Lecture 4

Huaxin Lin

June 9th, 2015,

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But first we will establish some Bott-map related existence theorems.

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$$[\Phi]|_{\mathcal{P}} = (\kappa + [h_1] + [h_2] \cdots + [h_k])|_{\mathcal{P}}. \quad \text{(e 0.2)}$$

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$$[L]|_{\mathcal{P}} = (\kappa + [h])|_{\mathcal{P}}, \tag{e 0.4}$$

where $N_1 = \sum_{i=1}^r (N(\delta, \mathcal{G}, \mathcal{P}, i) \pm \Lambda_i) \cdot |S_i.|$

Proof: Let $\psi_i^+ : G \to \mathbb{Z}$ be a homomorphism defined by $\psi_i^+(g_i) = 1$, $\psi_i^+(g_j) = 0$, if $j \neq i$, and $\psi_i^+|_{\mathrm{Tor}(G)} = 0$, and let $\psi_i^-(g_i) = -1$ and $\psi_i^-(g_j) = 0$, if $j \neq i$, and $\psi_i^-|_{\mathrm{Tor}(G)} = 0$, $i = 1, 2, ..., r$. Note that $\psi_i^- = -\psi_i^+$, $i = 1, 2, ..., r$. Let $\Lambda_i = |\psi_i^+([1_A])|$, $i = 1, 2, ..., r$.
Let $\kappa_i^+, \kappa_i^- \in Hom_\Lambda(\underline{K}(A), \underline{K}(\mathcal{K})$ be such that $\kappa_i^+|_G = \psi_i^+$ and $\kappa_i^- = \psi_i^-$, $i = 1, 2, ..., r$. Let $N_0(i) \geq 1$ (in place of N_0) be required by **??** for δ, $\mathcal{G}$, $J_0 = 1$ and $J_1 = M_i$. Define $N(\delta, \mathcal{G}, \mathcal{P}, i) = N_0(i)$, $i = 1, 2, ..., r$.
Let $\kappa \in Hom_\Lambda(\underline{K}(A), \underline{K}(\mathcal{K}))$. Then $\kappa|_G = \sum_{i=1}^r S_i \psi_i^+$, where $S_i = \kappa(g_i)$, $i = 1, 2, ..., r$.
By applying **4.1**, one obtains $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps $L_i^\pm : A \to M_{N_0(i) + \kappa_i^\pm([1_A])}$ and a homomorphism $h_i^\pm : A \to M_{N_0(i)}$ such that

$$[L_i^\pm]|_{\mathcal{P}} = (\kappa_i^\pm + [h_i^\pm])|_{\mathcal{P}}, \quad i = 1, 2, ..., r. \qquad \text{(e 0.5)}$$

Define $L = \sum_{i=1}^r L_i^{\pm,|S_i|}$, where $L^{\pm,|S_i|} : A \to M_{|S_i|N_0(i)}$ defined by

$$L^{\pm,|S_i|}(a) = \mathrm{diag}(\overbrace{L_i^{\pm}(a), ..., L_i^{\pm}(a)}^{|S_i|})$$

for all $a \in A$. One checks that $L : A \to M_{N_1}$, where $N_1 = \sum_{i=1}^r |S_i|(\Lambda_i' + N(\delta, \mathcal{G}, \mathcal{P}, i)$ and $\Lambda_i' = \psi_i^+([1_A])$, if $S_i > 0$, or $\Lambda_i' = -\psi_+^+([1_A])$, if $S_i < 0$, is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map and

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Examples: $\mathcal{C} \subset \mathcal{D}_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$.). Let $d \geq 1$, $X = \mathbb{I}^d$, the d-dimensional disk, $Z = \partial(X)$, F be a finite dimensional C^*-algebra and let $B \in \mathcal{D}_{d-1}$. Suppose that $\Gamma : B \to C(\partial(X), F)$ is a unital homomorphism. Define

$$A = \{(f, b) \in C(X, F) \oplus B : f|_{\partial(X)} = \Gamma(b)\}.$$

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All theorems stated for $PM_r(C(X))P$ so far works for C^*-algebras in $\mathcal{A}_d$ for all $d \geq 1$. **(Gong-L-Niu)**

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Lemma 4.3.
Let $A \in \mathcal{D}_s$ be a unital C^*-algebra and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be a positive map. For any $\epsilon > 0$ and any finite subset $\mathcal{F}$, there exist a finite subset $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$ and an integer $L \geq 1$ satisfying the following: For any unital homomorphism $\phi : A \to M_k$ and any unital homomorphism $\psi : A \to M_R$ for some $R \geq Lk$ such that

$$\mathrm{tr} \circ \psi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}, \tag{e 0.9}$$

there exist a unital homomorphism $\phi_0 : A \to M_{R-k}$ and a unitary $u \in M_R$ such that

$$\|\mathrm{Ad}\, u \circ \mathrm{diag}(\phi(f), \phi_0(f)) - \psi(f)\| < \epsilon \tag{e 0.10}$$

for all $f \in \mathcal{F}$.

