Math 251
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Assignment #3
Partial Solutions

From the Textbook:

Section 2.6:

12. Evaluate the limit

$$\lim_{x\to\infty}\sqrt{\frac{12x^3-5x+2}{1+4x^2+3x^3}}$$

Solution:

$$\lim_{x\to\infty}\sqrt{\frac{12x^3-5x+2}{1+4x^2+3x^3}} = \lim_{x\to\infty}\sqrt{\frac{(12x^3-5x+2)\frac{1}{x^3}}{(1+4x^2+3x^3)\frac{1}{x^3}}} = \lim_{x\to\infty}\sqrt{\frac{12-\frac{5}{x^2}+\frac{2}{x^3}}{\frac{1}{x^3}+\frac{4}{x}+3}}$$

$$= \sqrt{\lim_{x\to\infty}\frac{12-\frac{5}{x^2}+\frac{2}{x^3}}{\frac{1}{x^3}+\frac{4}{x}+3}} = \sqrt{\frac{\lim_{x\to\infty}(12-\frac{5}{x^2}+\frac{2}{x^3})}{\lim_{x\to\infty}(\frac{1}{x^3}+\frac{4}{x}+3)}}$$

$$= \sqrt{\frac{\lim_{x\to\infty}12-\lim_{x\to\infty}\frac{5}{x^2}+\lim_{x\to\infty}\frac{2}{x^3}}{\lim_{x\to\infty}\frac{1}{x^3}+\lim_{x\to\infty}\frac{4}{x}+\lim_{x\to\infty}3}}$$

$$= \sqrt{\frac{12-0+0}{0+0+3}} = \sqrt{4} = 2.$$

Additional Exercises:

1. This problem is meant to show you that when proving $\lim_{x\to a} f(x) = L$ using the ε δ definition, your choice of δ will depend on a.

   (a) Prove that $\lim_{x\to 0}\left(x^2+4x+2\right) = 2$ using the ε, δ definition of limits.

   Solution: Given $\varepsilon > 0$ set $\delta = \min\{1, e/5\}$ and assume $|x| < \delta$. Then we know

- $|x| < 1$ which implies

$$-1 < x < 1 \quad \Rightarrow \quad 3 < x+4 < 5 \quad \Rightarrow \quad |x+4| < 5 \quad (\star)$$

- $|x| < \varepsilon/5 \quad (\star\star)$

So we have

$$|(x^2+4x+2)-2| = |x^2+4x| = |x||x+4| \overset{(\star)}{<} 5|x| \overset{(\star\star)}{<} 5 \cdot \varepsilon/5 = \varepsilon$$

(b) Prove that $\lim_{x\to 1} \left(x^2+4x+2\right) = 7$ using the ε, δ definition of limits.

Solution: Given $\varepsilon > 0$ set $\delta = \min\{1, e/6\}$ and assume $|x-1| < \delta$. Then we know

- $|x-1| < 1$ which implies

$$-1 < x-1 < 1 \quad \Rightarrow \quad 4 < x+5 < 6 \quad \Rightarrow \quad |x+5| < 6 \quad (\star)$$

- $|x-1| < \varepsilon/6 \quad (\star\star)$

So we have

$$|(x^2+4x+2)-7| = |x^2+4x-5| = |x-1||x+5| \overset{(\star)}{<} 6|x-1| \overset{(\star\star)}{<} 6 \cdot \varepsilon/6 = \varepsilon$$

Extra Credit: Write down the rule of a function whose domain is all real numbers, but is nowhere continuous

[i.e. a function f such that $\lim_{x\to a} f(x) \neq f(a)$ for **every** real number a.]

Solution: Here's one example known as the Dirichlet function: Let f be the functions whose rule is given by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$$

Then $f(x)$ is defined for all real numbers (because every real number is either rational or irrational). Also $\lim_{x\to a} f(x)$ does not exist for any real number a (think about why this is true) . Therefore f is nowhere continuous.