Finite Geometries via Algebraic Affine Buildings

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The purpose of this note is (1) to give a vague description of algebraic affine buildings, (2) to give a brief description of arithmetic groups, and then (3) to combine these subjects in order to discuss finite quotients of algebraic affine buildings modulo suitable arithmetic groups.

1. Algebraic affine buildings

I'll start with an example:

Let V be $\mathbf{Q}^n$, with standard basis $v_1, \dots, v_n$. Let p be a prime, and write $\mathcal{O} = \left\{\frac{a}{b} \mid a, b \in \mathbf{Z}, p \not| \ b\right\}$; this is a subring of $\mathbf{Q}$ such that $\mathcal{O}/p\mathcal{O} \cong GF(p)$. Let $G(\mathbf{Q})$ denote the group $SL(n, \mathbf{Q})$ of $n \times n$ matrices over $\mathbf{Q}$ of determinant 1. The subgroup

$$B = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ p\mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ \dots & \dots & \dots & \dots \\ p\mathcal{O} & \dots & p\mathcal{O} & \mathcal{O} \end{pmatrix}$$

of $G(\mathbf{Q})$ is called an *Iwahori subgroup*. Note that B mod p is a Borel subgroup of $G(\mathcal{O} \bmod p) = G(GF(p)) = SL(n, p)$. (This type of notation is intended to be thought of in terms of matrix groups, in which entries are taken from the indicated rings.) Using $G(\mathbf{Q})$ and B, an affine building Δ is obtained as follows (cf. [10]):

Δ is a simplicial complex;

a simplex of Δ is a proper subgroup of $G(\mathbf{Q})$ containing B;

X is a face of Y iff $X \geq Y$.

The maximal simplexes (chambers) are just the conjugates of B. There are n types of vertices, which are the conjugates of the subgroups of matrices blocked as follows:

$$\begin{array}{r c} & \overbrace{\qquad}^{r} \quad \overbrace{\qquad}^{n-r} \\ \begin{array}{r} r\{ \\ n-r\{ \end{array} & \begin{pmatrix} \mathcal{O} & \frac{1}{p}\mathcal{O} \\ p\mathcal{O} & \mathcal{O} \end{pmatrix} \end{array}$$

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for $r = 1, \dots, n$; for example, when $r = n$ this group is just $SL(n, \mathcal{O}) = G(\mathcal{O})$. The above group is the stabilizer in $G(\mathbf{Q})$ of the *$\mathcal{O}$-lattice*

$$L_r = \sum_{i \le r} \mathcal{O} \frac{v_i}{p} + \sum_{i > r} \mathcal{O} v_i$$

and also of cL_r for all $c \in \mathbf{Q}^*$. (An *$\mathcal{O}$-lattice* is an $\mathcal{O}$-submodule of V generated by a basis.) Therefore, vertices correspond to *lattice classes* $[L] = \{cL \mid c \in \mathbf{Q}^*\}$; moreover, every $\mathcal{O}$-lattice is the image of a unique L_r under $G(\mathbf{Q})$.

Note that the star of $[L_r]$ is just the usual (spherical) building for L_r/pL_r, i.e., for $G\,(GF(p)) = SL(n,p)$. The corresponding diagram reflects this fact: it is the extended Dynkin diagram of type $\tilde{A}_n$, having n nodes. (Hide any node and observe that the diagram A_{n-1} is left: the diagram of the building of $SL(n,p)$.) There is also an obvious dihedral group of graph automorphisms, generated by the usual graph automorphism of the $SL(n,p)$ building together with an n-cycle (produced, for example, by the diagonal matrix $\mathrm{diag}(1, 1, \dots, 1, p)$).

All of the above goes through with the field $\mathbf{Q}_p$ of p-adic numbers in place of $\mathbf{Q}$, using the ring $\mathbf{Z}_p$ of p-adic integers in place of $\mathcal{O}$. *This produces an isomorphic building.*

This is an example of an *algebraic affine building*. The general case is as follows [10].

