Math 111 Exam 1 Solutions (Yellow)

For this exam assume all functions have domain and range some subset of the real numbers. Be sure to show all your work in a clear and organized manner, to to indicate you answers.

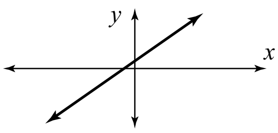
1. (3pts each) True or False

   (a) If f is the function whose rule is given by $f(x) = x + 1$, then $f^{-1}(x) = x - 1$.
   True, for setting $f(y) = x$ and solving for y we have

   $$y + 1 = x \quad \Rightarrow \quad y = x - 1.$$

   Thus, setting $f^{-1}(x) = y$ we have $f^{-1}(x) = x - 1$.

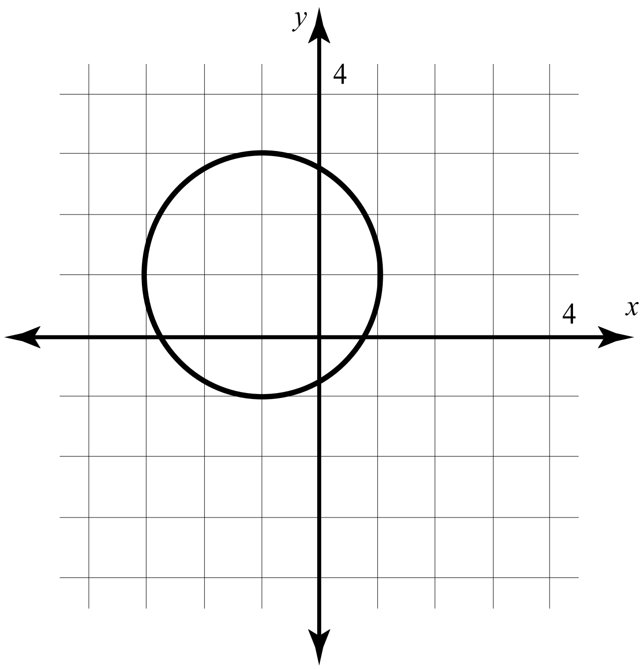
   (b) A linear function with positive slope is increasing on $(-\infty, \infty)$.
   True, for all lines with positive slope look something like



   (c) If f is an even function and the point $(2, 6)$ is on the graph of f, then the point $(-2, 6)$ must also be on the graph of f.
   True, because saying the point $(2, 6)$ is on the graph of f is the same as saying $f(2) = 6$. Since f is even we know $f(-2) = f(2) = 6$ so the point $(-2, 6)$ is also on the graph of f.

   (d) If $t = 0.75w$, then t is inversely proportional to w with constant of proportionality 0.75.
   False, for t is directly proportional to w with constant of proportionality 0.75.

2. (10pts) Give the equation of the of the circle pictured below.



Solution: The equation for a circle with center (x_1, y_1) and radius r is

$$(x - x_1)^2 + (y - y_1)^2 = r^2.$$

For the circle on pictured on the right, the center is $(-1, 1)$ and the radius is 2, so the equation of the circle is

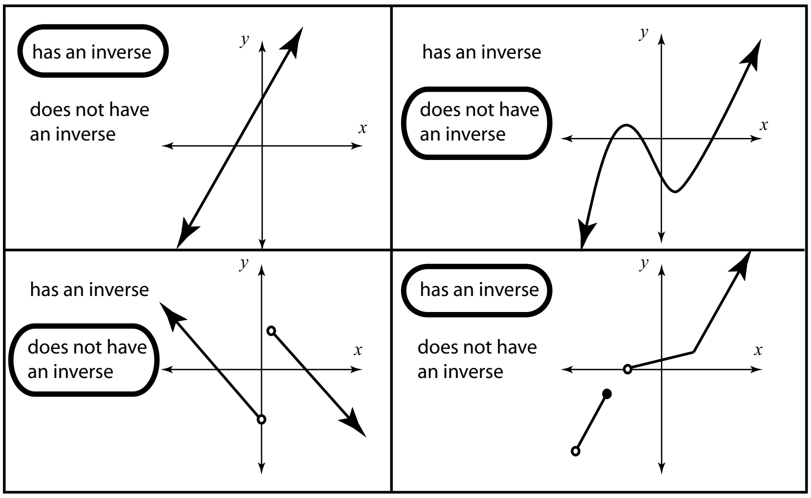
$$(x - (-1))^2 + (y - 1)^2 = 2^2$$

or cleaned up a bit

$$(x + 1)^2 + (y - 1)^2 = 4.$$

3. (8pts) Indicate whether or not the functions corresponding to the following graphs have inverses.

