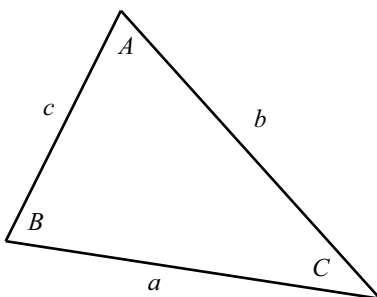


Math 112  
Jonny Comes  
Fall 2005  
Assignment #7  
Partial Solutions

**Additional Exercises:** (Be sure to justify all your answers)

For the following problems use the triangle



1. Solve the triangle (when possible), given:

(i)  $a = 2$ ,  $b = 2$ , and  $c = 2\sqrt{3}$ .

**Solution:** We need to find  $A$ ,  $B$ , and  $C$ . First let's find  $A$ .

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc \cos A \quad \Rightarrow \quad 2^2 = 2^2 + (2\sqrt{3})^2 - 2 \cdot 2 \cdot (2\sqrt{3}) \cdot \cos A \\ \Rightarrow \quad 4 &= 4 + 12 - 8\sqrt{3} \cos A \quad \Rightarrow \quad -12 = -8\sqrt{3} \cos A \\ \Rightarrow \quad \cos A &= \frac{-12}{-8\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{2 \cdot 3} = \frac{\sqrt{3}}{2}\end{aligned}$$

Since we know  $0^\circ < A < 180^\circ$  we have

$$A = \arccos\left(\frac{\sqrt{3}}{2}\right) = 30^\circ.$$

Now let's find  $B$ .

$$\begin{aligned}b^2 &= a^2 + c^2 - 2ac \cos B \quad \Rightarrow \quad 2^2 = 2^2 + (2\sqrt{3})^2 - 2 \cdot 2 \cdot (2\sqrt{3}) \cdot \cos B \\ \Rightarrow \quad 4 &= 4 + 12 - 8\sqrt{3} \cos B \quad \Rightarrow \quad -12 = -8\sqrt{3} \cos B \\ \Rightarrow \quad \cos B &= \frac{-12}{-8\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{2 \cdot 3} = \frac{\sqrt{3}}{2}\end{aligned}$$

Since we know  $0^\circ < B < 180^\circ$  we have

$$B = \arccos\left(\frac{\sqrt{3}}{2}\right) = 30^\circ.$$

[We just did the same exact calculation as we did to find  $A$ . Maybe it would have been better to say  $b = a$  so by symmetry  $B = A = 30^\circ$ ]

To find  $C$  we can use the fact that  $A + B + C = 180^\circ$  to get

$$C = 180^\circ - A - B = 180^\circ - 30^\circ - 30^\circ = 120^\circ.$$

(ii)  $a = 5$ ,  $b = 7$ , and  $c = 6$ .

**Solution:** We need to find  $A$ ,  $B$ , and  $C$ . First let's find  $A$ .

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A &\Rightarrow & 5^2 = 7^2 + 6^2 - 2 \cdot 7 \cdot 6 \cdot \cos A \\ \Rightarrow & 25 = 49 + 36 - 84 \cos A &\Rightarrow & -60 = -84\sqrt{3} \cos A \\ & &\Rightarrow & \cos A = \frac{-60}{-84} = \frac{5}{7} \end{aligned}$$

Since we know  $0^\circ < A < 180^\circ$  we have

$$A = \arccos\left(\frac{5}{7}\right).$$

Now let's find  $B$ .

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B &\Rightarrow & 7^2 = 5^2 + 6^2 - 2 \cdot 5 \cdot 6 \cdot \cos B \\ \Rightarrow & 49 = 61 - 60 \cos B &\Rightarrow & -12 = -60 \cos B \\ & &\Rightarrow & \cos B = \frac{-12}{-60} = \frac{1}{5} \end{aligned}$$

Since we know  $0^\circ < B < 180^\circ$  we have

$$B = \arccos\left(\frac{1}{5}\right).$$

To find  $C$  we can use the fact that  $A + B + C = 180^\circ$  to get

$$C = 180^\circ - A - B = 180^\circ - \arccos\left(\frac{5}{7}\right) - \arccos\left(\frac{1}{5}\right).$$

(iii)  $a = 2$ ,  $b = 6$ , and  $c = 3$ .

**Solution:** Well,  $a + c = 2 + 3 = 5 < 6 = b$  so the values given do not satisfy the Triangle Inequality. Thus there are no solutions.

