Isomorphisms of symplectic planes

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Abstract. Every nondesarguesian symplectic spread is also symplectic over its kernel. Any equivalence of nondesarguesian symplectic spreads preserves the resulting symplectic structures over the kernels.

1 Introduction

A *spread* of a $2n$-dimensional vector space V is a set Σ of n-dimensional subspaces such that each nonzero vector is in one and only one member of Σ. A spread determines an affine plane $\mathfrak{A}(\Sigma)$, whose points are the vectors and whose lines are the translates of the members of Σ. A spread Σ of V is called *symplectic* if there is a nondegenerate alternating bilinear form $(\ ,\ )$ on V such that each $X \in \Sigma$ is totally isotropic: $(X, X) = 0$. Then $\mathfrak{A}(\Sigma)$ is called a *symplectic plane*. Finite symplectic planes have been studied, for example, in [5, 2, 9, 6, 7, 8, 1, 4]; finite ones in characteristic 2 have properties not shared by planes defined using arbitrary spreads.

The *kernel* $\mathbf{K}(\mathfrak{A}(\Sigma))$ of the spread Σ or plane $\mathfrak{A}(\Sigma)$ is the set of all additive endomorphisms g of V such that $Xg \subseteq X$ for all $X \in \Sigma$. This is the largest skew field $\mathbf{K}$ over which V can be viewed as a vector space $V_{\mathbf{K}}$ such that Σ consists of subspaces. Note that $\mathfrak{A}(\Sigma)$ and Σ are nondesarguesian precisely when $\dim V_{\mathbf{K}} > 2$.

Two spreads of V are called *equivalent* if there is an invertible semilinear transformation sending one to the other. Our goal is to study this notion in the case of symplectic spreads.

Theorem 1. *Let Σ_1 and Σ_2 be nondesarguesian symplectic spreads in a K-space V equipped with a nondegenerate alternating bilinear form $(\ ,\ )$. Assume that $g \in \Gamma\mathrm{L}(V)$ sends Σ_1 to Σ_2.*

(i) *Then Σ_i is a symplectic spread of the $\mathbf{K}_i$-space $V = V_{\mathbf{K}_i}$, where the kernel $\mathbf{K}_i$ of Σ_i is commutative $(i = 1, 2)$.*

(ii) *Let $(\ ,\ )_{\mathbf{K}_i}$ be a nondegenerate alternating bilinear form on $V_{\mathbf{K}_i}$ with respect to which Σ_i is symplectic. Then g sends $\mathbf{K}_1$ to $\mathbf{K}_2$, $V_{\mathbf{K}_1}$ to $V_{\mathbf{K}_2}$ and preserves the symplectic structure: $(ug, vg)_{\mathbf{K}_2} = k_2[(u,v)_{\mathbf{K}_1}]^\sigma$ for some $k_2 \in \mathbf{K}_2^*$, some $\sigma \in \mathrm{Aut}(\mathbf{K}_2)$, and all $u, v \in V$.*

This theorem also holds trivially when the spreads are desarguesian and the kernels are fields (cf. Section 4).

For the situation in (ii), in [5, Theorem 3.5] we asked whether there is some $h \in \mathbf{K}_2^*$ such that gh^{-1} preserves the symplectic geometry over the *original* field K. In general we see no reason why this should occur, even when V is finite, but it does hold under suitable numerical assumptions:

Theorem 2. *Let Σ_1 and Σ_2 be spreads in a finite K-space V that are symplectic with respect to the nondegenerate alternating bilinear form $(\ ,\ )$. Assume that $g \in \Gamma\mathrm{L}(V)$ sends Σ_1 to Σ_2. If $\mathbf{K}_2$ is the kernel of Σ_2, and if either $|K|$ is even or $[\mathbf{K}_2 : K]$ is odd, then $g = sk_2$ with $k_2 \in \mathbf{K}_2^*$ and $s \in \Gamma\mathrm{L}(V)$ satisfying $(us, vs) = k(u,v)^\sigma$ for some $k \in K^*$, some $\sigma \in \mathrm{Aut}(K)$, and all $u, v \in V$.*

Versions of the last theorem were proved in [5, Theorem 3.5] and [9], assuming that $|K|$ is even or $\dim V = 4$, respectively. In practice, "usually" $\mathbf{K}_2 = K$, in which case the theorem is a special case of Theorem 1.

The elementary proofs of the above results use the associated symplectic polarities.

