ASYMPTOTIC RESULTS ON MODULAR REPRESENTATIONS OF SYMMETRIC GROUPS AND ALMOST SIMPLE MODULAR GROUP ALGEBRAS

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1. Main results

Let G be a group and F be a field. When is the lattice of the (two-sided) ideals of the group algebra FG as simple as possible? More precisely, every group algebra has the augmentation ideal. It is natural to ask when FG has no other proper nonzero ideals. In this case the group algebra FG is called *almost simple*. The following problem goes back to Kaplansky (1965).

Problem. For a field F, find a group-theoretic characterization of the groups G for which FG is almost simple.

In fact, this is a question about infinite simple groups because it is easy for G finite, and because a non-trivial normal subgroup gives rise to a non-trivial ideal different from the augmentation ideal. Also note that the problem reduces easily to the question on when the augmentation ideal is simple as a ring.

The first interesting class of groups with almost simple FG was discovered in [3]. This class is rather exotic and contains groups like the universal Hall group and algebraically closed groups. For locally finite groups G a substantial progress was achieved recently by using representation theory of finite groups, see [16, 17, 14], and others. The representation theory approach transforms the problems on ideals to certain problems on asymptotic behavior of representations of finite groups, which are often of independent interest. The method is more effective over fields of characteristic zero as the theory of ordinary representations is much better elaborated than the modular theory.

This paper is devoted to the modular aspect of the theory. One of our main results deals with the case where G is a direct limit of alternating groups and F is any field of characteristic $p > 2$. We note that the case where $\operatorname{char} F = 0$ was considered in [16].

Let $\mathbb{N}$ be the set of natural numbers. Denote by $\mathrm{Alt}(\Omega)$ and $\mathrm{Sym}(\Omega)$ (or, simply, A_n and Σ_n) the alternating and symmetric groups, respectively, on a set Ω with $|\Omega| = n$. Let

$$\mathrm{Alt}(\Omega_1) \subset \mathrm{Alt}(\Omega_2) \subset \cdots \subset \mathrm{Alt}(\Omega_i) \subset \dots \tag{1}$$

be a chain of (strict) embeddings of finite alternating groups. Then we can consider the union (more precisely, the direct limit)

$$G = \bigcup_{i=1}^{\infty} \mathrm{Alt}(\Omega_i),$$

which is a locally finite group. We call such G a *limit alternating group*. We emphasize that the embeddings in (1) do not have to be natural embeddings. If the embeddings are

Mathematics Subject Classification: 16S34, 20C20, 20C30.

The first author has been supported by INTAS (Project 93-183-Ext) and by the Institute of Mathematics of the National Academy of Sciences of Belarus in the framework of the state program "Mathematical structures". The second author was supported by the NSF.

natural then G is the finitary alternating group Alt_∞, which consists of all permutations of $\mathbb{N}$ fixing all but finitely many numbers. In general G has no natural permutation representation, like in the case of non-diagonal limit permutation groups defined below, see [16] for more details. We note that for diagonal G the natural permutation set may be identified with $\mathbb{N}$ but the action of each $\mathrm{Alt}(\Omega_i)$ has infinitely many non-trivial orbits unless G is the finitary alternating group.

Definition 1.1. (i) An embedding $\mathrm{Alt}(\Omega_1) \to \mathrm{Alt}(\Omega_2)$ is called *diagonal* if the orbits of $\mathrm{Alt}(\Omega_1)$ on Ω_2 have lengths 1 or $|\Omega_1|$.

(ii) A limit alternating group G is called *diagonal* if all but finitely many embeddings $\mathrm{Alt}(\Omega_i) \to \mathrm{Alt}(\Omega_{i+1})$ are diagonal. Otherwise G is called *non-diagonal*.

The next theorem explains the ring theoretic importance of non-diagonal groups.

Theorem A. *Let* $\mathrm{char}\, F \neq 2$*, and* G *be a limit alternating group. Then* FG *is almost simple if and only if* G *is non-diagonal.*

The proof of Theorem A (see also a more general Theorem 3.13) relies on four powerful tools. The first one is a general result of Passman and Zalesskii [13] which implies that the Jacobson radical of FG is trivial. The second one is the inductive systems techniques introduced by Zalesskii (see e.g. the exposition [18] and references there). These tools allow us to reduce the problem to some questions on the asymptotic behavior of representations of finite alternating groups. To answer these questions we need one of the main results of [14]. This result shows that if the embeddings in (1) are non-diagonal, then for any $n \in \mathbb{N}$ there exists $N > n$ such that, for any $i > N$, the group $\mathrm{Alt}(\Omega_n)$ has a *regular* orbit on Ω_i, i.e. an orbit of length $|\mathrm{Alt}(\Omega_n)|$. Finally, we need to prove some new asymptotic results on the *branching rules* for symmetric groups in characteristic p. We believe these results might be of independent interest so we present them in this section.

Recall that the representations $\mathbf{1}_{\Sigma_n} : g \mapsto 1$ and $\mathrm{sgn}_n : g \mapsto \mathrm{sign}(g)$ are the only 1-dimensional representations of Σ_n. All the other representations of Σ_n are faithful. Suppose first that $\mathrm{char}\, F = 0$. Then the natural permutation $F\Sigma_n$-module M (of dimension n) splits as a direct sum $M \cong \mathbf{1}_{\Sigma_n} \oplus V$ where V is the *natural irreducible* Σ_n-module. Denote by V^σ the module $V \otimes \mathrm{sgn}_n$. Using the classical branching rule one can easily deduce the following simple but useful fact.

Asymptotic Theorem. *Let* $\mathrm{char}\, F = 0$.

(i) *For any* $n \in \mathbb{N}$ *there exists* $M > n$ *such that, for every* $i \geq M$*, the restriction of any* Σ_i*-module to* Σ_n *has a* 1*-dimensional composition factor.*

(ii) *For any* $n \in \mathbb{N}$ *there exists* $N > n$ *such that, for every* $i \geq M$*, the restriction of any faithful irreducible* Σ_i*-module to* Σ_n *has either the natural irreducible module* V *or the module* V^σ *as a composition factor.*

It is the analogue of this theorem which we need to find in characteristic p in order to prove Theorem A. Note that in positive characteristic the result is wrong as stated.

From now on let $\mathrm{char}\, F = p > 0$.

If p does not divide n then the natural permutation module M splits, as in characteristic 0, into a direct sum of $\mathbf{1}_{\Sigma_n}$ and an irreducible module V of dimension $n-1$. Otherwise M is uniserial with composition factors $\mathbf{1}_{\Sigma_n}, V, \mathbf{1}_{\Sigma_n}$ where V is an irreducible module of dimension $n-2$. In any case we still call V the *natural irreducible* (Σ_n-)module.

We recall from [8] that the irreducible $F\Sigma_n$-modules are labelled by the p-regular partitions of n, i.e. the partitions of n in which each part appears with multiplicity $< p$. If λ is such a partition, we denote the corresponding irreducible module by D^λ. We make use of an important class of irreducible Σ_n-modules introduced in [10].

Definition 1.2. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0)$ be a p-regular partition. The number of rows s is called the *height* of λ. We say λ is *completely splittable* if $\lambda_1 - \lambda_s + s \leq p$. In this case the corresponding irreducible module D^λ is also called *completely splittable*. (Clearly, $1 \leq s \leq p-1$).

Observe that the natural irreducible module $V = D^{(n-1,1)}$ and the module V^σ are not completely splittable (for $n > p$). Also note that the 1-dimensional modules $\mathbf{1}_{\Sigma_n} = D^{(n)}$ and sgn_n are the only completely splittable modules of heights 1 and $p-1$, respectively (see Section 2). On the other hand, all completely splittable modules of heights $2, 3, \dots, p-2$ are faithful. These facts and the following result proved in [10] show that the straightforward generalization of the Asymptotic Theorem above to characteristic p is wrong.

