COMPLETELY SPLITTABLE REPRESENTATIONS OF SYMMETRIC GROUPS

ALEXANDER KLESHCHEV

Introduction

This note is motivated by papers [6] and [7].

Let K be a field of characteristic $p > 0$, P_n the set of all p-regular partitions of n, Σ_n the symmetric group on n letters, and D^λ the irreducible $K\Sigma_n$-module corresponding to a partition $\lambda \in P_n$ ([2]). For a composition $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ of n, the subgroup

$$\Sigma^\mu = \Sigma_{\mu_1} \times \Sigma_{\mu_2} \times \cdots \times \Sigma_{\mu_r} < \Sigma_n$$

is called a (standard) Young subgroup.

If $\lambda = (l_1 \geq l_2 \geq \cdots \geq l_m > 0)$ is a partition, let

$$h(\lambda) = m, \qquad \chi(\lambda) = l_1 - l_m + m.$$

In fact $\chi(\lambda)$ is the length of a special hook.

Example. Let $\lambda = (8,7,7,5,3,3)$. Then $\chi(\lambda) = 11$. The nodes of the corresponding hook are coloured in black in the following picture.

$$\begin{array}{cccccccc}
\circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\circ & \circ & \bullet & \circ & \circ & \circ & \circ & \\
\circ & \circ & \bullet & \circ & \circ & \circ & \circ & \\
\circ & \circ & \bullet & \circ & \circ & & & \\
\circ & \circ & \bullet & & & & & \\
\circ & \circ & \bullet & & & & &
\end{array}$$

In [7], a nice family of $K\Sigma_n$-modules for the case $p = 5$ is constructed. It consists of pairs, $\{D_n^+, D_n^-\}$, $n \geq 1$ (denoted by $\{R_n, R_n'\}$ in [7]). If $n = 1$, $D_n^+ \cong D_n^-$ is the trivial module over Σ_1, and D_n^+, D_n^- are non-isomorphic irreducible $K\Sigma_n$-modules for $n > 1$. The following remarkable property of the family is stated in [7]. The restriction of $D_n^\pm$ to any Young subgroup $\Sigma_{\mu_1} \times \Sigma_{\mu_2} \times \cdots \times \Sigma_{\mu_r}$ is semi-simple, and, moreover, every simple component of this restriction has form $D_{\mu_1}^\pm \otimes D_{\mu_2}^\pm \otimes \cdots \otimes D_{\mu_r}^\pm$.

The paper [6] is devoted to the description of the dimensions of the D^λ for λ satisfying $\chi(\lambda) \leq p$. The answer is obtained in terms of the paths in a modular version of the Young graph (see below).

1991 *Mathematics Subject Classification.* 20C05.

This modular Young graph has been also considered in [5]. It was proved there that

(1) $$\dim D^\lambda \geq \sharp\{\text{paths from } (1) \text{ to } \lambda \text{ in the modular Young graph}\}$$

for an *arbitrary* $\lambda \in P_n$. This inequality was obtained using results on the restrictions of modular irreducibles D^λ to Σ_{n-1}. It follows from the proof of Proposition 2.2 in [5] (cf. also Theorem 2.3 below) that we have equality in (1) if and only if the restriction $D^\lambda \downarrow_{\Sigma_i}$ is semi-simple for any $i < n$. This is equivalent to the condition that $D^\lambda \downarrow_{\Sigma^\mu}$ is semi-simple for any Young subgroup $\Sigma^\mu < \Sigma_n$, cf. Lemma 1.7 below. Thus we are led to the following definition.

Definition 0.1. An irreducible $K\Sigma_n$-module D^λ is called *completely splittable* if and only if the restriction $D^\lambda \downarrow_{\Sigma^\mu}$ to any Young subgroup $\Sigma^\mu < \Sigma_n$ is semi-simple.

In this paper we first describe the completely splittable representations. It turns out that D^λ is completely splittable if and only if $\chi(\lambda) \leq p$. As a consequence, we reprove Mathieu's result on $\dim D^\lambda$ for λ with $\chi(\lambda) \leq p$.