Lemma **4.4.**

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$$\mathrm{Bott}(\phi,\, u)|_{\mathcal{P}} = \kappa \circ \boldsymbol{\beta}|_{\mathcal{P}}. \tag{e 0.13}$$

Proof: To simplify notation, without loss of generality, we may assume that $\mathcal{F}$ is a subset of the unit ball. Let $\Delta_1 = (1/8)\Delta$ and $\Delta_2 = (1/16)\Delta$. Let $\epsilon_0 > 0$ and $\mathcal{G}_0 \subset A$ be a finite subset satisfy the following: If $\phi' : A \to B$ (for any unital C^*-algebra B) is a unital ϵ_0-$\mathcal{G}_0$-multiplicative contractive completely positive linear map and $u' \in B$ is a unitary such that

$$\|\phi'(g)u' - u'\phi'(g)\| < 4\epsilon_0 \text{ for all } g \in \mathcal{G}_0, \tag{e 0.14}$$

then $\mathrm{Bott}(\phi', u')|_{\mathcal{P}}$ is well defined. Moreover, if $\phi' : A \to B$ is another unital ϵ_0-$\mathcal{G}_0$-multiplicative contractive completely positive linear map then

$$\mathrm{Bott}(\phi', u')|_{\mathcal{P}} = \mathrm{Bott}(\phi'', u'')|_{\mathcal{P}}, \tag{e 0.15}$$

provided that

$$\|\phi'(g) - \phi''(g)\| < 4\epsilon_0 \text{ and } \|u' - u''\| < 4\epsilon_0 \text{ for all } g \in \mathcal{G}_0. \tag{e 0.16}$$

We may assume that $1_A \in \mathcal{G}_0$. Let

$$\mathcal{G}_0' = \{g \otimes f : g \in \mathcal{G}_0 \text{ and } f = \{1_{C(\mathbb{T})}, z, z^*\}.$$

where z is the identity function on the unit circle $\mathbb{T}$. We also assume that if $\Psi' : A \otimes C(\mathbb{T}) \to C$ (to some unital C^*-algebra C) is a $\mathcal{G}_0'$-ϵ_0-multiplicative contractive completely positive linear map, then there exist a unitary $u' \in C$ such that

$$\|\Psi'(1 \otimes z) - u'\| < 4\epsilon_0. \qquad \text{(e 0.17)}$$

Without loss of generality, we may assume that $\mathcal{G}_0$ is in the unital ball of A. Let $\epsilon_1 = \min\{\epsilon/64, \epsilon_0/512)$ and $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{G}_0$.
Let $\mathcal{H}_0 \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}$) be a finite subset and $L \geq 1$ be an integer required by **4.3** for ϵ_1 (in place of ϵ) and $\mathcal{F}_1$ (in place of $\mathcal{F}$) as well as Δ_2 (in place of Δ).
Let $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$ be finite subsets, $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$) be a finite subset, $\delta_1 > 0$ (in place of δ), $\mathcal{P}_1 \subset \underline{K}(A)$ (in place of $\mathcal{P}$) be a finite subset, $\mathcal{H}_2 \subset A_{s.a.}$ be a finite subset and $1 > \sigma > 0$ be required by **??** for ϵ_1 (in place of ϵ), $\mathcal{F}_1$ (in place of $\mathcal{F}$) and Δ_1. We may assume that $[1_A] \in \mathcal{P}_2$, $\mathcal{H}_2$ is in the unit ball of A and $\mathcal{H}_0 \subset \mathcal{H}_1$.

Without loss of generality, we may assume that $\delta_1, \sigma < \epsilon_1/16$ and $\mathcal{F}_1 \subset \mathcal{G}_1$. Let $\mathcal{P}_2 = \mathcal{P} \cup \mathcal{P}_1$.
Suppose that A has irreducible representations of rank $r_1, r_2, ..., r_k$. Fix one irreducible representation $\pi_0 : A \to M_{r_1}$. Let $N(p) \geq 1$ (in place of $N(\mathcal{P}_0)$) and $\mathcal{H}_0 \subset A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}$) be a finite subset required by **??** for $\{1_A\}$ (in place of $\mathcal{P}_0$) and $(1/3)\Delta$.
Let $G_0 = G \cap K_0(A)$ and write $G_0 = \mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2} \oplus \mathrm{Tor}(G_0)$, where $\mathbb{Z}^{s_2} \oplus \mathrm{Tor}(G_0) \subset \ker \rho_A$. Let $x_j = (\overbrace{0, ..., 0}^{j-1}, 1, 0, ..., 0) \in \mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2}$, $j = 1, 2, ..., s_2$. Note that $A \otimes C(\mathbb{T}) \in \mathcal{A}_s$ and $A \otimes C(\mathbb{T})$ has irreducible representations of rank $r_1, r_2, ..., r_k$. Let