Proposition 1.3. [10, 2.8] *Let $N > n > (p-1)^2$, and let λ be a completely splittable partition of height s. Then the restriction $D^\lambda\downarrow^{\Sigma_N}_{\Sigma_n}$ is a direct sum of completely splittable modules of height s.*

Proposition 1.3 shows that essentially the best result one can hope for as a modular analogue of the Asymptotic Theorem above is

Theorem B (Modular Asymptotic Theorem). *Let $n > (p-1)^2$.*

(i) *Assume $p > 2$. There exists $M = M(n) > n$ such that, for any $i \geq M$, the restriction of any Σ_i-module to Σ_n contains a completely splittable Σ_n-module as a composition factor.*

(ii) *Assume $p > 3$. There exists $N = N(n) > n$ such that, for any $i \geq N$, the restriction of any faithful irreducible Σ_i-module to Σ_n contains either the natural irreducible module V, or the module V^σ, or a faithful completely splittable Σ_n-module as a composition factor.*

(ii′) *Assume $p = 3$. There exists $N = N(n) > n$ such that, for any $i \geq N$, the restriction of any faithful irreducible Σ_i-module to Σ_n contains as a composition factor either the natural irreducible module V, or the module V^σ, or an irreducible Σ_n-module D^λ or $D^\lambda \otimes \mathrm{sgn}_n$ with $\lambda = (\lambda_1, \lambda_2)$, $2 \leq \lambda_1 - \lambda_2 \leq 7$.*

Remark. The proof of the Modular Asymptotic Theorem relies on the modular branching rules for symmetric groups [9] and the Mullineux Conjecture proved in [7] (see also [2]). The Mullineux bijection becomes trivial in characteristic 2. This is the reason why our methods fail in this case. In characteristic 3 we use the results of Sheth [15].

2. Inductive systems and asymptotic branching

In this section we prove the Modular Asymptotic Theorem.

Notation. Throughout the paper F is an arbitrary field of characteristic $p > 0$.

If G is a group and M is an FG-module, we denote by $\mathrm{Irr}M$ the set of the *isomorphism classes* of the composition factors of M. If G is a subgroup of a group H and $\Phi = \{M_1, M_2, \dots, M_k\}$ is a set of H-modules, we denote by $\Phi\downarrow_G$ the set $\bigcup_{i=1}^n \mathrm{Irr}(M_i\downarrow_G)$.

If $\lambda = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_s > 0)$ is a partition we write $h(\lambda)$ for s. We do not distinguish between λ and its Young diagram.

If A is a removable node of a partition λ, i.e. one obtains a Young diagram after removing A, we write λ_A for $\lambda \setminus \{A\}$, which is a partition of $n-1$.

Let $m, f \in \mathbb{N}$. Express m in the form

$$m = (p-1)d + r, \quad d \in \mathbb{Z}, \quad 0 < r \leq p-1.$$

We need to consider certain classes of partitions. Set

$$\beta(m, f) = ((f+d)^{p-1}, (f+d-1)^{p-1}, \dots, (f+1)^{p-1}, f^r),$$
$$\gamma(m, f) = (f + (m-1)(p-1), f + (m-2)(p-1), \dots, f + (p-1), f),$$

$\gamma'(m,f) = (f+(m-2)(p-1), f+(m-2)(p-1), f+(m-3)(p-1), \dots, f+(p-1), f)$. Obviously all these partitions are of height m. If $\lambda^i = (\lambda^i_1, \dots, \lambda^i_{t_i})$, $i = 1, \dots, k$, are partitions and $\lambda^i_{t_i} \geq \lambda^{i+1}_1$ for all $i < k$, put

$$(\lambda^1; \dots; \lambda^k) = (\lambda^1_1, \dots, \lambda^1_{t_1}, \dots, \lambda^k_1, \dots, \lambda^k_{t_k})$$

(glue λ_2 to the bottom of λ_1, λ_3 to the bottom of λ_2, and so on). We write $\beta(m,f;\mu)$ and $\gamma(m,f;\mu)$ instead of $(\beta(m,f);\mu)$ and $(\gamma(m,f);\mu)$, respectively.

Denote by ε_n the partition corresponding to the sign representation of Σ_n. By [7],

$$\varepsilon_n = (l^{a_1}, (l-1)^{a_2})$$

where l, a_1 and a_2 are determined from $n = (p-1)(l-1)+a_1$, $0 < a_1 \leq p-1$, $a_1+a_2 = p-1$. So

$$\begin{aligned} \beta(m,f) &= (\varepsilon_{k_1}; \dots; \varepsilon_{k_d}; (f^r)); \\ \gamma(m,f) &= ((n_1); (n_2); \dots; (n_m)); \\ \gamma'(m,f) &= ((n_2); (n_2); (n_3); \dots; (n_m)). \end{aligned}$$

where $k_i = (f+d+1-i)(p-1)$ and $n_j = f+(p-1)(m-j)$.

Some properties of the Mullineux bijection. For a p-regular partition λ we denote by $\mathbf{M}(\lambda)$ its Mullineux image, which is the p-regular partition defined by

$$D^\lambda \otimes \mathrm{sgn}_n \cong D^{\mathbf{M}(\lambda)}.$$

The map $\lambda \mapsto \mathbf{M}(\lambda)$ is called the Mullineux bijection. We shall use an explicit description of the bijection $\mathbf{M}$, see [7]. For a partition μ we denote by μ_1 the first part of μ.

Lemma 2.1. *Let $m \geq p-1$, $\varepsilon_m = (l^{a_1}, (l-1)^{a_2})$ and μ be a partition such that $\mu_1 \leq l-1$. If the partition $\alpha = (\varepsilon_m; \mu)$ is p-regular then $\mathbf{M}(\alpha) = ((m); \mathbf{M}(\mu))$.*

Proof. Recall that $\mathbf{M}(\varepsilon_m) = (m)$. We shall prove by induction on m that if μ satisfies the assumptions of the lemma, then $\mathbf{M}(\alpha) = (\mathbf{M}(\varepsilon_m); \mathbf{M}(\mu))$ and $\mathbf{M}(\mu)_1 \leq m-(p-1)$ where $\mathbf{M}(\mu)_1$ is the length of the first row in $\mathbf{M}(\mu)$. If $m = p-1$, then $\mu = \emptyset$ and everything is clear.

Assume that $p \leq m < 2p-1$. Then $l = 2$ and $\mu_1 \leq 1$, so $m = p-1+a_1$ and $\mu = (1^b)$. We may assume that $b > 0$. Then the Mullineux symbol $G(\alpha)$ of α is

$$G(\alpha) = \begin{pmatrix} p+b & m-p \\ p-1+b & m-p \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} p+1 \\ p \end{pmatrix}$$

(the second case arises if $m = p$, when b must be 1 since otherwise α is not p-regular). Therefore

$$G(\mathbf{M}(\alpha)) = \begin{pmatrix} p+b & m-p \\ 2 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} p+1 \\ 2 \end{pmatrix}.$$

We have $b \leq a_1 = m-(p-1)$. So $G(\mathbf{M}(\alpha))$ is the Mullineux symbol of the partition (m,b) in both cases. Thus $\mathbf{M}(\alpha) = (m,b)$.

Assume now that $m \geq 2p-1$. Let

$$\alpha' = \alpha \backslash \{p\text{-edge of } \alpha\}.$$

Observe that $\alpha' = (\varepsilon_{m-p}; \mu')$. Note that $m-p \geq p-1$. By inductive hypothesis,

$$\mathbf{M}(\alpha') = (\mathbf{M}(\varepsilon_{m-p}); \mathbf{M}(\mu')) = ((m-p); \mathbf{M}(\mu'))$$

and

$$\mathbf{M}(\mu')_1 \leq (m-p)-(p-1). \tag{2}$$

We have

$$G(\alpha) = \begin{pmatrix} p+a & G(\alpha') \\ p-1+b & \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} a & G(\mu') \\ b & \end{pmatrix} = G(\mu).$$

Therefore

$$G(\mathbf{M}(\alpha)) = \begin{pmatrix} p+a & G(\mathbf{M}(\alpha')) \\ 1+c & \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} a & G(\mathbf{M}(\mu')) \\ c & \end{pmatrix} = G(\mathbf{M}(\mu)).$$

Now, in view of (2), in constructing $\mathbf{M}(\alpha)$ one glues a nodes to the rows of $\mathbf{M}(\mu')$ (obtaining $\mathbf{M}(\mu)$) and p nodes to the first row (obtaining (m)). Therefore $\mathbf{M}(\alpha) = (\mathbf{M}(\varepsilon_m); \mathbf{M}(\mu))$, and $\mathbf{M}(\mu)_1 \leq m - (p-1)$, as required. □

Lemma 2.2. *Let $\alpha = (\varepsilon_{m_1}; \ldots; \varepsilon_{m_k}; \mu)$ be a p-regular partition with $m_i \leq m_{i+1} - (p-1)$ for all i. Then $\mathbf{M}(\alpha) = ((m_1, \ldots, m_k); \mathbf{M}(\mu))$.*

Proof. Apply Lemma 2.1 (k times). □

Lemma 2.3. *Let $m = d(p-1) + r$ where $0 \leq r < p-1$. Then $\mathbf{M}(\beta(m,f)) = \gamma(d, e; \nu)$ where $e = (f+1)(p-1)$, $\nu = \mathbf{M}((f^r))$.*

