Math 251 Exam 1 Solutions

1. (5pts) What is the definition of a derivative?

   [all I'm looking for here is the rule of $f'(x)$]

   **Solution:** $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ whenever the limit exists and is not $\infty$ or $-\infty$

2. (5pts) State the squeeze theorem.

   **Solution:** If $f, g$, and $h$ are functions with

$$f(x) \leq g(x) \leq h(x)$$

   for all $x$ near $a$ and

$$\lim_{x\to a} f(x) = L \qquad \text{and} \qquad \lim_{x\to a} h(x) = L$$

   Then

$$\lim_{x\to a} g(x) = L.$$

3. (10pts) State where the function $f$ is discontinuous, given that

$$f(x) = \begin{cases} x-3 & \text{if } x < 1, \\ x^2 - 2x - 1 & \text{if } 1 \leq x < 3, \\ 0 & \text{if } x = 3, \\ 2^x - 6 & \text{if } x > 3. \end{cases}$$

   **Solution:** The only possible discontinuities of $f$ occur at $x = 1$ and $x = 3$. First let's look at $x = 1$:

   We need to compare $f(1)$ with $\lim\limits_{x\to 1} f(x)$. Well, $f(1) = (1)^2 - 2(1) - 1 = -2$. Now to find $\lim\limits_{x\to 1} f(x)$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x-3) = 1 - 3 = -2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 - 2x - 1) = 1^2 - 2(1) - 1 = -2$$

   so $\lim\limits_{x\to 1} f(x) = -2$. Therefore

$$f(1) = -2 = \lim_{x\to 1} f(x)$$

   so $f$ is continuous at $x = 1$.

   We need to compare $f(3)$ with $\lim\limits_{x\to 3} f(x)$. Well, $f(3) = 0$. Now to find $\lim\limits_{x\to 3} f(x)$:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 - 2x - 1) = 3^2 - 2(3) - 1 = 2$$

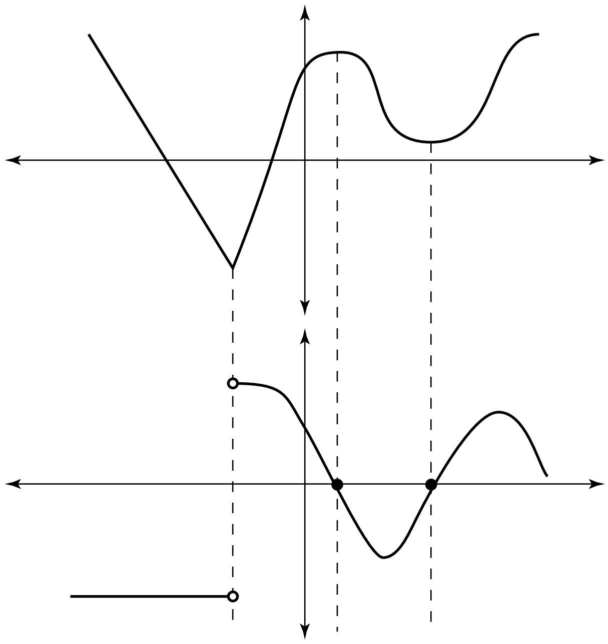
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2^x - 6) = 2^3 - 6 = 2$$

   so $\lim\limits_{x\to 3} f(x) = 2$. Therefore

$$f(3) = 0 \neq 2 = \lim_{x\to 3} f(x)$$

   so $f$ is not continuous at $x = 3$.

4. (10pts) Sketch a graph of $\frac{df}{dx}$ below the given graph of $f$.



5. (10pts) Prove that $\lim_{x \to -3} (4x + 1) = -11$ using the $\varepsilon$, $\delta$ definition of a limit.

**Solution:** Given $\varepsilon > 0$, set $\delta = \varepsilon/4$ and assume $|x + 3| < \delta$. Then

$$|(4x + 1) - (-11)| = |4x + 12| = |4(x + 3)| = 4|x + 3| < 4\delta = 4(\varepsilon/4) = \varepsilon.$$

Thus $|(4x + 1) - (-11)| < \varepsilon$ as desired.

6. Let $f(x) = \sqrt{3x-2}$

(a) (10pts) Find the rule of $f'(x)$.
**Solution:**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3(x+h)-2}-\sqrt{3x-2}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)-2}-\sqrt{3x-2}}{h} \cdot \frac{(\sqrt{3(x+h)-2}+\sqrt{3x-2})}{(\sqrt{3(x+h)-2}+\sqrt{3x-2})}$$

$$= \lim_{h\to 0} \frac{3(x+h)-2-(3x-2)}{h(\sqrt{3(x+h)-2}+\sqrt{3x-2})} = \lim_{h\to 0} \frac{3x+3h-2-3x+2}{h(\sqrt{3(x+h)-2}+\sqrt{3x-2})}$$

$$= \lim_{h\to 0} \frac{3h}{h(\sqrt{3(x+h)-2}+\sqrt{3x-2})} = \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)-2}+\sqrt{3x-2}}$$

$$= \frac{3}{\sqrt{3x-2}+\sqrt{3x-2}} = \frac{3}{2\sqrt{3x-2}}$$

So $f'(x) = \frac{3}{2\sqrt{3x-2}}$.

(b) (10pts) Give the equation of the tangent line to $f$ at the point $(9,5)$.
**Solution:** We know the slope of the tangent line at the point $(9,5)$ is

$$f'(9) = \frac{3}{2\sqrt{3(9)-2}} = \frac{3}{2\sqrt{25}} = \frac{3}{10}$$

So the equation of the tangent line is $y-5 = \frac{3}{10}(x-9)$ or

$$y = \frac{3}{10}\,x - \frac{27}{10} + 5$$

or

$$y = \frac{3}{10}\,x + \frac{23}{10}.$$

7. (10pts) Evaluate the following limit, if it exists.