We then develop the result of Ryba mentioned above describing all families of representations defined below.

Definition 0.2. Let $\Phi^n = \{D_1^n, D_2^n, \dots, D_{z(n)}^n\}$ be a set of irreducible $K\Sigma_n$-modules, $n = 1, 2, \dots$. The set $\Phi = \bigcup_{n \geq 1} \Phi^n$ is called a *semi-simple inductive system* if and only if the following two conditions hold.

(1) The restriction $D_j^n \downarrow_{\Sigma_{n-1}}$ is a *direct sum* of modules from Φ^{n-1} (for all $n = 2, 3, \dots$ and $j = 1, 2, \dots, z(n)$).

(2) For any element D_i^{n-1} of Φ^{n-1} there exists an element D_j^n of Φ^n such that D_i^{n-1} is a component of the restriction $D_j^n \downarrow_{\Sigma_{n-1}}$ $(n = 2, 3, \dots)$.

Remarks (1) If $\Phi = \bigcup_{n \geq 1} \Phi^n$ is a semi-simple inductive system as in 0.2 then it follows that any $D_j^n \in \Phi^n$ is a completely splittable module (cf. 0.1 and 1.7). Moreover, it follows that any simple component of the restriction $D_j^n \downarrow_{\Sigma(\mu_1, \mu_2, \dots, \mu_r)}$ to a Young subgroup has the form $D_{i_1}^{\mu_1} \otimes D_{i_2}^{\mu_2} \otimes \cdots \otimes D_{i_r}^{\mu_r}$ where $D_{i_v}^{\mu_v} \in \Phi^{\mu_v}$, $v = 1, \dots, r$.

(2) The system $D_n^\pm$ from [7] is an example of a semi-simple inductive system.

(3) I am grateful to A.E. Zalesskii for the following remarks. If, in Definition 0.2, we do not demand restrictions to be semi-simple we obtain the usual definition of an inductive system. This was introduced by A.E. Zalesskii as a tool for the investigation of the lattice of the two-sided ideals in group rings of the locally finite groups, cf. [10]. It follows from our classification of the semi-simple inductive systems given below that for $p > 3$ there exist those which do not contain trivial and sign modules for sufficiently large n. This fact can be interpreted in the following terms. Let A_∞ be the finitary alternating group (i.e. $A_\infty = \cup_{n \geq 1} A_n$ where $A_{n-1} < A_n$ is the standard embedding). For $p > 3$ there exists a two-sided ideal of the group algebra KA_∞ which is not contained in the augmentation ideal. It was known before that a similar result holds for $p = 2$. By contrast, if the characteristic of K is zero then *any* two-sided ideal of KA_∞ is contained in the augmentation ideal, cf. [9].

1. Preliminaries

Throughout $\lambda = (l_1 \geq l_2 \geq \cdots \geq l_m > 0)$ is an arbitrary p-regular partition of n. We write it in the form

$$\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_k^{a_k}),\ \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0,\ a_1, a_2, \ldots, a_k > 0. \tag{2}$$

Then $\chi(\lambda) = \lambda_1 - \lambda_k + \sum_{v=1}^{k} a_v$, $h(\lambda) = \sum_{v=1}^{k} a_v$. For $1 \leq i \leq j \leq k$ we put

$$\beta(i,j) = \lambda_i - \lambda_j + \sum_{v=i}^{j} a_v, \qquad \gamma(i,j) = \lambda_i - \lambda_j + \sum_{v=i+1}^{j} a_v.$$

We consider the Young diagram for λ as the set (denoted by the same symbol λ) of nodes

$$\lambda = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i \geq 1,\ 1 \leq j \leq l_i\}.$$

Let $A_i = (\sum_{v=1}^{i} a_v, \lambda_i)$, $i = 1, 2, \ldots, k$. The nodes A_i are by definition the *removable* nodes of λ. If $A = A_i$ is a removable node then we denote by λ_A the partition of $n-1$ with the Young diagram $\lambda \setminus A$. For any integer i with $1 \leq i \leq k$ we put

$$M_i = \{j \in \mathbb{Z} \mid 1 \leq j < i,\ \beta(j,i) \equiv 0 \pmod{p}\}.$$

Definition 1.1. A removable node A_i of λ is called *normal* if and only if for any $j \in M_i$ there exists an integer $d(j)$ such that

$$j < d(j) < i, \qquad \beta(j, d(j)) \equiv 0 \pmod{p}, \qquad |\{d(j) | j \in M_i\}| = |M_i|.$$

We state a result from [4] for future reference.

