

Math 251
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Assignment #9
Partial Solutions

From the Textbook:

Section 4.2

32. Prove the identity $2 \sin^{-1}(x) = \cos^{-1}(1 - 2x^2)$ for $x \geq 0$.

(We should require $0 \leq x \leq 1$, for $\sin^{-1}(x)$ is undefined if $x > 1$)

Proof: Set $f(x) = 2 \sin^{-1}(x)$ and $g(x) = \cos^{-1}(1 - 2x^2)$. Then we are required to show $f(x) = g(x)$ when $x \geq 0$. Well

$$f'(x) = 2 \frac{d}{dx} \sin^{-1}(x) = \frac{2}{\sqrt{1-x^2}}$$

and setting $u = 1 - 2x^2$ we see

$$\begin{aligned} g'(x) &= \frac{d}{dx} \cos^{-1}(u) = \frac{d}{du} \cos^{-1}(u) \frac{du}{dx} = \left(\frac{-1}{\sqrt{1-u^2}} \right) \frac{d}{dx} (1 - 2x^2) \\ &= \frac{4x}{\sqrt{1 - (1 - 2x^2)^2}} = \frac{4x}{\sqrt{1 - (1 - 4x^2 + 4x^4)}} = \frac{4x}{\sqrt{4x^2 - 4x^4}} \\ &= \frac{4x}{\sqrt{4x^2(1-x^2)}} \stackrel{(*)}{=} \frac{4x}{2x\sqrt{(1-x^2)}} = \frac{2}{\sqrt{(1-x^2)}} \end{aligned}$$

Notice that for the equality $(*)$ to hold we need $\sqrt{x^2}$ to be $x \dots$ this requires $x \geq 0$. Anyway, $f'(x) = g'(x)$ so by the corollary on page 294 we know that $f - g$ is a constant. Since

$$f(0) - g(0) = 2 \sin^{-1}(0) - \cos^{-1}(1 - 2(0)^2) = 2(0) - \cos^{-1}(1) = 0$$

we see that $f(x) - g(x) = 0$, or equivalently $f(x) = g(x)$.

Section 4.3

For problems 12 and 18 Find:

- (a) the intervals on which f is increasing and decreasing.

- (b) the local maximum and minimum values of f .
 (c) the intervals of concavity and inflection points of f .

12. $f(x) = 5 - 3x^2 + x^3$

Solution: For parts (a) and (b) we need to find the first derivative:

$$f'(x) = -6x + 3x^2 = 3x(x - 2)$$

Since the zeros of $f'(x)$ are 0 and 2 and f' is defined everywhere we need to look at the intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$. Here's the table I get:

interval	$3x$	$x - 2$	$f'(x)$	f
$x < 0$	-	-	+	increasing
$0 < x < 2$	+	-	-	decreasing
$x > 2$	+	+	+	increasing

This completes part (a).

Using the table and the first derivative test I see

- f has a local max at $x = 0$, the max value is $f(0) = 5$.
- f has a local min at $x = 2$, the min value is $f(2) = 1$.

This completes part (b).

For part (c) we need to find the second derivative:

$$f''(x) = -6 + 6x = 6(x - 1)$$

Since the only zero of $f''(x)$ is 1 and f'' is defined everywhere we need to look at the intervals $(-\infty, 1)$ and $(1, \infty)$. Here's the table I get:

interval	$f''(x) = 6(x - 1)$	f
$x < 1$	-	concave down
$x > 1$	+	concave up

Since the concavity of f changes at $x = 1$ we see

- f has an inflection point of $(1, 3)$.

This completes part (c).

18. $f(x) = x^2e^x$

Solution: For parts (a) and (b) we need to find the first derivative (using the product rule):

$$f'(x) = 2xe^x + x^2e^x = xe^x(2 + x)$$

Since e^x is always positive (i.e. never zero) the zeros of $f'(x)$ are 0 and -2. Also f' is defined everywhere, so we need to look at the intervals $(-\infty, -2)$, $(-2, 0)$, and $(0, \infty)$. Here's the table I get:

interval	x	e^x	$2 + x$	$f'(x)$	f
$x < -2$	-	+	-	+	increasing
$-2 < x < 0$	-	+	+	-	decreasing
$x > 0$	+	+	+	+	increasing

This completes part (a).