Proof. This follows from Lemma 2.2 with $\mu = (f^r)$. □

Lemma 2.4. *Let $\mu = (\mu_1 \geq \cdots \geq \mu_k)$ be a p-regular partition, and m be an integer satisfying $0 \leq m \leq \mu_k - (p-1)$. Set $\alpha = (\mu; (m))$. Then $\mathbf{M}(\alpha) = (\mathbf{M}(\mu); \varepsilon_m)$.*

Proof. The proof is similar to that of Lemma 2.1. Proceed by induction on m, the case $m = 0$ being trivial. Set

$$\alpha' = \alpha \backslash \{p\text{-edge of } \alpha\}.$$

Observe that $\alpha' = (\mu'; (m-p))$ (or μ' if $m \leq p$). Moreover,

$$\mu'_k - (p-1) \geq \mu_k - p - (p-1) \geq m - p$$

if $m > p$. Therefore by inductive hypothesis, $\mathbf{M}(\alpha') = (\mathbf{M}(\mu'); \varepsilon_{m-p})$ (or $\mathbf{M}(\mu')$ if $m \leq p$). We consider the generic case $m > p$ (the case $m \leq p$ being similar). We have

$$G(\alpha) = \begin{pmatrix} a+p & G(\alpha') \\ b+1 & \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} a & G(\mu') \\ b & \end{pmatrix} = G(\mu).$$

Note that the assumption $\mu_k - (p-1) \geq m > 0$ implies $\mu_k \geq p$, whence a is divisible by p. Therefore

$$G(\mathbf{M}(\alpha)) = \begin{pmatrix} a+p & G(\mathbf{M}(\alpha')) \\ a-b+p-1 & \end{pmatrix}$$

Now in constructing $\mathbf{M}(\alpha)$, one glues p nodes to the $p-1$ rows of ε_{m-p} (obtaining ε_m) and a nodes to the rows of $\mathbf{M}(\mu')$ (obtaining $\mathbf{M}(\mu)$). Therefore $\mathbf{M}(\alpha) = (\mathbf{M}(\mu); \varepsilon_m)$, as required. □

Lemma 2.5. *Let $\mu = (\mu_1 \geq \cdots \geq \mu_l)$ be a p-regular partition, $\alpha = (\mu; (m_1, \ldots, m_k))$ where $\mu_l - (p-1) \geq m_1$, and $m_i - (p-1) \geq m_{i+1}$ for $1 \leq i < k$. Then $\mathbf{M}(\alpha) = (\mathbf{M}(\mu); \varepsilon_{m_1}; \ldots; \varepsilon_{m_k})$.*

Proof. Apply Lemma 2.4 (k times). □

Lemma 2.6. *Let $f, d \in \mathbb{N}$, $e = f(p-1)$, and $\mu = (\mu_1, \ldots, \mu_k)$ be a p-regular partition with $\mu_k \geq e + d(p-1)$. Then*

$$\mathbf{M}((\mu; \gamma(d,e))) = (\mathbf{M}(\mu); \beta(d(p-1), f)).$$

In particular,

$$\mathbf{M}(\gamma'(d,e)) = (\nu; \beta((d-2)(p-1), f))$$

where $\nu = \mathbf{M}((l^2))$ *with* $l = e + (d-2)(p-1)$.

Proof. This follows from Lemma 2.5. ☐

Lemma 2.7. *Let* $k \in \mathbb{N}$, *and* $\lambda = (\lambda_1, \dots, \lambda_{p-1})$ *be a partition with* $\lambda_{p-1} \geq kp$ *and* $\lambda_1 - \lambda_{p-1} > 1$. *If* $\mathbf{M}(\lambda) = (\mu_1, \mu_2, \dots)$, *then* $\mu_{p+1} \geq k$.

Proof. Define $\lambda^{(1)} = \lambda$, $\lambda^{(i)} = \lambda^{(i-1)} \backslash \{p\text{-edge of } \lambda^{(i-1)}\}$ for $1 < i \leq k$. Since $\lambda_1 - \lambda_{p-1} > 1$, the number of nodes of the p-edge of $\lambda^{(1)}$ is $n_1 p$ with $n_1 \geq 2$. This implies $\lambda_1^{(2)} - \lambda_{p-1}^{(2)} > 1$. Also $\lambda_{p-1}^{(2)} \geq \lambda_{p-1}^{(1)} - p = (k-1)p$. Therefore we have

$$G(\lambda) = \begin{pmatrix} n_1 p & n_2 p & \cdots & n_k p & \cdots \\ p-1 & p-1 & & p-1 & \end{pmatrix}$$

where $n_i \geq 2$ $(1 \leq i \leq k)$. Hence

$$G(\mathbf{M}(\lambda)) = \begin{pmatrix} n_1 p & n_2 p & \cdots & n_k p & \cdots \\ (n_1 - 1)p + 1 & (n_2 - 1)p + 1 & & (n_k - 1)p + 1 & \end{pmatrix}.$$

Since $(n_i - 1)p + 1 \geq p + 1$ for $1 \leq i \leq k$, we have $\mu_{p+1} \geq k$. ☐

Lemmas on branching. The results of this section rely on the modular branching rules proved in [9, 11], especially Theorem 0.4(ii) from [9] and Theorem 1.4 from [11]. A version of the modular branching rule needed here is as follows (for the definition of *normal* removable nodes see [9, Definition 0.3]).

Theorem 2.8. *Let* A *be a removable node of a* p*-regular partition* λ *such that* λ_A *is also* p*-regular. Then* D^{λ_A} *is a composition factor of* $D^\lambda\downarrow_{\Sigma_{n-1}}$ *if and only if* A *is normal.*

The following two lemmas follow immediately from the modular branching rule.

Lemma 2.9. $D^{\gamma(d,f)} \in \operatorname{Irr}(D^{\gamma(d,f+1)}\downarrow_{\Sigma_n})$ *where* $n = |\gamma(d,f)|$.

Lemma 2.10. $D^{\gamma'(d,f)} \in \operatorname{Irr}(D^{\gamma(d,f)}\downarrow_{\Sigma_n})$ *where* $n = |\gamma'(d,f)|$.

Lemma 2.11 ([1, 4.10]). *Let* $\mu = (\mu_1, \mu_2, \dots)$ *be a* p*-regular partition (*μ *may be* $\emptyset$*),* $f > \mu_1 + 1$. *Set* $\beta = \beta(m, f; \mu)$, $\alpha = \beta(m, f-1; \mu)$, $n = |\alpha|$. *Then*

$$D^\alpha \in \operatorname{Irr}(D^\beta\downarrow_{\Sigma_n}).$$

Proposition 2.12. *Let* $\lambda = (\lambda_1, \dots, \lambda_k)$ *be a* p*-regular partition,* $m \in \{1, 2, \dots, k\}$, *and*

$$\lambda_m > 2 + \sum_{i=m+1}^{k} \lambda_i.$$

Set $f = \lambda_m - 1 - \sum_{i=m+1}^k \lambda_i$, $\beta = \beta(m,f)$, $n = |\beta|$, $\beta' = \beta(m, f+1; (1))$, $n' = |\beta'|$. *Then*

$$D^\beta \in \operatorname{Irr}(D^\lambda\downarrow_{\Sigma_n}).$$

Moreover, if $\lambda_{m+1} \neq 0$, *then*

$$D^{\beta'} \in \operatorname{Irr}(D^\lambda\downarrow_{\Sigma_{n'}}).$$

Proof. The first part is [1, 4.14]. The second part is contained in the proof of [1, 4.14]. ☐

Lemma 2.13. *Let $\mu = (\mu_1, \mu_2, \dots)$ be a non-empty p-regular partition, $f > \mu_1 + 1$, and let A be the top removable node of (the Young diagram of) μ such that μ_A is p-regular. Set $\gamma = \gamma(m, f; \mu)$, $\alpha = \gamma(m, f-1; \mu_A)$, $n = |\alpha|$. Then*

$$D^\alpha \in \mathrm{Irr}(D^\gamma\downarrow_{\Sigma_n}).$$

Proof. The proof is identical to [1, 4.11]. □

Lemma 2.14. *Let $\mu = (\mu_1, \mu_2, \dots)$ be a non-empty p-regular partition, $f > |\mu| + 1$, $e = f - |\mu|$, $n = |\gamma(m, e)|$, $n' = |\gamma(m, e+1; (1))|$. Then*

$$D^{\gamma(m,e+1;(1))} \in \mathrm{Irr}(D^{\gamma(m,f;\mu)}\downarrow_{\Sigma_{n'}}),$$

$$D^{\gamma(m,e)} \in \mathrm{Irr}(D^{\gamma(m,f;\mu)}\downarrow_{\Sigma_n}).$$

Proof. Apply Lemma 2.13 sufficiently many times. □

Lemma 2.15. *Let $\mu = (\mu_1, \dots, \mu_m)$ and $\lambda = (\lambda_1, \dots, \lambda_m, \dots, \lambda_h)$ be p-regular partitions. Set $c = \lambda_{m+1} + \dots + \lambda_h$. Assume that $D^\mu \in \mathrm{Irr}(D^\lambda\downarrow_{\Sigma_{|\mu|}})$. Then $\mu_m \leq \lambda_m + c$.*

Proof. If $D^\nu \in \mathrm{Irr}(D^\rho\downarrow_{\Sigma_{|\rho|-1}})$, then ν dominates ρ_A for some removable node A of ρ. Therefore

$$\nu_m + \nu_{m+1} + \dots \leq (\rho_A)_m + (\rho_A)_{m+1} + \dots \leq \rho_m + \rho_{m+1} + \cdots.$$

The lemma follows by induction. □

Inductive systems. Let G be a countable locally finite group. Then G can be considered as a union

$$G = \cup_{n=1}^\infty G_n$$

where $G_1 \subset G_2 \subset \dots$ are finite subgroups of G.