Theorem 1.2. [4] *The restriction $D^\lambda \downarrow_{\Sigma_{n-1}}$ is semi-simple if and only if $\gamma(i,j) \not\equiv 0 \pmod{p}$ for any $i < j$ with A_i and A_j normal. In this case λ_{A_v} is p-regular for any normal A_v, and*

$$D^\lambda \downarrow_{\Sigma_{n-1}} \cong \bigoplus \{D^{\lambda_{A_v}} | A_v \text{ is normal}\}.$$

Corollary 1.3. *Let $\chi(\lambda) \leq p$. Then $D^\lambda \downarrow_{\Sigma_{n-1}}$ is semi-simple. Moreover*

$$D^\lambda \downarrow_{\Sigma_{n-1}} \cong \bigoplus_{v=1}^{k} D^{\lambda_{A_v}} \quad \textit{if} \quad \chi(\lambda) < p,$$

and

$$D^\lambda \downarrow_{\Sigma_{n-1}} \cong \bigoplus_{v=1}^{k-1} D^{\lambda_{A_v}} \quad \textit{if} \quad \chi(\lambda) = p.$$

Proof. Clearly $\chi(\lambda) = \beta(1,k)$. Let $1 \leq i < j \leq k$. Then $\gamma(i,j) > 0$, and $\gamma(i,j) < \beta(1,k)$ implies $\gamma(i,j) < p$. So $\gamma(i,j) \not\equiv 0 \pmod{p}$. By Theorem 1.2, $D^\lambda \downarrow_{\Sigma_{n-1}}$ is semi-simple, and

$$D^\lambda \downarrow_{\Sigma_{n-1}} \cong \bigoplus \{D^{\lambda_{A_v}} | A_v \text{ is normal}\}.$$

Since $\beta(i,j) < \beta(1,k) = p$ unless $(i,j) = (1,k)$, we get $\beta(i,j) \not\equiv 0 \pmod{p}$ for all $1 \leq i < j \leq k$, $(i,j) \neq (1,k)$. If $\beta(1,k) < p$ then by Definition 1.1 all nodes A_v,

$v = 1, 2, \ldots, k$, are normal. If $\beta(1, k) = p$ then similarly all of them but A_k are normal. $\square$

Lemma 1.4. *Let $\lambda \neq (n)$.*

(i) $\chi(\lambda_{A_v}) > \chi(\lambda)$ *if and only if* $v = k$ *and* $\lambda_k > 1$. *In this case* $\chi(\lambda_{A_v}) = \chi(\lambda) + 1$.

(ii) $\chi(\lambda_{A_v}) < \chi(\lambda)$ *if and only if* $v = k$, $\lambda_k = 1$ *or* $v = 1$, $a_1 = 1$.

Proof. This follows immediately from the definitions. $\square$

Corollary 1.5. *If $\chi(\lambda) \leq p$ and D^μ is a simple component of $D^\lambda \downarrow_{\Sigma_{n-1}}$ then $\chi(\mu) \leq p$.*

Proof. By 1.3,

$$D^\lambda \downarrow_{\Sigma_{n-1}} \cong \bigoplus_{v=1}^{x} D^{\lambda_{A_v}}$$

where $x = k$ if $\chi(\lambda) < p$, and $x = k - 1$ if $\chi(\lambda) = p$. (Note that $k > 1$ if $\chi(\lambda) = p$ since λ is p-regular throughout the paper.)

