Math 112
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Assignment #4
Partial Solutions

Additional Exercises:

1. Use the half angle identity to show:

   (a) $\sin(\pi/8) = \frac{\sqrt{2-\sqrt{2}}}{2}$

   Solution: Using the half angle identity for sine we have

$$\sin\left(\frac{\pi}{8}\right) = \sin\left(\frac{\left(\frac{\pi}{4}\right)}{2}\right) = \pm\sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$= \pm\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{2}.$$

   Since $\frac{\pi}{8}$ is in the first quadrant we know $\sin(\pi/8) > 0$ so that

$$\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

   (b) $\sin(\pi/16) = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$

   Solution: Using the half angle identity for sine we have

$$\sin\left(\frac{\pi}{16}\right) = \sin\left(\frac{\left(\frac{\pi}{8}\right)}{2}\right) = \pm\sqrt{\frac{1-\cos\left(\frac{\pi}{8}\right)}{2}}.$$

   So, in order to calculate $\sin(\pi/16)$ we need to know the value of $\cos(\pi/8)$. Using the Pythagorean identity and the fact that $\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}$ [from part (a)] you can show

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

[For more detail, see the solution of additional problem 4 from homework #2] Therefore

$$\sin\left(\frac{\pi}{16}\right) = \pm\sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2}}}{2}}{2}}$$

$$= \pm\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \pm\sqrt{\frac{\frac{2-\sqrt{2+\sqrt{2}}}{2}}{2}}$$

$$= \pm\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}} = \pm\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}.$$

Since $\frac{\pi}{16}$ is in the first quadrant we know $\sin(\pi/16) > 0$ so that

$$\sin\left(\frac{\pi}{16}\right) = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}.$$

There is a pattern forming here:

$$\begin{aligned}
\sin\left(\tfrac{\pi}{4}\right) &= \tfrac{\sqrt{2}}{2} \\
\sin\left(\tfrac{\pi}{8}\right) &= \tfrac{\sqrt{2-\sqrt{2}}}{2} \\
\sin\left(\tfrac{\pi}{16}\right) &= \tfrac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2} \\
\sin\left(\tfrac{\pi}{32}\right) &= \tfrac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \\
\sin\left(\tfrac{\pi}{64}\right) &= \tfrac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2} \\
\vdots \quad & \quad \vdots
\end{aligned}$$

2. (a) Use the addition & subtraction identities to calculate the exact value of $\sin(\pi/12)$.
[Hint: $\frac{\pi}{12} = \frac{4\pi}{12} - \frac{3\pi}{12}$]

Solution:

$$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

(b) Now use a half-angle identity to calculate $\sin(\pi/12)$.

Solution:

$$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\left(\frac{\pi}{6}\right)}{2}\right) = \pm\sqrt{\frac{1-\cos\left(\frac{\pi}{6}\right)}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \pm\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{3}}{2}}{2}} = \pm\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \pm\sqrt{\frac{2-\sqrt{3}}{4}} = \pm\frac{\sqrt{2-\sqrt{3}}}{2}.$$

Since $\frac{\pi}{12}$ is in the first quadrant we know $\sin(\pi/12) > 0$ so that

$$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

Bonus: If you want to, ON A SEPARATE SHEET OF PAPER, show that your answers in parts (a) and (b) are in fact equal.

[this is extra credit for a reason, do not be discouraged if you cannot do it]

Solution: So we are trying to show $\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2-\sqrt{3}}}{2}$. First, I claim that both of these numbers are positive. To see this notice $\sqrt{6} > \sqrt{2}$ so that $\sqrt{6}-\sqrt{2} > 0$ and thus $\frac{\sqrt{6}-\sqrt{2}}{4} > 0$. Also $2 > \sqrt{3}$ so that $2-\sqrt{3} > 0$ which implies $\sqrt{2-\sqrt{3}} > 0$ and thus $\frac{\sqrt{2-\sqrt{3}}}{2} > 0$. Now since both numbers are positive real numbers, it is enough to show that their squares are equal, i.e. that $\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2 = \left(\frac{\sqrt{2-\sqrt{3}}}{2}\right)^2$.

[This requires that both numbers are positive, for $3^2 = (-3)^2$ but $3 \neq -3$.]

Well,

$$\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2 = \frac{\left(\sqrt{6}-\sqrt{2}\right)^2}{4^2} = \frac{\left(\sqrt{6}-\sqrt{2}\right)\left(\sqrt{6}-\sqrt{2}\right)}{16}$$

$$= \frac{6+2-\sqrt{12}-\sqrt{12}}{16} = \frac{8-2\sqrt{12}}{16} = \frac{8-2\sqrt{4\cdot 3}}{16}$$

$$= \frac{8 - 2 \cdot 2\sqrt{3}}{16} = \frac{8 - 4\sqrt{3}}{16} = \frac{4(2-\sqrt{3})}{16} = \frac{2-\sqrt{3}}{4},$$

and

$$\left(\frac{\sqrt{2-\sqrt{3}}}{2}\right)^2 = \frac{\left(\sqrt{2-\sqrt{3}}\right)^2}{2^2} = \frac{2-\sqrt{3}}{4}.$$

3. Given that $\tan t = 2/5$ and $\pi < t < 3\pi/2$ find the exact values of $\sin(t)$ and $\cos(t)$.

   [Hint: On the back of this sheet there is an example of a similar problem]

   Solution: We use the Pythagorean identity, just not the one we usually use. Since the problem gives us $\tan t$ we use the identity

$$1 + \tan^2 t = \sec^2 t.$$

   [Which identity should you use for problem 3?]

   Now substituting $2/5$ in for $\tan t$ we have

$$1 + \left(\frac{2}{5}\right)^2 = \sec^2 t$$

$$\Rightarrow 1 + \frac{4}{25} = \sec^2 t$$

$$\Rightarrow \frac{25}{25} + \frac{4}{25} = \sec^2 t$$

$$\Rightarrow \frac{29}{25} = \sec^2 t$$

$$\Rightarrow \pm\sqrt{\frac{29}{25}} = \sec t$$

$$\Rightarrow \pm\frac{\sqrt{29}}{5} = \sec t.$$

   Since $\pi < t < 3\pi/2$ we know $\sec t < 0$ so we have

$$\sec t = -\frac{\sqrt{29}}{5}.$$

Now we know $\cos t = \frac{1}{\sec t}$ (because $\sec t = \frac{1}{\cos t}$) so

$$\cos t = \frac{1}{\sec t} = \frac{1}{\left(-\frac{\sqrt{29}}{5}\right)} = -\frac{5}{\sqrt{29}}.$$

To find $\sin t$ we use our knowledge of $\tan t$ and $\cos t$ as follows:

$$\tan t = \frac{\sin t}{\cos t}$$

$$\Rightarrow \sin t = \tan t \cos t = \left(\frac{2}{5}\right)\left(-\frac{5}{\sqrt{29}}\right) = -\frac{2}{\sqrt{29}}.$$