True or False Solutions

1. If $\cos\theta > 0$ and $\sin\theta < 0$ then $\tan\theta$ must be negative.

   Solution: TRUE. We know $\tan\theta = \frac{\sin\theta}{\cos\theta}$, and a negative number divided by a positive number is always negative.

2. $\cos(t)$ is always less than or equal to 1.

   Solution: TRUE. $\cos(t)$ is defined as an x-coordinate on the unit circle, and all the x-coordinates on the unit circle are less than or equal to 1.

3. If $\sin\theta = \sin\phi$ then θ and ϕ must be coterminal angles.

   Solution: FALSE. There are many counterexamples, here's one:

   $\sin 0 = 0$ and $\sin\pi = 0$ but 0 and π do not represent coterminal angles.

4. The point $(\pi, 3)$ is on the graph of the function f where

   $$f(t) = \sin(t)\cos(t) + 3 - \tan(t).$$

   Solution: TRUE.

   $$f(\pi) = \sin(\pi)\cos(\pi) + 3 - \tan(\pi) = 0 \cdot (-1) + 3 - 0 = 3,$$

   and saying $f(\pi) = 3$ is the same as saying the point $(\pi, 3)$ is on the graph of the function f.

5. If $g(t) = \sin(t) + \cos(t)\sin^2(t)$, then $g(-t) = -g(t)$ for all real numbers t.

   Solution: FALSE. For example let $t = \pi/4$. Then

   $$g\left(-\frac{\pi}{4}\right) = \sin\left(-\frac{\pi}{4}\right) + \cos\left(-\frac{\pi}{4}\right)\sin^2\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)^2 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{8}}{8}.$$

   Whereas

   $$-g\left(\frac{\pi}{4}\right) = -\left[\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\sin^2\left(\frac{\pi}{4}\right)\right] = -\frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)^2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{8}}{8}.$$

   and $-\frac{\sqrt{2}}{2} + \frac{\sqrt{8}}{8} \neq -\frac{\sqrt{2}}{2} - \frac{\sqrt{8}}{8}$.
   [It is important that you know why $\sin^2(-t) \neq -\sin t$. This is because $\sin^2(-t) = (\sin(-t))^2 = (-\sin t)^2 = \sin^2 t$. Without this, you might try to show the statement is true.]

6. If $|\sin(t)| = 1$, then $\tan(t)$ is undefined.

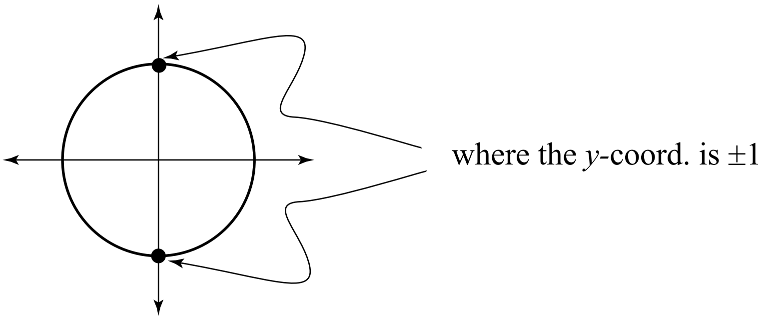
   Solution 1: TRUE. Whenever $|\sin(t)| = 1$ we have $\sin t = \pm 1$. Thus, the Pythagorean identity gives us

   $$\sin^2 t + \cos^2 t = 1 \quad \Rightarrow \quad (\pm 1)^2 + \cos^2 t = 1$$

   $$\Rightarrow \quad 1 + \cos^2 t = 1 \quad \Rightarrow \quad \cos^2 t = 0 \quad \Rightarrow \quad \cos t = 0.$$

   Therefore $\tan t = \frac{\sin t}{\cos t} = \frac{\pm 1}{0}$ is undefined.

   Solution 2: TRUE. Looking at the unit circle, whenever $|\sin(t)| = 1$ we know $\cos t = 0$

   Therefore when $|\sin(t)| = 1$ we have $\tan t = \frac{\sin t}{\cos t} = \frac{\sin t}{0}$ is undefined.

7. If $\csc t = 31$ then we know $\sin t = \frac{1}{31}$.

   Solution: TRUE. $\csc t = \frac{1}{\sin t}$, which implies $\sin t = \frac{1}{\csc t} = \frac{1}{31}$.

8. If $f(t) = 201 \sin(7t - 3)$ then we know the graph of f has period 7.

   Solution: FALSE. The period of the graph of f is $\frac{2\pi}{7}$.