We claim that $\chi(\lambda_{A_v}) \leq p$ for any v such that $D^{\lambda_{A_v}}$ is a component of $D^\lambda \downarrow_{\Sigma_{n-1}}$. Indeed, by Lemma 1.4(i), $\chi(\lambda_{A_v}) > p \geq \chi(\lambda)$ implies $v = k$, $\lambda_k > 1$, and $\chi(\lambda) = p$. But if $\chi(\lambda) = p$ then $D^{\lambda_{A_k}}$ is not a component of $D^\lambda \downarrow_{\Sigma_{n-1}}$. $\square$

Lemma 1.6. *Let G_1 and G_2 be finite groups, and M be a $K[G_1 \times G_2]$-module such that $M \downarrow_{G_v}$ is semi-simple for $v = 1, 2$. Assume K is algebraically closed. Then M is semi-simple.*

Proof. The proof goes along the lines of the proof of Theorem 5 in [8]. Since G_2 centralises G_1 then G_2 preserves the G_1-homogeneous components of M. So we may assume that

$$M \downarrow_{G_1} \cong \underbrace{L \oplus \cdots \oplus L}_{t \text{ times}}$$

where L is an irreducible KG_1-module. Then the centraliser of G_1 in M is $GL(W)$ where W is the t-dimensional vector space, and, moreover, $M \cong L \otimes W$ as a module over $G_1 \times GL(W)$. It remains to notice that $G_2 < GL(W)$ acts completely reducibly on W. $\square$

Lemma 1.7. *The following conditions are equivalent.*

(i) $D^\lambda \downarrow_{\Sigma^\mu}$ *is semi-simple for any Young subgroup* $\Sigma^\mu < \Sigma_n$.

(ii) $D^\lambda \downarrow_{\Sigma_i}$ *is semi-simple for any $i < n$ (here Σ_i is considered as the subgroup of Σ_n consisting of the permutations of the first i letters).*

Proof. (i) implies (ii) because Σ_i is the Young subgroup $\Sigma^{(i,1,\ldots,1)}$. Assume that $D^\lambda \downarrow_{\Sigma_i}$ is semi-simple. Then the restriction of D^λ to any subgroup consisting of the permutations of a fixed set of i letters is semi-simple since such a subgroup is conjugate to Σ_i. So (ii) implies (i) in view of Lemma 1.6 (and the fact that $GF(p)$ is a splitting field for Σ_n so we may assume that K is algebraically closed). $\square$

2. Main Results

Theorem 2.1. *D^λ is completely splittable if and only if $\chi(\lambda) \leq p$.*

Proof. Proceed by induction on n. If $n < p$ the assertion of the theorem is clear. Let $n \geq p$ and suppose the result holds for $n-1$.

Assume $\chi(\lambda) \leq p$. Then $D^\lambda \downarrow_{\Sigma_{n-1}}$ is semi-simple by 1.3. Moreover, in view of 1.5, $\chi(\mu) \leq p$ as soon as D^μ is a component of $D^\lambda \downarrow_{\Sigma_{n-1}}$. By the inductive hypothesis, D^μ is completely splittable. Now it suffices to apply Lemma 1.7 to prove that D^λ is completely splittable.

In the other direction, assume $\chi(\lambda) > p$. If $D^\lambda \downarrow_{\Sigma_{n-1}}$ is not semi-simple we are done. So we may suppose that

$$D^\lambda \downarrow_{\Sigma_{n-1}} \cong \bigoplus \{D^{\lambda_{A_v}} | A_v \text{ is normal}\},$$

cf. Theorem 1.2. If there exists a normal node A with $\chi(\lambda_A) > p$ then by the inductive hypothesis D^{λ_A} is not completely splittable, hence D^λ is not completely splittable.

So we may assume that $\chi(\lambda_A) \leq p$ for any normal node A of λ. Note that A_1 is normal, by definition, so by Lemma 1.4 either $k = 1$, $\lambda_1 = 1$ or $a_1 = 1$. In the former case $\lambda = (1^n)$ is not p-regular because $n > p$, giving a contradiction. So $a_1 = 1$. If $\lambda = (n)$ then $\chi(\lambda) = 1$ which contradicts the assumption $\chi(\lambda) > p$. So $k \geq 2$. Moreover, $\chi(\lambda_{A_1}) = \chi(\lambda) - 1$ so $\chi(\lambda) = p + 1$.