Using the table and the first derivative test I see

- f has a local max at $x = -2$, the max value is $f(-2) = 4e^{-2}$.
- f has a local min at $x = 0$, the min value is $f(0) = 0$.

This completes part (b).

For part (c) we need to find the second derivative (product rule...twice):

$$f''(x) = (2e^x + 2xe^x) + (2xe^x + x^2e^x) = 2e^x + 4xe^2 + x^2e^x = e^x(x^2 + 4x + 2)$$

Since e^x is never zero, we need to find all zeros of $x^2 + 4x + 2$. Setting $x^2 + 4x + 2 = 0$ and solving for x (either by completing the square or using the quadratic formula) gives

$$x = -2 \pm \sqrt{2}$$

So we need to look at the intervals $(-\infty, -2 - \sqrt{2})$, $(-2 - \sqrt{2}, -2 + \sqrt{2})$, and $(-2 + \sqrt{2}, \infty)$. Here's the table I get:

interval	e^x	$x^2 + 4x + 2$	$f''(x)$	f
$x < -2 - \sqrt{2}$	+	+	+	concave up
$-2 - \sqrt{2} < x < -2 + \sqrt{2}$	+	-	-	concave down
$x > -2 + \sqrt{2}$	+	+	+	concave up

Since the concavity of f changes at both $x = -2 - \sqrt{2}$ and $x = -2 + \sqrt{2}$ we see

- f has an inflection point of $(-2 - \sqrt{2}, (6 + 4\sqrt{2})e^{-2-\sqrt{2}})$.
- f has an inflection point of $(-2 + \sqrt{2}, (6 - 4\sqrt{2})e^{-2+\sqrt{2}})$.

This completes part (c).

22. Find the local max and min values of f using both the first and second derivative tests where

$$f(x) = \frac{x}{x^2 + 4}$$

Which one do you like better?

Solution: We begin by finding all critical points of f . To do this we need to find the first derivative (quotient rule):

$$f'(x) = \frac{(x^2 + 4) - x(2x)}{(x^2 + 4)^2} = \frac{4 - x^2}{(x^2 + 4)^2} = \frac{(2 - x)(2 + x)}{(x^2 + 4)^2}$$

Solving for $x^2 + 4 = 0$ gives $x = \pm\sqrt{-4}$ which is not real, so f' is defined for all real numbers. So the critical numbers of f are the zeros of $f'(x)$, which are -2 and 2 .

1st derivative test Let's make a table:

interval	$2 - x$	$2 + x$	$(x^2 + 4)^2$	$f'(x)$	f
$x < -2$	+	-	+	-	decreasing
$-2 < x < 2$	+	+	+	+	increasing
$x > 2$	-	+	+	-	decreasing

Looking at the table we see

- f has a local min at $x = -2$, the min value is $f(-2) = -\frac{1}{4}$.
- f has a local max at $x = 2$, the max value is $f(2) = \frac{1}{4}$.

2nd derivative test Let's find the second derivative (quotient rule on $f'(x) = \frac{4-x^2}{(x^2+4)^2}$):

$$f''(x) = \frac{-2x(x^2 + 4)^2 - (4 - x^2) \left(\frac{d}{dx}(x^2 + 4)^2\right)}{(x^2 + 4)^4}$$

I'm not going to simplify this, because all I care about is $f''(\pm 2)$ and the term $(4 - x^2) \left(\frac{d}{dx}(x^2 + 4)^2\right)$ is zero when $x = \pm 2$. So

$$f''(-2) = \frac{-2(-2)((-2)^2 + 4)^2 - 0}{((-2)^2 + 4)^4} = \frac{4}{((-2)^2 + 4)^2} > 0$$

and

$$f''(2) = \frac{-2(2)(2^2 + 4)^2 - 0}{(2^2 + 4)^4} = \frac{-4}{(2^2 + 4)^2} < 0$$

So by the second derivative test we see

- f has a local min at $x = -2$, the min value is $f(-2) = -\frac{1}{4}$.
- f has a local max at $x = 2$, the max value is $f(2) = \frac{1}{4}$.