2   Background

We refer to [3] for the standard background concerning spreads, translation planes and their kernels. For example, any isomorphism $\mathfrak{A}(\Sigma_1) \to \mathfrak{A}(\Sigma_2)$ that sends 0 to 0 is induced by a semilinear transformation sending Σ_1 to Σ_2. We refer to [10] for background concerning symplectic geometry. If $(\ ,\ )$ is a nondegenerate alternating bilinear form on V, then $\Gamma\mathrm{Sp}(V) = \{g \in \Gamma\mathrm{L}(V) \mid (ug, vg) = k(u,v)^\sigma$ for some $k \in K^*$ and $\sigma \in \mathrm{Aut}(K)$, and all $u, v \in V\}$ (cf. the statements of both of the above theorems). Associated with $(\ ,\ )$ there is a *symplectic polarity* $\theta\colon W \to W^\theta = \{v \in V \mid (v, W) = 0\}$ of the projective geometry $\mathrm{PG}(V)$ of V, and the corresponding centralizer $C_{\mathrm{P}\Gamma\mathrm{L}(V)}(\theta) = \mathrm{P}\Gamma\mathrm{Sp}(V)$ is $\Gamma\mathrm{Sp}(V)$ modulo scalars.

3   Uniqueness

Before proving the theorems, we record a very simple observation (implicit in [5, Theorem 3.5] and reappearing in Section 5):

Proposition 1. *Suppose that Σ is a spread in a K-space V, where K is the kernel of Σ. Then Σ is symplectic with respect to at most one symplectic polarity of V.*

Proof. If Σ is symplectic with respect to the symplectic polarities θ and θ', then θ and θ' send each member of Σ to itself. Hence, so does $\theta\theta'$. The collineation $\theta\theta'$ of $\mathrm{PG}(V)$ is induced by a semilinear transformation that fixes every member of Σ and so is in the kernel K. Then $\theta\theta'$ fixes every member of $\mathrm{PG}(V)$, and hence $\theta\theta' = 1$. $\square$

In general, if $\mathbf{K}$ is the kernel of Σ then θ' lies in the coset $\theta(\mathbf{K}^*/K^*)$.

4   Lifting symplectic structures

We will be viewing a vector space V over more than one skew field. As in the statement of Theorem 1, if one of these skew fields is called F, and we wish to emphasize that we are viewing V as an F-space, we will write V_F instead of V.

Theorem 3. *Let Σ be a nondesarguesian symplectic spread of the K-space V, with kernel $\mathbf{K} \supseteq K$. Then $\mathbf{K}$ is commutative, and there is a nondegenerate alternating bilinear form $(\ , \ )_{\mathbf{K}}$ on $V_{\mathbf{K}}$ such that*

 (i) *Σ is a symplectic spread of $V_{\mathbf{K}}$ with respect to $(\ , \ )_{\mathbf{K}}$,*
 (ii) *$f((\ , \ )_{\mathbf{K}})$ is the given nondegenerate alternating bilinear form on V for some nonzero K-linear functional $f\colon \mathbf{K} \to K$, and*
(iii) *$(\ , \ )_{\mathbf{K}}$ and f are uniquely determined up to multiplication by elements of $\mathbf{K}^*$ and K^*, respectively.*

Proof. (i) Let G denote the group of all automorphisms and all anti-automorphisms of $\mathrm{PG}(V)$ that fix each member of Σ. Then G contains the given polarity θ and has a normal subgroup $\mathbf{K}^*/K^*$ consisting of the elements of $\mathrm{P\Gamma L}(V)$ that induce the identity on Σ. It follows that θ permutes the set of $\mathbf{K}$-invariant subspaces of V; these are precisely the subspaces of $V_{\mathbf{K}}$. Consequently, by restricting θ to the subspaces of $V_{\mathbf{K}}$ we obtain a polarity $\theta_{\mathbf{K}}$ of $\mathrm{PG}(V_{\mathbf{K}})$. Clearly, Σ is still a spread of $V_{\mathbf{K}}$, while each member of Σ is still sent to itself by $\theta_{\mathbf{K}}$. Each point of $\mathrm{PG}(V_{\mathbf{K}})$ lies in some member of Σ and hence is perpendicular to itself. Thus, since we are assuming that $\dim V_{\mathbf{K}} > 2$, by [10, p. 53] $\mathbf{K}$ is commutative and $\theta_{\mathbf{K}}$ is a symplectic polarity determined by a nondegenerate alternating bilinear form $(\ , \ )_{\mathbf{K}}$ on $V_{\mathbf{K}}$, as required.