Assume first that $k = 2$. Then $\beta(1,2) = \chi(\lambda) = p + 1$, hence A_2 is normal. Moreover $\gamma(1,2) = \beta(1,2) - a_1 = p$. Therefore $D^\lambda \downarrow_{\Sigma_{n-1}}$ is not semi-simple by 1.2.

So we may assume $k \geq 3$. Note that

$$\beta(1,2) = \beta(1,3) - (\lambda_2 - \lambda_3) - a_3 \leq \beta(1,3) - 2 \leq \beta(1,k) - 2 = \chi(\lambda) - 2 = p - 1.$$

So $\beta(1,2) \not\equiv 0 \pmod p$ and hence A_2 is normal. However $\chi(\lambda_{A_2}) = \chi(\lambda) = p + 1$, by virtue of Lemma 1.4, giving a contradiction. $\square$

Definition 2.2. We define a graph Y as follows. Vertices of Y are labelled by the p-regular partitions λ of all $n \geq 1$ with $\chi(\lambda) \leq p$. There exists an arrow $\mu \to \lambda$ if and only if $\mu = \lambda_A$ for some removable node A of λ.

Remark. The graph Y just defined is a complete subgraph of the p-Young graph defined in [5].

The following result is proved in [6] by different methods.

Proposition 2.3. *Let $n \geq 2$, $\chi(\lambda) \leq p$. Then*

$$\dim D^\lambda = \sharp\{\textit{paths from } (1) \textit{ to } \lambda \textit{ in } Y\}.$$

Proof. It follows from 1.3, 1.4 and 2.2 that

$$D^\lambda \downarrow_{\Sigma_{n-1}} \cong \bigoplus \{D^\mu | \text{there exists an arrow } \mu \to \lambda \text{ in } Y\}.$$

Now the proposition is obtained by induction. $\square$

We reformulate Proposition 2.3 in terms of λ-tableaux. If t is a λ-tableau we denote by $t(i,j)$ the entry of t in the i-th row and j-th column. A λ-tableau is called *standard* if its entries increase in rows from left to right and in columns from top to bottom. We shall call a standard λ-tableau *p-standard* if for any two of its entries $t(i,j)$ and $t(i',j')$ with $i > i'$, $j < j'$, and $i + j' + 1 - i' - j = p$ one has $t(i',j') > t(i,j)$.

Corollary 2.4. *Let $n \geq 2$, $\chi(\lambda) \leq p$. Then*

$$\dim D^\lambda = \sharp\{p\text{-standard } \lambda\text{-tableaux}\}.$$

Proof. The proof can be obtained by induction similarly to the proof of 2.3. Alternatively, one may notice that there exists a 1-1-correspondence between paths from (1) to λ in Y and p-standard λ-tableaux and apply 2.3. $\square$

Notation. For $s = 1, 2, \dots, p-1$ put

$$\xi^s_n = \{D^\mu | \mu \in P_n,\ h(\mu) = s,\ \chi(\mu) \leq p\},$$
$$\omega^s_n = \{D^\mu | \mu = (\mu_1, \mu_2, \dots, \mu_r) \in P_n,\ h(\mu) < s,\ \mu_1 \leq p - s\}.$$

Of course, $\xi^s_n = \emptyset$ if and only if $n < s$, and $\omega^s_n = \emptyset$ if and only if $n > (p-s)(s-1)$. Set

$$\Phi^s_n = \xi^s_n \cup \omega^s_n,$$
$$\Phi(s) = \bigcup_{n \geq 1} \Phi^s_n.$$

For any $M \subseteq \{1, 2, \dots, p-1\}$ put

$$\Phi(M) = \bigcup_{s \in M} \Phi(s) = \bigcup_{n \geq 1} (\bigcup_{s \in M} \Phi^s_n).$$

