Math 111–Midterm 2 Solutions

These are not the only methods for solving these problems. In general there are many different ways of solving the same problem.

1. [10pts] Find the remainder when $2x^3 - 2x + 3$ is divided by $x^2 - 3x + 2$.
Solution:

$$\begin{array}{r|l} & 2x + 6 \\ \hline x^2 - 3x + 2 & 2x^3 + 0x^2 - 2x + 3 \\ & -(2x^3 - 6x^2 + 4x) \\ \hline & 6x^2 - 6x + 3 \\ & -(6x^2 - 18x + 12) \\ \hline & 12x - 9 \end{array}$$

Therefore the remainder is $r(x) = 12x - 9$.

2. [10pts] Write the following expression as a single logarithm.

$$3\ln(x) + 4\ln(y) - \ln(x^3) - 2\ln(z)$$

Solution:

$$\begin{aligned} & 3\ln(x) + 4\ln(y) - \ln(x^3) - 2\ln(z) \\ &= \ln(x^3) + \ln(y^4) - \ln(x^3) - \ln(z^2) \\ &= \ln(y^4) - \ln(z^2) \\ &= \ln\left(\frac{y^4}{z^2}\right). \end{aligned}$$

3. [15pts] Let $f(x)$ be the function whose graph is a parabola with vertex $(-4, 5)$, which passes through the point $(0, 1)$. Find the rule for $f(x)$.

Solution: First, since the graph of $f(x)$ is a parabola, we know f is a quadratic function. Thus for some a, h, and k we have

$$f(x) = a(x - h)^2 + k,$$

where (h, k) is the vertex of the graph of $f(x)$. Since the vertex of the graph of $f(x)$ is $(-4, 5)$, we have

$$f(x) = a(x + 4)^2 + 5.$$

Since the point $(0, 1)$ is on the graph of $f(x)$ we know $f(0) = 1$, which implies

$$1 = f(0) = a(0 + 4)^2 + 5$$

$$\Rightarrow 1 = a \cdot 4^2 + 5$$

$$\Rightarrow 1 = 16a + 5$$

$$\Rightarrow -4 = 16a$$

$$\Rightarrow -1/4 = a.$$

Therefore

$$f(x) = -1/4(x + 4)^2 + 5.$$

4. [15pts] Let

$$f(x) = \frac{3x^2 - 6x}{x^3 - 6x^2 + 8x}.$$

(i) Find all the holes, vertical asymptotes, and horizontal asymptotes in the graph of $f(x)$.

Solution: first notice the degree of the numerator is less than the degree of the denominator. Thus there is a **horizontal asymptote** at $y = 0$ (the x-axis). Next we factor both the numerator and the denominator:

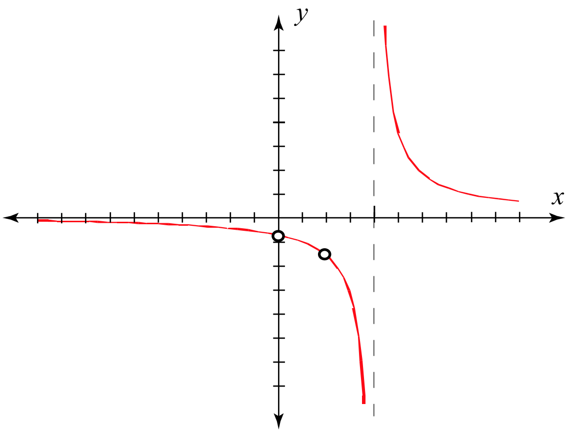
$$f(x) = \frac{3x^2 - 6x}{x^3 - 6x^2 + 8x} = \frac{3x(x-2)}{x(x-2)(x-4)}.$$

Notice that $f(x)$ is not defined when $x = 0, x = 2$, or $x = 4$. Thus the graph of $f(x)$ has discontinuities (holes or vert. asymptotes) at $x = 0, x = 2$, and $x = 4$. When we simplify we see

$$\frac{3x(x-2)}{x(x-2)(x-4)} = \frac{3}{x-4}.$$

Therefore the graph of $f(x)$ has **holes** at $x = 0$ and $x = 2$, and a **vertical asymptote** at $x = 4$.

(ii) Sketch a complete graph of $f(x)$.



