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Generalized Quadrangles Associated with $G_2(q)$*

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If $q \equiv 2 \pmod 3$, a generalized quadrangle with parameters q, q^2 is constructed from the generalized hexagon associated with the group $G_2(q)$.

1. INTRODUCTION

In addition to the generalized quadrangles arising naturally from small-dimensional symplectic, unitary, or orthogonal geometries, the only known finite examples are the following quadrangles and their duals: ones with parameters q, q (where $q = 2^e > 8$) or q^2, q (where $q = 2^{2e+1} > 8$) due to Tits [4, p. 304]; and others with parameters $q-1$, $q+1$ (for prime powers q) due to Hall [5] and Ahrens and Szekeres [1]. In this note we will discuss a procedure for constructing all but the last of the above examples, and others as well. In particular, we will prove the following result.

THEOREM 1. *If q is a prime power such that $q \equiv 2 \pmod 3$ and $q > 2$, then there exists a general quadrangle with parameters q, q^2 not isomorphic to any of the aforementioned ones.*

The only surprising feature of these quadrangles is that they arise from the groups $G_2(q)$, which are themselves associated with generalized hexagons. The automorphism group of each quadrangle is isomorphic to the stabilizer in Aut $G_2(q)$ of a line of the corresponding hexagon.

The precise relationship between the generalized quadrangles and hexagons is given in Section 2. In view of the restrictions forced on q, there does not seem to be any geometric proof of the theorem (cf. (#) in Section 2).

The algebraic proof of the theorem occupies Sections 3–5. In particular, Section 3 contains an elementary construction procedure. Analogous procedures can be easily obtained for generalized hexagons and octagons,

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although they involve many more axioms. However, we have not yet been able to use them to construct any examples other than the known ones.

2. Geometric Description

Suppose that $q \equiv 2 \pmod 3$, and fix a line L of the generalized hexagon $\mathscr{H}$ associated with $G_2(q)$. (There are two dual choices for $\mathscr{H}$. The relevant one has L a line of an $O(7,q)$ geometry; cf. Tits [6].) We use the metric defined on the union of the sets of points and lines of $\mathscr{H}$.

Now define Points and Lines as follows.

Points: points of L; lines at distance 4 from L.

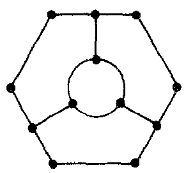
Lines: L; lines at distance 6 from L; points at distance 3 from L.

A Point of L is defined as being on L and all Lines at distance 2 from it. A Point not on L is on all Lines at distance 1 or 2 from it.

This defines a generalized quadrangle with parameters q, q^2. When $q = 2$, it is the unique quadrangle with these parameters. For $q > 2$, it is the quadrangle mentioned in Theorem 1.

Remark. Starting with a line L of a generalized hexagon with parameters s, s, an incidence structure of Points and Lines can be defined as above. This produces a generalized quadrangle with parameters s, s^2, provided that the following condition ($\#$) holds in the hexagon.

($\#$) There is *no* configuration of 12 points and 10 lines as in the figure (where L has been drawn as a circle for reasons of symmetry).



It seems likely that ($\#$) characterizes the $G_2(q)$ hexagons, where $q \equiv 2 \pmod 3$.

3. Construction Procedure

Let Q be a finite group, and $\mathscr{F}$ a family of subgroups of Q. With each $A \in \mathscr{F}$ is associated another subgroup A^*.

AXIOMS. *For every three-element subset* $\{A, B, C\}$ *of* $\mathscr{F}$, *and some integers* s *and* t,

(i) $|Q| = st^2$, $|\mathscr{F}| = s+1$, $|A| = t$, $|A^*| = st$, $A < A^*$;
(ii) $Q = A^*B$, $A^* \cap B = 1$; *and*
(iii) $AB \cap C = 1$.

Construction. Let $A \in \mathscr{F}$ and $q \in Q$ be arbitrary.

Point. Symbol $[A]$; coset Aq.

Line. $\mathscr{F}$; coset A^*q; element q.