Lemma 2.5. *Let $D^\mu \in \xi^s_n$, $D^\theta \in \xi^s_N$. If $\mu = (m_1, \dots, m_s)$, $\theta = (t_1, \dots, t_s)$, and $m_v \leq t_v$ for $v = 1, 2, \dots, s$ then D^μ is a component of $D^\theta \downarrow_{\Sigma_n}$.*

Proof. This follows from 1.3 (by induction on $N - n$). $\square$

Lemma 2.6. *Let $D^\mu \in \xi^s_n$, $D^\theta \in \xi^s_N$. If $N \gg 0$, then D^μ is a component of $D^\theta \downarrow_{\Sigma_n}$.*

Proof. Let $\mu = (m_1, \dots, m_s)$, $\theta = (t_1, \dots, t_s)$. If N is big enough then it follows from the assumptions that $m_v \leq t_v$, $v = 1, 2, \dots, s$. Now apply Lemma 2.5. $\square$

Lemma 2.7. *Let $n \leq (p-s)(s-1)$, $D^\mu \in \omega^s_n$, $D^\theta \in \Phi^s_N$. If $N \gg 0$, then D^μ is a component of $D^\theta \downarrow_{\Sigma_n}$.*

Proof. Let $\eta = ((p-s)^{(s-1)}, 1)$, $d = (p-s)(s-1)+1$. Then $D^\eta \in \xi^s_d$ so by Lemma 2.6, D^η is a component of $D^\theta \downarrow_{\Sigma_d}$. Now it remains to notice that by 1.3, D^μ is a component of $D^\eta \downarrow_{\Sigma_n}$. $\square$

Theorem 2.8. (i) *Any $\Phi(s)$, $s = 1, 2, \dots, p-1$ is a semi-simple inductive system.*

(ii) *For any $M \subseteq \{1, 2, \dots, p-1\}$, $\Phi(M)$ is a semi-simple inductive system.*

(iii) *Any semi-simple inductive system coincides with* $\Phi(M)$ *for some subset* $M \subseteq \{1, 2, \dots, p-1\}$.

Proof. (i) Let $D^\mu \in \Phi_n^s$. Then $\chi(\mu) \leq p$. By 2.1, the restriction $D^\mu \downarrow_{\Sigma_{n-1}}$ is semisimple. Let D^ν be any of its components. We prove that $D^\nu \in \Phi_{n-1}^s$.

If $D^\mu \in \omega_n^s$ then it follows from 1.3 that $D^\nu \in \omega_{n-1}^s \subseteq \Phi_{n-1}^s$.

Let $D^\mu \in \xi_n^s$ and $\mu = (\mu_1, \mu_2, \dots, \mu_s)$. If $\mu_s > 1$ or $\mu_s = 1$, $\nu \neq (\mu_1, \mu_2, \dots, \mu_{s-1})$ then by 1.3 and 1.5 we have $D^\nu \in \xi_{n-1}^s \subseteq \Phi_{n-1}^s$. Assume $\mu_s = 1$ and $\nu = (\mu_1, \mu_2, \dots, \mu_{s-1})$. Then it follows from 1.3 that $\mu_1 + s - 1 = \chi(\mu) < p$, i.e. $\mu_1 \leq p - s$. Thus $D^\nu \in \omega_{n-1}^s \subseteq \Phi_{n-1}^s$.

Now we prove that there exists $D^\eta \in \Phi_{n+1}^s$ such that D^μ is a component of $D^\eta \downarrow_{\Sigma_n}$. In view of what has already been proved, it suffices to show that for some $N > n$ there is $D^\theta \in \Phi_N^s$ such that D^μ is a component of $D^\theta \downarrow_{\Sigma_n}$. But this follows from 2.6 and 2.7.

(ii) follows from (i).

(iii) Let $\Phi = \cup_{n \geq 1} \Phi_n$ be an arbitrary semi-simple inductive system. By 2.1, $D^\mu \in \Phi_n$ implies $\chi(\mu) \leq p$. In particular, $h(\mu) \leq p-1$ since μ is p-regular. Define

$$M = \{s | h(\mu) = s \text{ for infinitely many } \mu \text{ with } D^\mu \in \Phi\}.$$

Then $M \subseteq \{1, 2, \dots, p-1\}$. We shall prove that $\Phi = \Phi(M)$.