(iv)  $a = 6$ ,  $b = 3\sqrt{2}$ , and  $C = 45^\circ$ .

**Solution:** We need to find  $A$ ,  $B$ , and  $c$ . First let's find  $c$ .

$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab \cos C = 6^2 + (3\sqrt{2})^2 - 2 \cdot 6 \cdot (3\sqrt{2}) \cdot \cos 45^\circ \\ \Rightarrow c^2 &= 36 + 18 - 36\sqrt{2} \cdot \left(\frac{\sqrt{2}}{2}\right) = 54 - 36 \left(\frac{\sqrt{2}\sqrt{2}}{2}\right) = 54 - 36 = 18\end{aligned}$$

Since we know  $c$  is positive

$$c = \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}.$$

The shortest way to proceed from here is to notice that  $c = b$  so we know  $B = C = 45^\circ$  and thus

$$A = 180^\circ - B - C = 180^\circ - 45^\circ - 45^\circ = 90^\circ.$$

2. Given that  $a = 2$ ,  $b = 3$ , and  $C = 45^\circ$  find  $c$ .

**Solution:** Using the law of cosines we have

$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab \cos C = 2^2 + 3^2 - 2(2)(3) \cos(45^\circ) \\ &= 4 + 9 - 12 \left(\frac{\sqrt{2}}{2}\right) = 13 - 6\sqrt{2}\end{aligned}$$

Thus  $c^2 = 13 - 6\sqrt{2}$ , and since we know  $c$  is positive we have

$$c = \sqrt{13 - 6\sqrt{2}}.$$

3. Given that  $b = 4$ ,  $c = 1$ , and  $A = 120^\circ$  find  $a$ .

**Solution:** Using the law of cosines we have

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc \cos A = 4^2 + 1^2 - 2(4)(1) \cos(120^\circ) \\ &= 16 + 1 - 8 \left(\frac{1}{2}\right) = 17 - 4 = 13\end{aligned}$$

Thus  $a^2 = 13$ , and since we know  $a$  is positive we have

$$a = \sqrt{13}.$$

4. Given that  $a = 6$ ,  $c = 4$ , and  $B = 150^\circ$  find  $a$ .

**Solution:** Well,  $a = 6$  as given. I meant to ask for  $b$ ... oh well.

5. Given that  $b = 4$ ,  $c = 5$ , and  $A = 15^\circ$  find  $a$ .

[Hint: use half-angle id]

**Solution:** Using the law of cosines we have

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A = 4^2 + 5^2 - 2(4)(5) \cos(15^\circ) \\ &= 16 + 25 - 40 \cos(15^\circ) = 41 - 40 \cos(15^\circ). \end{aligned}$$

Now to find  $\cos(15^\circ)$  we use the half angle id to get

$$\begin{aligned} \cos(15^\circ) &= \cos\left(\frac{30^\circ}{2}\right) = \pm \sqrt{\frac{1 + \cos(30^\circ)}{2}} = \pm \sqrt{\frac{\left(1 + \frac{\sqrt{3}}{2}\right)}{2}} \\ &= \pm \sqrt{\frac{\left(\frac{2+\sqrt{3}}{2}\right)}{2}} = \pm \sqrt{\frac{2 + \sqrt{3}}{4}} = \pm \frac{\sqrt{2 + \sqrt{3}}}{2} \end{aligned}$$

Since  $\cos(15^\circ)$  is positive we have  $\cos(15^\circ) = \frac{\sqrt{2+\sqrt{3}}}{2}$ . Thus

$$a^2 = 41 - 40 \left( \frac{\sqrt{2 + \sqrt{3}}}{2} \right) = 41 - 20\sqrt{2 + \sqrt{3}}.$$

Since we know  $a$  is positive we have

$$a = \sqrt{41 - 20\sqrt{2 + \sqrt{3}}}.$$

6. Given that  $a = 12$ ,  $A = 45^\circ$ , and  $B = 30^\circ$  find  $b$  and  $C$ .

**Solution:** To find  $C$  we use

$$A + B + C = 180^\circ \Rightarrow C = 180^\circ - A - B \Rightarrow C = 105^\circ$$

To find  $b$  we use the law of sines to get

$$\begin{aligned} \frac{b}{\sin B} &= \frac{a}{\sin A} \Rightarrow \frac{b}{\sin 30^\circ} = \frac{12}{\sin 45^\circ} \\ \Rightarrow b &= \frac{12 \sin 30^\circ}{\sin 45^\circ} = \frac{12 \left(\frac{1}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)} = \frac{12}{\left(\frac{\sqrt{2}}{2}\right)} = \frac{12}{1} \cdot \frac{2}{\sqrt{2}} = \frac{24}{\sqrt{2}} \\ &= \frac{24}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{24\sqrt{2}}{2} = 12\sqrt{2}. \end{aligned}$$