Incidence. $[A]$ is on $\mathscr{F}$ and A^*q; all other incidences are obtained via inclusion.

Notation. $\mathscr{Q}(Q, \mathscr{F})$ is the geometry of points and lines just constructed.

THEOREM 2. *$\mathscr{Q}(Q, \mathscr{F})$ is a generalized quadrangle with parameters s, t.*

The proof is just a straightforward check.

All classical examples (and their duals) arise from Theorem 2; so do Tits' examples. Before discussing this, we will need some notation concerning automorphism groups.

Consider any generalized quadrangle $\mathscr{Q}$ with parameters s, t. If L is a line, let U_L denote the group of all automorphisms fixing each line meeting L. Then $|U_L| \leqslant s$ (since U_L acts semiregularly on the points outside L of each such line).

If x is a point, U_x is defined in a dual manner.

For distinct intersecting lines L, M, let U_{LM} denote the group of all automorphisms fixing every point of L, every point of M, and every line on $L \cap M$. Then $|U_{LM}| \leqslant t$. (An element of U_{LM} fixing a line not on $L \cap M$ must fix pointwise a subquadrangle with parameters s, t, and hence must be 1.)

We now turn to examples of Theorem 2.

EXAMPLE 1. $s = t = q$, Q is a three-dimensional vector space over $GF(q)$, $\mathscr{F}$ is an oval, and A^* is the tangent line to $\mathscr{F}$ at A. This example is due to Tits [4, p. 304].

Note that $A = U_{[A]}$. If q is even then $U_{\mathscr{F}}$ is the knot of $\mathscr{F}$.

EXAMPLE 2. $s = q^2$, $t = q$, Q is a four-dimensional vector space over $GF(q)$, $\mathscr{F}$ is an ovoid, and A^* is the tangent plane to $\mathscr{F}$ at A. This example is also due to Tits [4, p. 304]. $\mathscr{Q}(Q, \mathscr{F})$ is the $SU(4, q)$ quadrangle if and only if $\mathscr{F}$ is a quadric.

Note that $A = U_{[A]}$ again. Suppose that $\mathscr{F}$ is not a quadric. Then Aut

$\mathscr{Q}(Q, \mathscr{F})$ fixes the line $\mathscr{F}$ and acts on the group Q generated by the groups $U_{[A]}$. Certainly, Q is its own centralizer in Aut $\mathscr{Q}(Q, \mathscr{F})$. Also, $\mathscr{F}$ produces an inversive plane from which the vector space Q can be reconstructed [4, pp. 265–268]. Thus, Aut $\mathscr{Q}(Q, \mathscr{F})$ is a *semidirect product of Q with the subgroup of $\Gamma L(4, q)$ stabilizing $\mathscr{F}$.*

Remark. The $Sp(4, q)$ and $SU(4, q)$ quadrangles arose in Examples 1 and 2. The $O(5, q)$, $O^-(6, q)$, $SU(5, q)$, and dual $SU(5, q)$ quadrangles all arise as examples of Theorem 2. A description of the $O^-(6, q)$ quadrangle will be of use in Theorem 1.

EXAMPLE 3. $s = q$, $t = q^2$. The $O^-(6, q)$ quadrangle can be obtained as follows. Let Q consist of all triples (α, c, β) with α, $\beta \in GF(q^2)$ and $c \in GF(q)$, and define

$$(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \operatorname{Tr} \bar{\beta}\alpha', \beta + \beta')$$

(where $\bar{\beta} = \beta^q$ and $\operatorname{Tr} \gamma = \gamma + \bar{\gamma}$). Then Q is a group of order q^5, with center consisting of all $(0, c, 0)$, $c \in GF(q)$. Let $\mathscr{F}$ consist of the groups $A(\infty) = \{(0, 0, \beta) \mid \beta \in GF(q^2)\}$ and $A(t) = \{(\alpha, \alpha\bar{\alpha}t, \alpha t) \mid \alpha \in GF(q^2)\}$ for $t \in GF(q)$. If $A \in \mathscr{F}$, write $A^* = AZ(Q)$. Then Theorem 2 applies.