Personally, I like the second derivative test better.

For problems 34, 36, and 44 Find:

- Find the intervals of increase or decrease.
- Find the local maximum and minimum values.
- Find the intervals of concavity and inflection points.
- Use the information from parts (a)-(c) to sketch the graph.

34. $f(x) = 2 + 3x - x^3$

Solution: For parts (a) and (b) we need to find the first derivative:

$$f'(x) = 3 - 3x^2 = 3(1 - x^2) = 3(1 - x)(1 + x)$$

Since the zeros of $f'(x)$ are -1 and 1 and f' is defined everywhere we need to look at the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. Here's the table I get:

interval	$3(1 - x)$	$1 + x$	$f'(x)$	f
$x < -1$	+	-	-	decreasing
$-1 < x < 1$	+	+	+	increasing
$x > 1$	-	+	-	decreasing

This completes part (a).

Using the table and the first derivative test I see

- f has a local min at $x = -1$, the min value is $f(-1) = 0$.
- f has a local max at $x = 1$, the max value is $f(1) = 4$.

This completes part (b).

For part (c) we need to find the second derivative:

$$f''(x) = -6x$$

Since the only zero of $f''(x)$ is 0 and f'' is defined everywhere we need to look at the intervals $(-\infty, 0)$ and $(0, \infty)$. Here's the table I get:

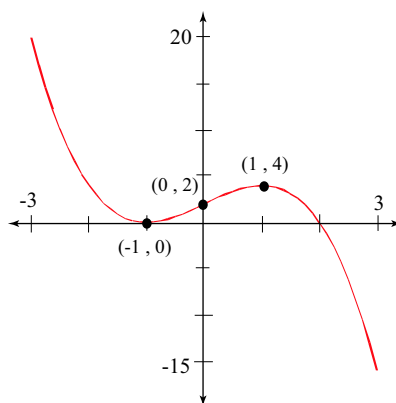
interval	$f''(x) = -6x$	f
$x < 0$	+	concave up
$x > 0$	-	concave down

Since the concavity of f changes at $x = 0$ we see

- f has an inflection point of $(0, 2)$.

This completes part (c).

Here's the graph of f :



36. $g(x) = 200 + 8x^3 + x^4$

Solution: For parts (a) and (b) we need to find the first derivative:

$$g'(x) = 24x^2 + 4x^3 = 4x^2(6 + x)$$

Since the zeros of $g'(x)$ are 0 and -6 and g' is defined everywhere we need to look at the intervals $(-\infty, -6)$, $(-6, 0)$, and $(0, \infty)$. Here's the table I get:

interval	$4x^2$	$6 + x$	$g'(x)$	g
$x < -6$	+	-	-	decreasing
$-6 < x < 0$	+	+	+	increasing
$x > 0$	+	+	+	increasing

This completes part (a).

Using the table and the first derivative test I see

- g has a local min at $x = -6$, the min value is $f(-6) = -232$.

This completes part (b).

For part (c) we need to find the second derivative:

$$g''(x) = 48x + 12x^2 = 12x(4 + x)$$

Since the zeros of $g''(x)$ are 0 and -4 and g'' is defined everywhere we need to look at the intervals $(-\infty, -4)$, $(-4, 0)$, and $(0, \infty)$. Here's the table I get:

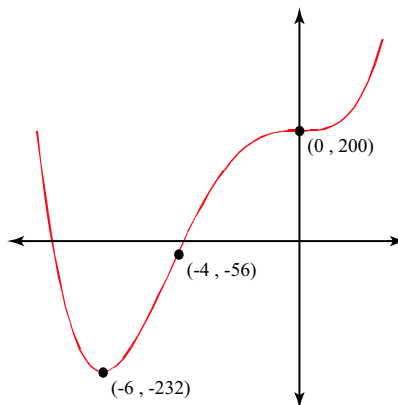
interval	$12x$	$4 + x$	$g''(x)$	g
$x < -4$	-	-	+	concave up
$-4 < x < 0$	-	+	-	concave down
$x > 0$	+	+	+	concave up

Since the concavity of g changes at both $x = -4$ and $x = 0$ we see

- g has an inflection point of $(-4, -56)$.
- g has an inflection point of $(0, 200)$.

