Math 251 Exam 2 Solutions

1. (2pts each) Write the rules for each of the following making sure to use correct notation. Assume f and g are differentiable and c is a constant.

(a) $\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$

(b) $\frac{d}{dx}(cf(x)) = cf'(x)$

(c) $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x)+f(x)g'(x)$

(d) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$

(e) $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

2. (10pts) Prove the following equality. Show all work, and justify steps when appropriate.

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

Solution: Set $y = \arctan(x)$. Then we know $\tan(y) = x$. Differentiating with respect to x gives us

$$\frac{d}{dx}(\tan(y)) = \frac{d}{dx}(x) \quad \Rightarrow \quad \sec^2(y)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2(y)}.$$

If we divide both sides of the Pythagorean identity $\cos^2 y + \sin^2 y = 1$ by $\cos^2 y$ we get $1+\tan^2 y = \sec^2 y$. So we can substitute $1+\tan^2 y$ for $\sec^2 y$ in our expression for $\frac{dy}{dx}$ to get $\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$. But $\tan y = x$, so $\tan^2 y = x^2$. Therefore we have

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

3. (10pts) Find the rule of $f'(x)$ where

$$f(x) = (1+x^2)\arctan(x)$$

[Hint: you can use the result from problem 2 even if you did not finish problem 2]

Solution: Using the product rule we have

$$f'(x) = \frac{d}{dx}((1+x^2))\arctan(x) + (1+x^2)\frac{d}{dx}(\arctan(x))$$

$$= 2x\arctan(x) + (1+x^2)\left(\frac{1}{1+x^2}\right) = 2x\arctan(x)+1.$$

$$\boxed{f'(x) = 2x\arctan(x)+1}$$

4. (10pts) Find the global (absolute) maximum and minimum of the function f on the interval $[1, 3]$ where

$$f(x) = \frac{x}{x^2+4}$$

Solution: Since $x^2 + 4 = 0 \quad \Rightarrow \quad x = \pm\sqrt{-4}$ and $\sqrt{-4}$ is not a real number, we see f is continuous on $[1, 3]$. Therefore we only need to check the values $f(1)$, $f(3)$, and $f(c)$ for every critical number c in $(1, 3)$. So let's find all critical numbers:

$$f'(x) = \frac{(x^2+4)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+4)}{(x^2+4)^2} = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}.$$

Again, since $x^2 + 4 \neq 0$ for any real number x, we see $f'(x)$ exists for all real x. (You really should mention this!!!) Thus, the critical points are all zeros of $f'(x)$. For $f'(x)$ to be zero, the numerator of $\frac{4-x^2}{(x^2+4)^2}$ must be zero...

$$4 - x^2 = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = 2 \quad \text{or} \quad x = -2$$

Since -2 is not in the interval $[1, 3]$, 2 is the only critical number which might give an absolute max or min. Well,

$$f(1) = \frac{1}{5}, \quad f(3) = \frac{3}{13}, \quad f(2) = \frac{2}{8} = \frac{1}{4}.$$

Since $\frac{1}{5} = \frac{3}{15} < \frac{3}{13} < \frac{3}{12} = \frac{1}{4}$ we see

$$\boxed{f \text{ has a global max value of } \tfrac{1}{4} \text{ and it occurs at } x = 2}$$

$$\boxed{f \text{ has a global min value of } \tfrac{1}{5} \text{ and it occurs at } x = 1}$$

5. (10pts each) Find the derivative of the following functions.

(a) $f(x) = \cos(2e^x)$

Solution: (Use the chain rule) If we set $u = 2e^x$, then $\frac{du}{dx} = 2\frac{d}{dx}(e^x) = 2e^x$ so $f'(x) = \frac{d}{du}(\cos(u))\frac{du}{dx} = -\sin(u)(2e^x) = -2e^x\sin(2e^x)$.