If α and β are restricted to $GF(q)$, the result is a subgroup R of Q of order q^3. The groups $A \cap R$ and $A^* \cap R$ produce a subquadrangle with parameters q, q, namely, the $O(5, q)$ quadrangle.

The maps $(\alpha, c, \beta) \to (\alpha, c - \alpha\bar{\alpha}b, \beta - \alpha b)$, $b \in GF(q)$, and $(\alpha, c, \beta) \to (-\beta, c - \operatorname{Tr} \bar{\alpha}\beta, \alpha)$ generate a subgroup of Aut Q isomorphic to $SL(2, q)$. They map $\mathscr{F}$ to itself (inducing $A(t) \to A(t + b)$ and $A(t) \to A(-1/t)$, respectively). Note that this $SL(2, q)$ normalizes R. (In fact, the $SL(2, q)$ normalizes $q + 1$ subgroups of Q of order q^3). Let S' denote the commutator subgroup of a group S, and write $S'' = (S')'$. Then

(*) *If $q > 3$ and S is the stabilizer of the line $\mathscr{F}$ of the $O^-(6, q)$ quadrangle, then $S'' \cong Q \rtimes SL(2, q)$ with $|Q| = q^5$, and S'' has a normal subgroup of order q^3.*

For further information concerning groups Q arising as above, as well as their normalizers, see [3, Sect. 3].

4. $G_2(q)$

Let q be a prime power. Since the generalized quadrangle allegedly constructed in Section 2 has the stabilizer of L in $G_2(q)$ acting on it, it will be necessary to study that stabilizer. The required information is o pp. 244–245 of Tits [7]. However, we will make this somewhat more explicit in order to facilitate later calculations.

Let Q consist of all quintuples $(\alpha, \beta, \gamma, \delta, \varepsilon) \in GF(q^5)$, this time with $GF(q)^5$ the operation

$$(\alpha, \beta, \gamma, \delta, \varepsilon)(\alpha', \beta', \gamma', \delta', \varepsilon') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \alpha'\varepsilon - 3\beta'\delta, \delta + \delta', \varepsilon + \varepsilon').$$

Let x_i denote the element with ith coordinate x and all others 0, and let X_i denote the set of all such elements x_i. Then $X_i \cong GF(q)^+$ via $x \to x_i$. Also, $Q = X_1X_2X_3X_4X_5$ and $X_3 = Q' \leqslant Z(Q)$. If $3 \nmid q$ (which is the situation we will eventually have), then $Q' = Z(Q)$.

Define the following functions on Q:

$$x_6\colon\ (\alpha, \beta, \gamma, \delta, \varepsilon) \to (\alpha, \beta + \alpha x, \gamma - 3\beta^2 x - 3\alpha\beta x^2 - \alpha^2 x^3, \delta + 2\beta x + \alpha x^2, \varepsilon + 3\delta x + 3\beta x^2 + \alpha x^3),$$
$$j\colon\ (\alpha, \beta, \gamma, \delta, \varepsilon) \to (\varepsilon, -\delta, \gamma - \alpha\varepsilon + 3\beta\delta, \beta, -\alpha),$$
$$h_{a,b}\colon\ (\alpha, \beta, \gamma, \delta, \varepsilon) \to (\alpha b^3, \beta a b^2, \gamma a^3 b^3, \delta a^2 b, \varepsilon a^3),$$
$$s_a\colon\ (\alpha, \beta, \gamma, \delta, \varepsilon) \to (\alpha a, \beta a, \gamma a^2, \delta a, \varepsilon a).$$

Each of these is in Aut Q. Let $X_6 = \{x_6 \mid x \in GF(q)\}$. Then $X_6 \cong GF(q)^+$, and $\langle X_6, j\rangle \cong SL(2, q)$. Note that Aut $GF(q)$ acts on Q componentwise. Let G^2 be the subgroup of Aut Q generated by X_6, j, and all automorphisms $h_{a,b}$ and s_a, and set $G = \langle G^2, \text{Aut } GF(q)\rangle$. Then QG^2 is the stabilizer in $G_2(q)$ of the line L appearing in Section 2, while QG is the stabilizer of L in Aut $G_2(q)$. Moreover, $G^2 \cong GL(2, q)$, while $G \cong \Gamma L(2, q)$.