First we prove that $s \in M$ implies $\Phi(s) \subseteq \Phi$. Let $D^\mu \in \Phi_n^s$ for some $s \in M$. Since every Φ_d^s is finite then by definition of M, for any d there is $N > d$ and $\theta \in P_N$ with $h(\theta) = s$, $\chi(\theta) \leq p$, $D^\theta \in \Phi_N$. By definition of inductive system, any component of $D^\theta \downarrow_{\Sigma_n}$ belongs to Φ_n. But D^μ is such a component provided N is big enough, in view of Lemmas 2.6 and 2.7.

Now we prove that $\Phi \subseteq \cup_{s \in M} \Phi(s)$. Let $D^\mu \in \Phi_n$. Then for any $N > n$ we have that D^μ is a component of $D^\theta \downarrow_{\Sigma_n}$ for some $D^\theta \in \Phi_N$. However it follows from the definition of the set M that for $N \gg 0$, $D^\theta \in \Phi_N$ implies $h(\theta) \in M$. Since we also have $\chi(\theta) \leq p$, this means that $D^\theta \in \cup_{s \in M} \Phi(s)$. Hence $D^\mu \in \cup_{s \in M} \Phi(s)$. □

Remarks (1) The system Φ^1 is clearly the trivial inductive system, i.e. $\Phi_n^1 = \{D^{(n)}\}$ for any n. The system Φ^{p-1} is the sign system, i.e. $\Phi_n^{p-1} = \{\mathrm{sgn}_n\}$ where sgn_n is the sign representation of Σ_n. Indeed, $\Phi_n^{p-1} = \{D^{\varepsilon_n}\}$ where $\varepsilon_n = ((d+1)^r, d^{p-1-r})$ if $n = (p-1)d + r$, $d, r \in \mathbb{Z}$, $0 \leq r < p-1$. The fact that $D^{\varepsilon_n} \cong \mathrm{sgn}_n$ can be found in [3] or easily deduced from the main result of [1].

(2) In view of the previous remark and Theorem 2.8 there are only three semi-simple inductive systems in case $p = 3$, namely, the trivial one, the sign one, and the union of the trivial and the sign systems. They do not provide non-trivial inductive systems for the alternating groups. That is why for $p = 3$ the question about two-sided ideals mentioned in Remark (3) after Definition 0.2 remains open.

(3) The system $D_n^\pm$ from [7] coincides with $\Phi(2)$ for $p = 5$.

References

1. G. D. James. On the decomposition matrices of the symmetric groups, II, *J. Algebra* 43(1976), 45-54.
2. G. D. James. *The Representation Theory of the Symmetric Groups*, volume 682 of Lecture Notes in Mathematics, Springer, 1978.
3. J. C. Jantzen and G. M. Seitz. On the representation theory of the symmetric group, *Proc. London Math. Soc* 65(1992), 475-504.
4. A. S. Kleshchev. Branching rules for modular representations of symmetric groups, II. *J. Reine angew. Math.* 459(1995), 163-212.
5. A. S. Kleshchev. Branching rules for modular representations of symmetric groups, III. *J. London Math. Soc. (2)*, 1995 *(to appear)*.
6. O. Mathieu. On the dimension of some modular irreducible representations of the symmetric group, *preprint*, 1994.
7. A. Ryba. Fibonacci representations of the symmetric groups, *J. Algebra* 170(1994), 678-686.
8. G. M. Seitz. Abstract homomorphisms of algebraic groups. *Proc. London. Math. Soc. (to appear)*.
9. A. E. Zalesskii. Group rings of inductive limits of alternating groups, *Leningrad Math. J.* 2(1991), 1287-1303 *(in Russian)*.
10. A. E. Zalesskii. Group rings of simple locally finite groups. In *"Finite and Locally Finite Groups"*, Proc. Istanbul Summer Inst., 1994, Kluwer *(to appear)*.

Department of Mathematics, University of Oregon, Eugene OR97403, U.S.A.
E-mail address: klesh@math.uoregon.edu