b\tau_i v + cv) + cv + c\sigma\tau_i v \\ &= \pi_i\theta(v) + b\pi_i v + c\tau_i\sigma v + cv + c\sigma\tau_i v, \end{aligned}$$

whence $b\pi_i v + c(\tau_i\sigma + \sigma\tau_i + 1)v = 0$. In particular, $c \neq 0$. If $b \neq 0$ we have

$$(b\pi_i + c\tau_i\sigma + c\sigma\tau_i + c)v = 0. \qquad (v \in D)$$

Conjugating we get

$$(b(1,2)+c(1,2,3)+c(1,3,2)+c)v=0. \qquad (v \in D)$$

Lemma 3.5 now leads to a contradiction as $D \not\cong \mathbf{1}_{\Sigma_n}$. So we may assume that $b=0$. Then we get the identity

$$(\tau_i\sigma+\sigma\tau_i+1)v=0. \qquad (v \in D)$$

But $\tau_i\sigma =: \gamma$ is a 3-cycle and $\sigma\tau_i=\gamma^2$. So it suffices to show that there is a non-zero $v \in D$ such that $\gamma v = v$. But this follows from Lemma 3.4, provided $D \not\cong S$. Finally if $D \cong S$ and $n=2l$ is even, then $S = D^{(l+1,l-1)}$. By [14], $D^{(l+1,l-1)}{\downarrow_{\Sigma_{n-1}}} \cong D^{(l,l-1)}$ and $D^{(l,l-1)}{\downarrow_{\Sigma_{n-2,2}}}$ is reducible (and self-dual). So the Hom-spaces under consideration have dimensions 1 and 2, and the theorem is true. □

Corollary 3.7. *Let $n \geq 4$ and D be an irreducible $F\Sigma_n$-module with $\dim D > 1$. If $p=2$ and n is odd, assume that $D \not\cong S$. Then*

$$\dim \operatorname{Hom}_{\Sigma_n}(M^{(n-2,2)}, \operatorname{End}(D)) > \dim \operatorname{Hom}_{\Sigma_n}(M^{(n-1,1)}, \operatorname{End}(D)).$$

Proof. Follows from Theorems 3.3, 3.6 and the isomorphisms

$$\operatorname{Hom}_{\Sigma_n}(M^{\nu}, \operatorname{End}(D)) \cong \operatorname{Hom}_{\Sigma_\nu}(\mathbf{1}_{\Sigma_\lambda}, \operatorname{End}(D)) \cong \operatorname{End}_{\Sigma_\lambda}(D{\downarrow_{\Sigma_\lambda}}).$$

□

Lemma 3.8. *Let $n \geq 4$ and D be an irreducible $F\Sigma_n$-module with $\dim D > 1$. If $p = 2$, assume additionally that n is odd and $D \not\cong S$. Then $\operatorname{End}(D)$ contains either $Y^{(n-2,2)}$, $(S^{(n-2,2)})^*$ or both as submodules.*

Proof. Assume first that $p>2$, $n \not\equiv 1,2 \pmod p$ or $p=2$, $n \equiv 3 \pmod 4$. Then the result follows from Lemmas 1.2(i), 1.3(ii) and Corollary 3.7. Now let $p>2$ and $n \equiv 1 \pmod p$. Then $M^{(n-1,1)}$ splits as $\mathbf{1} \oplus D^{(n-1,1)}$. By Lemma 1.2(ii), there is a surjection $M^{(n-2,2)} \to M^{(n-1,1)}$, and by Corollary 3.7, there must exist a homomorphism $\theta : M^{(n-2,2)} \to \operatorname{End}(D)$ which does not factor through this surjection. If the restriction $\theta|Y^{(n-2,2)}$ is injective then $Y^{(n-2,2)}$ is a submodule of $\operatorname{End}(D)$. Otherwise, in view of Lemma 1.2(ii), the kernel of $\theta|Y^{(n-2,2)}$ is $\mathbf{1}$. But $Y^{(n-2,2)}/\mathbf{1} \cong (S^{(n-2,2)})^*$, so $(S^{(n-2,2)})^*$ embeds into $\operatorname{End}(D)$. The cases $p>2$, $n \equiv 2 \pmod p$ and $p=2$, $n \equiv 1 \pmod 4$ are considered similarly to the case $n \equiv 1 \pmod p$ using 1.2(iii), 1.3(i) and 3.7. □