K	field with a complete, discrete valuation	(such as $\mathbf{Q}_p$)
$\mathcal{O}$	corresponding valuation ring	($\mathbf{Z}_p$ in the case of $\mathbf{Q}_p$)
π	canonical prime (uniformizer)	(p in the case of $\mathbf{Q}_p$)
$k = \mathcal{O}/\pi\mathcal{O}$	residue field	($GF(p)$ in the case of $\mathbf{Q}_p$)
$G(K)$	an absolutely simple, simply connected algebraic group over K of rank $\ell \ge 2$	
$G(\mathcal{O})$	the corresponding group over $\mathcal{O}$: think in terms of matrix entries; this is assumed to be a maximal subgroup of $G(K)$	
$G(\mathcal{O}) \to G(\mathcal{O} \bmod \pi) = G(k)$		
B	*Iwahori subgroup*, the preimage in $G(K)$ of a Borel subgroup of $G(k)$	
Δ	*affine building* of $G(K)$	

As above, Δ is a simplicial complex whose simplexes are the proper subgroups of $G(K)$ containing a conjugate of B, with X a face of Y iff $X \ge Y$. Chambers are the conjugates of B. The rank of Δ is $\ell + 1$: there are $\ell + 1$ different types of vertices. The requirement that $G(O)$ is special amounts to the fact that the associated diagram is the extended Dynkin diagram corresponding to the Dynkin diagram of the finite group $G(k)$; the group $G(K)$ always contains such subgroups [10]. In the example given above, all the vertex stabilizers are special.

A second example may help in wading through these definitions. Let f be a quadratic form on a vector space V over $\mathbf{Q}_p$, of Witt index $\ell \geq 2$. The corresponding algebraic group $G(K)$ is the commutator subgroup of the orthogonal group of f. (Actually, in order to conform with the above definitions I should in fact take $G(K)$ to be the associated spin group. However, the center of that spin group acts trivially on Δ, and hence can be ignored for our purposes.) Here, vertices correspond to classes $[L]$ of *suitable* $\mathbf{Z}_p$-lattices L in V. (In unpublished work, Rehmann and Scharlau have obtained necessary and sufficient conditions in order that a lattice L produces a vertex – not just for this orthogonal situation, but for all the classical groups.)

More concretely, consider the quadratic from $f = \sum_1^4 x_i x_{i+4}$ on $\mathbf{Q}_p^8$, with orthogonal group $\Omega^+(8, \mathbf{Q}_p)$. Let $e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4$ be the standard basis, so that with respect to the underlying bilinear form $(e_i, e_j) = 0 = (f_i, f_j)$ and $(e_i, f_j) = \delta_{ij}$. The vertices of Δ are the lattice classes containing the lattices

$$
\begin{aligned}
L_1 &= \langle e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4 \rangle_{\mathbf{Z}_p} \\
L_2 &= \langle \frac{e_1}{p}, e_2, e_3, e_4, pf_1, f_2, f_3, f_4 \rangle_{\mathbf{Z}_p} \\
L_3 &= \langle \frac{e_1}{p}, \frac{e_2}{p}, e_3, e_4, f_1, f_2, f_3, f_4 \rangle_{\mathbf{Z}_p} \\
L_4 &= \langle \frac{e_1}{p}, \frac{e_2}{p}, \frac{e_3}{p}, \frac{e_4}{p}, f_1, f_2, f_3, f_4 \rangle_{\mathbf{Z}_p} \\
L_5 &= \langle \frac{e_1}{p}, \frac{e_2}{p}, \frac{e_3}{p}, \frac{f_4}{p}, f_1, f_2, f_3, e_4 \rangle_{\mathbf{Z}_p}
\end{aligned}
$$

The diagram is $\tilde{D}_4$, extended D_4: with central node arising from L_3.

2. Arithmetic groups

Let $G(K)$ be as above. A subgroup G of $G(K)$ is *discrete* if G_x is finite for each vertex x of Δ; and G is *cocompact* if it has a finite number of chamber-orbits (or, equivalently, a finite number of orbits on some type of vertices). These definitions are motivated by the topology on Δ inherited from that of the field K.