$$\bar{r} = \max\{|(\pi_0)_{*0}(x_j)| : 0 \leq j \leq s_1 + s_2\}.$$

Let $\mathcal{P}_3 \subset \underline{K}(A \otimes C(\mathbb{T}))$ be a finite subset set containing $\mathcal{P}_2$, $\{\beta(g_j) : 1 \leq j \leq r\}$ and a finite subset which generates $\beta(\mathrm{Tor}(G_1))$.
Choose $\delta_2 > 0$ and finite subset

$$\overline{\mathcal{G}} = \{g \otimes f : g \in \mathcal{G}_2, \ \ f \in \{1, z, z^*\}\}$$

in $A \otimes C(\mathbb{T})$, where $\mathcal{G}_2 \subset A$ is a finite subset such that, for any unital δ_2-$\overline{\mathcal{G}}$-multiplicative contractive completely positive linear map $\Phi' : A \otimes C(\mathbb{T}) \to C$ (for any unital C^*-algebra C with $T(C) \neq \emptyset$), $[\Phi']|_{\mathcal{P}_3}$ is well defined and

$$[\Phi']|_{\mathrm{Tor}(G_0) \oplus \boldsymbol{\beta}(\mathrm{Tor}(G_1)} = 0. \tag{e 0.18}$$

We may assume $\mathcal{G}_2 \supset \mathcal{G}_1 \cup \mathcal{F}_1$.
Let $\sigma_1 = \min\{\Delta_2(\hat{h}) : h \in \mathcal{H}_1\}$. Note $K_0(A \otimes C(\mathbb{T})) = K_0(A) \oplus \boldsymbol{\beta}(K_1(A))$ and $\underline{K}(A \otimes C(\mathbb{T})) = \underline{K}(A) \oplus \boldsymbol{\beta}(\underline{K}(A))$. Consider the subgroup of $K_0(A \otimes C(\mathbb{T}))$:

$$\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2} \oplus \mathbb{Z}^r \oplus \mathrm{Tor}(K_0(A) \oplus \boldsymbol{\beta}(\mathrm{Tor}(K_1(A)).$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. Let $N(\delta_3, \overline{\mathcal{G}}, \mathcal{P}_3, i)$ and Λ_i, $i = 1, 2, ..., s_1 + s_2 + r$, be required by **4.2.** (for $A \otimes C(\mathbb{T})$). Choose an integer $n_1 \geq N(p)$ such that

$$\frac{(\sum_{i=1}^{s_1+s_2+r} N(\delta_3, \overline{\mathcal{G}}, \mathcal{P}_3, i) + 1 + \Lambda_i)N(p)}{n_1 - 1} < \min\{\sigma/16, \sigma_1/2\}. \tag{e 0.19}$$

Choose $n > n_1$ such that

$$\frac{n_1+2}{n} < \min\{\sigma/16, \sigma_1/2, 1/(L+1)\}. \tag{e 0.20}$$

Let $\epsilon_2 > 0$ and let $\mathcal{F}_2 \subset A$ be a finite subset such that $[\Psi]|_{\mathcal{P}_2}$ is well defined.
Let $\epsilon_3 = \min\{\epsilon_2/2, \epsilon_1\}$ and $\mathcal{F}_3 = \mathcal{F}_1 \cup \mathcal{F}_2$.
Let $\delta_4 > 0$ (in place of δ), $\mathcal{G}_3 \subset A$ (in place of $\mathcal{G}$) be a finite subset and let $\mathcal{H}_3 \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}_2$) required by Cor. 2.5 for ϵ_3 (in place of ϵ), $\mathcal{F}_3 \cup \mathcal{H}_1$ (in place of $\mathcal{F}$), $\delta_3/2$ (in place of ϵ_0), $\mathcal{G}_2$ (in place of $\mathcal{G}_0$), Δ, $\mathcal{H}_1$ (in place of $\mathcal{H}$), $\min\{\sigma/16, \sigma_1/2\}$ (in place of σ) and n^2 (in place of K) required by Cor. 2.5 (with $L_1 = L_2$).
Let $\mathcal{G} = \mathcal{F}_3 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ and let $\delta = \min\{\epsilon_3/16, \delta_4, \delta_3/16\}$. Let $\mathcal{G}_5 = \{g \otimes f : g \in \mathcal{G}_4,\ \ f \in \{1, z, z^*\}\}$.
Let $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_0$. Define
$N_0 = (n+1)N(p)(\sum_{i=1}^{s_1+s_2+r} N(\delta_3, \mathcal{G}_0, \mathcal{P}_3, i) + \Lambda_i + 1)$ and define $N = N_0 + N_0\bar{r}$. Fix any $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ with

$$K = \max\{|\kappa(\beta(g_i)| : 1 \le j \le r\}.$$

Let $R > N(K+1)$. Suppose that $\phi : A \to M_R$ is a unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear map such that

$$\mathrm{tr} \circ \phi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}. \tag{e 0.21}$$