9. $\sin(a) + \sin(b) = \sin(a + b)$ for all real numbers a and b.

   Solution: FALSE. For example Let $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$. Then we have

   $$\sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1 + \sqrt{3}}{2}.$$

   On the other hand

   $$\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{6} + \frac{2\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

   But $\frac{1+\sqrt{3}}{2} \neq 1$.

10. $\cos(a-b) = \cos(a) - \cos(b)$ for all real numbers a and b.

    Solution: FALSE. For example if we let $a = \pi$ and $b = 0$ then we have

    $$\cos(a-b) = \cos(\pi - 0) = \cos(\pi) = -1$$

    whereas

    $$\cos(a) - \cos(b) = \cos(\pi) - \cos(0) = -1 - 1 = -2.$$

11. If $\tan(a) = \tan(b)$, then we know $a = b$.

    Solution: FALSE. There are many counter examples, here's one: $\tan(0) = 0 = \tan(\pi)$ but $0 \neq \pi$.

12. There are angles θ and ϕ such that $\sin(\theta) = \sin(\phi)$ but θ and ϕ are not coterminal.

    Solution: TRUE. Here is one such example: $\sin(\pi) = 0 = \sin(0)$ but π and 0 are not coterminal.

13. It is possible for $\sin\theta = 5/13$ and $\tan\theta = -5/12$.

    Solution: TRUE. If we assume $\sin\theta = 5/13$, then using the Pythagorean identity we see

    $$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \Rightarrow \left(\frac{5}{13}\right)^2 + \cos^2\theta &= 1 \\ \Rightarrow \cos^2\theta &= 1 - \frac{25}{169} \\ \Rightarrow \cos^2\theta &= \frac{144}{169} \\ \Rightarrow \cos\theta &= \pm\frac{12}{13}. \end{aligned}$$

    Thus $\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{-5/13}{12/13} = \pm\frac{5}{12}$. Since this is the only restriction on $\tan\theta$, it is possible for $\tan\theta = -5/12$.

14. The domain of the function f whose rule is $f(t) = \tan(t)$ is all real numbers.

    Solution: FALSE. For example $\tan(\pi/2)$ is undefined (because $\cos(\pi/2) = 0$ and we cannot divide by zero). Thus $\pi/2$ is a real number which is not in the domain of f.

    In fact, any number which represents an angle coterminal to either $\pi/2$ or $-\pi/2$ is not in the domain of f.

15. The point $(\pi/3, 2)$ is on the graph of the function h whose rule is $h(t) = (\sin(t) + \tan(t)\cos^2(t))^2$.

Solution: FALSE. To say that $(\pi/3, 2)$ is on the graph of h is equivalent to saying $h(\pi/3) = 2$, so all we need to do is find $h(\pi/3)$. Well,

$$h(\pi/3) = \big(\sin(\pi/3) + \tan(\pi/3)\cos^2(\pi/3)\big)^2$$

$$\left(\frac{\sqrt{3}}{2} + \left(\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}\right) \cdot \left(\frac{1}{2}\right)^2\right)^2$$

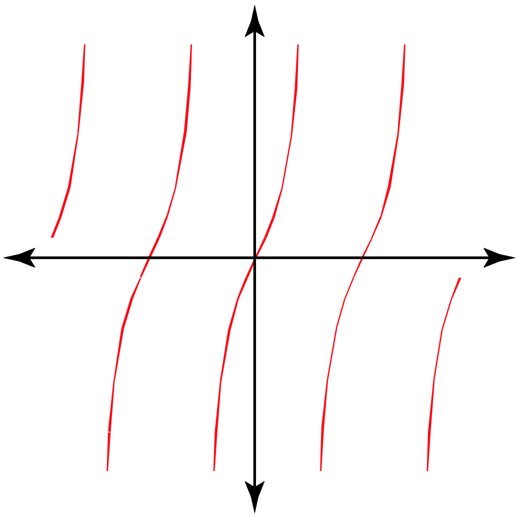
$$= \left(\frac{\sqrt{3}}{2} + \left(\sqrt{3}\right) \cdot \left(\frac{1}{4}\right)\right)^2 = \left(\frac{2\sqrt{3}}{4} + \left(\frac{\sqrt{3}}{4}\right)\right)^2$$

$$= \left(\frac{3\sqrt{3}}{4}\right)^2 = \frac{27}{16}.$$

However, $\frac{27}{16} \neq 2$, so the point $(\pi/3, 2)$ is NOT on the graph of h.