This completes part (c).

Here's the graph of g :



44. $f(t) = t + \cos t$ where $-2\pi \leq t \leq 2\pi$.

Solution: Throughout this problem, keep in mind we only care about what happens on the interval $[-2\pi, 2\pi]$. For parts (a) and (b) we need to find the first derivative:

$$f'(t) = 1 - \sin t$$

Setting $f'(t) = 0$ and solving for t we get $\sin t = 1$ which implies $t = \frac{\pi}{2}$ or $t = -\frac{3\pi}{2}$ (these are the only solutions in the interval $[-2\pi, 2\pi]$). Since f' is defined everywhere we need to look at the intervals $[-2\pi, -\frac{3\pi}{2})$, $(-\frac{3\pi}{2}, \frac{\pi}{2})$, and $(\frac{\pi}{2}, 2\pi]$. Here's the table I get:

interval	$f'(t) = 1 - \sin t$	f
$-2\pi \leq t < -\frac{3\pi}{2}$	+	increasing
$-\frac{3\pi}{2} < t < \frac{\pi}{2}$	+	increasing
$\frac{3\pi}{2} < t \leq 2\pi$	+	increasing

The easiest way to get this table is to notice $\sin t \leq 1$ for all t , so $-\sin t \geq -1$ for all t , so $1 - \sin t \geq 0$ for all t . This completes part (a).

Using the table and the first derivative test we see that f has no local max or min values. This completes part (b).

For part (c) we need to find the second derivative:

$$f''(t) = -\cos t$$

Since the zeros of $f''(t)$ are $\pm\frac{3\pi}{2}$ and $\pm\frac{\pi}{2}$ and f'' is defined everywhere we need to look at the intervals $[-2\pi, 2\pi]$. Since f' is defined everywhere we need to look at the intervals $[-2\pi, -\frac{3\pi}{2})$, $(-\frac{3\pi}{2}, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{3\pi}{2})$, and $(\frac{3\pi}{2}, 2\pi]$. Here's the table I get:

interval	$\cos t$	$f''(t) = -\cos t$	f
$-2\pi \leq t < -\frac{3\pi}{2}$	+	-	concave down
$-\frac{3\pi}{2} < t < -\frac{\pi}{2}$	-	+	concave up
$-\frac{\pi}{2} < t < \frac{\pi}{2}$	+	-	concave down
$\frac{\pi}{2} < t < \frac{3\pi}{2}$	-	+	concave up
$\frac{3\pi}{2} < t \leq 2\pi$	+	-	concave down

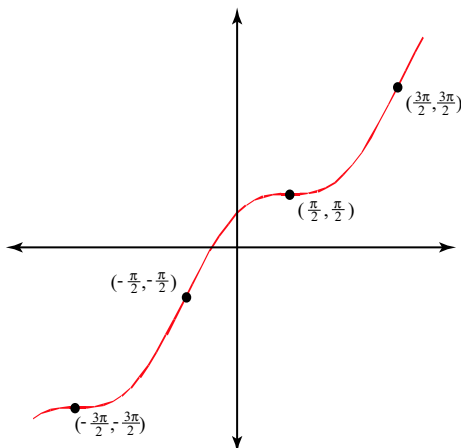
To get this table, I use the unit circle. Since the concavity of f changes at each zero of $f''(t)$ we see

- f has an inflection point of $(-\frac{3\pi}{2}, -\frac{3\pi}{2})$.
- f has an inflection point of $(-\frac{\pi}{2}, -\frac{\pi}{2})$.

- f has an inflection point of $(\frac{\pi}{2}, \frac{\pi}{2})$.
- f has an inflection point of $(\frac{3\pi}{2}, \frac{3\pi}{2})$.

This completes part (c).

