Examples for 2.4

Comments: When you write up problems from section 2.4 I don't want to see how you "found δ." In the following examples I have a *Finding* δ section and a *Solution* section. You should only turn in the *Solution* section.

1. (this is similar to problems 1 and 2 in section 2.4) How close to 3 do we have to take x so that $2x - 12$ is within a distance of 0.1 of -6?

   Finding δ: This problem is asking for a number $\delta > 0$ such that $|(2x-12)-(-6)| < 0.1$ whenever $|x-3| < \delta$. Notice that

   $$|(2x-12)-(-6)| = |2x-6| = |2(x-3)| = 2|x-3|.$$

   Because of this, asking for

   $$|(2x-12)-(-6)| < 0.1 \quad \text{whenever} \quad |x-3| < \delta$$

   is the same as asking for

   $$2|x-3| < 0.1 \quad \text{whenever} \quad |x-3| < \delta$$

   or (dividing by 2)

   $$|x-3| < 0.05 \quad \text{whenever} \quad |x-3| < \delta.$$

   Seeing this, it's probably a good idea to take $\delta = 0.05$.

   Solution: Set $\delta = 0.05$ and assume $|x-3| < \delta$. Then we know

   $$|(2x-12)-(-6)| = |2x-6| = |2(x-3)| = 2|x-3| < 2\delta = 2(0.05) = 0.1$$

   Thus $|(2x-12)-(-6)| < 0.1$ as desired.

2. Set $f(x) = x^2 + 2x - 4$. Prove $\lim_{x\to 2} f(x) = 4$ using the ε, δ definition of a limit.

Finding δ: Given any $\varepsilon > 0$ we want a number $\delta > 0$ such that $|f(x) - 4| < \varepsilon$ whenever $|x - 2| < \delta$. Since

$$|f(x)-4| = |(x^2+2x-4)-4| = |x^2+2x-8| = |(x+4)(x-2)| = |x+4||x-2|$$

we see that in addition to bounding $|x - 2|$, we also need to bound $|x + 4|$. We can do so by assuming $|x - 2| < 1$. Then we know

$$-1 < x - 2 < 1 \quad \Rightarrow \quad 5 < x + 4 < 7$$

so that $|x + 4| < 7$. Thus

$$|x + 4||x - 2| < 7|x - 2|$$

Now we want

$$|f(x) - 4| < \varepsilon \quad \text{whenever} \quad |x - 2| < \delta$$

and since we have shown $|f(x)-4| = |x+4||x-2| < 7|x-2|$ it suffices to show

$$7|x - 2| < \varepsilon \quad \text{whenever} \quad |x - 2| < \delta$$

or

$$|x - 2| < \varepsilon/7 \quad \text{whenever} \quad |x - 2| < \delta$$

This shows us we should take $|x - 2| < \varepsilon/7$. But wait minute, we also needed $|x - 2| < 1$, so we should take $\delta = \min\{1, \varepsilon/7\}$.

Solution: Given $\varepsilon > 0$, set $\delta = \min\{1, \varepsilon/7\}$. Now assume $|x - 2| < \delta$. Then we know

- $|x - 2| < 1$ which gives us

  $$-1 < x - 2 < 1 \quad \Rightarrow \quad 5 < x + 4 < 7 \quad \Rightarrow \quad |x + 4| < 7 \quad (\star)$$

- $|x - 2| < \varepsilon/7 \quad (\star\star)$

So we have

$$|f(x) - 4| = |(x^2 + 2x - 4) - 4| = |x^2 + 2x - 8| = |(x + 4)(x - 2)|$$

$$= |x + 4||x - 2| \overset{(\star)}{<} 7|x - 2| \overset{(\star\star)}{<} 7 \cdot \varepsilon/7 = \varepsilon$$

as desired