Then, by Cor. 2.5, there exists mutually orthogonal projections $e_0, e_1, e_2, ..., e_n \in M_R$ such that $e_1, e_2, ..., e_n$ are equivalent, $\mathrm{tr}(e_0) < \min\{\sigma/64, \sigma_1/4\}$ and $e_0 + \sum_{i=1}^n e_i = 1_{M_R}$, and there exists a unital $\delta_3/2$-$\mathcal{G}_2$-multiplicative contractive completely positive linear map $\psi_0 : A \to e_0 M_R e_0$ and a unital homomorphism $\psi : A \to e_1 M_R e_1$ such that

$$\|\phi(f) - (\psi_0(f) \oplus \overbrace{\psi(f), \psi(f), ..., \psi(f)}^{n^2})\| < \epsilon_3 \text{ for all } f \in \mathcal{F}_3 \text{ and} \tag{e 0.22}$$

$$\mathrm{tr} \circ \psi(h) \geq \Delta(\hat{h})/3n \text{ for all } h \in \mathcal{H}_1. \tag{e 0.23}$$

Let $\alpha \in Hom_\Lambda(\underline{K}(A \otimes C(\mathbb{T})), \underline{K}(M_r))$ be define as follows: $\alpha|_{\underline{K}(A)} = [\pi_0]$ and $\alpha|_{\boldsymbol{\beta}(\underline{K}(A))} = \kappa|_{\boldsymbol{\beta}(\underline{K}(A))}$. Let

$$\max\{|\kappa \circ \boldsymbol{\beta}(g_i)| : i = 1, 2, ..., r, |\pi_0(x_j)| : 1 \leq j \leq s_1 + s_2\} \leq \max\{K, \bar{r}\}.$$

Applying we obtain a unital δ_3-$\mathcal{G}$-multiplicative contractive completely positive linear map $\Psi : A \otimes C(\mathbb{T}) \to M_{N_1'}$, where

$N_1' \le N_1 = \sum_{j=1}^{s_1+s_2+r} N(\delta_3, \mathcal{G}_0, \mathcal{P}_3, j) + \Lambda_i)\max\{K, \bar{r}\}$, and a homomorphism $H_0 : A \otimes C(\mathbb{T}) \to H_0(1_A)M_{N_1'}H_0(1_A)$ such that such that

$$[\Psi]|_{\mathcal{P}_3} = (\alpha + [H_0])|_{\mathcal{P}_3}. \tag{e 0.24}$$

In particular, since $[1_A] \in \mathcal{P}_2 \subset \mathcal{P}_3$,

$$\mathrm{rank}\Psi(1_A) = r_1 + \mathrm{rank}(H_0).$$

Note that

$$\frac{N_1' + N(p)}{R} \le \frac{N_1 + N(p)}{N(K+1)} < 1/(n+1). \tag{e 0.25}$$

Let $R_1 = \mathrm{rank}\, e_1$. Then $R_1 \ge R/(n+1)$. So, from (e 0.25) $R_1 \ge N_1 + N(p)$. In other words, $R_1 - N_1' \ge N(p)$. Note that

$$t \circ \psi(\hat{g}) \ge (1/3)\Delta(\hat{g}) \text{ for all } g \in \mathcal{H}_0,$$

where t is the tracial state on M_{R_1}. By applying to the case that $\phi = \pi_0 \oplus H_0$ and $\mathcal{P}_0 = \{[1_A]\}$, we obtain a unital homomorphism

$h_0 : A \otimes C(\mathbb{T}) \to M_{nR_1 - N_1'}$. Define $\psi_0' : A \otimes C(\mathbb{T}) \to e_0 M_R e_0$ by $\psi_0'(a \otimes f) = \psi_0(a) \cdot f(1) \cdot e_0$ for all $a \in A$ and $f \in C(\mathbb{T})$, where $1 \in \mathbb{T}$. Define $\psi' : A \otimes C(\mathbb{T}) \to e_1 M_R e_1$ by $\psi'(a \otimes f) = \psi(a) \cdot f(1) \cdot e_0$ for all $a \in A$ and $f \in C(\mathbb{T})$. Let $E_1 = \mathrm{diag}(e_1, e_2, ..., e_{nn_1})$.
Define $L_1 : A \to E_1 M_R E_1$ by

$L_1(a) = \pi_0(a) \oplus H_0|_A(a) \oplus h_0(a \otimes 1) \oplus (\overbrace{\psi(f), ..., \psi(f)}^{n(n_1-1)})$ for $a \in A$ and define $L_2 : A \to E_1 M_R E_1$ by