Recall that d_1 and d_2 denote the number of G-orbits on Ω and the 2-subsets of Ω, respectively.

Theorem 3.9. *Let $n \geq 4$, $G \leq \Sigma_n$ be a subgroup, and D be a simple $F\Sigma_n$-module with $\dim D > 1$. If $p=2$, assume additionally that n is odd and $D \not\cong S$. Then the restriction D_G is reducible whenever $d_1 < d_2$.*

Proof. Suppose D_G is irreducible and $d_1 < d_2$. Then $\operatorname{End}_G(D_G) \cong \operatorname{End}(D)^G$ is 1-dimensional by Schur's lemma. We consider the following two cases.

Case 1: $p>2$, $n \not\equiv 1 \pmod p$ or $p=2$, $n \equiv 3 \pmod 4$. Then by Lemmas 1.2(i),(iii), 1.3(ii) and 3.8, either $\mathbf{1}_{\Sigma_n} \oplus Y^{(n-2,2)} \subseteq \operatorname{End}(D)$ or $\mathbf{1}_{\Sigma_n} \oplus (S^{(n-2,2)})^* \subseteq \operatorname{End}(D)$ (or both). By Lemma 1.7, $\operatorname{End}(D)^G$ is at least 2-dimensional, giving a contradiction.

Case 2: $p>2, n \equiv 1 \pmod p$ or $p=2$, $n \equiv 1 \pmod 4$. By Lemma 3.8 again, either $Y^{(n-2,2)} \subseteq \operatorname{End}(D)$ or $(S^{(n-2,2)})^* \subseteq \operatorname{End}(D)$. In the first case Lemma 1.7 implies that $\operatorname{End}(D)^G$ is at least 2-dimensional. In the second case $(S^{(n-2,2)})^* = D^{(n-2,2)}|\mathbf{1}$ implies that $(S^{(n-2,2)})^* \oplus \mathbf{1}_{\Sigma_n} \subseteq \operatorname{End}(D)$. Now apply Lemma 1.7 as in Case 1. □

Theorem 3.10. *Let $n \geq 4$, $G \leq \Sigma_n$ be a subgroup, and D be a simple $F\Sigma_n$-module with $\dim D > 1$. If $p = 2$, assume additionally that n is odd and $D \not\cong S$. If the restriction D_G is irreducible then either $G \leq \Sigma_{n-1}$ or G is 2-transitive.*

Proof. Let D_G be irreducible. Assume first that G is intransitive. Then, up to a conjugation, G is contained in a standard Young subgroup $\Sigma_{n-k} \times \Sigma_k$ for some $1 \leq k \leq n/2$. If $k > 1$, then $d_1 = 2 < d_2 = 3$ with respect to the action of $\Sigma_{n-k} \times \Sigma_k$. Hence $D{\downarrow_{\Sigma_{n-k}\times\Sigma_k}}$ is reducible by Theorem 3.9 applied to the subgroup $\Sigma_{n-k} \times \Sigma_k$. This gives a contradiction. Therefore $k = 1$ and so $G \leq \Sigma_{n-1}$. We may now assume that G is transitive. If G is not 2-homogeneous, then $d_1 = 1 < d_2$ and we get a contradiction by Theorem 3.9. Thus, G is 2-homogeneous. Now, if G is not 2-transitive apply Theorem 2.5. $\square$

Finally, observe that the Main Theorem is a special case of Theorem 3.10.

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