Definition 2.16. [18, 1.1] Let Φ_n be a non-empty set of the isomorphism classes of irreducible FG_n-modules ($n = 1, 2, \dots$). We say that the collection $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ is an *inductive system* (for G) if for any $n \in \mathbb{N}$ we have

$$\Phi_n = \mathrm{Irr}(\Phi_{n+1}\downarrow_{G_n}).$$

The following result explains our interest in inductive systems. Note that G always has at least two inductive systems: the inductive system $\Gamma = \{\Gamma_n\}_{n\in\mathbb{N}}$ with $\Gamma_n = \mathrm{Irr}\, G_n$ for all n and the *unitary* inductive system with $\Phi_n = \{\mathbf{1}_{G_n}\}$ for all n. Any inductive system distinct from Γ is called *proper*. The following result was proved by the third author, see [18].

Theorem 2.17. *Let $G = \cup_{n=1}^\infty G_n$ be a locally finite group as above. Assume additionally that G is simple. Then the group algebra FG is almost simple if and only there are no proper inductive systems for G, except for the unitary one.*

Proof. Assume there are no proper inductive systems, except for the unitary one. Then by [17, Theorem 1], (see also [18, 1.25]) every proper subideal of the augmentation ideal is contained in the Jacobson radical of FG. But this radical is trivial by a theorem of Passman and Zalesskii [13, 12]. Now since the augmentation ideal has codimension 1, we conclude that FG is almost simple. The converse follows immediately from [18, 1.25]. □

We denote by Σ_∞ the group of all finitary permutations of $\mathbb{N}$. Clearly, Σ_∞ is a locally finite group and

$$\Sigma_\infty = \cup_{n>N}\Sigma_n$$

where Σ_n is the group of all permutations of the set $\{1, 2, \dots, n\}$, and N is any natural number. So we may consider the inductive systems for Σ_∞.

Definition 2.18. Let $\Phi = \{\Phi_n\}_{n>N}$ be an inductive system for Σ_∞. Define the inductive system $\Phi^\sigma = \{\Phi^\sigma_n\}_{n>N}$ by

$$\Phi^\sigma_n = \{D^\lambda \otimes \mathrm{sgn}_n \mid D^\lambda \in \Phi_n\}.$$

One can easily check that Φ^σ so defined is indeed an inductive system.

We shall use some special inductive systems introduced in [10, 1]. For $n > (p-1)^2$ let $\Phi(s)_n$ be the set of all completely splittable Σ_n-modules of height s (see Definition 1.2), and let $\Phi(s) = \{\Phi(s)_n\}_{n>(p-1)^2}$. It has been proved in [1] that $\Phi(1), \Phi(2), \dots, \Phi(p-1)$ are the minimal inductive systems for $\Sigma_\infty = \cup_{n>(p-1)^2}\Sigma_n$.

Lemma 2.19. [1, 5.6] *Let $s \in \{1, 2, \dots, p-1\}$. Then $\Phi(s)^\sigma = \Phi(p-s)$.*

Note that $\Phi(1)_n = \{\mathbf{1}_{\Sigma_n}\}$. It follows from Lemma 2.19 that $\Phi(p-1)_n = \{\mathrm{sgn}_n\}$.

It is convenient to call the inductive systems $\Phi(1)$, $\Phi(p-1)$, and $\Phi(1) \cup \Phi(p-1)$ (for Σ_∞) *trivial*. One of our goals in the next subsection is to describe the *minimal non-trivial* inductive systems for Σ_∞.

Put

$$\Pi_n = \{D^{(n)}, D^{(n-1,1)}\}, \quad n > (p-1)^2.$$

The following result is straightforward.

Lemma 2.20. $\Pi = \{\Pi_n\}_{n>(p-1)^2}$ *is an inductive system for Σ_∞.*

The inductive systems for Σ_∞ consisting of modules D^λ with λ having at most two parts have been classified by Sheth [15]. We need a very special corollary of his result:

Lemma 2.21. [15, Theorem 7.7] *Let $p > 2$. Set*

$$\Theta_n = \{D^{(l_1,l_2)} \mid l_1 + l_2 = n, \quad 0 \le l_1 - l_2 \le p^2 - 2\}.$$

Then $\Theta = \{\Theta_n\}_{n\in\mathbb{N}}$ is an inductive system for Σ_∞, $\Theta \supset \Phi(2)$, and there are no inductive systems between Θ and $\Phi(2)$.

Minimal non-trivial inductive systems. Set

$$\begin{aligned}
\Phi(\beta, m)_n &= \cup_{f>n} \mathrm{Irr}(\beta(m,f){\downarrow_{\Sigma_n}}),\\
\Phi(\beta, m; (1))_n &= \cup_{f>n} \mathrm{Irr}(\beta(m,f;(1)){\downarrow_{\Sigma_n}});\\
\Phi(\beta, m) &= \{\Phi(\beta, m)_n\}_{n>(p-1)^2},\\
\Phi(\beta, m; (1)) &= \{\Phi(\beta, m; (1))_n\}_{n>(p-1)^2}.
\end{aligned}$$

It follows from Lemma 2.11 that $\Phi(\beta, m)$ and $\Phi(\beta, m; (1))$ are inductive systems for the group $\Sigma_\infty = \cup_{n>(p-1)^2}\Sigma_n$ (for any $m \in \mathbb{N}$). Recall the systems $\Phi(1), \Phi(2), \dots, \Phi(p-1)$, and Π, introduced before.

Lemma 2.22. *Let $1 \le m \le p-1$. Then $\Phi(\beta, m) = \Phi(m)$.*

Proof. Clearly, $D^{\beta(m,f)} \in \Phi(m)$ for any f. So the definition of $\Phi(\beta, m)$ ensures that $\Phi(\beta, m) \subseteq \Phi(m)$. Now the result follows from the minimality of $\Phi(m)$. $\square$

Lemma 2.23. $\Phi(\beta, p-1; (1)) = \Pi^\sigma$.

Proof. Let $n = |\beta(p-1, f; (1))|$. By Lemma 2.1 we have $\mathbf{M}(\beta(p-1, f; (1))) = (n-1, 1)$. The lemma follows. $\square$

Definition 2.24. *Let $\Phi = \{\Phi_n\}_{n>N}$ be an inductive system for $\Sigma_\infty = \cup_{n>N}\Sigma_n$. We define its height $h(\Phi)$ as*

$$h(\Phi) = \sup\{h(\lambda) \mid D^\lambda \in \Phi_n \quad \text{for some } n > N\}.$$

Proposition 2.25 ([1, 5.2]). *Let $\Phi = \{\Phi_n\}_{n>N}$ be an inductive system for the group $\Sigma_\infty = \cup_{n>N}\Sigma_n$. Assume that $h(\Phi) = +\infty$. Then $\Phi_n = \operatorname{Irr}\Sigma_n$ for all $n > N$.*

The proposition above shows that all proper inductive systems for Σ_∞ have finite heights. So assume from now on that Φ is an inductive system with $h(\Phi) < +\infty$. It follows that for any $M \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and $D^\lambda \in \Phi_n$ such that $\lambda_1 > M$ (in other words, the first part of the partitions involved gets arbitrarily large). This shows that the set

$$V(\Phi) = \{k \in \mathbb{N} \mid \text{ for any } M \in \mathbb{N} \text{ there exists } n \in \mathbb{N} \text{ and } D^\lambda \in \Phi_n \\ \text{with } \lambda = (\lambda_1, \lambda_2 \dots) \text{ such that } \lambda_k > M\}$$

is not empty (it contains $k = 1$). Since also $V(\Phi) \subseteq \{1, 2, \dots, h(\Phi)\}$ we can define

$$m(\Phi) = \max V(\Phi).$$

It follows from the definition of $m = m(\Phi)$ and $h = h(\Phi)$ that there exists $c = c(\Phi) \in \mathbb{N}$ such that

$$\sum_{j=m+1}^{h} \lambda_j \leq c \tag{3}$$

for any $D^\lambda \in \Phi_n$ (for any n). Let $N_0 \geq c + 2$. By definition of m, there exists $n \in \mathbb{N}$ and $D^\lambda \in \Phi_n$ such that $\lambda_m > N_0$. Set

$$\begin{aligned} f &= \lambda_m - \sum_{j=m+1}^{h} \lambda_j - 1, \\ \beta &= \beta(m, f), \quad n = |\beta|, \\ \beta' &= \beta(m, f+1; (1)), \quad n' = |\beta'|. \end{aligned}$$