More notation is needed in order to describe the basic example of a discrete cocompact subgroup of $G(K)$; in parentheses is the special case of the rational field itself. (Note, however, that there is an additional possibility being ignored here, in which the ground field F has nonzero characteristic.)

F	finite extension of $\mathbf{Q}$	
v	place (valuation)	(p or ∞ in the case of $\mathbf{Q}$)

$K = F_v$ completion of F at v ($\mathbf{Q}_p$ or $\mathbf{R}$ in the case of $\mathbf{Q}$)
$\mathcal{O}_v$ corresponding valuation ring if v is finite ($\mathbf{Z}_p$ in the case of $\mathbf{Q}$)
$\mathcal{O}^v = \{a \in F \mid v'(a) \geq 0$ for all finite $v' \neq v\}$ ($\mathbf{Z}[\frac{1}{p}]$ in the case of $\mathbf{Q}$)
$G(F)$ is embedded in $G(F_v)$ in the obvious manner.
$G = G(\mathcal{O}^v)$

Assume that $G(F_{v'})$ is compact for all infinite places v'. *Fix* a finite place v. *Assume* that $G(F_v) = G(K)$ is noncompact and has rank $\ell \geq 2$. Then: *G is discrete and cocompact in $G(F_v)$.*

For example, if f is a quadratic form over $\mathbf{Q}$ then $G(\mathbf{Q})$ is the (derived subgroup of the) orthogonal group; the only infinite place is ∞, with $\mathbf{Q}_\infty = \mathbf{R}$. Then the compactness of $G(\mathbf{Q}_\infty) = G(\mathbf{R})$ simply asserts that f is a definite quadratic form (over $\mathbf{R}$). The group G is $G(\mathbf{Z}[\frac{1}{p}])$ in case v corresponds to the prime p.

Returning to the general case, a subgroup Γ of $G(F_v)$ is called an *arithmetic group* if $\Gamma \cap G(\mathcal{O}^v)$ has finite index in both Γ and $G(\mathcal{O}^v)$. A fundamental theorem of Margulis asserts that every discrete cocompact group Γ arises in essentially this manner (this requires the assumption that $\ell \geq 2$; cf. [14]). While there are even more general definitions of this sort, here it will only be necessary to consider $G(\mathcal{O}^v)$.

Thinking in terms of classical groups (although the following is true in general), it is always possible to choose bases so that the stabilizer of a given vertex of Δ looks like $G(\mathcal{O}_v)$ [10]. This motivates considering the number of orbits of $G(\mathcal{O}^v)$ on the coset space $G/G(\mathcal{O}_v)$. This number is an integer h that is *independent* of the choice of v – subject to the condition that $G(F_v)$ is noncompact and has rank ≥ 2 [5]. This integer is called the *class number* of $G(F)$. This is a standard concept in the case of quadratic forms (where, however, it is called the spinor class number: the class number has a similar but slightly different meaning [3]).

Borel and Prasad [2] recently have shown that, given an integer h, there are only a finite number of pairs F, $G(F)$, for which the class number is h. In the special case of orthogonal groups this was proved in [9].

The case of greatest interest in finite group theory and finite geometry occurs when $h = 1$. Here, $G(\mathcal{O}^v)$ is *transitive* on one of the vertex types of the building. Note the bizarre fact that, whereas the buildings arising from different places v are drastically different (the residue fields have different characteristics, and hence so do stars within the buildings), nevertheless transitivity for one v implies transitivity for all (suitable) v!

Example: Let $F = \mathbf{Q}$, $f = \sum_1^6 x_i^2$, $V = \mathbf{Q}_p^6$. Then $h = 1$. This form is clearly positive definite over $\mathbf{R}$. The resulting building has diagram if $p \equiv 1 \pmod 4$ and otherwise. In either case there are obvious graph automorphisms possible. In fact, the *full* group of transformations over $\mathbf{Z}[\frac{1}{p}]$ preserving f *projectively* is transitive on the set of *all* vertices if $p \equiv 1 \pmod 4$, and on the set of non-end-node vertices for the remaining primes p. This is precisely the sort of example discussed in [6] in an *ad hoc* manner.

