Math 111–Exam 2 Solutions

1. (3pts each) True or False:

   (a) A polynomial function of degree 7 can have 6 real roots.
   True, for a polynomial function of degree 7 has at most 7 real roots, and $6 < 7$.

   (b) If $x + 1$ is a factor of a polynomial function $f(x)$, then $x - 1$ is also a factor of $f(x)$.
   False, for example the polynomial functions with rules $f(x) = x^2 + x$, $f(x) = (x + 1)(x - 2)^2$, and $f(x) = x + 1$ all have $x + 1$ as a factor but do not have $x - 1$ as a factor. (There are many more counterexamples).

   (c) $x + 1$ is a factor of $x^4 - 15x - 16$.
   True, for $(-1)^4 - 15(-1) - 16 = 0$ so -1 is a root of $x^4 - 15x - 16$. Thus $x - (-1) = x + 1$ is a factor of $x^4 - 15x - 16$.

   (d) The range of some quadratic functions is $(-\infty, \infty)$.
   False, for all quadratic functions have either a minimum or a maximum value (the y-coordinate of the vertex). If the function has a minimum value, then no real number less than that value is in the range. Similarly if the function has a maximum value, then no real number greater than that value is in the range.

2. (10pts) For what values of x is $|x^2 - 1| \leq 8$? Give your answer in interval notation.

Solution: First we consider the equation $|x^2 - 1| = 8$. To solve for x we must solve the two equations

$$x^2 - 1 = 8 \qquad \text{and} \qquad x^2 - 1 = -8.$$

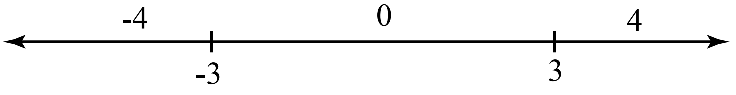
Solving for x in the first equation we see

$$x^2 - 1 = 8 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3.$$

Solving for x in the second equation we see

$$x^2 - 1 = -8 \quad \Rightarrow \quad x^2 = -7 \quad \Rightarrow \quad x = \pm\sqrt{-7}$$

which is not a real number, thus there are no real solutions to the second equation. Now we must check one point in each of the intervals $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$ in the original inequality. I will check $-4, 0$, and 4.



Since

$$|(-4)^2 - 1| = 15 \not\leq 8 \quad \text{and} \quad |(0)^2 - 1| = 1 \leq 8 \quad \text{and} \quad |(4)^2 - 1| = 15 \not\leq 8$$

we see the solution is

or in interval notation $[-3, 3]$.

3. (10pts) Suppose that I invest \$400 at 13% compounded quarterly and I do not withdraw any money. How much money will I have after 3 years?

   (You do NOT need to simplify your answer)

   Solution: The rule

   $$f(t) = 400\left(1 + \frac{.13}{4}\right)^t$$

   gives the amount of money in my account after t quarters. Since there are 12 quarters in 3 years we see that after 3 years I will have

   $$f(12) = 400\left(1 + \frac{.13}{4}\right)^{12}$$

   dollars in my account.

4. (10pts) Recall that the area A of a rectangle is given by $A = lw$ where l is the length and w is the width. If a rectangle has the property that the sum of its length and 3 times its width is 12, what is the maximal area of the rectangle?

   (You must show work to justify your answer)

   Solution: Since $l + 3w = 12$ we know $l = 12 - 3w$, so we have

   $$A = lw = (12 - 3w)w = -3w^2 + 12w.$$

   To find the maximal area, we must find the vertex of the parabola which is the graph of the equation $A = -3w^2 + 12w$. Notice

   $$\begin{aligned} A &= -3w^2 + 12w \\ \Rightarrow \frac{A}{-3} &= w^2 - 4w \\ \Rightarrow 4 + \frac{A}{-3} &= w^2 - 4w + 4 \\ \Rightarrow 4 + \frac{A}{-3} &= (w - 2)^2 \\ \Rightarrow \frac{A}{-3} &= (w - 2)^2 - 4 \\ \Rightarrow A &= -3(w - 2)^2 + 12. \end{aligned}$$

   Thus the vertex is $(2, 12)$, which tells us the maximum area occurs when the width is 2, and the maximal area is 12.

5. (10pts) Let $f(x) = x^4 + 6x^3 - 7x^2 - 60x$. Use long division and the fact that $f(-4) = 0$ to find all the roots f.