Solution: A function has an inverse exactly when it is a one-to-one function. One-to-one functions are those functions whose graphs pass the horizontal line test. Only two of the following graphs pass the horizontal line, so the functions corresponding to those two graphs have inverses, while the other two do not.



4. (10pts) Consider the functions f and g whose rules are given below. Give the domain of each in interval notation.

$$f(x) = 2\sqrt{2x-1} + 5$$

Solution: Since we cannot take the square root of negative numbers we see the only restriction on our input x is that

$$2x - 1 \geq 0 \quad \Rightarrow \quad 2x \geq 1 \quad \Rightarrow \quad x \geq \frac{1}{2}.$$

Therefore the domain of f is all real numbers greater than or equal to $\frac{1}{2}$, in interval notation

$$\left[\frac{1}{2} , \infty \right).$$

$$g(x) = \frac{3x-2}{x^2 - 5x + 4}$$

Solution: Since we cannot divide by zero we see the only restriction on our input x is that

$$x^2 - 5x + 4 \neq 0 \quad \Rightarrow \quad (x-4)(x-1) \neq 0 \quad \Rightarrow \quad x - 4 \neq 0 \text{ and } x - 1 \neq 0$$

$$\Rightarrow \quad x \neq 4 \text{ and } x \neq 1.$$

Therefore the domain of g is all real numbers except 1 and 4, in interval notation

$$(-\infty , 1) \cup (1 , 4) \cup (4 , \infty).$$

5. (10pts) Recall the difference quotient is given by

$$\frac{f(x+h)-f(x)}{h}$$

Compute and simplify the difference quotient of the function f where

$$f(x) = 3x^2 + 7$$

Solution: Since $f(x+h) = 3(x+h)^2+7 = 3(x^2+2xh+h^2)+7 = 3x^2+6xh+3h^2+7$ we see

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(3x^2+6xh+3h^2+7)-(3x^2+7)}{h} \\ &= \frac{3x^2+6xh+3h^2+7-3x^2-7}{h} \\ &= \frac{6xh+3h^2}{h} = \frac{h(6x+3h)}{h} \\ &= 6x+3h.\end{aligned}$$

6. (10pts) Find the rule of f^{-1} where f is the one-to-one function whose rule is given by

$$f(x) = \frac{x^3}{3-x^3}$$

Solution: To find the rule of f^{-1} we first set $f(y) = x$:

$$f(y) = \frac{y^3}{3-y^3} = x.$$

Now we solve for y as follows:

$$\frac{y^3}{3-y^3} = x \quad \Rightarrow \quad (3-y^3)\frac{y^3}{3-y^3} = x(3-y^3) \quad \Rightarrow \quad y^3 = 3x - xy^3$$

$$\Rightarrow \quad y^3 + xy^3 = 3x \quad \Rightarrow \quad y^3(1+x) = 3x \quad \Rightarrow \quad \frac{y^3(1+x)}{1+x} = \frac{3x}{1+x}$$

$$\Rightarrow \quad y^3 = \frac{3x}{1+x} \quad \Rightarrow \quad y = \sqrt[3]{\frac{3x}{1+x}}.$$

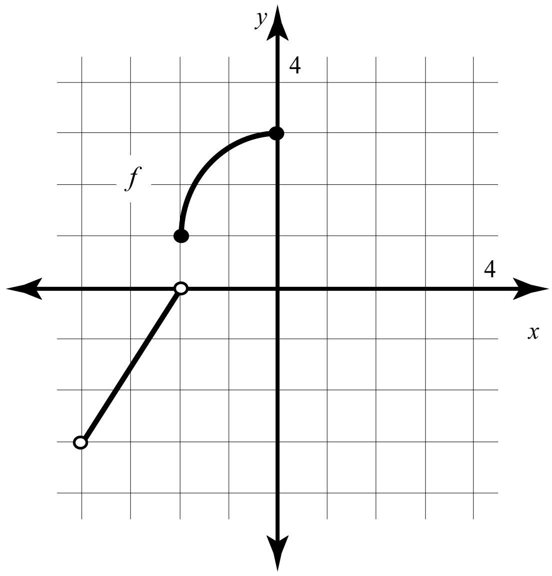
Finally we set $f^{-1}(x) = y$ and we have

