Math 251
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Winter 2006
Assignment #10
Partial Solutions

From the Textbook:

Section 4.5 I will follow steps A-H in the book. Keep in mind that some of these steps are very hard... so I will omit some.

7. Sketch a graph of $y = 2x^5 - 5x^2 + 1$

   Solution:

   A. The domain is all real numbers.

   B. When $x = 0$, $y = 1$ so the y-intercept is $(0, 1)$. Setting $y = 0$ and solving for x is too hard.

   C. No symmetry.

   D. Since $\lim_{x\to\infty}(2x^5 - 5x^2 + 1) = \infty$ and $\lim_{x\to-\infty}(2x^5 - 5x^2 + 1) = -\infty$, there are no horizontal asymptotes. Since the domain is all real numbers, there are no vertical asymptotes.

   E. Notice $y' = 10x^4 - 10x = 10x(x^3 - 1)$, so the critical numbers are $x = 0$ and $x = 1$. Here's a table

interval	$10x$	$x^3 - 1$	y'	y
$x < 0$	$-$	$-$	$+$	increasing
$0 < x < 1$	$+$	$-$	$-$	decreasing
$1 < x$	$+$	$+$	$+$	increasing

   F. Looking at the table above and using the first derivative test we see

   - There is a local max at $x = 0$, the max value is $y = 1$.
   - There is a local min at $x = 1$, the max value is $y = -2$.

   G. Notice $y'' = 40x^3 - 10$. Let's set $y'' = 0$ and solve for x:

$$40x^3 - 10 = 0 \quad \Rightarrow \quad 40x^3 = 10 \quad \Rightarrow \quad x^3 = \frac{1}{4} \quad \Rightarrow \quad x = \sqrt[3]{\frac{1}{4}} = \frac{1}{\sqrt[3]{4}}.$$

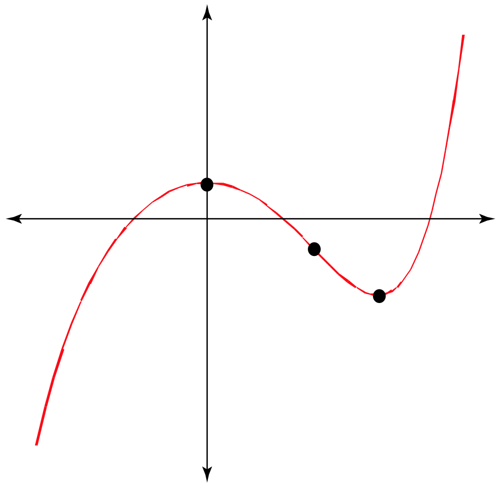
   Here's a table

interval	y''	y
$x < \frac{1}{\sqrt[3]{4}}$	$-$	concave down
$\frac{1}{\sqrt[3]{4}} < x$	$+$	concave up

   Looking at the table above we see

   - There is an inflection point of $(\frac{1}{\sqrt[3]{4}}, 1 - \frac{9}{4\sqrt[3]{2}})$ [to sketch a graph, this is not that helpful... so if we use a calculator to approximate, we see the inflection point is at about $(0.63, -0.79)$

H. Here's a picture of the curve where the max/min and inflection points are labelled with dots:



11. Sketch a graph of $y = \frac{1}{x^2-9}$

Solution:

A. The domain is all real numbers except ± 3.

B. When $x = 0$, $y = -\frac{1}{9}$ so the y-intercept is $(0, -\frac{1}{9})$. Since the numerator of $\frac{1}{x^2-9}$ is never zero, we see y is never zero, so there are no x-intercepts.

C. Since $f(-x) = \frac{1}{(-x)^2-9} = \frac{1}{x^2-9} = f(x)$ we see f is even, so it has symmetry about the y-axis.

D. Since $\lim_{x\to\pm\infty} \frac{1}{x^2-9} = 0$ we see there is a horizontal asymptote of $y = 0$. Since $\lim_{x\to 3^+} \frac{1}{x^2-9} = \infty$ and $\lim_{x\to 3^-} \frac{1}{x^2-9} = -\infty$ we see there is a vertical asymptote of $x = 3$. By symmetry we know $\lim_{x\to -3^+} \frac{1}{x^2-9} = -\infty$ and $\lim_{x\to -3^-} \frac{1}{x^2-9} = \infty$ so there is also a vertical asymptote of $x = 3$.

E. Notice $y' = \frac{-2x}{(x^2-9)^2}$, so the critical numbers are $x = 0$, $x = -3$ and $x = 3$. Here's a table

interval	$-2x$	$(x^2-9)^2$	y'	y
$x < -3$	$+$	$+$	$+$	increasing
$-3 < x < 0$	$+$	$+$	$+$	increasing
$0 < x < 3$	$-$	$+$	$-$	decreasing
$3 < x$	$-$	$+$	$-$	decreasing

F. Looking at the table above and using the first derivative test we see

- There is a local max at $x = 0$, the max value is $y = -\frac{1}{9}$.