This situation suggests that one should look for some kinds of classification theorems. In a series of papers, using very detailed case arguments, G. L. Watson determined all definite quadratic forms over $\mathbf{Q}$ of class number 1 in dimension $n \geq 5$ (as indicated above, the notion of class number he used is not quite the same as what I have been using in the general case; but class number 1 for Watson implies class number 1 for us; the converse is probably true when $n \geq 5$, but this has yet to be proved). Watson showed that $n \leq 10$: see the references in [13] for the cases $n \geq 7$; his results for $n = 5$ and 6 remained unpublished at the time of his death in 1988.

In more recent work still in progress, R. Scharlau has considered the corresponding problem of determining those quadratic forms over algebraic number fields for which $h = 1$ (with h as defined above). He used [9] to show that $n \leq 14$ – strengthening the estimates $n \leq 34$ in [9] and $n \leq 18$ in [5] – and that the possibilities for the field F are severely limited if $n \geq 6$. This work did not use buildings, groups, or geometry. However, it should eventually have interesting applications to finite geometry, for reasons to be explained in the next section.

The above example with $F = \mathbf{Q}$ and $f = \sum_1^6 x_i^2$ is especially interesting when $p = 2$. Here, $G(\mathbf{Z}_2)$ is the stabilizer in $G(\mathbf{Q}_2)$ of the lattice $L = \mathbf{Z}_2^6$. Then the stabilizer in $G = G(\mathbf{Z}[\frac{1}{2}])$ of L is just $G(\mathbf{Z})$, which is a finite group of the form $2^5 A_6$. This acts on $L/2L$, which is a vector space over $GF(2)$. The form f mod 2 is actually linear, with kernel $H = \{(x_i) \in L \mid \Sigma x_i \equiv 0 \pmod 2\}$. There is then a quadratic form $\frac{1}{2}f$ mod 2 induced on $H/2L$. This form is preserved by $G(\mathbf{Z})$, the induced group being $A_6 \cong O(5,2)'$. Thus, $G(\mathbf{Z})$ is chamber-transitive on the building produced on $H/2L$, which is in turn the star of the vertex $[L]$. In view of the transitivity of G on the vertices of the same type as $[L]$, it follows that G is chamber-transitive. This is one of the examples of chamber-transitive groups classified in the result in [7], which to a large extent is subsumed by the results contained in Meixner's paper for this conference.

3. Finite geometries

After all of the infinite groups and complexes appearing in the previous sections, it is now time to describe some implications for finite geometry.

If Δ and Δ' are two simplicial complexes then a map $\phi : \Delta \to \Delta'$ is called a *cover* if it is simplicial, onto, and for each vertex x the restriction $\phi_{St(x)} : St(x) \to St(x\phi)$ is an isomorphism. (There is a more general notion of 2-*cover* [11], but covers will suffice for the present purposes.)

The case of concern here is the one in which Δ is an affine building obtained as in Section 1. Then Δ has rank $\ell+1 \geq 3$. Note that Δ' will be a complex that locally looks exactly like Δ: stars in Δ' will be isomorphic to stars in Δ. What do all the finite complexes Δ' look like? There is a discrete automorphism group A of Δ such that $\Delta' \cong \Delta/A$; that is, Δ' can be identified in a natural manner with the set of orbits of A on Δ. In particular, A is cocompact; it is the group of deck transformations of the cover [11].