$L_2(a) = \Psi(a \otimes 1) \oplus h_0(a \otimes 1) \oplus (\overbrace{\psi(f), ..., \psi(f)}^{n(n_1-1)})$ for $a \in A$. Note that

$$[L_1]|_{\mathcal{P}_1} = [L_2]|_{\mathcal{P}_1} \qquad \text{(e 0.26)}$$

$$\mathrm{tr} \circ L_1(h) \geq \Delta_1(\hat{h}), \ \ \mathrm{tr} \circ L_2(h) \geq \Delta_1(\hat{h}) \ \text{ for all } h \in \mathcal{H}_1 \ \text{ and} \qquad \text{(e 0.27)}$$

$$|\mathrm{tr} \circ L_1(g) - \mathrm{tr} \circ L_2(g)| < \sigma \ \text{ for all } g \in \mathcal{H}_2. \qquad \text{(e 0.28)}$$

It follows from **??** that there exists a unitary $w_1 \in E_1 M_R E_1$ such that

$$\|\mathrm{ad}\, w_1 \circ L_2(a) - L_1(a)\| < \epsilon_1 \ \text{ for all } a \in \mathcal{F}_1. \qquad \text{(e 0.29)}$$

Define $E_2 = (e_1 + e_2 + \cdots + e_{n^2})$ and define $\Phi : A \to E_2 M_R E_2$ by

$$\Phi(f)(a) = \mathrm{diag}(\overbrace{\psi(a), \psi(a), ..., \psi(a)}^{n^2}) \text{ for all } a \in A. \tag{e 0.30}$$

Then

$$\mathrm{tr} \circ \Phi(h) \geq \Delta_2(\hat{h}) \text{ for all } h \in \mathcal{H}_0 \tag{e 0.31}$$

By (e 0.20), $\frac{n}{n_1+2} > L + 1$. By applying **4.3**, we obtain a unitary $w_2 \in E_2 M_R E_2$ and a unital homomorphism $H_1 : A \to (E_2 - E_1) M_R (E_2 - E_1)$ such that

$$\|\mathrm{ad}\, w_2 \circ \mathrm{diag}(L_1(a), H_1(a)) - \Phi(a)\| < \epsilon_1 \text{ for all } a \in \mathcal{F}_1. \tag{e 0.32}$$

Put

$$w = (e_0 \oplus w_1 \oplus (E_2 - E_1))(e_0 \oplus w_2) \in M_R.$$

Define $H_1' : A \otimes C(\mathbb{T}) \to (E_2 - E_1) M_R (E_2 - E_1)$ by $H_1'(a \otimes f) = H_1(a) \cdot f(1) \cdot (E_2 - E_1)$ for all $a \in A$ and $f \in C(\mathbb{T})$. Define

$\Psi_1 : A \to M_R$ by

$$\Psi_1(f) = \psi_0'(f) \oplus \Psi(f) \oplus h_0 \oplus \overbrace{\psi'(f), ..., \psi'(f)}^{n_1-1}) \oplus H_1'(f) \qquad (e\,0.33)$$

for all $f \in A \otimes C(\mathbb{T})$. It follows from (e 0.29), (e 0.32) and (e 0.22) that

$$\|\phi(a) - w^*\Psi_1(a \otimes 1)w\| < \epsilon_1 + \epsilon_1 + \epsilon_3 \text{ for all } a \in \mathcal{F}. \qquad (e\,0.34)$$

Now let $v \in M_R$ be a unitary such that

$$\|\Psi_1(1 \otimes z) - v\| < 4\epsilon_1. \qquad (e\,0.35)$$

Put $u = w^* v w$. Then, we estimate that

$$\|[\phi(a), u]\| < \min\{\epsilon, \epsilon_0) \text{ for all } a \in \mathcal{F}_1. \qquad (e\,0.36)$$

Moreover, by (e 0.29),(e 0.24) and (e 0.15),

$$\mathrm{Bott}(\phi, u)|_{\mathcal{P}} = \kappa \circ \boldsymbol{\beta}|_{\mathcal{P}}. \qquad (e\,0.37)$$

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$$\|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon, \ \ \text{for all } f \in \mathcal{F}. \tag{e 0.42}$$

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Proof:

Write $F_1 = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$ and $F_2 = M_{r_1} \oplus M_{r_2} \oplus \cdots \oplus M_{r_l}$. We may assume that, $\mathrm{ker}\phi_0 \cap \mathrm{ker}\phi_1 = \{0\}$.
We may assume that $\phi_0|_{M_{n_i}}$ is injective, $i = 1, 2, ..., k(0)$ with $k(0) \le k$, $\phi_0|_{M_{n_i}} = 0$ if $i > k(0)$, and $\phi_1|_{M_{n_i}}$ is injective, $i = k(1), k(1)+1, ..., k$ with $k(1) \le k$, $\phi_1|_{M_{n_i}} = 0$, if $i < k(1)$. Write $F_{1,0} = \bigoplus_{i=1}^{k(0)} M_{n_i}$ and $F_{1,1} = \bigoplus_{j=k(1)}^{k} M_{n_j}$. Note that $k(1) \le k(0) + 1$, $\phi_0|_{F_{1,0}}$ and $\phi_1|_{F_{1,1}}$ are injective. Note $\phi_0(F_{1,0}) = \phi_0(F_1) = \pi_0(C)$ and $\phi_1(F_{1,1}) = \phi_1(F_1) = \pi_1(C)$. Let $\psi_0 = (\phi_0|_{F_{1,0}})^{-1}$ and $\psi_1 = (\phi_1|_{F_{1,1}})^{-1}$.
For each fixed $a \in A$,, since $L_i(a) \in C$ $(i = 0, 1)$, there are elements

$$g_{a,i} = g_{a,i,1} \oplus g_{a,i,2} \oplus \cdots \oplus g_{a,i,k(0)} \oplus \cdots \oplus g_{a,i,k} \in F_1,$$

such that $\phi_0(g_{a,i}) = \pi_0 \circ L_i(a)$ and $\phi_1(g_{a,i}) = \pi_1 \circ L_i(a)$, $i = 1, 2.$, where $g_{a,i,j} \in M_{n_j}$, $j = 1, 2, ..., k$ and $i = 1, 2$. Note that such $g_{a,i}$ is unique since $\mathrm{ker}\phi_0 \cap \mathrm{ker}\phi_1 = \{0\}$. Since $w_0 \in \pi_0(C) = \phi_0(F_1)$, there is a unitary

$$u_0 = u_{0,1} \oplus u_{0,2} \oplus \cdots \oplus u_{0,k(0)} \oplus \cdots \oplus u_{0,k}$$

such that $\phi_0(u_0) = w_0$.

Note that the first $k(0)$ components of u_0 is uniquely determined by w_0 (since ϕ_0 is injective on this part) and the components after $k(0)$'s components can be chosen arbitrarily (since $\phi_0 = 0$ on this part). Similarly there exist

$$u_1 = u_{1,1} \oplus u_{1,2} \oplus \cdots \oplus u_{1,k(1)} \oplus \cdots \oplus u_{1,k}$$

such that $\phi_1(u_1) = w_1$
Now by e 0.47 and e 0.48, we have

$$\begin{aligned} \|\phi_0(u_0)^*\phi_0(g_{a,1})\phi_0(u_0) - \phi_0(g_{a,2})\| &< \epsilon \text{ and} && \text{(e 0.51)} \\ \|\phi_1(u_1)^*\phi_1(g_{a,1})\phi_1(u_1) - \phi_1(g_{a,2}))\| &< \epsilon \text{ for all } a \in \mathcal{F}. && \text{(e 0.52)} \end{aligned}$$

Since ϕ_0 is injective on F_1^i for $i \le k(0)$ and ϕ_1 is injective on F_1^i for $i > k(0)$ (note that we use $k(1) \le k(0) + 1$), we have

$$\|(u_{0,i})^*(g_{a,1,i})u_{0,i} - (g_{a,2,i})\| < \epsilon \ \forall \ i \le k(0) \text{ and} \qquad \text{(e 0.53)}$$

$$\|(u_{1,i})^*(g_{a,1,i})u_{1,i} - (g_{a,2,i})\| < \epsilon \ \forall \ i > k(0) \qquad \text{(e 0.54)}$$

for all $a \in \mathcal{F}$.

Let $u = u_{0,1} \oplus \cdots \oplus u_{0,k(0)} \oplus u_{1,k(0)+1} \oplus \cdots \oplus u_{1,k} \in F_1$—that is for the first $k(0)$'s components of u, we use u_0's corresponding components, and for the last $k - k(0)$ components of u, we use u_1's. From e 0.53 and e 0.53. we have

$$\|u^* g_{a,1} u - g_{a,2}\| < \epsilon \quad \text{for all } a \in \mathcal{F}.$$

Apply ϕ_0 and ϕ_1 to the above inequality, we get e 0.49 and e 0.50 as desired.

Proof of Theorem 4.5. There is n_0 such that $n_0 x = 0$ for all $x \in K_i(A \otimes C(\mathbb{T}))$, $i = 0, 1$. Set $N = n_0!$. Put Δ_1 be defined above for the given Δ.

Let $\mathcal{H}_1' \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$) for $\epsilon/32$ (in place of ϵ) and $\mathcal{F}$ required by **3.5**.

Let $\delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$) be a finite subset and let $\mathcal{P}_0 \subset \underline{K}(A)$ (in place of $\mathcal{P}$) be a finite subset required by **3.5** for $\epsilon/32$ (in place of ϵ), $\mathcal{F}$ and Δ_1. We may assume that $\delta_1 < \epsilon/32$ and $(2\delta_1, \mathcal{G}_1)$ is a KK-pair.