By Proposition 2.12,

$$D^\beta \in \operatorname{Irr}(D^\lambda\downarrow_{\Sigma_n}) \subseteq \Phi_n.$$

Moreover, if $\lambda_{m+1} \neq 0$, then

$$D^{\beta'} \in \operatorname{Irr}(D^\lambda\downarrow_{\Sigma_{n'}}).$$

So we have

Lemma 2.26. *Let Φ be an inductive system for Σ_∞, and let $m = m(\Phi)$. Then $\Phi \supseteq \Phi(\beta, m)$. Moreover, if for each $M \in \mathbb{N}$ there exists a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $D^\lambda \in \Phi_{|\lambda|}$, $\lambda_m > M$, and $\lambda_{m+1} \neq 0$, then $\Phi \supseteq \Phi(\beta, m; (1))$.*

Lemma 2.27. *Let $p > 2$, $m = m(\Phi)$. Assume that $m = d(p-1) + r$ for $d \geq 1$ and $0 < r < p - 1$. Then $\Phi \supset \Phi(\beta, d(p-1))$.*

Proof. By Lemma 2.26, $\Phi \supseteq \Phi(\beta, m)$. It follows from Lemma 2.3 that the inductive system Φ^σ contains $D^{\gamma(d,(f+1)(p-1);\mu(f))}$ for all $f \in \mathbb{N}$, where $\mu(f) = \mathbf{M}((f^r))$. Note that $|\mu(f)| = fr$, whence $(f+1)(p-1) > |\mu(f)| + 1$. So, by Lemma 2.14, Φ^σ contains $D^{\gamma(d,(f+1)(p-1)-fr)}$ for all $f \in \mathbb{N}$. Since $(f+1)(p-1) - fr \to +\infty$ as $f \to +\infty$, Lemma 2.9 implies that Φ^σ contains all $D^{\gamma(d,e)}$, $e \in \mathbb{N}$. In particular, Φ^σ contains all $D^{\gamma(d,f(p-1))}$, $f \in \mathbb{N}$. Therefore, by Lemma 2.6, the inductive system $\Phi = (\Phi^\sigma)^\sigma$ contains all $D^{\beta(d(p-1),f)}$, $f \in \mathbb{N}$, so $\Phi \supset \Phi(\beta, d(p-1))$. □

Lemma 2.28. *Let $p > 2$, $m = m(\Phi) = d(p-1)$ for $d \geq 2$. Then $\Phi \supset \Phi(\beta, m - p)$.*

Proof. As in the proof of Lemma 2.27 we see that Φ^σ contains $D^{\gamma(d,f(p-1))}$ for all $f \in \mathbb{N}$. By Lemma 2.10, Φ^σ contains all $D^{\gamma'(d,f(p-1))}$, $f \in \mathbb{N}$. Set

$$\mu(f) = \mathbf{M}(\gamma'(d, f(p-1)).$$

By Lemma 2.6,

$$\mu(f) = (\nu(f); \beta((d-2)(p-1), f))$$

where

$$\nu(f) = \mathbf{M}((l^2)) \in \Phi(2)^\sigma, \quad l = f(p-1) + (d-2)(p-1).$$

By Lemma 2.19, $h(\nu(f)) = p-2$, so

$$h(\mu(f)) = p - 2 + (d-2)(p-1) = m - p.$$

Observe that the length of the last row of $\mu(f)$ is f. By Proposition 2.12 (applied with $m = k$), $D^{\beta(m-p,f-1)} \in \mathrm{Irr}(D^{\mu(f)}{\downarrow_{\Sigma_{|\beta(m-p,f-1)|}}})$. Therefore $\Phi \supset \Phi(\beta, m-p)$. $\square$

Theorem 2.29. *Let F be a field of characteristic $p > 2$.*

(i) *Let $p > 3$. Then any non-trivial inductive system contains one of the following systems:*

$$\Pi, \quad \Pi^\sigma, \quad \Phi(s)\ (1 < s < p-1). \tag{4}$$

(ii) *Let $p = 3$. Then any non-trivial inductive system contains one of the following systems:*

$$\Pi, \quad \Pi^\sigma, \quad \Theta, \quad \Theta^\sigma. \tag{5}$$

Proof. Let Φ be a proper non-trivial inductive system. In view of Lemma 2.19, it suffices to prove that either Φ or Φ^σ contains one of the inductive systems (4) (or (5) if $p = 3$). So we may assume without loss of generality that $m(\Phi) \geq m(\Phi^\sigma)$. Set $m = m(\Phi)$. Consider the following cases.

Case 1: $m = 1$. Since $\Phi \neq \Phi(1)$, we have $h(\Phi) > 1$. Therefore for each i there exists a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $D^\lambda \in \Phi_i$ and $\lambda_2 \neq 0$. Hence by Lemma 2.26, $\Phi \supseteq \Phi(\beta, 1; (1)) = \Pi$.

Case 2: $1 < m < p-1$. By Lemmas 2.26 and 2.22, we have $\Phi \supseteq \Phi(\beta, m) = \Phi(m)$.

Case 3: $m = p-1$. We may assume that Φ does not contain $\Phi(\beta, p-1; (1)) = \Pi^\sigma$ (see Lemma 2.23 for the last equality). So, in view of Lemma 2.26, there exists $N_1 \in \mathbb{N}$ such that for any partition $\lambda = (\lambda_1, \lambda_2, \dots)$ with $D^\lambda \in \Phi_{|\lambda|}$ and $\lambda_p \neq 0$ we have $\lambda_{p-1} < N_1$.

Furthermore, take $D^\lambda \in \Phi_{|\lambda|}$ with λ_{p-1} very large. By Lemma 2.7, $D^\lambda \otimes \mathrm{sgn} = D^\mu$ with μ_{p+1} very large, unless $\lambda_1 - \lambda_{p-1} \leq 1$. But Φ^σ may not contain D^μ with arbitrarily large μ_{p+1} since then $m(\Phi^\sigma) \geq p+1 > m(\Phi)$ which contradicts our assumptions. Thus, there exists $N_2 > N_1$ such that for each λ with $D^\lambda \in \Phi_{|\lambda|}$ and $\lambda_{p-1} \geq N_2$ we have $\lambda_1 - \lambda_{p-1} \leq 1$, and so $\lambda = \varepsilon_{|\lambda|}$, as in addition $\lambda_p = 0$.

By Lemma 2.26, $\Phi \supseteq \Phi(\beta, p-1)$. But $\Phi(\beta, p-1)_n = \{\mathrm{sgn}_n\}$. Set

$$\Psi_n = \Phi_n \backslash \{\mathrm{sgn}_n\}.$$

We are going to show that $\Psi = \{\Psi_n\}_{n>N}$ is an inductive system for $\Sigma_\infty = \cup_{n>N} \Sigma_n$, provided N is large enough. It suffices to check that $\mathrm{sgn}_j \notin \mathrm{Irr}(\Psi_i{\downarrow_{\Sigma_j}})$ for any pair $i > j > N$. Assume not, i.e. there exists $\lambda = (\lambda_1, \lambda_2, \dots)$ with $D^\lambda \in \Psi_i$ such that $\mathrm{sgn}_j \in \mathrm{Irr}(D^\lambda{\downarrow_{\Sigma_j}})$. Let $c = c(\Phi)$, see (3). Since N is large enough, we may assume that the length of the $(p-1)$-st row of ε_j is at least $N_2 + c$. So, by Lemma 2.15, we have $\lambda_{p-1} \geq N_2$. Hence by the choice of N_2, $D^\lambda = \mathrm{sgn}_i \notin \Psi_i$. This yields a contradiction. So Ψ is an inductive system. Observe that $\Psi \cup \Phi(p-1) = \Phi$. Therefore Ψ is non-trivial. Since

$\lambda_{p-1} < N_2$ for each partition λ with $D^\lambda \in \Psi_{|\lambda|}$, we have $m(\Psi) < p-1$. So Ψ (and Φ) contains one of the systems (4) by what has already been proved.

