Math 251
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Assignment #2
Partial Solutions

From the Textbook:

Section 2.4:

30. Prove $\lim_{x\to 3}(x^2+x-4)=8$ using the ε, δ definition of a limit.

    Solution: Given $\varepsilon > 0$ set $\delta = \min\{1, \varepsilon/8\}$ and assume $|x-3| < \delta$. Then we know

    - $|x-3| < 1$ which implies

    $$-1 < x-3 < 1 \quad \Rightarrow \quad 6 < x+4 < 8 \quad \Rightarrow \quad |x+4| < 8 \quad (\star)$$

    - $|x-3| < \varepsilon/8 \quad (\star\star)$

    We want to show $|f(x)-8| < \varepsilon$. Well

    $$|f(x)-8| = |x^2+x-4-8| = |x^2+x-12| = |(x+4)(x-3)|$$

    $$= |x+4||x-3| \overset{(\star)}{<} 8\cdot|x-3| \overset{(\star\star)}{<} 8\cdot\varepsilon/8 = \varepsilon$$

    as desired.

32. Prove $\lim_{x\to 2} x^3 = 8$ using the ε, δ definition of a limit.

    Solution: Given $\varepsilon > 0$ set $\delta = \min\{1, \varepsilon/19\}$ and assume $|x-2| < \delta$. Then we know

    - $|x-2| < 1$ which implies

    $$-1 < x-2 < 1 \quad \Rightarrow \quad 1 < x < 3 \quad \Rightarrow \quad 1^2+2\cdot1+4 < x^2+2x+4 < 3^2+2\cdot3+4$$

    $$\Rightarrow \quad 7 < x^2+2x+4 < 19 \quad \Rightarrow \quad |x^2+2x+4| < 19 \quad (\star)$$

    - $|x-2| < \varepsilon/19 \quad (\star\star)$

We want to show $|f(x) - 8| < \varepsilon$. Well

$$|f(x) - 8| = |x^3 - 8| = |(x-2)(x^2 + 2x + 4)|$$

$$= |x-2||x^2 + 2x + 4| \overset{(\star)}{<} |x-2| \cdot 19 \overset{(\star\star)}{<} \varepsilon/19 \cdot 19 = \varepsilon.$$

Additional Exercises:

1. Let $f(x) = x^2 - 3x - 3$. Fix $\varepsilon > 0$ and suppose $0 < \delta < \min\{1, \varepsilon/8\}$. Show that $|f(x) - 7| < \varepsilon$ whenever $|x - 5| < \delta$.

   Solution: Since $|x - 5| < \delta = \min\{1, \varepsilon/8\}$ we know

   - $|x - 5| < 1$ which implies

     $$-1 < x - 5 < 1 \quad \Rightarrow \quad 6 < x + 2 < 8 \quad \Rightarrow \quad |x + 2| < 8 \quad (\star)$$

   - $|x - 5| < \varepsilon/8 \quad (\star\star)$

   We want to show $|f(x) - 7| < \varepsilon$. Well

   $$|f(x) - 7| = |x^2 - 3x - 3 - 7| = |x^2 - 3x - 10| = |(x-5)(x+2)|$$

   $$= |x-5||x+2| \overset{(\star)}{<} |x-5| \cdot 8 \overset{(\star\star)}{<} \varepsilon/8 \cdot 8 = \varepsilon$$

   just like we wanted.

2. Let $f(x) = x^3$. Fix $\varepsilon > 0$ and suppose $0 < \delta < \min\{1, \varepsilon/7\}$. Show that $|f(x) - 1| < \varepsilon$ whenever $|x - 1| < \delta$.

   Solution: Since $|x - 1| < \delta = \min\{1, \varepsilon/7\}$ we know

   - $|x - 1| < 1$ which implies

     $$-1 < x - 1 < 1 \quad \Rightarrow \quad 0 < x < 2 \quad \Rightarrow \quad 0^2 + 0 + 1 < x^2 + x + 1 < 2^2 + 2 + 1$$

     $$\Rightarrow \quad 1 < x^2 + x + 1 < 7 \quad \Rightarrow \quad |x^2 + x + 1| < 7 \quad (\star)$$

   - $|x - 1| < \varepsilon/7 \quad (\star\star)$

   We want to show $|f(x) - 1| < \varepsilon$. Well

   $$|f(x) - 1| = |x^3 - 1| = |(x-1)(x^2 + x + 1)|$$

   $$= |x-1||x^2 + x + 1| \overset{(\star)}{<} |x-5| \cdot 7 \overset{(\star\star)}{<} \varepsilon/7 \cdot 7 = \varepsilon.$$