However, not all discrete cocompact groups A produce a simplicial complex Δ/A: just consider the case in which A happens to be transitive on some type of vertices of Δ. This is one of the reasons for replacing simplicial complexes by chamber systems when discussing building-like geometries [11] (cf. [4]). Nevertheless, for purposes of finite geometry complexes are the appropriate things to aim for. Fortunately, there is no difficulty finding such complexes – and unfortunately there are simply too many of them:

If A is a discrete cocompact subgroup of $G(K)$ then

1. *So is every subgroup of finite index;*
2. *A is residually finite: the intersection of all the normal subgroups of finite index is* 1*; and*
3. *There is a constant M (depending on $G(F)$) such that, if D is a subgroup of A of index at least M, then Δ/D is a simplicial complex and $\Delta \to \Delta/D$ is a cover.*

Here, 1 is easy, 2 is not difficult, and 3 is an observation of Tits [12]. The net effect of these facts is that, as already stated, there are too many possibilities for the simplicial complex Δ'. What is needed is some way to narrow the study of such complexes. Not surprisingly, transitivity properties provide at least one way to do this.

The following simple construction produces finite complexes Δ' with large induced groups. Start with a discrete group G transitive on some vertex type of Δ. Take any normal subgroup A of G such that G/A is finite and not too small. Then, by 2 and 3 above, $\Delta' = \Delta/A$ will be a simplicial complex on which G/A acts, and G/A will also be transitive on a vertex type.

Example: Let f be a definite quadratic form on a vector space over $\mathbf{Q}$, let $G(\mathbf{Q})$ be as usual, and assume that $G(\mathbf{Q})$ has class number 1. Let p be a prime, and assume that f has Witt index ≥ 2 over $\mathbf{Q}_p$ and that $G(\mathbf{Z}_p)$ is the stabilizer of a vertex of Δ. Then $G = G(\mathbf{Z}[\frac{1}{p}])$ is transitive on that vertex type. Now if $m > 1$ is an integer then let $A(m) = \{g \in G \mid g \equiv 1 \pmod{m}\}$. This is a normal subgroup of G, and $G/A(m)$ is usually just $G(\mathbf{Z}/m\mathbf{Z})$. If m is sufficiently large then $\Delta/A(m)$ will be a complex that locally looks exactly like Δ: stars in $\Delta/A(m)$ will be isomorphic to stars in Δ.

Problem: What properties does $\Delta' = \Delta/A$ have, either in the preceding situation, or in the more general situation in which A is a normal subgroup of a group G transitive on some vertex type, or in even more general contexts?

Since there are evidently large numbers of finite geometries Δ' obtained even in the preceding example, it is not at all clear what sorts of properties one should look for. Perhaps there are some kinds of configuration theorems possible. At the moment, the only results are essentially asymptotic in nature. These are motivated by [8], which contains analogous results when Δ is the affine building (a tree) for $SL(2, \mathbf{Q}_p)$. Assume that $G = G(\mathcal{O}^v)$ is transitive on some vertex type.

1. *There is a constant C_G such that, if $A \triangleleft G$, G/A is finite, and Δ/A is a complex, then the diameter of Δ/A is $\leq C_G \log_2(\#)$, where $\#$ is the number of vertices, or alternatively the number of chambers, of Δ (the constant C_G depends on which definition of $\#$ is used).*

Here, *diameter* refers to the usual diameter of a graph: the 1-skeleton of Δ/A, or the chamber-graph of Δ/A. This result is asymptotically best possible, as a simple counting argument shows; and it is an easy consequence of results in [1]. However, even a bound on the constant C_G seems to be very difficult to compute. By contrast, the following seems to be true in general; I have only verified it in the case of quadratic and hermitian forms:

2. *There is a computable constant C'_G such that, for all A as above, the geometric girth of Δ/A is $\geq C'_G \log_2(\#)$.*

Here, the *geometric girth* is the length of the shortest circuit *not* homotopic to 0; homotopy refers either to the usual simplicial concept or to the one for chambers [11]. For example, in the simplicial case of the Example in Section 3, when $p = 2$ one can take $C'_G = \frac{1}{30}$.

At this point it should be clear that this subject is in its infancy, at least from the point of view of finite geometry. It is not yet clear what the most important questions are; there certainly are few tools available to study the geometries $\Delta' = \Delta/A$.

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