(b) $f(x) = \frac{4x+7}{3\ln(x)}$

Solution: (Use the quotient rule)

$$f'(x) = \frac{d}{dx}\left(\frac{4x+7}{3\ln(x)}\right) = \left(\frac{1}{3}\right)\frac{d}{dx}\left(\frac{4x+7}{\ln(x)}\right) = \left(\frac{1}{3}\right)\left(\frac{\ln(x)\frac{d}{dx}(4x+7)-(4x+7)\frac{d}{dx}(\ln(x))}{(\ln(x))^2}\right)$$

$$= \left(\frac{1}{3}\right)\left(\frac{4\ln(x)-(4x+7)\left(\frac{1}{x}\right)}{(\ln(x))^2}\right) = \left(\frac{1}{3}\right)\left(\frac{4\ln(x)-(4x+7)\left(\frac{1}{x}\right)}{(\ln(x))^2}\right)\left(\frac{x}{x}\right) = \frac{4x\ln(x)-4x+7}{3x(\ln(x))^2}.$$

(c) $f(x) = (8\sqrt{x})^{2-x}$

Solution: (Use log. diff.) Set $y = (8\sqrt{x})^{2-x}$. Then

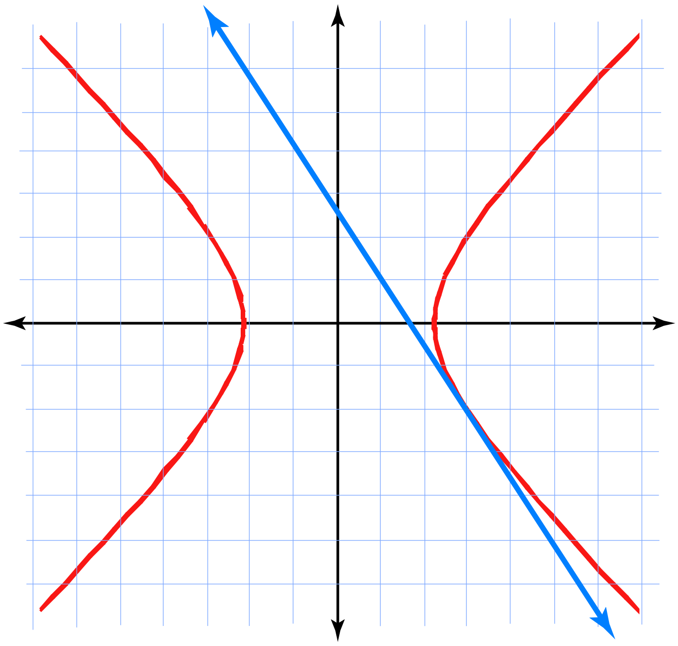
$$\ln(y) = \ln((8\sqrt{x})^{2-x}) = (2-x)\ln(8\sqrt{x}).$$

Now, differentiating each side with respect to x gives

$$\frac{y'}{y} = \frac{d}{dx}((2-x))\ln(8\sqrt{x}) + (2-x)\frac{d}{dx}(\ln(8\sqrt{x})) = -\ln(8\sqrt{x}) + \frac{4x^{-1/2}}{8\sqrt{x}} = -\ln(8\sqrt{x}) + \frac{1}{2x}$$

$$\Rightarrow f'(x) = y' = y\left(-\ln(8\sqrt{x}) + \frac{1}{2x}\right) = (8\sqrt{x})^{2-x}\left(\frac{1}{2x} - \ln(8\sqrt{x})\right).$$

6. (a) (2pts) Here's a picture of the curve $x^2 - y^2 = 5$. As accurately as you can, sketch the tangent line to the curve at the point $(3, -2)$.