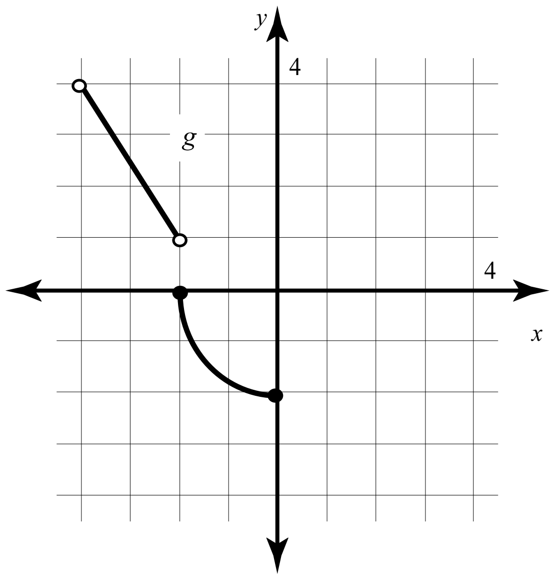
$$f^{-1}(x) = \sqrt[3]{\frac{3x}{1+x}}.$$

7. (10pts) Let f be the function whose graph is given on the left. On the right, sketch a graph of the function g whose rule is given by

$$g(x) = -f(x) + 1$$

Solution: The graph of g is the graph of f first reflected about the y-axis, and then shifted up one unit (notice that the order of the transformations matters). The graph of g is given on the left.

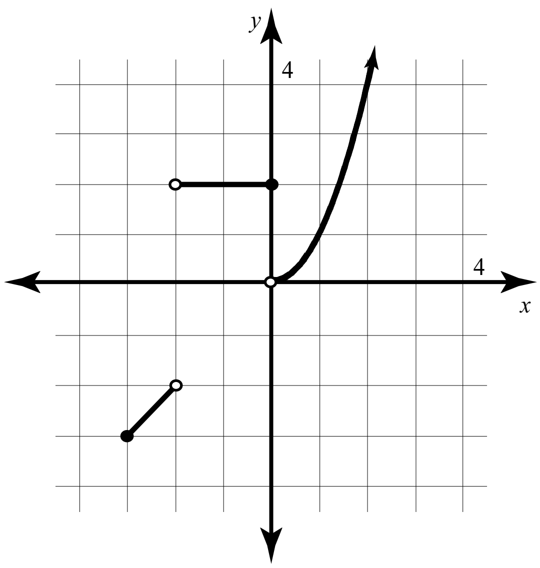




8. (10pts) Sketch a graph of the function f whose rule is given by

$$f(x) = \begin{cases} x, & \text{if } -3 \leq x < -2; \\ 2, & \text{if } -2 < x \leq 0; \\ x^2, & \text{if } x > 0. \end{cases}$$

Solution:



9. (10pts) Let f be the function whose rule is given by

$$f(x) = 2x^3 - 5x + 1$$

Is f even, odd, or neither? (You will receive no credit unless you justify your answer)

Solution: To see if f is even, odd, or neither we need to compare $f(x), f(-x)$, and $-f(x)$. Since

$$f(-x) = 2(-x)^3 - 5(-x) + 1 = -2x^3 + 5x + 1,$$

and

$$-f(x) = -(2x^3 - 5x + 1) = -2x^3 + 5x - 1$$

we see that $f(-x) \neq f(x)$ so f is not even, and $f(-x) \neq -f(x)$ so f is not odd. Thus f is neither even nor odd.

10. (10pts) Suppose f and g are one-to-one functions which satisfy the following

$$f(1) = 3, \qquad f(3) = -2, \qquad g(3) = 4, \qquad g(1) = 2.$$

Solution:

$$(g \circ f)(1) = g(f(1)) = g(3) = 4.$$

$$(f + g)(3) = f(3) + g(3) = -2 + 4 = 2.$$

$$(g \circ f^{-1})(3) = g(f^{-1}(3)) = g(1) = 2.$$

For the last computation we used the fact that $f(1) = 3$ implies $f^{-1}(3) = 1$.

Bonus (10pts) Let f and g be as in problem 10. Suppose further that g is an odd function. Compute

$$\left(g \circ f^{-1} \circ g\right)(-1)$$

Solution: Since g is odd we have $g(-1) = -g(1) = -2$, so we see

$$\left(g \circ f^{-1} \circ g\right)(-1) = g(f^{-1}(g(-1))) = g(f^{-1}(-2)) = g(3) = 4.$$

As in problem 10 we used the fact that $f(3) = -2$ implies $f^{-1}(-2) = 3$.