7. Given that  $c = 3$ ,  $B = 120^\circ$ , and  $C = 45^\circ$  find  $b$  and  $A$ .

**Solution:** To find  $A$  we use

$$A = 180^\circ - B - C = 180^\circ - 120^\circ - 45^\circ = 15^\circ.$$

To find  $b$  we use the law of sines to get

$$\begin{aligned}\frac{b}{\sin B} &= \frac{c}{\sin C} \quad \Rightarrow \quad \frac{b}{\sin 120^\circ} = \frac{3}{\sin 45^\circ} \\ \Rightarrow \quad b &= \frac{3 \sin 120^\circ}{\sin 45^\circ} = \frac{3 \left( \frac{\sqrt{3}}{2} \right)}{\left( \frac{\sqrt{2}}{2} \right)} = \left( \frac{3}{1} \right) \left( \frac{\sqrt{3}}{2} \right) \left( \frac{2}{\sqrt{2}} \right) = \frac{3\sqrt{3}}{\sqrt{2}}\end{aligned}$$

8. Given that  $a = 7$ ,  $A = 30^\circ$ , and  $B = 30^\circ$  find  $b$  and  $C$ .

**Solution:** To find  $C$  we use

$$C = 180^\circ - A - B = 180^\circ - 30^\circ - 30^\circ = 120^\circ.$$

To find  $b$  we use the law of sines to get

$$\begin{aligned}\frac{b}{\sin B} &= \frac{a}{\sin A} \quad \Rightarrow \quad \frac{b}{\sin 30^\circ} = \frac{7}{\sin 30^\circ} \\ \Rightarrow \quad b &= \frac{7 \sin 30^\circ}{\sin 30^\circ} = 7\end{aligned}$$

9. Given that  $b = 4$ ,  $B = 135^\circ$ , and  $C = 30^\circ$  find  $c$  and  $A$ .

**Solution:** To find  $A$  we use

$$A = 180^\circ - B - C = 180^\circ - 135^\circ - 30^\circ = 15^\circ.$$

To find  $c$  we use the law of sines to get

$$\begin{aligned}\frac{c}{\sin C} &= \frac{b}{\sin B} \quad \Rightarrow \quad \frac{c}{\sin 30^\circ} = \frac{4}{\sin 135^\circ} \\ \Rightarrow \quad c &= \frac{4 \sin 30^\circ}{\sin 135^\circ} = \frac{4 \left( \frac{1}{2} \right)}{\left( \frac{\sqrt{2}}{2} \right)} = \left( \frac{4}{1} \right) \left( \frac{1}{2} \right) \left( \frac{2}{\sqrt{2}} \right) = \frac{4}{\sqrt{2}} \\ &= \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}.\end{aligned}$$

10. Given that  $a = 6$ ,  $A = 120^\circ$ , and  $C = 30^\circ$  find  $c$  and  $B$ .

**Solution:** To find  $B$  we use

$$B = 180^\circ - A - C = 180^\circ - 120^\circ - 30^\circ = 30^\circ.$$

To find  $c$  we use the law of sines to get

$$\begin{aligned}\frac{c}{\sin C} &= \frac{a}{\sin A} \Rightarrow \frac{c}{\sin 30^\circ} = \frac{6}{\sin 120^\circ} \\ \Rightarrow c &= \frac{6 \sin 30^\circ}{\sin 120^\circ} = \frac{6 \left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \left(\frac{6}{1}\right) \left(\frac{1}{2}\right) \left(\frac{2}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} \\ &= \frac{6}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}.\end{aligned}$$

**Note:** For problems 11-17, there may be 0, 1, or 2 solutions.

11. Given that  $a = 4\sqrt{3}$ ,  $b = 4$ , and  $B = 30^\circ$  find  $A$  and  $C$ .

**Solution:** First we use the law of sines to find  $A$ :

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \sin A = \frac{a \sin B}{b} = \frac{4\sqrt{3} \sin 30^\circ}{4} = \frac{4\sqrt{3} \left(\frac{1}{2}\right)}{4} = \frac{\sqrt{3}}{2}$$