Case 4: $p-1 < m < 2(p-1)$. Set $r = m-(p-1)$. By Lemma 2.26, $\Phi \supseteq \Phi(\beta, m)$. It follows from Lemma 2.3 that Φ^σ contains all $D^{\gamma(1,(f+1)(p-1);\mu(f))}$, $f \in \mathbb{N}$, where $\mu(f) = \mathbf{M}((f^r))$. We have $|\mu(f)| = fr$. Hence

$$(f+1)(p-1) > |\mu(f)| + 1.$$

So Lemma 2.14 implies that Φ^σ contains all $D^{\gamma(1,n(f);(1))}$, $f \in \mathbb{N}$, where $n(f) = (f+1)(p-1) - fr + 1$. Note that

$$D^{\gamma(1,n(f);(1))} = D^{(n(f),1)} \in \Pi_{n(f)+1}.$$

Since $n(f) \to +\infty$ as $f \to +\infty$, $\Phi^\sigma \supset \Pi$, so $\Phi \supset \Pi^\sigma$, as desired.

Case 5: $p > 3$ and $m = 2(p-1)$. Then by Lemmas 2.28 and 2.22, $\Phi \supseteq \Phi(\beta, p-2) = \Phi(p-2)$, as desired.

Case 6: $p = 3$ and $m = 2(p-1) = 4$. By Lemma 2.26, $\Phi \supseteq \Phi(\beta, 4)$. Recall that $\Phi(\beta, 4)$ is generated by all $\beta(4, f) = (\varepsilon_{2(f+1)}; \varepsilon_{2f})$, $f \in \mathbb{N}$. Therefore by Lemma 2.2, $\Phi(\beta, 4)^\sigma$ is generated by all partitions $\eta(f) = (2f+2, 2f)$, $f \in \mathbb{N}$. We shall use Lemma 2.21. Note that

$$D^{\eta(f)} \in (\Theta_{4f+2} \backslash \Phi(2)_{4f+2})$$

for all f. Since there are no inductive systems between Θ and $\Phi(2)$, we have $\Phi(\beta, 4)^\sigma = \Theta$. It follows that $\Phi \supseteq \Theta^\sigma$, as required.

Case 7: $m > 2(p-1)$. Represent m in the form $m = d(p-1) + r$ where $0 \leq r < p-1$. Assume that $r = 0$. Then $d \geq 3$. By Lemma 2.28, $\Phi(\beta, m-p) \subset \Phi$. Since $m(\Phi(\beta, m-p)) = m - p > p-1$, $\Phi(\beta, m-p)$ is non-trivial, and we can apply induction. If $r \neq 0$, then the theorem follows from Lemma 2.27. $\square$

Set

$$\mathcal{T}_n = \begin{cases} \cup_{s=2}^{p-2} \Phi(s)_n \cup \{D^{(n-1,1)}, D^{(n-1,1)} \otimes \mathrm{sgn}_n\}, & \text{if } p > 3 \\ \Theta_n \cup \Theta_n^\sigma \cup \{D^{(n-1,1)}, D^{(n-1,1)} \otimes \mathrm{sgn}_n\}, & \text{if } p = 3. \end{cases}$$

Corollary 2.30. *Let $p > 2$ and $n > (p-1)^2$. Then there exists $M = M(n) > n$ such that, for any $i \geq M$, the restriction of any faithful irreducible Σ_i-module to Σ_n contains a composition factor $D \in \mathcal{T}_n$.*

Proof. For each $i \geq n$ denote by Ψ_i the set of all $D^\lambda \in \operatorname{Irr} \Sigma_i$ such that

$$\operatorname{Irr}(D^\lambda {\downarrow}_{\Sigma_n}) \cap \mathcal{T}_n = \emptyset$$

Observe that for all $i \geq n$ we have $\operatorname{Irr}(\Psi_{i+1} {\downarrow}_{\Sigma_i}) \subseteq \Psi_i$. For $i \geq n$ we set

$$\Phi_i = \cap_{j \geq i} \operatorname{Irr}(\Psi_j {\downarrow}_{\Sigma_i})$$

We show that $\Phi = \{\Phi_i\}_{i \geq n}$ is an inductive system. Let $i \geq n$. Note that

$$\Psi_i \supseteq \operatorname{Irr}(\Psi_{i+1} {\downarrow}_{\Sigma_i}) \supseteq \operatorname{Irr}(\Psi_{i+2} {\downarrow}_{\Sigma_i}) \supseteq \dots$$

Since the set Ψ_i is finite, there exists $k = k(i)$ such that for any $j \geq k$ we have

$$\Phi_i = \operatorname{Irr}(\Psi_j {\downarrow}_{\Sigma_i}). \tag{6}$$

Choose $j \geq \max(k(i), k(i+1))$. Then

$$\operatorname{Irr}(\Phi_{i+1} {\downarrow}_{\Sigma_i}) = \operatorname{Irr}(\operatorname{Irr}(\Psi_j {\downarrow}_{\Sigma_{i+1}}) {\downarrow}_{\Sigma_i}) = \operatorname{Irr}(\Psi_j {\downarrow}_{\Sigma_i}) = \Phi_i,$$

so Φ is an inductive system.

If the corollary is false, there exists $j \geq k(n)$ such that Ψ_j contains a faithful module D. In view of (6), Φ_n contains a faithful module. Therefore Φ is non-trivial. By Theorem 2.29,

Φ contains one of the systems (4). Therefore $\Phi_n \cap \mathcal{T}_n \neq \emptyset$. On the other hand, since $\Phi_n \subseteq \Psi_n$,

$$\Phi_n \cap \mathcal{T}_n \subseteq \Psi_n \cap \mathcal{T}_n = \emptyset.$$

The contradiction obtained proves the corollary. $\square$

Now parts (ii) and (iii) of the Modular Asymptotic Theorem stated in Section 1 follow from Corollary 2.30. Part (i) follows from the following proposition.

Proposition 2.31. *Let $p > 2$ and $n > (p-1)^2$. There exists $N = N(n) > n$ such that, for any $i \geq N$, the restriction of any simple Σ_i-module to Σ_n contains a completely splittable D^λ as a composition factor (see Section 1).*

Proof. The proof is similar to that of Corollary 2.30, but uses [1, 5.7] instead of Theorem 2.29. $\square$

3. Diagonal and Non-diagonal groups

Throughout this section we denote by M_n and V_n the natural permutation module (of dimension n) and the natural irreducible module, respectively, for both FA_n and $F\Sigma_n$. If G and H are two groups, L is an FG-module and M is an FH-module we write $L \boxtimes M$ for the *outer* tensor product of L and M (which is a module over $G \times H$). If N is another FG-module we write $L \otimes N$ for the *inner* tensor product of L and N (which is a G-module).

Diagonal embeddings. In this subsection we prove the easier direction of Theorem A.

Any diagonal embedding $A_n \to A_m$ is obtained by composing a diagonal embedding $A_n \to \underbrace{A_n \times \cdots \times A_n}_{k \text{ times}}$, the embedding $\underbrace{A_n \times \cdots \times A_n}_{k \text{ times}} \to A_{kn}$ as a 'Young subgroup', and the natural embedding $A_{kn} \to A_m$ ($m \geq kn$). So

Lemma 3.1. *Let $\varphi : A_n \to A_m$ be a diagonal embedding. Then the only possible composition factors of the restriction $V_m{\downarrow}_{A_n}$ are V_n and $\mathbf{1}_{A_n}$. Moreover, V_n always occurs.*

Proof. Note that the restriction of V_m to A_{kn} has composition factors V_{kn} and $\mathbf{1}_{A_{kn}}$, and V_{kn} always occurs ($\mathbf{1}_{A_{kn}}$ also occurs providing $kn < m$). It now suffices to observe that all composition factors of the restriction of V_{kn} to $A_n \times \cdots \times A_n$ are of the form

$$\mathbf{1}_{A_n} \boxtimes \cdots \boxtimes \mathbf{1}_{A_n} \quad \text{or} \quad \mathbf{1}_{A_n} \boxtimes \cdots \boxtimes \mathbf{1}_{A_n} \boxtimes V_n \boxtimes \mathbf{1}_{A_n} \boxtimes \cdots \boxtimes \mathbf{1}_{A_n},$$

and modules of the second form occur. $\square$

The next lemma is only needed in the proof of Proposition 3.10.