Solution: Since $f(-4) = 0$, -4 is a root of f. Therefore $x + 4$ is a factor of $f(x)$, so we divide $f(x)$ by $x + 4$ as follows

$$\begin{array}{rr}
 & x^3 + 2x^2 - 15x \qquad\quad \\
x + 4 & \big)\overline{\, x^4 + 6x^3 - 7x^2 - 60x} \\
 & -\underline{(x^4 + 4x^3)} \qquad\qquad\qquad \\
 & 2x^3 - 7x^2 - 60x \\
 & -\underline{(2x^3 + 8x^2)} \qquad\quad \\
 & -15x^2 - 60x \\
 & -\underline{(-15x^2 - 60x)} \\
 & 0 \quad
\end{array}$$

Thus

$$\begin{aligned}
f(x) &= (x + 4)(x^3 + 2x^2 - 15x) \\
&= x(x + 4)(x^2 + 2x - 15) \\
&= x(x + 4)(x + 5)(x - 3).
\end{aligned}$$

Therefore the roots of $f(x)$ are $0, -4, -5$, and 3.

6. (10pts) Suppose the graph of quadratic function f has vertex $(4, 1)$ and passes through the point $(2, -11)$. Write down the rule of $f(x)$.

Solution: We know $f(x) = a(x - h)^2 + k$ where the (h, k) is the vertex. Since the vertex is $(4, 1)$ we see $h = 4$ and $k = 1$ so that

$$f(x) = a(x - 4)^2 + 1.$$

Since the point $(2, -11)$ is on the graph of f we know $f(2) = -11$ so that

$$-11 = f(2) = a(2 - 4)^2 + 1$$

$$\Rightarrow -11 = a(-2)^2 + 1$$

$$\Rightarrow -11 = a(4) + 1$$

$$\Rightarrow -12 = a(4)$$

$$\Rightarrow -3 = a.$$

So the rule of f is

$$f(x) = -3(x - 4)^2 + 1.$$

7. (3pts each) Let f be the rational function whose rule is given by

$$f(x) = \frac{4(x+2)(x-3)^3(x+1)^2}{8(x+2)^2(x-3)^2(x+1)^3}.$$

Notice:

$$f(x) = \frac{4(x+2)(x-3)^3(x+1)^2}{8(x+2)^2(x-3)^2(x+1)^3} = \frac{(x-3)}{2(x+2)(x+1)}.$$

(a) State the domain of f.

Solution: Since we cannot divide by zero, the only real numbers which are not in the domain are the roots of the denominator, namely

$$x \neq -2, \quad x \neq 3, \quad x \neq -1.$$

(b) State the horizontal asymptote of the graph of f.

Solution: Since the degree of the numerator is 6 and the degree of the denominator is 7, we know the horizontal asymptote is

$$y = 0 \quad \text{(or the } x\text{-axis)}$$

(c) Find all vertical asymptotes of the graph of f (if any exist).

Solution: Since the only factors remaining in the denominator after simplifying are $x+2$ and $x+1$ we know the vertical asymptotes are

$$x = -2 \quad \text{and} \quad x = -1.$$

(d) Find all x-values of holes in the graph of f (if any exist).

Solution: Since the factor $x-3$ appears in the denominator of the rule, but does not appear in the denominator after simplifying the rule, we know there is a hole when

$$x = 3.$$

(e) Give the (x, y)-coordinates of all the holes of the the graph of f (if the graph has any holes).

Solution: To find the y-value of the hole corresponding to $x = 3$, we evaluate the simplified rule at $x = 3$ and we get $\frac{(3-3)}{2(3+2)(3+1)} = 0$ so that the hole in the graph of f is at the point

$$(3, 0).$$

(f) Find the y-intercept of the graph of f (if one exists).

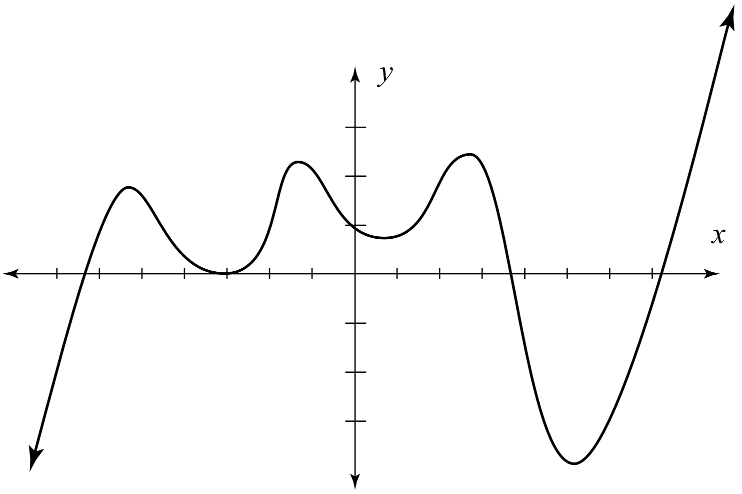
Solution: To find the y-intercept we set $x = 0$ (which is ok since 0 is in the domain of f). Thus the y-intercept is

$$f(0) = \frac{(0-3)}{2(0+2)(0+1)} = \frac{-3}{4}.$$

(g) Find the x-intercept(s) of the graph of f (if any exist).