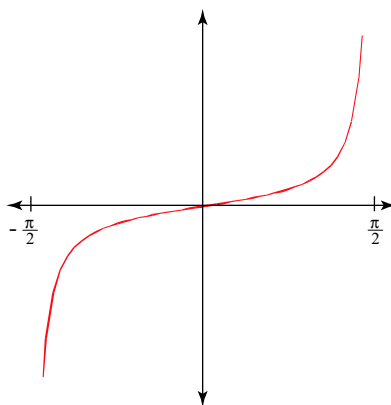
Here's the graph of f :



48. Let $f(x) = x \tan(x)$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

- (a) Find the vertical and horizontal asymptotes.

Solution: Since we are only interested in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ it doesn't make sense to talk about horizontal asymptotes, so let's find the vertical asymptotes. Since x is defined everywhere, and $\tan(x)$ is undefined at $\pm\frac{\pi}{2}$ the only possibilities for vertical asymptotes are at $x = \pm\frac{\pi}{2}$. Looking at the graph of $\tan(x)$



we see $\lim_{x \rightarrow \frac{\pi}{2}^-} \tan(x) = \infty$ and $\lim_{x \rightarrow -\frac{\pi}{2}^+} \tan(x) = -\infty$. Thus

$$\lim_{x \rightarrow \frac{\pi}{2}^-} x \tan(x) = \left(\lim_{x \rightarrow \frac{\pi}{2}^-} x \right) \left(\lim_{x \rightarrow \frac{\pi}{2}^-} \tan(x) \right) = \left(\frac{\pi}{2} \right) (\infty) = \infty$$

and

$$\lim_{x \rightarrow -\frac{\pi}{2}^+} x \tan(x) = \left(\lim_{x \rightarrow -\frac{\pi}{2}^+} x \right) \left(\lim_{x \rightarrow -\frac{\pi}{2}^+} \tan(x) \right) = \left(-\frac{\pi}{2} \right) (-\infty) = \infty$$

So f has vertical asymptotes of $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$.

- (b) Find the intervals of increase or decrease.

Solution: We need the first derivative (product rule):

$$f'(x) = \tan(x) + x \sec^2(x)$$

Ok, first I notice that $f(0) = \tan(0) + (0) \sec^2(0) = 0 + 0 \cdot 1 = 0$. Now I will show that $f(x) > 0$ whenever $0 < x < \pi/2$: Well, suppose $0 < x < \pi/2$. Then we know $\tan x > 0$ (look at it's graph above) and $x > 0$, and $\sec^2 x > 0$ (because the square of any number is positive). Thus $f'(x)$ is the sum of two positive numbers, which is positive. Similarly I will show that $f(x) < 0$ whenever $-\pi/2 < x < 0$: Well, suppose $-\pi/2 < x < 0$. Then we know $\tan x < 0$ and $x < 0$, and $\sec^2 x > 0$. Thus $f'(x)$ is the sum of two negative numbers, which is negative. Let's sum up these results in a table:

interval	$f'(x)$	f
$-\pi/2 \leq x < 0$	-	decreasing
$0 < x < \pi/2$	+	increasing

- (c) Find the local max and min values.

Solution:

Looking at the table above and using the first derivative test we see

- f has a local min at $x = 0$, the min value is $f(0) = 0$.

- (d) Find the intervals of concavity and the inflection points.

Solution: Let's find the second derivative (product and chain rule):

$$f''(x) = \sec^2(x) + (1) \sec^2(x) + x \frac{d}{dx} \sec^2(x)$$

$$= 2 \sec^2 x + 2x \sec(x)(\sec(x) \tan(x)) = 2 \sec^2 x(1 + x \tan x)$$

I will show that $f''(x)$ is positive for all x in the interval $(-\pi/2, \pi/2)$. Well suppose $-\pi/2 < x < \pi/2$, then $\tan x$ and x are either both positive or both negative, in either case $x \tan x$ is positive. So $1 + x \tan x$ is positive. Since $\sec^2(x)$ is always positive, we see $f''(x)$ is the product of positive numbers, and is therefore positive. So f is concave up on $(-\pi/2, \pi/2)$. This also tells us f has no inflection points.

(e) Here's a graph of f