5. [10pts] Let a and b be positive real numbers. Simplify the following expression:

$$\frac{\sqrt[6]{a^{-3}b}}{a^{-2}b^{1/6}a}$$

Solution:

$$\frac{\sqrt[6]{a^{-3}b}}{a^{-2}b^{1/6}a} = \frac{(a^{-3}b)^{1/6}}{a^{-2}ab^{1/6}} = \frac{a^{-1/2}b^{1/6}}{a^{-1}b^{1/6}}$$

$$= a^{-1/2-(-1)}b^{1/6-1/6} = a^{1/2}b^0$$

$$= \sqrt{a}.$$

6. [10pts] Solve the following inequalities and express your answers in interval notation.

(i)

$$|4x + 2| < 1$$

Solution:

$$-1 < 4x + 2 < 1$$
$$\Rightarrow -3 < 4x < -1$$
$$\Rightarrow -3/4 < x < -1/4$$

Therefore x is any number in the interval $(-3/4, -1/4)$.

(ii)

$$|3x + 2| \geq 4$$

Solution:

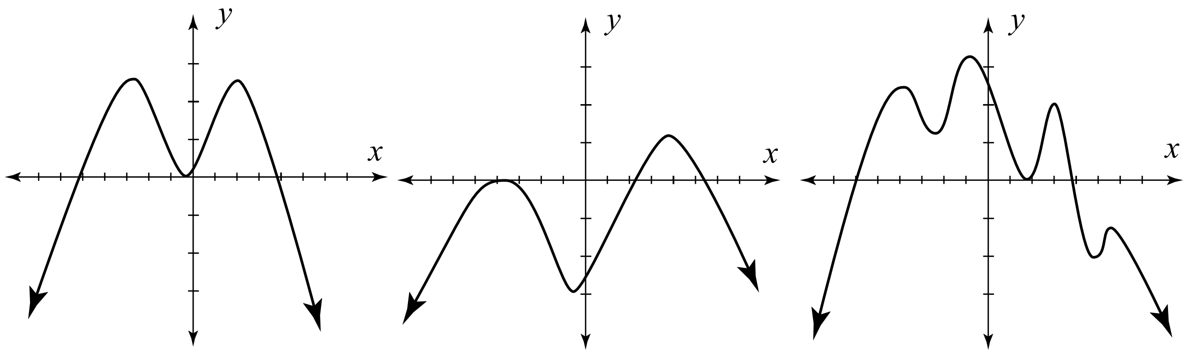
$$\begin{array}{rcl} 3x + 2 \geq 4 & \text{or} & 3x + 2 \leq -4 \\ 3x \geq 2 & \text{or} & 3x \leq -6 \\ x \geq 2/3 & \text{or} & x \leq -2 \end{array}$$

Therefore x is any number in the set $(-\infty, -2] \cup [2/3, \infty)$.

7. [10pts] Sketch a complete graph of a polynomial function $f(x)$ which satisfies ALL of the following three conditions.

(i) The degree of $f(x)$ is even.
(ii) The leading coefficient of $f(x)$ is negative.
(iii) $f(x)$ has three roots, exactly one of which is of even multiplicity.

Solution: Here are three such graphs, there are many.



8. [10pts] What amount of money, A, will be in a savings account after 2 years if the initial deposit was \$317, and the interest rate is 13% compounded continuously?
(You do NOT need to simplify your answer)

Solution: Using the formula $A = Pe^{rt}$ we have

$$A = 317e^{.13(2)} = 317e^{.26}.$$

9. [10pts] Let

$$f(x) = 3x^{23} - 2x^2 - x.$$

(i) Is $x - 1$ a factor of $f(x)$? (Justify your answer)

Solution: YES. The factor theorem tells us $x-1$ is a factor of $f(x)$ exactly when $f(1) = 0$, and

$$f(1) = 3(1)^{23} - 2(1)^2 - 1 = 3 - 2 - 1 = 0.$$

(ii) Is $x + 1$ a factor of $f(x)$? (Justify your answer)

Solution: NO. The factor theorem tells us $x+1$ is a factor of $f(x)$ exactly when $f(-1) = 0$, and

$$f(-1) = 3(-1)^{23} - 2(-1)^2 - (-1) = -3 - 2 + 1 = -4 \neq 0.$$

10. [10pts] (i) For any positive real numbers b and v, with $b \neq 1$, find the value of

$$b^{\log_b(v)}$$

Solution:

$$b^{\log_b(v)} = v$$

(ii) Give a clear justification (in words) for your answer in part (i).

Solution: $\log_b(v)$ is the power of b which equals v, so b to that power equals v.