(b) (10pts) Find the equation of the tangent line to the point $(3, -2)$ on the curve

$$x^2 - y^2 = 5.$$

[You should be able to check your answer with part (a)]

Solution: (Use implicit diff.) Differentiating each side with respect to x and solving for $\frac{dy}{dx}$ gives

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(5) \quad \Rightarrow \quad 2x - 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad -2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}.$$

To find the slope of the tangent line at $(3, -2)$ we must evaluate $\frac{dy}{dx}$ at $x = 3$ and $y = -2$. Thus the desired slope is $-\frac{3}{2}$. So the equation of the tangent line is

$$y - (-2) = -\frac{3}{2}(x - 3) \quad \text{or equivalently} \quad y = -\frac{3}{2}x + \frac{5}{2}.$$

Notice this matches the picture above.

7. (10pts) The volume V of a cylindrical can with radius r and height h is given by $V = \pi r^2 h$. Suppose the radius of the can is increasing at a rate of 2 inches per second, and the height of the can is decreasing at a rate of 3 inches per second. How fast is the volume changing when both the radius and the height are 1 inch?

Solution: Well using the product and chain rule we have

$$\frac{dV}{dt} = \frac{d}{dt}(\pi r^2 h) = \pi \frac{d}{dt}(r^2 h) = \pi \left(\frac{d}{dt}(r^2)h + r^2 \frac{d}{dt}(h) \right) = \pi \left(2rh\frac{dr}{dt} + r^2 \frac{dh}{dt} \right).$$

The problem tells us $\frac{dr}{dt} = 2$ and $\frac{dh}{dt} = -3$ (negative because the height is decreasing). So when r and h are both 1 we have

$$\frac{dV}{dt} = \pi \left(2(1)(1)(2) + (1)^2(-3)\right) = \pi.$$

So, at that moment the volume is increasing at a rate of π inches^3 per second.

8. (10pts) Evaluate the following limit. Be sure to show all your work using correct notation.

$$\lim_{x \to 0} \left(\frac{(2x)^3}{\sin^2(x)\sin(4x)} \right)$$

Solution: We will use the fact that $\lim\limits_{h \to 0} \frac{\sin h}{h} = 1 = \lim\limits_{h \to 0} \frac{h}{\sin h}$.

$$\begin{aligned}
\lim_{x \to 0} \left(\frac{(2x)^3}{\sin^2(x)\sin(4x)} \right) &= \lim_{x \to 0} \left[\left(\frac{x}{\sin(x)} \right) \left(\frac{x}{\sin(x)} \right) \left(\frac{4x}{\sin(4x)} \right) \left(\frac{(2x)^3}{4x^3} \right) \right] \\
&= \lim_{x \to 0} \left(\frac{x}{\sin(x)} \right) \lim_{x \to 0} \left(\frac{x}{\sin(x)} \right) \lim_{x \to 0} \left(\frac{4x}{\sin(4x)} \right) \lim_{x \to 0} \left(\frac{(2x)^3}{4x^3} \right) \\
&= (1)(1)(1) \lim_{x \to 0} \left(\frac{(2x)^3}{4x^3} \right) = \lim_{x \to 0} \left(\frac{8x^3}{4x^3} \right) = \lim_{x \to 0} (2) = 2.
\end{aligned}$$

Bonus: (10pts) Suppose f is a function with inverse function f^{-1} and both f and f^{-1} are differentiable everywhere. Prove

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

Solution 1: (compare to problem 3) Set $y = f^{-1}(x)$. Then we know $f(y) = x$. Differentiating with respect to x gives us

$$\frac{d}{dx}(f(y)) = \frac{d}{dx}(x) \quad \Rightarrow \quad f'(y)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{f'(y)}.$$

But $y = f^{-1}(x)$, thus

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

Solution 2: One property of inverse functions is that

$$f(f^{-1}(x)) = x$$

(this is half of what is sometimes referred to as the Round-Trip theorem). Differentiating both sides with respect to x, and using the chain rule gives

$$f'(f^{-1}(x))\frac{d}{dx}\left(f^{-1}(x)\right) = 1 \quad \Rightarrow \quad \frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$