All of these properties of G may be verified by direct computation. The action of G^2 on Q/X_3 is given on p. 497 of [2], so only the X_3 component needs to be checked.

However, a direct approach not assuming prior familiarity with $G_2(q)$ is as follows. In the next section we will define a family $\mathscr{F}$ of $q + 1$ subgroups of Q. It is straightforward to check that G sends $\mathscr{F}$ to itself; in fact, G is the stabilizer of $\mathscr{F}$ in Aut Q if $q > 2$ (cf. Section 5, Remark 2).

5. Proof of Theorem 1

Let Q and G be as in Section 4. Define $A(\infty)$, $A(t)$, $A^*(\infty)$ and $A^*(t)$ as follows, for $t \in GF(q)$:

$$A(\infty) = X_4X_5,$$
$$A(t) = \{(\alpha, \alpha t, -\alpha^2 t^3, \alpha t^2, \alpha t^3)(0, \beta, -3\beta^2 t, 2\beta t, 3\beta t^2) \mid \alpha, \beta \in GF(q)\},$$
$$A^*(\infty) = X_3 A(\infty),\ A^*(t) = X_3 A(t).$$

Note that $A(0)=X_1X_2$. Let $\mathscr{F}=\{A(\infty), A(t) \mid t \in GF(q)\}$. Then G maps $\mathscr{F}$ to itself, and acts on it as it does on the projective line. For example, x_6 induces the map $A(t) \to A(t+x)$ and j induces $A(t) \to A(-1/t)$, for $t \in \{\infty\} \cup GF(q)$.

We must verify the axioms of Section 3. Axiom (i) is obvious. For the remaining ones, we can use G in order to assume that $A=A(0)$, $B=A(\infty)$, and $C=A(1)$. Then (ii) is obvious. A typical element of $A(1)$ has the form

$$(\alpha,\ \alpha+\beta, -\alpha^2-3\alpha\beta-3\beta^2,\ \alpha+2\beta,\ \alpha+3\beta).$$

The requirement $AB \cap C=1$ then states that $-\alpha^2-3\alpha\beta-3\beta^2 \neq 0$ unless $\alpha=\beta=0$. Thus, we must require that the polynomial $(x+1)^2+(x+1)+1$ is irreducible.

It remains to show that the resulting quadrangle is new if $q>2$. First of all, the duals of the quadrangles constructed in Section 3, Example 2, using nonquadric ovoids do not contain a group such as QG in their automorphism groups.

Now note that $(QG)'' \cong Q \rtimes SL(2,q)$, where $SL(2,q)$ acts irreducibly on $Q/Z(Q)$. Thus, $(QG)''$ has no normal subgroup of order q^3. In view of (*) in Section 3, this completes the proof of Theorem 1.

Remark 1. It is well known that (up to isomorphism) there is only one generalized quadrangle with $s=2$ and $t=4$. Here, $S/Q \cong S_3 \times S_3$.

Remark 2. We will show that, if $q>2$, then $\operatorname{Aut} \mathscr{Q}(Q, \mathscr{F})= QG \cong Q \rtimes \Gamma L(2,q)$.

First of all, $U_{\mathscr{F},A^*} \geqslant A$, while $|A|=q \geqslant |U_{\mathscr{F},A^*}|$. Thus, $U_{\mathscr{F},A^*}=A$, so that Q is generated by those groups $U_{\mathscr{F},L}$ such that L is a line meeting the line $\mathscr{F}$ once. Also, $U_{\mathscr{F}}=Z(Q)$.