Moreover, we may assume that δ_1 is so small that if $\|uv - vu\| < 3\delta_1$, then the Exel formula

$$\tau(\mathrm{bott}_1(u, v)) = \frac{1}{2\pi\sqrt{-1}}(\tau(\log(u^* vuv^*))$$

holds for any pair of unitaries u and v in any unital C^*-algebra C with tracial rank zero and any $\tau \in T(C)$ (see Theorem 3.6 of [**?**]). Moreover if $\|v_1 - v_2\| < 3\delta_1$, then

$$\mathrm{bott}_1(u, v_1) = \mathrm{bott}_1(u, v_2).$$

Let $g_1, g_2, ..., g_{k(A)} \in U(M_{m(A)}(A))$ ($m(A) \geq 1$ is an integer) be a finite subset such that $\{\bar{g_1}, \bar{g_2}, ..., \bar{g}_{k(A)}\} \subset J_c(K_1(A))$ and such that $\{[g_1], [g_2], ..., [g_{k(A)}]\}$ forms a set of generators for $K_1(A)$. Let $\mathcal{U} = \{\bar{g_1}, \bar{g_2}, ..., \bar{g}_{k(A)}\} \subset J_c(K_1(A))$ be a finite subset.
Let $\mathcal{U}_0 \subset A$ be a finite subset such that

$$\{g_1, g_2, ..., g_{k(A)}\} = \{(a_{i,j}) : a_{i,j} \in \mathcal{U}_0\}.$$

Let $\delta_u = \min\{1/256m(A)^2, \delta_1/16m(A)^2\}$, $\mathcal{G}_u = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{U}_0$ and let $\mathcal{P}_u = \mathcal{P}_0$.
Let $\delta_2 > 0$ (in place of δ), let $\mathcal{G}_2 \subset A$ (in place of $\mathcal{G}$) and let $\mathcal{H}_2' \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}$) and let $N_1 \geq 1$ (in place of N) be an integer required by **4.4** for δ_u (in place of ϵ), $\mathcal{G}_u$ (in place of $\mathcal{F}$), $\mathcal{P}_u$ (in place of $\mathcal{P}$) and Δ and with $\bar{g}_j$ (in place of g_j), $j = 1, 2, ..., k(A)$ (with $k(A) = r$).
Let $d = \min\{\Delta(\hat{h}) : h \in \mathcal{H}_2'\}$. Let $\delta_3 > 0$ and let $\mathcal{G}_3 \subset A \otimes C(\mathbb{T})$ be finite subset satisfying the following: For any δ_3-$\mathcal{G}_3$-multiplicative contractive completely positive linear map $L' : A \otimes C(\mathbb{T}) \to C'$ (for any unital C^*-algebra C' with $T(C') \neq \emptyset$),

$$|\tau([L](\beta(\bar{g}_j))| < d/8, \ \ j = 1, 2, ..., k(A). \tag{e 0.55}$$

Without loss of generality, we may assume that

$$\mathcal{G}_3 = \{g \otimes z : g \in \mathcal{G}_3' \text{ and } z \in \{1, z, z^*\}\},$$

where $\mathcal{G}_3' \subset A$ is a finite subset (by choosing a smaller δ_3 and large $\mathcal{G}_3'$). Let $\epsilon_1'' = \min\{d/27m(A)^2, \delta_u/2, \delta_2/2m(A)^2, \delta_3/2m(A)^2\}$ and let $\bar{\epsilon}_1 > 0$ (in place of δ) and $\mathcal{G}_4 \subset A$ (in place of $\mathcal{G}$) be a finite subset required by **??** for ϵ_1'' (in place of ϵ) and $\mathcal{G}_u \cup \mathcal{G}_3'$. Put

$$\epsilon_1 = \min\{\epsilon_1', \epsilon_1'', \bar{\epsilon}_1\}.$$

Let $\mathcal{G}_5 = \mathcal{G}_u \cup \mathcal{G}_3' \cup \mathcal{G}_4$.
Let $\mathcal{H}_3' \subseteq A^+$ (in place of $\mathcal{H}_1$), $\delta_4 > 0$ (in place of δ), $\mathcal{G}_6 \subset A$ (in place of $\mathcal{G}$), $\mathcal{H}_4' \subset A_{s.a.}$ (in place of $\mathcal{H}_2$), $\mathcal{P}_1 \subset \underline{K}(A)$ (in place of $\mathcal{P}$) and $\sigma_4 > 0$ (in place of σ_2) be the finite subsetc and constants required by Theorem 2.1 $\epsilon_1/4$ (in place ϵ) and $\mathcal{G}_5$ (in place of $\mathcal{F}$) and Δ.
Let $N_2 \geq N_1$ such that $(k(A)+1)/N_2 < d/8$. Choose $\mathcal{H}_5' \subset A_+ \setminus \{0\}$ and $\delta_5 > 0$ and a finite subset $\mathcal{G}_7 \subset A$ such that, for any M_m and unital δ_5-$\mathcal{G}_7$-multiplicative contractive completely positive linear map $L' : A \to M_m$, if $\mathrm{tr} \circ L'(h) > 0$ for all $h \in \mathcal{H}_5'$, then $m \geq N_2((8/d)+1)$.