So we have  $\sin A = \frac{\sqrt{3}}{2}$ , and we know  $0 \leq A \leq 180^\circ$  so either

$$\text{Case 1 : } A = \arcsin\left(\frac{\sqrt{3}}{2}\right) = 60^\circ$$

or

$$\text{Case 2 : } A = 180^\circ - \arcsin\left(\frac{\sqrt{3}}{2}\right) = 180^\circ - 60^\circ = 120^\circ.$$

**Case 1:** If  $A = 60^\circ$  then (using  $A + B + C = 180^\circ$ ) we know

$$C = 180^\circ - 30^\circ - 60^\circ = 90^\circ.$$

**Case 2:** If  $A = 120^\circ$  then (using  $A + B + C = 180^\circ$ ) we know

$$C = 180^\circ - 30^\circ - 120^\circ = 30^\circ.$$

So our final solutions are

$$A = 60^\circ \quad \text{and} \quad C = 90^\circ$$

or

$$A = 120^\circ \quad \text{and} \quad C = 30^\circ$$

12. Given that  $b = 2$ ,  $c = 6$ , and  $B = 150^\circ$  find  $A$  and  $C$ .

**Solution:** First we use the law of sines to find  $C$ :

$$\frac{\sin C}{c} = \frac{\sin B}{b} \Rightarrow \sin C = \frac{c \sin B}{b} = \frac{6 \sin 150^\circ}{2} = \frac{6 \left(\frac{1}{2}\right)}{2} = \frac{3}{2}$$

So we have  $\sin C = \frac{3}{2}$ , but this is impossible because  $\sin x$  cannot be more than 1. Thus there are **no solutions**.

13. Given that  $b = \frac{\sqrt{3}}{3}$ ,  $c = \frac{1}{2}$ , and  $C = 60^\circ$  find  $A$  and  $C$ .

**Solution:**  $C$  is given to be  $60^\circ$ , so all we need to do is find  $A$ . First we use the law of sines to find  $B$ :

$$\begin{aligned} \frac{\sin B}{b} = \frac{\sin C}{c} &\Rightarrow \sin B = \frac{b \sin C}{c} = \frac{\left(\frac{\sqrt{3}}{3}\right) \sin 60^\circ}{\left(\frac{1}{2}\right)} = \frac{\left(\frac{\sqrt{3}}{3}\right) \left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} \\ &= \left(\frac{\sqrt{3}}{3}\right) \left(\frac{\sqrt{3}}{2}\right) \left(\frac{2}{1}\right) = \frac{3}{3} = 1 \end{aligned}$$

So we have  $\sin B = 1$ , and we know  $0 \leq B \leq 180^\circ$  so either

$$\text{Case 1: } B = \arcsin(1) = 90^\circ$$

or

$$\text{Case 2: } B = 180^\circ - \arcsin(1) = 180^\circ - 90^\circ = 90^\circ.$$

Both cases are the same and we have:

$$A = 180^\circ - B - C = 180^\circ - 90^\circ - 60^\circ = 30^\circ.$$

14. Given that  $a = 4$ ,  $b = \sqrt{2}$ , and  $B = 135^\circ$  find  $A$  and  $C$ .

**Solution:** First we use the law of sines to find  $A$ :

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \sin A = \frac{a \sin B}{b} = \frac{4 \sin 135^\circ}{\sqrt{2}} = \frac{4 \left(\frac{\sqrt{2}}{2}\right)}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$$

So we have  $\sin A = 2$ , but this is impossible because  $\sin x$  cannot be more than 1. Thus there are **no solutions**.

15. Given that  $b = \sqrt{3}$ ,  $c = \sqrt{2}$ , and  $B = 60^\circ$  find  $A$  and  $C$ .

**Solution:** First we use the law of sines to find  $C$ :

$$\begin{aligned}\frac{\sin C}{c} &= \frac{\sin B}{b} \Rightarrow \sin C = \frac{c \sin B}{b} = \frac{\sqrt{2} \sin 60^\circ}{\sqrt{3}} = \frac{\sqrt{2} \left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}} \\ &= \left(\frac{\sqrt{2}}{1}\right) \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{3}}\right) = \frac{\sqrt{2}}{2}\end{aligned}$$

So we have  $\sin C = \frac{\sqrt{2}}{2}$ , and we know  $0 \leq C \leq 180^\circ$  so either

$$\text{Case 1: } C = \arcsin\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$$

or

$$\text{Case 2: } C = 180^\circ - \arcsin\left(\frac{\sqrt{2}}{2}\right) = 180^\circ - 45^\circ = 135^\circ.$$

**Case 1:** If  $C = 45^\circ$  then (using  $A + B + C = 180^\circ$ ) we know

$$A = 180^\circ - 60^\circ - 45^\circ = 75^\circ.$$

**Case 2:**  $C = 135^\circ$  then (using  $A + B + C = 180^\circ$ ) we know

$$A = 180^\circ - 60^\circ - 135^\circ = -15^\circ.$$

But this is not possible.