16. There is a number M such that $\tan\theta$ is always less than M.

Solution: FALSE. One way of convincing yourself of this is to look at the graph of $f(t) = \tan t$



Notice that the graph is not bounded above.

Another way to see this, is given any number M, there is a line with slope $2M$ through the origin. If we let θ be the angle between the x-axis and this line (in standard position), then $\tan\theta = 2M > M$. So I have found an angle θ such that $\tan\theta \not< M$.

17. The graph of $f(t) = 12\cos(2t-4)$ passes through the t-axis when $t = \pi/4 + 2$. [Just to clarify, the t-axis is what you might normally think of as the x-axis]

    Solution: TRUE. Saying the graph of f passing through the t-axis when $t = \pi/4 + 2$ is the same as saying $f(\pi/4+2) = 0$. So we just need to show $f(\pi/4+2) = 0$. Well

$$f\left(\frac{\pi}{4}+2\right) = 12\cos\left(2\left(\frac{\pi}{4}+2\right)-4\right) = 12\cos\left(\frac{\pi}{2}+4-4\right)$$

$$= 12\cos\left(\frac{\pi}{2}\right) = 12\cdot 0 = 0.$$

18. For any real number t, $\sin(\csc(t)) = t$.

    Solution: FALSE. There are many possible values of t such that $\sin(\csc(t)) \neq t$. [in fact, I don't know how to find a t where the equality does hold] Here are a couple such counterexamples with some justification.

    - Let $t = 1000$. Because $\sin(\theta) \leq 1$ for all real numbers θ and $\csc(1000)$ is just some real number, we see $\sin(\csc(1000)) \leq 1$ and therefore $\sin(\csc(1000))$ cannot equal 1000.

      [The same argument would work for any value of t greater than 1... A similar argument will work for any value of t less than -1.]
    - Let $t = 0$. Because $\csc(0) = \frac{1}{\sin(0)} = \frac{1}{0}$ is undefined, we see $\sin(\csc(0))$ is undefined and is thus not equal to 0.

      [The same argument would work for any number which is not in the domain of $\csc(t)$.]

19. If $\sin\theta = \frac{\sqrt{2}}{2}$ and $\cos\theta = \frac{\sqrt{2}}{2}$, then we know θ must be $\frac{\pi}{4}$.

    Solution: FALSE. θ could be any of the following:

$$\ldots, -\frac{15\pi}{4}, -\frac{7\pi}{4}, \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \ldots$$

That is to say, θ could be any angle which is coterminal to $\pi/4$.

20. If $\tan(x) = 4$, then $\tan(2x) = 8$.

    Solution: FALSE. Using the double angle identity for tangent we have
    $$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x} = \frac{2 \cdot 4}{1 - 4^2} = -\frac{8}{15} \neq 8.$$

21. If $f(x) = \sin^2(x)\sec x - \cos x$, then $f(-x) = f(x)$.

22. For any real number t in the domain of the tangent function, the following is an identity.
    $$\tan t = \frac{\sin(t + \pi)}{\cos(t + 2\pi)}$$

    Solution: FALSE. There are many possible counterexamples, here's one:

    Take $t = \frac{\pi}{4}$. Then
    $$\tan t = \tan\left(\frac{\pi}{4}\right) = 1$$
    whereas
    $$\frac{\sin(t + \pi)}{\cos(t + 2\pi)} = \frac{\sin\left(\frac{\pi}{4} + \pi\right)}{\cos\left(\frac{\pi}{4} + 2\pi\right)} = \frac{\sin\left(\frac{5\pi}{4}\right)}{\cos\left(\frac{9\pi}{4}\right)} = \frac{\left(-\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)} = -1.$$

23. We know $\arccos(\cos(x)) = x$ for any real number x which lies between 0 and π.

    Solution: TRUE. By definition $\arccos(u)$ is the unique number between 0 and π such that $\cos(\arccos(u)) = u$. Therefore [setting $u = \cos(x)$] we have $\arccos(\cos(x))$ is the unique number between 0 and π such that
    $$\cos[\arccos(\cos(x))] = \cos(x).$$
    But x is already between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so
    $$\arccos(\cos(x)) = x.$$

24. We know $\arccos(\cos(x)) = x$ for ANY real number x.

    Solution: FALSE. This is false whenever x is not between 0 and π. For example
    $$\arccos(\cos(-\pi)) = \arccos(-1) = \pi \neq -\pi.$$

25. We know $\cos(\arccos(x)) = x$ for any real number x with $-1 \leq x \leq 1$.

    Solution: TRUE. This is part of the definition of $\arccos(x)$.