Lemma 3.2. *$M_k \boxtimes F \boxtimes F$ is a subfactor of the restriction $V_{3k}{\downarrow}^{\Sigma_{3k}}_{\Sigma_k \times \Sigma_k \times \Sigma_k}$.*

Proof. Let $\{e_1, e_2, \ldots, e_{3k}\}$ be the natural basis of M_{3k}. Set L to be the 1-dimensional submodule $F(e_1+e_2+\cdots+e_{3k})$ if the characteristic p divides $3k$, and set $L = (0)$ otherwise. Denote $\bar{e}_j = e_j + L \in M_{3k}/L$. Then V_{3k} is a submodule of M_{3k}/L (of codimension 1) generated by the elements $\bar{e}_i - \bar{e}_{3k}$, $1 \leq i \leq 3k-1$. Set

$$N_1 = \mathrm{span}\{\bar{e}_i - \bar{e}_{3k} \mid 2k+1 \leq i \leq 3k-1\}, \quad N_2 = \mathrm{span}\{N_1, \bar{e}_i - \bar{e}_{3k} \mid 1 \leq i \leq k\}.$$

Then $N_1 \subset N_2$ are submodules of $V_{3k}{\downarrow}^{\Sigma_{3k}}_{\Sigma_k \times \Sigma_k \times \Sigma_k}$, and $N_2/N_1 \cong M_k \boxtimes F \boxtimes F$. $\square$

Proposition 3.3. *Assume that G is a diagonal limit alternating group. Then FG is not almost simple.*

Proof. By Theorem 2.17, it is enough to construct a proper inductive system for G, different from the unitary one. We may assume that all embeddings in (1) are diagonal (just take the union starting from i large enough, which does not change the group G). Set $\Phi_n = \{\mathbf{1}_{\mathrm{Alt}(\Omega_n)}, V_{|\Omega_n|}\}$ for all n. Then $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ is a (proper non-unitary) inductive system, thanks to Lemma 3.1. $\square$

Non-diagonal embeddings. The main result of this subsection is Theorem 3.12, which claims that a non-diagonal limit alternating group G has no proper inductive systems different from the unitary one. Since G is simple, Theorem A follows from Proposition 3.3, Theorem 3.12 and Theorem 2.17.

Lemma 3.4. *Let X be a finite group. Then each irreducible $\Sigma_n \times X$-module is of the form $D \boxtimes L$ with $D \in \operatorname{Irr}\Sigma_n$ and $L \in \operatorname{Irr} X$.*

Proof. Since any field is a splitting field for Σ_n, the lemma follows from [5, 7.10 and 10.38]. $\square$

We cite an important result of [14]

Lemma 3.5. [14, Theorem 1.7] *Let $\mathrm{Alt}(\Omega_1) \subset \mathrm{Alt}(\Omega_2) \subset \cdots \subset \mathrm{Alt}(\Omega_i) \subset \dots$ be non-diagonal embeddings. Then for every $i \in \mathbb{N}$ there exists $N = N(i) > i$ such that, for any $n > N$, the group $\mathrm{Alt}(\Omega_i)$ has a regular orbit on Ω_n.*

We shall use the following result of Bryant and Kovacs [4] (the proof can also be found in [6, III.2.16]).

Lemma 3.6. [4] *Let X be a finite group and let V_g be an FX-module on which g does not act as a scalar, for all $g \in X \setminus \{1\}$. Then the regular module FX is a direct summand of $\otimes_{g\in X\setminus\{1\}} V_g$.*

Lemma 3.7. *Let $k, n \in \mathbb{N}$, $n > (p-1)^2$, and λ be a completely splittable partition of kn of height s. Denote by H the Young subgroup $\underbrace{\Sigma_n \times \cdots \times \Sigma_n}_{k \text{ times}} < \Sigma_{kn}$. Then any composition factor of the restriction $D^\lambda{\downarrow}_H$ is of the form*

$$D^{\lambda(1)} \boxtimes \cdots \boxtimes D^{\lambda(k)}$$

where $\lambda(1), \dots \lambda(k)$ are completely splittable partitions of n of height s.

Proof. It is proved in [10] that the composition factors of the restriction $D^\lambda{\downarrow}_{\Sigma_n}$ are of the form D^μ where μ is a completely splittable partition of n of height s. Hence the composition factors of the restriction $D^\lambda{\downarrow}_{\Sigma_n\times\cdots\times\Sigma_n}$ are as claimed. $\square$

The case $p = 3$ in the Asymptotic Theorem is somewhat exceptional. We need the following lemma for this case only.

Lemma 3.8. *Let $p = 3$; $k, n \in \mathbb{N}$; $n > 9$. We say a partition λ is nice if $\lambda = (\lambda_1, \lambda_2)$ and $1 < \lambda_1 - \lambda_2 \leq 7$. Let λ be a nice partition of kn. Denote by H the Young subgroup $\underbrace{\Sigma_n \times \cdots \times \Sigma_n}_{k \text{ times}} < \Sigma_{kn}$. Then the restriction $D^\lambda{\downarrow}_H$ has a composition factor of the form $D^{\lambda(1)} \boxtimes \cdots \boxtimes D^{\lambda(k)}$ where $\lambda(1), \dots, \lambda(k)$ are nice.*

Proof. Without loss of generality we may assume that $k = 2$. If $\lambda = (\lambda_1, \lambda_2)$ is a partition of r and $s = \lambda_1 - \lambda_2$, we shall write (in this proof only) $D_s^{(r)}$ for D^λ. Also let $N \sim m_1 D_1 + \cdots + m_l D_l$ mean that $D_1, \dots, D_l$ are the composition factors of the module N

with multiplicities $m_1, \dots, m_l$, respectively. We use the following special case of the main result from [15]: if $r > 9$ then

$$\begin{aligned}
D_0^{(r)}\downarrow_{\Sigma_{r-1}} &\sim D_1^{(r-1)};\\
D_1^{(r)}\downarrow_{\Sigma_{r-1}} &\sim D_0^{(r-1)};\\
D_2^{(r)}\downarrow_{\Sigma_{r-1}} &\sim D_1^{(r-1)} + 2D_3^{(r-1)} + D_7^{(r-1)};\\
D_3^{(r)}\downarrow_{\Sigma_{r-1}} &\sim D_2^{(r-1)} + D_4^{(r-1)};\\
D_4^{(r)}\downarrow_{\Sigma_{r-1}} &\sim D_3^{(r-1)};\\
D_5^{(r)}\downarrow_{\Sigma_{r-1}} &\sim D_4^{(r-1)} + 2D_6^{(r-1)};\\
D_6^{(r)}\downarrow_{\Sigma_{r-1}} &\sim D_5^{(r-1)} + D_7^{(r-1)};\\
D_7^{(r)}\downarrow_{\Sigma_{r-1}} &\sim D_6^{(r-1)}.
\end{aligned}$$

It follows that the composition factors of $D^\lambda\downarrow_{\Sigma_n\times\Sigma_n}$ are all of the form $D_{s_1}^{(n)} \boxtimes D_{s_2}^{(n)}$ where $s_1, s_2 \le 7$. Moreover, for any $1 < s \le 7$ and $9 \le m < 2n$, we get by induction on $2n - m$ that the restriction $D_s^{(2m)}\downarrow_{\Sigma_m}$ contains more composition factors $D_t^{(m)}$ with $1 < t \le 7$ than composition factors $D_t^{(m)}$ with $0 \le t \le 1$. Since $D_1^{(m)}$ and $D_0^{(m)}$ are the sign modules sgn_m (for m odd and even, respectively), they are 1-dimensional. Now we can conclude that some $D_{s_1}^{(n)} \boxtimes D_{s_2}^{(n)}$ with $1 < s_1, s_2 \le 7$ is a composition factor of $D^\lambda\downarrow_{\Sigma_n\times\Sigma_n}$. □

Let $\varphi_1, \dots, \varphi_{d-1} : X \to \Sigma_k$ be embeddings of a group X of order d into symmetric groups and $\varphi : X \to \Sigma_r$ be any homomorphism. Consider the embedding

$$X \to \underbrace{\Sigma_k \times \cdots \times \Sigma_k}_{d-1 \text{ times}} \times \Sigma_r,\ g \mapsto (\varphi_1(g), \dots, \varphi_{d-1}(g), \varphi(g)). \tag{7}$$

Then the following lemma holds.

