Math 112
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Assignment #5
Partial Solutions

Additional Exercises: (Be sure to justify all your answers)

Here are the main two properties of logs

$$\text{I. } \log_b(b^k) = k$$
$$\text{II. } b^{\log_b(k)} = k$$

Also recall the 3 steps for finding the rule of f^{-1}

Step 1. Set $f(y) = x$.
Step 2. Solve for y.
Step 3. Set $f^{-1}(x) = y$

1. Find the rule of f^{-1} for all of the following.

   (a) $f(x) = \ln(2x)$

   Solution:

   Step 1. $f(y) = \ln(2y)$, so we set

   $$x = \ln(2y)$$

   Step 2. Now we solve for y:

   $$\begin{aligned} x &= \ln(2y) \\ \Rightarrow e^x &= e^{\ln(2y)} \\ \Rightarrow e^x &= 2y \qquad \text{(using prop. II)} \\ \Rightarrow \frac{e^x}{2} &= y \end{aligned}$$

   Step 3. Setting $f^{-1}(x) = y$ we see

   $$f^{-1}(x) = \frac{e^x}{2}.$$

(b) $f(x) = 3\log(x-4)$

Solution:

Step 1. $f(y) = 3\log(y-4)$, so we set

$$x = 3\log(y-4)$$

Step 2. Now we solve for y:

$$\begin{aligned} x &= 3\log(y-4) \\ \Rightarrow \frac{x}{3} &= \log(y-4) \\ \Rightarrow 10^{x/3} &= 10^{\log(y-4)} \\ \Rightarrow 10^{x/3} &= y-4 \qquad \text{(using prop. II)} \\ \Rightarrow 10^{x/3} + 4 &= y \end{aligned}$$

Step 3. Setting $f^{-1}(x) = y$ we see

$$f^{-1}(x) = 10^{x/3} + 4.$$

(c) $f(x) = 12\log_2(3x+2)$

Solution:

Step 1. $f(y) = 12\log_2(3y+2)$, so we set

$$x = 12\log_2(3y+2)$$

Step 2. Now we solve for y:

$$\begin{aligned} x &= 12\log_2(3y+2) \\ \Rightarrow \frac{x}{12} &= \log_2(3y+2) \\ \Rightarrow 2^{x/12} &= 2^{\log_2(3y+2)} \\ \Rightarrow 2^{x/12} &= 3y+2 \qquad \text{(using prop. II)} \\ \Rightarrow 2^{x/12} - 2 &= 3y \\ \Rightarrow \frac{2^{x/12} - 2}{3} &= y \end{aligned}$$

Step 3. Setting $f^{-1}(x) = y$ we see

$$f^{-1}(x) = \frac{2^{x/12} - 2}{3}.$$

(d) $f(x) = 5^{x+3}$

Solution:

Step 1. $f(y) = 5^{y+3}$, so we set

$$x = 5^{y+3}$$

Step 2. Now we solve for y:

$$\begin{aligned} x &= 5^{y+3} \\ \Rightarrow \log_5(x) &= \log_5\left(5^{y+3}\right) \\ \Rightarrow \log_5(x) &= y + 3 \qquad \text{(using prop. I)} \\ \Rightarrow \log_5(x) - 3 &= y \end{aligned}$$

Step 3. Setting $f^{-1}(x) = y$ we see

$$f^{-1}(x) = \log_5(x) - 3.$$

(e) $f(x) = 9 \cdot 10^{3x}$

Solution:

Step 1. $f(y) = 9 \cdot 10^{3y}$, so we set

$$x = 9 \cdot 10^{3y}$$

Step 2. Now we solve for y:

$$\begin{aligned} x &= 9 \cdot 10^{3y} \\ \Rightarrow x/9 &= 10^{3y} \\ \Rightarrow \log(x/9) &= \log\left(10^{3y}\right) \\ \Rightarrow \log(x/9) &= 3y \qquad \text{(using prop. I)} \\ \Rightarrow \frac{\log(x/9)}{3} &= y \end{aligned}$$

Step 3. Setting $f^{-1}(x) = y$ we see

$$f^{-1}(x) = \frac{\log(x/9)}{3}.$$

(f) $f(x) = 12 \cdot e^{9x-14}$

Solution:

Step 1. $f(y) = 12 \cdot e^{9y-14}$, so we set

$$x = 12 \cdot e^{9y-14}$$

Step 2. Now we solve for y:

$$x = 12 \cdot e^{9y-14}$$

$$\Rightarrow x/12 = e^{9y-14}$$

$$\Rightarrow \ln(x/12) = \ln\left(e^{9y-14}\right)$$

$$\Rightarrow \ln(x/12) = 9y-14 \qquad \text{(using prop. I)}$$

$$\Rightarrow \ln(x/12) + 14 = 9y$$

$$\Rightarrow \frac{\ln(x/12) + 14}{9} = y$$

Step 3. Setting $f^{-1}(x) = y$ we see

$$f^{-1}(x) = \frac{\ln(x/12) + 14}{9}.$$

2. Use the round trip theorem to check you answers for each part of problem 1.

Solution:

1(a) We need to show $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Well,

$$f(f^{-1}(x)) = f\left(\frac{e^x}{2}\right) = \ln\left(2 \cdot \frac{e^x}{2}\right) = \ln\left(e^x\right) \overset{\text{by I}}{=} x$$

and

$$f^{-1}(f(x)) = f^{-1}\left(\ln(2x)\right) = \frac{e^{\ln(2x)}}{2} \overset{\text{by II}}{=} \frac{2x}{2} = x.$$