26. We know $\cos(\arccos(x)) = x$ for ANY real number x.

    Solution: FALSE. This is false whenever x is either greater than 1 or less than -1. For example $\arccos(-107)$ is undefined, so $\cos(\arccos(-107))$ is also undefined, and therefore

    $$\cos(\arccos(-107)) \neq -107.$$

27. We know $\arctan(\tan(x)) = x$ for any real number x which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

    Solution: TRUE. By definition $\arctan(u)$ is the unique number between $-\pi/2$ and $\pi/2$ such that $\tan(\arctan(u)) = u$. Therefore [setting $u = \tan(x)$] we have $\arctan(\tan(x))$ is the unique number between $-\pi/2$ and $\pi/2$ such that

    $$\tan[\arctan(\tan(x))] = \tan(x).$$

    But x is already between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so

    $$\arctan(\tan(x)) = x.$$

28. We know $\arctan(\tan(x)) = x$ for ANY real number x.

    Solution: FALSE. This is false whenever x is not between $-\pi/2$ and $\pi/2$. For example

    $$\arctan(\tan(\pi)) = \arctan(0) = 0 \neq \pi.$$

29. We know $\tan(\arctan(x)) = x$ for ANY real number x.

    Solution: TRUE. This is part of the definition of $\arctan(x)$.

30. We know $\arcsin(\sin(x)) = x$ for any real number x which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

    Solution: TRUE. By definition $\arcsin(u)$ is the unique number between $-\pi/2$ and $\pi/2$ such that $\sin(\arcsin(u)) = u$. Therefore [setting $u = \sin(x)$] we have $\arcsin(\sin(x))$ is the unique number between $-\pi/2$ and $\pi/2$ such that

    $$\sin[\arcsin(\sin(x))] = \sin(x).$$

    But x is already between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so

    $$\arcsin(\sin(x)) = x.$$

31. We know $\arcsin(\sin(x)) = x$ for ANY real number x.

    Solution: FALSE. This is false whenever x is not between $-\pi/2$ and $\pi/2$. For example

    $$\arcsin(\sin(\pi)) = \arcsin(0) = 0 \neq \pi.$$

32. We know $\sin(\arcsin(x)) = x$ for any real number x with $-1 \leq x \leq 1$.

    Solution: TRUE. This is part of the definition of $\arcsin(x)$.

33. We know $\sin(\arcsin(x)) = x$ for ANY real number x.

    Solution: FALSE. This is false whenever x is either greater than 1 or less than -1. For example $\arcsin(17)$ is undefined, so $\sin(\arcsin(17))$ is also undefined, and therefore

    $$\sin(\arcsin(17)) \neq 17.$$

34. If c is some real number, then we know the equation $\tan x = c$ has infinitely many solutions.

    Solution: TRUE. For ANY real number c we know

    $$\tan x = c \quad \Rightarrow \quad x = \arctan(c) + \pi k$$

    for any integer k. Thus there are infinitely many solutions, one for each integer.

35. If c is some real number, then we know the equation $\sin x = c$ has infinitely many solutions.

    Solution: FALSE. For example, set $c = 2$. We know $\sin x = 2$ has zero solutions.

    [Any c-value greater then 1 or less than -1 will give a counterexample.]

36. If c is some real number, then we know the equation $\sin x = c$ has either zero or infinitely many solutions.

    Solution: TRUE. For any real number c between -1 and 1 we know

    $$\sin x = c \quad \Longrightarrow \begin{cases} x = \arcsin(c) + 2\pi k \\ \text{or} \\ x = \pi - \arcsin(c) + 2\pi k \end{cases}$$

    for any integer k. Thus there are infinitely many solutions when $-1 \leq c \leq 1$, one for each integer. On the other hand, when c is greater than 1 or less than -1 the equation $\sin x = c$ has no solution. So there is either zero or infinitely many solutions.

37. If $\sec x = c$ where $c > 1$, then we know

$$x = \frac{1}{\arccos(c)} + 2\pi k \qquad \text{or} \qquad x = -\frac{1}{\arccos(c)} + 2\pi k$$

Solution: FALSE. For example if $c = 2$, then $\sec x = 2$ implies $\cos x = 1/2$, thus

$$x = \pm\frac{\pi}{3} + 2\pi k$$

for any integer k. On the other hand, $\arccos(2)$ is undefined, so $\pm\frac{1}{\arccos(c)} + 2\pi k$ is undefined for all integers k.