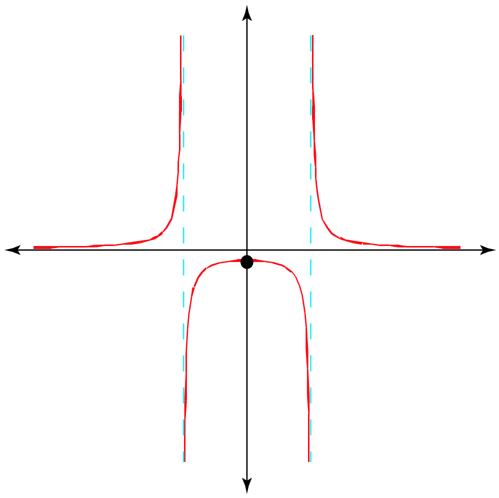
G. Notice

$$y'' = -2\left(\frac{(x^2-9)^2 - x(2(x^2-9)(2x))}{(x^2-9)^4}\right) = -2\left(\frac{(x^2-9) - x(2)(2x)}{(x^2-9)^3}\right) = \frac{6x^2+18}{(x^2-9)^3}$$

Since the numerator $6x^2 + 18$ is always positive, we see y'' is never zero, so there are no inflection points. To check concavity we only need to consider the intervals $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$. Here's a table

interval	y''	y
$x < -3$	$+$	concave up
$-3 < x < 3$	$-$	concave down
$x > 3$	$+$	concave up

H. Here's a picture of the curve where the max point is labelled with a dot:



13. Sketch a graph of $y = \frac{x}{x^2+9}$

Solution:

A. Since $x^2 + 9$ is always positive (i.e. never zero) the domain is all real numbers.

B. When $x = 0$, $y = 0$ so the y-intercept is $(0, 0)$. Since the numerator of $\frac{x}{x^2+9}$ is zero exactly when $x = 0$, we see the only x-intercept is $(0, 0)$.

C. Since $f(-x) = \frac{-x}{(-x)^2+9} = \frac{-x}{x^2+9} = -f(x)$ we see f is odd, so it has symmetry about the origin.

D. Since $\lim_{x\to\pm\infty} x = \pm\infty$ and $\lim_{x\to\pm\infty}(x^2 + 9) = \infty$ we can use L'Hospital's rule to see $\lim_{x\to\pm\infty} \frac{x}{x^2+9} \overset{L}{==} \lim_{x\to\pm\infty} \frac{1}{2x} = 0$ so there is a horizontal asymptote of $y = 0$. Since the domain is all real numbers, there are no vertical asymptotes.

E. Using the quotient rule we see $y' = \frac{9-x^2}{(x^2+9)^2}$, so the critical numbers are $x = -3$ and $x = 3$. Here's a table

interval	$9 - x^2$	$(x^2 + 9)^2$	y'	y
$x < -3$	$-$	$+$	$-$	decreasing
$-3 < x < 3$	$+$	$+$	$+$	increasing
$3 < x$	$-$	$+$	$-$	decreasing

F. Looking at the table above and using the first derivative test we see

- There is a local min at $x = -3$, the min value is $y = -\frac{1}{6}$.
- There is a local max at $x = 3$, the max value is $y = \frac{1}{6}$.

G. Using the quotient rule again we see

$$y'' = \frac{-2x(x^2+9)^2 - 2(x^2+9)(2x)(9-x^2)}{(x^2+9)^4} = \frac{-2x(x^2+9) - 2(2x)(9-x^2)}{(x^2+9)^3}$$
$$= \frac{2x^3 - 54x}{(x^2+9)^3} = \frac{2x(x^2-27)}{(x^2+9)^3}$$

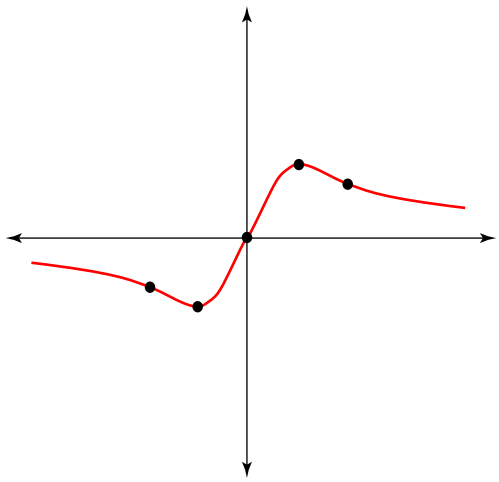
Setting the numerator equal to zero and solving for x we see the roots of y'' are 0, $3\sqrt{3}$, and $-3\sqrt{3}$. Here's a table

interval	$2x$	x^2-27	$(x^2+9)^3$	y''	y
$x < -3\sqrt{3}$	$-$	$+$	$+$	$-$	concave down
$-3\sqrt{3} < x < 0$	$-$	$-$	$+$	$+$	concave up
$0 < x < 3\sqrt{3}$	$+$	$-$	$+$	$-$	concave down
$x > 3\sqrt{3}$	$+$	$+$	$+$	$+$	concave up

Looking at the table above we see

- There is an inflection point of $(-3\sqrt{3}, -\frac{\sqrt{3}}{12})$
- There is an inflection point of $(0,0)$
- There is an inflection point of $(3\sqrt{3}, \frac{\sqrt{3}}{12})$

H. Here's a picture of the curve where the min/max and inflection points are labelled with dots:



55. Find an equation for the slant asymptote to the curve

$$y = \frac{x^2+1}{x+1}$$

Solution: We do long division:

$$\begin{array}{r|l} & \quad x - 1 \\ \hline x+1 & x^2 \qquad + 1 \\ & -x^2 - x \\ \hline & \quad -x + 1 \\ & \quad\ \ x + 1 \\ \hline & \qquad\quad 2 \end{array}$$

So we have

$$\frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1} \quad \Rightarrow \frac{x^2+1}{x+1} - (x-1) = \frac{2}{x+1}$$

Thus

$$\lim_{x\to\pm\infty} \left(\frac{x^2+1}{x+1} - (x-1) \right) = \lim_{x\to\pm\infty} \frac{2}{x+1} = 0$$

This tells us that the line $y = x - 1$ is a slant asymptote to the curve $y = \frac{x^2+1}{x+1}$.

Here is a picture where the slant asymptote $y = x - 1$ is a dashed line.

(you were not expected to sketch a graph on these problems)