1(b) We need to show $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Well,

$$f(f^{-1}(x)) = f\left(10^{x/3} + 4\right) = \ln\left(2 \cdot \frac{e^x}{2}\right) = \ln\left(e^x\right) \overset{\text{by I}}{=} x$$

and

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(3\log(x-4)\right) = 10^{\frac{3\log(x-4)}{3}} + 4 \\ &= 10^{\log(x-4)} + 4 \overset{\text{by II}}{=} x - 4 + 4 = x. \end{aligned}$$

1(c) We need to show $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Well,

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{2^{x/12} - 2}{3}\right) = 12\log_2\left(3 \cdot \frac{2^{x/12} - 2}{3} + 2\right) \\ &= 12\log_2\left(2^{x/12} - 2 + 2\right) = 12\log_2\left(2^{x/12}\right) \overset{\text{by I}}{=} 12 \cdot \frac{x}{12} = x \end{aligned}$$

and

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(12\log_2(3x+2)\right) = \frac{2^{\frac{12\log_2(3x+2)}{12}} - 2}{3} \\ &= \frac{2^{\log_2(3x+2)} - 2}{3} \overset{\text{by II}}{=} \frac{3x + 2 - 2}{3} = \frac{3x}{3} = x. \end{aligned}$$

1(d) Again, we need to show $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Well,

$$f(f^{-1}(x)) = f\left(\log_5(x) - 3\right) = 5^{\log_5(x) - 3 + 3} = 5^{\log_5(x)} \overset{\text{by II}}{=} x,$$

and

$$f^{-1}(f(x)) = f^{-1}\left(5^{x+3}\right) = \log_5\left(5^{x+3}\right) - 3 \overset{\text{by I}}{=} x + 3 - 3 = x.$$

1(e) Again, we need to show $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Well,

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{\log(x/9)}{3}\right) = 9 \cdot 10^{3 \cdot \frac{\log(x/9)}{3}} = 9 \cdot 10^{\log(x/9)} \\ &\overset{\text{by II}}{=} 9 \cdot \left(\frac{x}{9}\right) = x, \end{aligned}$$

and

$$f^{-1}(f(x)) = f^{-1}\left(9 \cdot 10^{3x}\right) = \frac{\log\left(\frac{9 \cdot 10^{3x}}{9}\right)}{3} = \frac{\log\left(10^{3x}\right)}{3} \overset{\text{by I}}{=} \frac{3x}{3} = x.$$

1(f) One more time, we need to show $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Well,

$$f(f^{-1}(x)) = f\left(\frac{\ln(x/12)+14}{9}\right) = 12 \cdot e^{9\cdot\frac{\ln(x/12)+14}{9}-14}$$

$$= 12 \cdot e^{\ln(x/12)+14-14} = 12 \cdot e^{\ln(x/12)} \overset{\text{by II}}{=} 12 \cdot \left(\frac{x}{12}\right) = x,$$

and

$$f^{-1}(f(x)) = f^{-1}\left(12 \cdot e^{9x-14}\right) = \frac{\ln\left(\frac{12\cdot e^{9x-14}}{12}\right)+14}{9}$$

$$= \frac{\ln\left(e^{9x-14}\right)+14}{9} \overset{\text{by I}}{=} \frac{9x-14+14}{9} = \frac{9x}{9} = x.$$

3. (a) Explain why $\arccos(\cos(x)) = x$ for any real number x which lies between 0 and π.

   Solution: By definition $\arccos(u)$ is the unique number between 0 and π such that $\cos(\arccos(u)) = u$. Therefore [setting $u = \cos(x)$] we have $\arccos(\cos(x))$ is the unique number between 0 and π such that

$$\cos[\arccos(\cos(x))] = \cos(x).$$

   But x is already between 0 and π, so

$$\arccos(\cos(x)) = x.$$

   (b) Find a number x such that $\arccos(\cos(x))$ is NOT equal to x.

   Solution: There are many such numbers. Here are a couple:

$$\arccos(\cos(-\pi)) = \arccos(-1) = \pi \neq -\pi,$$

$$\arccos(\cos(2\pi)) = \arccos(1) = 0 \neq 2\pi.$$

   (c) Find the following values

   [Hint: some should be undefined]

   i. $\cos(\arccos(0)) = 0$
   ii. $\cos(\arccos(3))$ is undefined
   iii. $\cos\left(\arccos\left(\frac{1}{2}\right)\right) = \frac{1}{2}$
   iv. $\cos(\arccos(-2))$ is undefined

4. True/False [Use the previous problem as a guide]

(a) We know $\arcsin(\sin(x)) = x$ for any real number x which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Solution: TRUE. By definition $\arcsin(u)$ is the unique number between $-\pi/2$ and $\pi/2$ such that $\sin(\arcsin(u)) = u$. Therefore [setting $u = \sin(x)$] we have $\arcsin(\sin(x))$ is the unique number between $-\pi/2$ and $\pi/2$ such that

$$\sin[\arcsin(\sin(x))] = \sin(x).$$

But x is already between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so

$$\arcsin(\sin(x)) = x.$$

(b) We know $\arcsin(\sin(x)) = x$ for ANY real number x.

Solution: FALSE. This is false whenever x is not between $-\pi/2$ and $\pi/2$. For example

$$\arcsin(\sin(\pi)) = \arcsin(0) = 0 \neq \pi.$$

(c) We know $\sin(\arcsin(x)) = x$ for ANY real number x.

Solution: FALSE. This is false whenever x is either greater than 1 or less than -1. For example $\arcsin(17)$ is undefined, so $\sin(\arcsin(17))$ is also undefined, and therefore

$$\sin(\arcsin(17)) \neq 17.$$