Solution: To find the y-intercept we set $f(x) = 0$. In doing so we only need to find the roots of the numerator. But none of the roots of the numerator are in the domain, so there are no x-intercepts.

8. (3pts each)



Above is a complete graph of the polynomial function f. Circle the correct answer.

(a) The degree of $f(x)$ is

  i. even
  ii. odd
  iii. neither even nor odd

  Answer: **ii**, because of the end behavior of the graph.

(b) The leading coefficient of $f(x)$ is

  i. positive
  ii. negative
  iii. zero

  Answer: **i**, because of the end behavior of the graph.

(c) The degree of $f(x)$ COULD be

  i. 4
  ii. 5
  iii. 8
  iv. 11
  v. none of the above

  Answer: **iv**, for the degree must be at least 7 and odd.

(d) How many roots of odd multiplicity does f have?

  i. 0
  ii. 1
  iii. 2
  iv. 3
  v. 4
  vi. none of the above

  Answer: **iv**, the only root of even multiplicity is root at $x = 3$, the other three are of odd multiplicity.

9. (10pts) Simplify (a and b are positive real numbers)

$$\frac{(9a^2b^{-3})^{1/2}}{b^{-2}(27a)^{-1/3}}$$

Solution:

$$\begin{aligned}\frac{(9a^2b^{-3})^{1/2}}{b^{-2}(27a)^{-1/3}} &= (9a^2b^{-3})^{1/2}b^2(27a)^{1/3} = 9^{1/2}(a^2)^{1/2}(b^{-3})^{1/2}b^2 27^{1/3}a^{1/3} \\ &= 3ab^{-3/2}b^2 3a^{1/3} = 3\cdot 3a\cdot a^{1/3}b^{-3/2}b^2 = 9a^{1+1/3}b^{-3/2+2} = 9a^{4/3}b^{1/2}.\end{aligned}$$

Bonus: (10pts) Find all real numbers x which satisfy the following equation.

$$x^4 + x = 2x^3.$$

Solution: Subtracting $2x^3$ from each side of the above equation we have

$$x^4 - 2x^3 + x = 0.$$

Factoring out an x we see $x(x^3-2x^2+1) = 0$. Notice that the coefficients of x^3-2x^2+1 sum to 0, so that 1 is a root of the polynomial $x^3 - 2x^2 + 1$ (you can see this directly by setting $x = 1 \ldots 1^3 - 2(1)^2 + 1 = 1 - 2 + 1 = 0$). Therefore $x - 1$ is a factor of $x^3 - 2x^2 + 1$. Dividing by $x - 1$ we have

$$\begin{array}{r l}
 & \quad x^2 - x - 1 \\
x-1 & \overline{\big)\; x^3 - 2x^2 + 0x + 1} \\
 & -\underline{(x^3 - x^2)} \\
 & \qquad -x^2 + 0x + 1 \\
 & \qquad -\underline{(-x^2 + x)} \\
 & \qquad\qquad -x + 1 \\
 & \qquad\qquad -\underline{(-x + 1)} \\
 & \qquad\qquad\qquad 0
\end{array}$$

Thus $x^4 - 2x^3 + x = x(x-1)(x^2-x-1)$. So the roots of $x^4 - 2x^3 + x$ are 0, 1, and the roots of $x^2 - x - 1$. To find the roots of $x^2 - x - 1$ we solve the equation $x^2 - x - 1 = 0$ as follows

$$\begin{aligned} x^2 - x - 1 = 0 \quad &\Rightarrow \quad x^2 - x = 1 \\ &\Rightarrow x^2 - x + \left(-\frac{1}{2}\right)^2 = 1 + \left(-\frac{1}{2}\right)^2 \\ &\Rightarrow \left(x - \frac{1}{2}\right)^2 = 1 + \frac{1}{4} \\ &\Rightarrow \left(x - \frac{1}{2}\right)^2 = \frac{5}{4} \\ &\Rightarrow x - \frac{1}{2} = \frac{\pm\sqrt{5}}{2} \\ &\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}. \end{aligned}$$

So x is either 0,1, $\frac{1+\sqrt{5}}{2}$, or $\frac{1-\sqrt{5}}{2}$.