Lemma 3.9. *Let $D_1, \dots, D_{d-1}$ be faithful irreducible Σ_k-modules ($k > 4$), and D be a Σ_r-module. Then the restriction $(D_1 \boxtimes \cdots \boxtimes D_{d-1} \boxtimes D)\downarrow_X$ contains the regular FX-module as a direct summand.*

Proof. We have

$$D_1 \boxtimes \cdots \boxtimes D_{d-1} \boxtimes D\downarrow_X \cong (D_1\downarrow_{\varphi_1(X)}) \otimes \cdots \otimes (D_{d-1}\downarrow_{\varphi_{d-1}(X)}) \otimes (D\downarrow_{\varphi(X)}).$$

Since D_i is faithful, Σ_k is a subgroup of $\operatorname{End} D_i$. As Σ_k has trivial center, each $1 \ne g \in X$ does not act as a scalar on D_i. Therefore by Lemma 3.6, $(D_1\downarrow_{\varphi_1(X)}) \otimes \cdots \otimes (D_{d-1}\downarrow_{\varphi_{d-1}(X)})$ contains the regular X-module as a direct summand. □

Proposition 3.10. *Let $p > 2$ and let X be a subgroup of $\mathrm{Sym}(\Omega)$, $d = |X|$ and $d > (p-1)^2$. Assume that X has at least $d-1$ regular orbits on the set Ω, and $|\Omega| \gg d(d-1)$. Then for any faithful $\mathrm{Sym}(\Omega)$-module D we have $\mathrm{Irr}(D\downarrow_X) = \mathrm{Irr}(X)$.*

Proof. Set $n = |\Omega|$. We may assume that $\Omega = \{1, 2, \dots, n\}$, and

$$\{1, 2, \dots, d\},\ \{d+1, \dots, 2d\},\ \dots,\ \{(d-2)d+1, \dots, (d-1)d\}$$

are regular orbits of X on Ω.

Assume first that $p > 3$. By Theorem B, the restriction $D\downarrow_{\Sigma_{d(d-1)}}$ contains (as a composition factor) either the natural module $V = V_{d(d-1)}$ or V^σ or a completely splittable module C of height s with $1 < s < p-1$. So, if we put $r = n - d(d-1)$ then the restriction of D to the Young subgroup $\Sigma_{d(d-1)} \times \Sigma_r$ must contain a module of the form $V \boxtimes B$ or $V^\sigma \boxtimes B$ or $C \boxtimes B$ where B is some Σ_r-module.

In the first two cases it follows from Lemma 3.2 that the restriction of D to the Young subgroup $H = \underbrace{\Sigma_d \times \cdots \times \Sigma_d}_{(d-1) \text{ times}} \times \Sigma_r$ contains a subfactor $M_d \boxtimes F \boxtimes \cdots \boxtimes F \boxtimes B$ or $M_d^\sigma \boxtimes F \boxtimes \cdots \boxtimes F \boxtimes B$. Note that the image of X in $\mathrm{Sym}(\Omega)$ is contained in H. Since $\{1, 2, \ldots, d\}$ is a regular orbit of X, the restriction of D to X contains a tensor product of the regular module with some other module, hence contains the regular module as a direct summand. In particular, we get the claim of the lemma.

In the second case, Proposition 1.3 shows that the restriction of D to H contains a composition factor of the form $C_1 \boxtimes \cdots \boxtimes C_{d-1} \boxtimes B$ where all modules C_i are faithful completely splittable. So the result follows from Lemma 3.9.

If $p = 3$ then, by Theorem B, we have two possibilities: either the restriction $D{\downarrow}_{\Sigma_{d(d-1)}}$ contains the natural module $V = V_{d(d-1)}$ or V^σ or one of the modules module D^λ, $D^\lambda \otimes \mathrm{sgn}_n$ where $\lambda = (\lambda_1, \lambda_2)$ with $1 < \lambda_1 - \lambda_2 \leq 7$. The first case is considered as for $p > 3$. In the second case Lemma 3.8 ensures that the restriction of D to H contains a composition factor of the form $C_1 \boxtimes \cdots \boxtimes C_{d-1} \boxtimes B$ with faithful modules $C_1, \ldots, C_{d-1}$. So we again can argue as in the case $p > 3$. ☐

Corollary 3.11. *Let $p > 2$, and let X be a subgroup of $\mathrm{Alt}(\Omega)$, $d = |X|$ and $d > (p-1)^2$. Assume that X has at least $2(d-1)$ regular orbits on the set Ω, and $|\Omega| \gg 2d(d-1)$. Then, for any faithful $\mathrm{Alt}(\Omega)$-module E we have $\mathrm{Irr}(E{\downarrow}_X) = \mathrm{Irr}(X)$.*

Proof. Obviously, there are X-invariant subsets Ω_1, Ω_2, and Ω_3 of Ω such that $|\Omega_1| = |\Omega_2| = n \gg d(d-1)$, Ω_i $(i = 1, 2)$ involves at least $d - 1$ regular X-orbits, and the permutation representations $X \to \mathrm{Sym}(\Omega_1)$ and $X \to \mathrm{Sym}(\Omega_2)$ are equivalent. Therefore

$$X < H = \{(\sigma, \sigma, \tau) \mid \sigma \in \Sigma_n, \quad \tau \in \mathrm{Alt}(\Omega_3)\} < \mathrm{Alt}(\Omega).$$

Denote $\Sigma = \{(\sigma, \sigma, 1) \mid \sigma \in \Sigma_n\}$. Then $\Sigma \cong \Sigma_n$ and $H = \Sigma \times \mathrm{Alt}(\Omega_3)$. Clearly, the restriction $E{\downarrow}_\Sigma$ has a faithful composition factor. Therefore the restriction of E to H contains a subfactor $D \boxtimes B$ with faithful $D \in \mathrm{Irr}\,\Sigma$. Now by Proposition 3.10, $\mathrm{Irr}(D{\downarrow}_X) = \mathrm{Irr}\,X$. Therefore $\mathrm{Irr}(E{\downarrow}_X) = \mathrm{Irr}\,X$, as required. ☐

Theorem 3.12. *Let $p > 2$ and G be a non-diagonal limit alternating group. Then G has no proper inductive systems, different from the unitary one.*

Proof. Let $G_i = \mathrm{Alt}(\Omega_i)$, i=1,2, ..., and $\Phi = \{\Phi_i\}_{i \in \mathbb{N}}$ be an inductive system for $G = \bigcup_{i \in \mathbb{N}} G_i$. Assume that Φ_i contains a faithful G_i-module (i.e. a module different from $\mathbf{1}_{G_i}$). Then Φ_j contains a faithful G_j-module for any $j \geq i$. We have to prove that in this case $\Phi_i = \mathrm{Irr}(G_i)$ for every $i \in \mathbb{N}$. We may assume that i is large enough since for any $k < i$ we have $\mathrm{Irr}((\mathrm{Irr}\,G_i){\downarrow}_{G_k}) = \mathrm{Irr}(G_k)$. By Lemma 3.5, the group G_i has arbitrarily many regular orbits on Ω_j, provided j is big enough. Now Corollary 3.11 shows that the restriction of any faithful irreducible G_j-module to G_i contains all irreducible G_i-modules as composition factors, provided j is big enough. ☐

It is well known that any countably infinite, locally finite, simple group G is a union of finite subgroups $\{G_i \mid i \in \mathbb{N}\}$ such that for each i we have $G_i \subset G_{i+1}$ and $G_i \cap N_{i+1} = \{1\}$ for some maximal normal subgroup N_{i+1} of G_{i+1}. The set of pairs $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbb{N}\}$ is called a *Kegel sequence* for G. Assume that $G_i/N_i \cong \mathrm{Alt}(\Omega_i)$ for all i. We say that G is *diagonal* if there exists $n \in \mathbb{N}$ such that for all $i > n$ the embedding $G_i \subset G_{i+1}$ is diagonal, i.e. for every point $\alpha \in \Omega_{i+1}$ the index $|G_i : (G_i)_\alpha N_i|$ is 1 or $|\Omega_i|$, and *non-diagonal* otherwise. Here $(G_i)_\alpha$ denotes the stabilizer of α in the natural action of G_i on Ω_{i+1}. Assume that $\mathrm{char}\,F = p > 0$ and all N_i are p-groups. Then N_i acts trivially on any irreducible G_i-module, in particular $\mathrm{Irr}\,G_i$ can be identified with $\mathrm{Irr}\,\mathrm{Alt}(\Omega_i)$. Clearly, if G is diagonal then it has a proper inductive system different from the unitary one (cf.

Proposition 3.3), so FG is not almost simple. The converse is also true. We have the following general result.

Theorem 3.13. *Let $\{(G_i, N_i) \mid i \in \mathbb{N}\}$ be a Kegel sequence for G with $G_i/N_i \cong \mathrm{Alt}(\Omega_i)$ for all i. Assume that G is non-diagonal and one of the following holds:*

(a) $\operatorname{char} F = 0$ *and each N_i is abelian;*
(b) $\operatorname{char} F = p > 0$, *each N_i is abelian and is not a p-group;*
(c) $\operatorname{char} F = p > 2$ *and each N_i is a p-group.*

Then FG is almost simple.

Proof. The cases (a) and (b) have been proved in [14, 1.6]. Consider the case (c). By [14, 1.7 and 3.1], for each i and k there exists $n > i$ such that G_i has at least k regular orbits on Ω_n. Now, using Corollary 3.11 as in the proof of Theorem 3.12, we conclude that G has no proper inductive systems different from the unitary one, so FG is almost simple. □

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