(ii) If W is any $\mathbf{K}$-subspace then $W\theta_{\mathbf{K}} = W\theta$. Hence, if $u, v \in V$ and $(u,v)_{\mathbf{K}} = 0$ then $(u, v) = 0$. For fixed $u \neq 0$ it follows that $f_u((u,v)_{\mathbf{K}}) := (u,v)$ is a well-defined map $f_u\colon \mathbf{K} \to K$, and hence is a K-linear functional. Similarly, any two such maps f_u agree on each $v \in V$. This implies (ii).

(iii) Each symplectic polarity arises from an alternating bilinear form that is uniquely determined up to a scalar. $\square$

This result and Theorem 1 also hold in the desarguesian case if $\mathbf{K}$ is commutative. When $\mathbf{K}$ is noncommutative, there is no symplectic structure possible on $V_{\mathbf{K}}$, but it is not clear whether or not, in this case, a desarguesian spread can be symplectic when viewed over a subfield of $\mathbf{K}$; it is also not clear that this is even an interesting question.

Proof of Theorem 1. Theorem 1(i) and Theorem 3(i) are the same. For Theorem 1(ii), observe that the uniqueness part of Theorem 3(iii) implies that g must send the form $(\ ,\ )_{\mathbf{K}_1}$ to $(\ ,\ )_{\mathbf{K}_2}$ up to a scalar and a field automorphism (since g is semilinear but not necessarily linear). $\qquad\square$

5 Proof of Theorem 2

Let θ be the symplectic polarity of $\mathrm{PG}(V)$ associated with $(\ ,\ )$; it is the identity on both Σ_1 and Σ_2. Then θ and $\theta^g = g^{-1}\theta g$ are symplectic polarities each of which is the identity on Σ_2. It follows that $\theta\theta^g$ arises from an element of $\Gamma\mathrm{L}(V)$ that is the identity on Σ_2, and hence lies in $\mathbf{K}_2$. Since $(|\mathbf{K}_2| - 1)/(|K| - 1)$ is odd by hypothesis, $\theta\theta^g$ is an element of $\mathrm{P}\Gamma\mathrm{L}(V)$ of odd order. Then $\theta^{\bar h} = \theta^g$ for some $\bar h$ in the dihedral group $\langle\theta, \theta^g\rangle$, and hence $\bar g\bar h^{-1}$ commutes with θ, where $\bar g$ is the element of $\mathrm{P}\Gamma\mathrm{L}(V)$ induced by g. Consequently, $\bar g\bar h^{-1} \in \mathrm{P}\Gamma\mathrm{Sp}(V)$.

As in Proposition 1, $\bar h$ is induced by some $h \in \mathbf{K}_2^*$. Thus, $g = skh$ with $s \in \Gamma\mathrm{Sp}(V)$ and $k \in K^*$, and then $kh \in Kh \subseteq \mathbf{K}_2$. $\qquad\square$

Remarks. 1. When $\mathbf{K}_1 = \mathbf{K}_2 = K$ the theorem states that $(us, vs) = k(u, v)^\sigma$ for some $k \in K^*$, $\sigma \in \mathrm{Aut}(K)$ and all $u, v \in V$; that is, $g \in \Gamma\mathrm{Sp}(V)$. The above argument is simply that $\theta^g = \theta$ as in Proposition 1, and hence $g \in \Gamma\mathrm{Sp}(V)$. This is already contained in Theorem 1(ii). For nondesarguesian Σ, this is exactly the case considered in [9].

2. An immediate consequence of Theorem 2 is that, under the stated numerical hypotheses regarding the kernel, *if Σ_1 and Σ_2 are equivalent then some element of $\Gamma\mathrm{L}(V)$ sends Σ_1 to Σ_2 while preserving the symplectic structure.*

Kernels can be nontrivial [5] or difficult [7] to determine. When the characteristic is 2 there is no need to find the kernel in order to use the above consequence; and this was the crucial use made of Theorem 2 in those references. However, it seems doubtful that the preceding consequence holds (for finite V) without any numerical assumptions.

3. It is natural to ask: how does the above proof differ from the argument in [5, Theorem 3.5]? Here we have used the fact that $\theta\theta^g$ lies in the kernel of the spread as a *projective* transformation of V, which enabled some of the problems in the odd characteristic case to disappear. This should have been noticed 20 years ago. It wasn't. Therefore, only the characteristic 2 case was handled at that time.

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