38. If $\sec x = c$ where $c > 1$, then we know

$$x = \arccos\left(\frac{1}{c}\right) + 2\pi k \qquad \text{or} \qquad x = -\arccos\left(\frac{1}{c}\right) + 2\pi k$$

Solution: TRUE. $\sec x = c$ implies $\cos x = \frac{1}{c}$ which implies

$$x = \frac{1}{\arccos(c)} + 2\pi k \qquad \text{or} \qquad x = -\frac{1}{\arccos(c)} + 2\pi k$$

for any integer k.

39. $\tan(\arctan(31)) = 31$.

Solution: TRUE. By definition, $\tan(\arctan(x)) = x$ for ANY real number. Since 31 is a real number it is true that $\tan(\arctan(31)) = 31$.

40. It is possible for the sides of a triangle to have lengths 3, 5, & 10.

Solution: FALSE. $3 + 5 = 8 < 10$, so these lengths do not satisfy the triangle inequality.

41. It is possible for the sides of a triangle to have lengths 2, 5, & 6.

Solution: TRUE. All we need to check is that the triangle inequality holds. It does hold, because the sum of any two lengths is greater than the third:

$$2 + 5 = 7 > 6,$$

$$5 + 6 = 11 > 2,$$

$$2 + 6 = 8 > 5.$$

42. It is possible for the interior angles of a triangle to have measures $63°$, $74°$, and $42°$.

    Solution: FALSE. We know the interior angle of a triangle must sum to $180°$ and
    $$63° + 74° + 42° = 179° \neq 180°.$$

43. It is possible for the equation $\cos x = c$ to have zero solutions, where c is some constant.

    Solution: TRUE. For example, when $c = 201$ we know $\cos x = 201$ has zero solutions.

    [Any c-value greater then 1 or less than -1 will work, because $\arccos(c)$ is only defined for $-1 \leq c \leq 1$.]

44. It is possible for the equation $\tan x = c$ to have zero solutions, where c is some constant.

    Solution: FALSE. $\tan x = c$ implies $x = \arctan(c) + \pi k$ for some integer k. This makes sense because $\arctan(c)$ is defined for ANY real number c.

45. If $f(x)$ is a polynomial with real coefficients and $f(2i + 3) = 0$, then we know $f(3 - 2i) = 0$.

    Solution: TRUE. The conjugate root theorem tells us that if $2i + 3$ is a root then its conjugate $\overline{2i + 3} = \overline{3 + 2i} = 3 - 2i$ is also a root.

46. If $f(x)$ is a polynomial with real coefficients and $f(3 - 2i) = 0$, then we know $f(2i - 3) = 0$.

    Solution: FALSE. $\overline{3 - 2i} = 3 + 2i \neq 2i - 3$. So the conjugate root theorem only guarantees $f(3 + 2i) = 0$. For example, let
    $$f(x) = (x - (3 - 2i))(x - (3 + 2i)) = x^2 - 6x + 13.$$

    Then the roots of $f(x)$ do not include $2i - 3$.

47. Every non-constant polynomial with real coefficients has at least one real root.

    Solution: FALSE. For example $x^2 + 1$ has no real root.

48. Every non-constant polynomial with real coefficients has at least one complex root.

    Solution: TRUE. This is the Fundamental Theorem of Algebra.

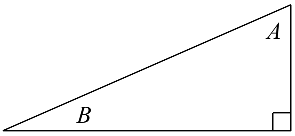
49. Every non-constant polynomial with real coefficients of odd degree has at least one real root.

    Solution: TRUE. The conjugate root theorem tells us if $a + bi$ is a root, then so is $a - bi$. So the non-real roots of a polynomial come in pairs. Thus the only polynomials which have no real roots are of even degree. This last sentence is the same as saying every odd degree polynomial has a real root.

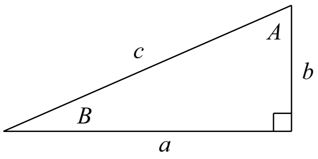
50. $x + 1$ is a factor of the polynomial $f(x) = x^{94} + 3x^{21} + 2$.

    Solution: TRUE. $x + 1$ is a factor of $f(x)$ exactly when $f(-1) = 0$, and $f(-1) = (-1)^{94} + 3(-1)^{21} + 2 = 1 + 3(-1) + 2 = 1 - 3 + 2 = 0$.

51. In the following triangle $\sin A$ equals $\cos B$.



Solution: TRUE. If we label the sides of the triangle as follows



then using SohCahToa we see

$$\sin A = \frac{a}{c} = \cos B.$$