So the final solution is

$$C = 45^\circ \quad \text{and} \quad A = 75^\circ$$

16. Given that  $b = \frac{\sqrt{2}}{2}$ ,  $c = 1$ , and  $C = 45^\circ$  find  $A$  and  $C$ .

**Solution:**  $C$  is given to be  $45^\circ$ , so all we need to do is find  $A$ . First we use the law of sines to find  $B$ :

$$\begin{aligned}\frac{\sin B}{b} &= \frac{\sin C}{c} \Rightarrow \sin B = \frac{b \sin C}{c} = \frac{\left(\frac{\sqrt{2}}{2}\right) \sin 45^\circ}{1} \\ &= \left(\frac{\sqrt{2}}{2}\right) \left(\frac{\sqrt{2}}{2}\right) = \frac{2}{4} = \frac{1}{2}\end{aligned}$$

So we have  $\sin B = \frac{1}{2}$ , and we know  $0 \leq B \leq 180^\circ$  so either

$$\text{Case 1 : } B = \arcsin\left(\frac{1}{2}\right) = 30^\circ$$

or

$$\text{Case 2 : } B = 180^\circ - \arcsin\left(\frac{1}{2}\right) = 180^\circ - 30^\circ = 150^\circ.$$

**Case 1:** If  $B = 30^\circ$  then (using  $A + B + C = 180^\circ$ ) we know

$$A = 180^\circ - 45^\circ - 30^\circ = 105^\circ.$$

**Case 2:**  $B = 150^\circ$  then (using  $A + B + C = 180^\circ$ ) we know

$$A = 180^\circ - 45^\circ - 150^\circ = -15^\circ.$$

But this is not possible.

So the final solution is

$$A = 105^\circ$$

17. Given that  $a = 17\sqrt{2}$ ,  $c = 17$ , and  $C = 30^\circ$  find  $A$  and  $B$ .

**Solution:** First we use the law of sines to find  $A$ :

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin C}{c} \quad \Rightarrow \quad \sin A = \frac{a \sin C}{c} = \frac{17\sqrt{2} \sin 30^\circ}{17} \\ &= \sqrt{2} \sin 30^\circ = \sqrt{2} \left(\frac{1}{2}\right) = \frac{\sqrt{2}}{2} \end{aligned}$$

So we have  $\sin A = \frac{\sqrt{2}}{2}$ , and we know  $0 \leq A \leq 180^\circ$  so either

$$\text{Case 1 : } A = \arcsin\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$$

or

$$\text{Case 2 : } B = 180^\circ - \arcsin\left(\frac{\sqrt{2}}{2}\right) = 180^\circ - 45^\circ = 135^\circ.$$

**Case 1:** If  $A = 45^\circ$  then (using  $A + B + C = 180^\circ$ ) we know

$$B = 180^\circ - 30^\circ - 45^\circ = 105^\circ.$$

**Case 2:**  $A = 135^\circ$  then (using  $A + B + C = 180^\circ$ ) we know

$$A = 180^\circ - 30^\circ - 135^\circ = 15^\circ.$$

So our final solutions are

$$A = 45^\circ \quad \text{and} \quad B = 105^\circ$$

or

$$A = 135^\circ \quad \text{and} \quad B = 15^\circ$$

18. True or False:

(a) It is possible for the sides of a triangle to have lengths 3, 5, & 10.

**Solution:** FALSE.  $3 + 5 = 8 < 10$ , so these lengths do not satisfy the triangle inequality.

(b) It is possible for the sides of a triangle to have lengths 2, 5, & 6.

**Solution:** TRUE. All we need to check is that the triangle inequality holds. It does hold, because the sum of any two lengths is greater than the third:

$$2 + 5 = 7 > 6,$$

$$5 + 6 = 11 > 2,$$

$$2 + 6 = 8 > 5.$$

(c) It is possible for the interior angles of a triangle to have measures  $63^\circ$ ,  $74^\circ$ , and  $42^\circ$ .

**Solution:** FALSE. We know the interior angle of a triangle must sum to  $180^\circ$  and

$$63^\circ + 74^\circ + 42^\circ = 179^\circ \neq 180^\circ.$$