Math 112
Jonny Comes
Fall 2005
Assignment #8
Partial Solutions

Additional Exercises: (Be sure to justify all your answers)

1. Divide $f(x) = x^4 - 3x^3 + 5x^2 - 9x + 6$ by $x^2 + 3$. Use your answer to find all the roots of $f(x)$.

   Solution: Well

$$
\begin{array}{r|l}
 & x^2 - 3x + 2 \\
\cline{2-2}
x^2 + 3 & \;x^4 - 3x^3 + 5x^2 - 9x + 6 \\
 & -x^4 - 3x^2 \\
\cline{2-2}
 & - 3x^3 + 2x^2 - 9x \\
 & 3x^3 + 9x \\
\cline{2-2}
 & 2x^2 + 6 \\
 & - 2x^2 - 6 \\
\cline{2-2}
 & 0
\end{array}
$$

   so $f(x) = (x^2 + 3)(x^2 - 3x + 2)$. So the roots of $f(x)$ are the roots of $x^2 + 3$ and the roots of $x^2 - 3x + 2$.

   The roots of $x^2 + 3$ are found by setting $x^2 + 3 = 0$ which implies $x^2 = -3$ or $x = \pm 3i$.

   The roots of $x^2 - 3x + 2 = (x - 2)(x - 1)$ are 2 and 1.

   Thus the roots of $f(x)$ are $1, 2, 3i$, and $-3i$.

2. Let $f(x) = x^6 - 6x^5 + 7x^4 + 28x^3 - 189x^2 + 34x - 195$. Given $f(i) = 0$ and $f(2 - 3i) = 0$

   (a) Write $f(x)$ as a product of linear factors (there should be 6 of them)

   [Hint: you'll need to do long division twice.]

   Solution: Since $2 - 3i$ is a root of $f(x)$ we know its conjugate $2 + 3i$ is also a root of $f(x)$ (by the conjugate root theorem).

Similarly we know i and $-i$ are roots. So the relationship between roots and factors give us

Roots		Linear Factors	
$2-3i$	$\longleftrightarrow$	$x-(2-3i)$	$=x-2+3i$
$2+3i$	$\longleftrightarrow$	$x-(2+3i)$	$=x-2-3i$
i	$\longleftrightarrow$	$x-i$	
$-i$	$\longleftrightarrow$	$x+i$	

Thus we know $f(x) = (x-2+3i)(x-2-3i)(x-i)(x+i)(\qquad ? \qquad)$.
Since

$$(x-2+3i)(x-2-3i) = x^2-2x-3ix-2x+4+6i+3ix-6i-9i^2 = x^2-4x+13$$

and

$$(x-i)(x+i) = x^2 - ix + ix - i^2 = x^2 + 1$$

we actually have $f(x) = (x^2-4x+13)(x^2+1)(\qquad ? \qquad)$. To find the missing factor (we can either foil $(x^2-4x+13)(x^2+1)$ and do long division once or) we can do long division twice as follows:

$$\begin{array}{r|rrrrrrr}
 & & & x^4 & -6x^3 & +6x^2 & +34x & -195 \\ \hline
x^2+1 & x^6 & -6x^5 & +7x^4 & +28x^3 & -189x^2 & +34x & -195 \\
 & -x^6 & & -x^4 & & & & \\ \cline{2-4}
 & & -6x^5 & +6x^4 & +28x^3 & & & \\
 & & 6x^5 & & +6x^3 & & & \\ \cline{3-5}
 & & & 6x^4 & +34x^3 & -189x^2 & & \\
 & & & -6x^4 & & -6x^2 & & \\ \cline{4-6}
 & & & & 34x^3 & -195x^2 & +34x & \\
 & & & & -34x^3 & & -34x & \\ \cline{5-7}
 & & & & & -195x^2 & & -195 \\
 & & & & & 195x^2 & & +195 \\ \cline{6-8}
 & & & & & & & 0
\end{array}$$

so $f(x) = (x^2+1)(x^4 - 6x^3 + 6x^2 + 34x - 195)$ and

$$\begin{array}{r|l}
 & x^2 \quad - 2x \quad - 15 \\
x^2 - 4x + 13 & x^4 - 6x^3 + 6x^2 + 34x - 195 \\
 & -x^4 + 4x^3 - 13x^2 \\
 & \quad -2x^3 - 7x^2 + 34x \\
 & \quad 2x^3 - 8x^2 + 26x \\
 & \qquad -15x^2 + 60x - 195 \\
 & \qquad 15x^2 - 60x + 195 \\
 & \qquad\qquad 0
\end{array}$$

so $x^4 - 6x^3 + 6x^2 + 34x - 195 = (x^2 - 4x + 13)(x^2 - 2x - 15)$ which implies

$$f(x) = (x^2+1)(x^2 - 4x + 13)(x^2 - 2x - 15)$$

Now $x^2 - 2x - 15 = (x-5)(x+3)$ So we have

$$f(x) = (x - 2 + 3i)(x - 2 - 3i)(x - i)(x + i)(x - 5)(x + 3)$$

as a product of linear factors.

(b) Use your answer for part (a) to list all the roots of $f(x)$.

Solution:

So the roots of $f(x)$ are

$$2 - 3i, \quad 2 + 3i, \quad i, \quad -i, \quad 5, \quad -3.$$

3. TRUE or FALSE

(a) If $f(x)$ is a polynomial with real coefficients and $f(2i+3) = 0$, then we know $f(3-2i) = 0$.

Solution: TRUE. The conjugate root theorem tells us that if $2i + 3$ is a root then its conjugate $\overline{2i+3} = \overline{3+2i} = 3 - 2i$ is also a root.

(b) If $f(x)$ is a polynomial with real coefficients and $f(3-2i) = 0$, then we know $f(2i-3) = 0$.

Solution: FALSE. $\overline{3-2i} = 3 + 2i \neq 2i - 3$. So the conjugate root theorem only guarantees $f(3+2i) = 0$. For example, let

$$f(x) = (x - (3-2i))(x - (3+2i)) = x^2 - 6x + 13.$$

Then the roots of $f(x)$ do not include $2i - 3$.

(c) Every non-constant polynomial with real coefficients has at least one real root.

Solution: FALSE. For example $x^2 + 1$ has no real root.

(d) Every non-constant polynomial with real coefficients has at least one complex root.

Solution: TRUE. This is the Fundamental Theorem of Algebra.

(e) Every non-constant polynomial with real coefficients of odd degree has at least one real root.

Solution: TRUE. The conjugate root theorem tells us if $a + bi$ is a root, then so is $a - bi$. So the non-real roots of a polynomial come in pairs. Thus the only polynomials which have no real roots are of even degree. This last sentence is the same as saying every odd degree polynomial has a real root.