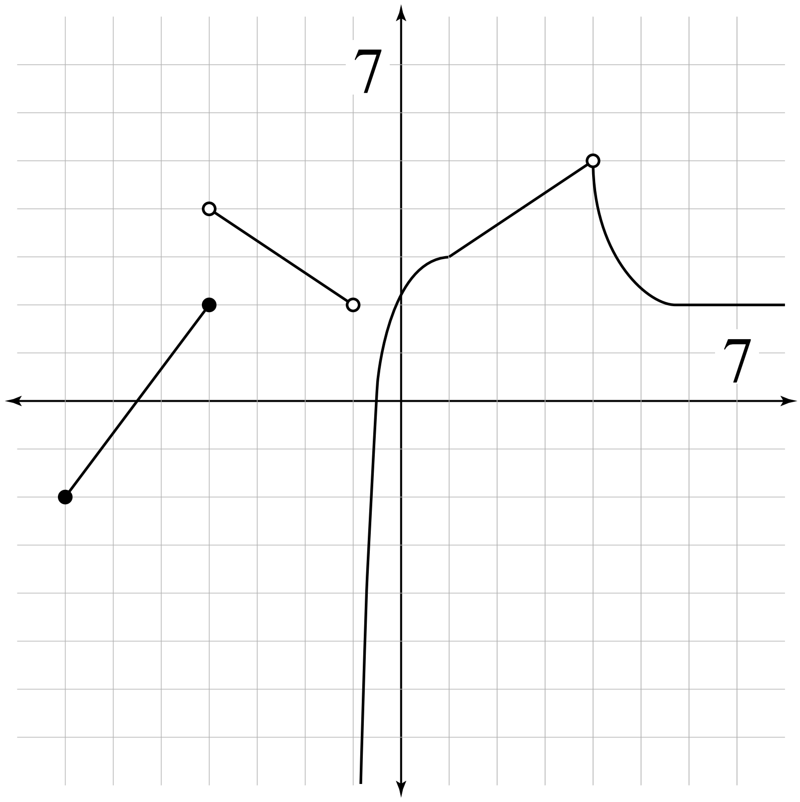
$$\lim_{x\to\infty} \frac{12x^2-5x+108}{x^3+3x}$$

**Solution:**

$$\lim_{x\to\infty} \frac{12x^2-5x+108}{x^3+3x} = \lim_{x\to\infty} \frac{12x^2-5x+108}{x^3+3x} \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{12-\frac{5}{x}+\frac{108}{x^2}}{x+\frac{3}{x}}$$

$$= \frac{12-\frac{5}{\infty}+\frac{108}{(\infty)^2}}{\infty+\frac{3}{\infty}} = \frac{12-0+0}{\infty+0} = \frac{12}{\infty} = 0.$$

8. (10pts) Consider the following graph of the function $f$




State the following values, if they exists.

$\lim_{x\to\infty} f(x) = 2$

$\lim_{x\to -1} f(x)$ does not exist

$\lim_{x\to -1^+} f(x) = -\infty$

$\lim_{x\to 4} f(x) = 5$

$\lim_{x\to -4^-} f(x) = 2$

9. (10pts) Show that $f(c) = 0$ for some number $c$ between 0 and 2 where

$$f(x) = 3^x + \sqrt[3]{4x} - 7.$$

**Solution:** $f$ is continuous on it's domain which is all real numbers. Thus $f$ is continuous on the interval $[0, 2]$. Now,

$$f(0) = 3^0 + \sqrt[3]{4(0)} - 7 = 1 + 0 - 7 = -6$$

$$f(2) = 3^2 + \sqrt[3]{4(2)} - 7 = 9 + 2 - 7 = 4$$

Since $f(0) < 0 < f(2)$ by the intermediate value theorem we know there exists a $c$ between 0 and 2 with $f(c) = 0$.

10. (10pts) Suppose $f$ is a function which satisfies

$$-x \leq f(x) \leq \frac{x+1}{x^2+3x+2}$$

whenever $-2 < x < 0$. Use the squeeze theorem to find $\lim_{x\to -1} f(x)$.

**Solution:** Well, $\lim_{x\to -1}(-x) = 1$ and

$$\lim_{x\to -1} \frac{x+1}{x^2+3x+2} = \lim_{x\to -1} \frac{x+1}{(x+1)(x+2)} = \lim_{x\to -1} \frac{1}{x+2} = \frac{1}{-1+2} = 1$$

So by the squeeze theorem we know $\lim_{x\to -1} f(x) = 1$.

Extra Credit (10pts) A fixed point of a function $f$ is a number $a$ such that $f(a) = a$. Show that the function with rule $f(x) = x^6 - 2^x + \sqrt{x} + 3x$ has a fixed point.

**Solution:** To show $f(a) = a$ is the same as showing $f(a) - a = 0$. Now

$$f(a) - a = a^6 - 2^a + \sqrt{a} + 3a - a = a^6 - 2^a + \sqrt{a} + 2a$$

Let $g(x) = x^6 - 2^x + \sqrt{x} + 2x$. Notice that $g$ is continuous on its domain which is $[0, \infty)$ so it's continuous on $[0, 1]$. Now $g(0) = 0^6 - 2^0 + \sqrt{0} + 2(0) = -1$ and $g(1) = 1^6 - 2^1 + \sqrt{1} + 2(1) = 2$ so by the intermediate value theorem there exists an $a$ between 0 and 1 with $g(a) = 0$. Thus $f(a) - a = g(a) = 0$, which implies $f(a) = a$.