Let J denote the stabilizer of the line $\mathscr{F}$ in $I=\operatorname{Aut} \mathscr{Q}(Q,\mathscr{F})$, and let J_1 denote the stabilizer of the line 1 in J. By the preceding paragraph, $Q \lhd J$. Also, $J=QJ_1$, $J_1 \geqslant G$ and J_1 is the stabilizer of $\mathscr{F}$ in $\operatorname{Aut} Q$. It is easy to check that (for $q>2$) G already contains all element of J_1 inducing $GF(q)$-semilinear transformations on $\bar{Q}=Q/Z(Q)$.

In order to show that $J=Q \rtimes G$, it thus suffices to prove the purely group theoretic fact that $\operatorname{Aut} Q$ acts $GF(q)$-semilinearly on $\bar{Q}$. If $u \in Q$, let u^* denote the preimage in Q of the 1-space of $\bar{Q}$ spanned by $\bar{u}=uZ(Q)$. Note that $(\bar{u}, \bar{v})=[u,v]$ defines a nonsingular alternating $GF(q)$-bilinear form on $\bar{Q}$; if q is even, $\bar{u} \to u^2$ defines a quadratic form on $\bar{Q}$, associated with $(\ ,\ )$. Thus, $\bar{Q}$ is equipped with a symplectic or orthogonal geometry. If $[u,v]=1$ then $[u^*,v^*]=1$. Thus, the maximal elementary abelian subgroups of Q are preimages of the totally isotropic (or singular) 2-spaces of $\bar{Q}$. Consequently $\operatorname{Aut} Q$ acts on the aforementioned geometry, and hence acts $GF(q)$ semilinearly, as asserted.

Finally, we must show that $I = J$. If $I > J$, then I has an element moving $\mathscr{F}$ to $A^* = A^*(\infty)$. In particular, $|U_{A^*}| = q$. Here, $U_{A^*} \leqslant J_1 = G = \Gamma L(2, q)$, so that $U_{A^*} \cap X_6 \neq 1$. However, if $x \neq 0$ then x_6 fixes only $q + 1$ lines on the point A (namely, the lines A^* and y_2, $y \in GF(q)$). This contradiction completes the proof that $I = J = Q \rtimes G$.

Remark 3. In view of Remark 2, it is straightforward to reconstruct the $G_2(q)$ hexagon from the quadrangle $\mathscr{Q}(Q, \mathscr{F})$.

Remark 4. The groups Q used in Theorem 1 and Example 3 are isomorphic. To see this, start with the group Q in Example 3, write $\alpha = a_1 + a_2\theta$ with $a_i \in GF(q)$ and a suitable, fixed θ in $GF(q^2)$, and compute.

When q is even, the groups $A^*/Z(Q)$ arising in Theorem 1 and Example 3 form a regulus of $Q/Z(Q)$.

Remark 5. $G_2(q)$ has a class of subgroups $S = SU(3, q)$, each of which has an orbit of $q^3 + 1$ lines. In the context of Section 2, let L be one of those lines. Then the remaining q^3 lines are Lines of the quadrangle. This yields a family of $q^3 + 1$ Lines which partition the Points of the quadrangle. Note that $S \cap Q$ has order q^3 and is transitive on the aforementioned set of q^3 Lines.

The lines of the unital for S correspond to reguli of the underlying $O(7, q)$ geometry, but do not seem to have a nice interpretation in the quadrangle.

Remark 6. There do not appear to be subquadrangles with parameters q, q arising in the following manner: for some subgroup R of Q of order q^3, the family $\{A \cap R \mid A \in \mathscr{F}\}$ satisfies the conditions of Theorem 2. Equivalently, there is probably no subgroup R of order q^3 such that the points $[A]$ and Ar ($A \in \mathscr{F}$, $r \in R$) are the points of a subquadrangle. This should be compared to the situation in Example 3.

If q is an odd prime, an elementary computation reveals that no such R exists.

Remark 7. Infinite analogs of our constructions obviously exist. If K is a field, then $G_2(K)$ produces a generalized quadrangle as in Section 2 precisely when K does not have characteristic 3 and the map $x \to x^3$, $x \in K$, is bijective.

Remark 8. It would be interesting to have a geometric relationship between the generalized quadrangles constructed here and the translation planes discussed in [2, Theorem 2].

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