Let $\delta = \min\{\epsilon_1/16, \delta_4/4m(A)^2, \delta_5/4m(A)^2\}$, let $\mathcal{G} = \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7$ and let $\mathcal{P} = \mathcal{P}_u \cup \mathcal{P}_1$. Let

$$\mathcal{H}_1 = \mathcal{H}_1' \cup \mathcal{H}_2' \cup \mathcal{H}_3' \cup \mathcal{H}_4' \cup \mathcal{H}_6'$$

and let $\mathcal{H}_2 = \mathcal{H}_4'$. Let $\gamma_1 = \sigma_4$ and let
$0 < \gamma_2 < \min\{d/16m(A)^2, \delta_u/9m(A)^2, 1/256m(A)^2\}$.
Now suppose that $C \in \mathcal{C}$ and $\phi, \psi : A \to C$ be two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps satisfying the assumption for the above given Δ, $\mathcal{H}_1$, δ, $\mathcal{G}$, $\mathcal{P}$, $\mathcal{H}_2$, γ_1, γ_2 and $\mathcal{U}$.
Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition so that

$$\|\pi_t \circ \phi(g) - \pi_{t'} \circ \phi(g)\| < \epsilon_1/16 \text{ and} \tag{e 0.56}$$

$$\|\pi_t \circ \psi(g) - \pi_{t'} \circ \psi(g)\| < \epsilon_1/16 \tag{e 0.57}$$

for all $g \in \mathcal{G}$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, ..., n$.
We write $C = A(F_1, F_2, h_0, h_1)$, $F_1 = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_{F(1)}}$ and $F_2 = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_{F(2)}}$. By the choice of $\mathcal{H}_5'$,

$$n_j \geq N_2(8/d + 1) \text{ and } m_s \geq N_2(8/d + 1), \tag{e 0.58}$$

$1 \le j \le F(2), \;\; 1 \le s \le F(1)$. By applying Theorem 2.1,there exists a unitary $w_i \in F_2$, if $0 < i < n$, $w_0 \in h_0(F_1)$, if $i = 0$, and $w_1 \in h_1(F_1)$, if $i = 1$, such that

$$\|w_i \pi_{t_i} \circ \phi(g) w_i^* - \pi_{t_i} \circ \psi(g)\| < \epsilon_1/16 \text{ for all } g \in \mathcal{G}_5. \qquad (\mathrm{e}\,0.59)$$

It follows from 0.8 that we may assume that there is a unitary $w_e \in F_1$ such that $h_0(w_e) = w_0$ and $h_1(w_e) = w_n$.
By (e 0.41), let $\omega_j \in M_{m(A)}(C)$ be a unitary such that
$\omega_j \in CU(M_{m(A)}(C))$ and

$$\|\langle(\phi \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rangle\langle(\psi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle - \omega_j\| < \gamma_2, \;\; j = 1, 2, ..., k(A).$$

Write

$$\omega_j = \prod_{l=1}^{e(j)} \exp(\sqrt{-1} a_j^{(l)})$$

for some selfadjoint element $a_j^{(l)} \in M_{m(A)}(C)$, $l = 1, 2, ..., e(j)$, $j = 1, 2, ..., k(A)$. Write

$$a_j^{(l)} = (a_j^{(l,1)}, a_j^{(l,2)}, ..., a_j^{(l, n_{F(2)})}) \text{ and } \omega_j = (\omega_{j,1}, \omega_{j,2}, ..., \omega_{j,F(2)})$$

in $C([0,1],F_2)=C([0,1],M_{n_1})\oplus\cdots\oplus C([0,1],M_{n_{F(2)}})$, where $\omega_{j,s}=\exp(\sqrt{-1}a_j^{(l,s)})$, $s=1,2,...,F(2)$. Then

$$\sum_{l=1}^{e(j)}\frac{n_s(t_s\otimes \mathrm{Tr}_{m(A)})(a_j^{(l,s)}(t))}{2\pi}\in\mathbb{Z},\quad t\in[0,1],$$

where t_s is the normalized trace on M_{n_s}, $s=1,2,...,F(2)$. In particular,

$$\sum_{l=1}^{e(j)} n_s(t\otimes \mathrm{Tr}_{m(A)})(a_j^{(l,s)}(t))=\sum_{l=1}^{e(j)} n_s(t\otimes \mathrm{Tr}_{m(A)})(a_j^{(l,s)}(t')) \quad (e\,0.60)$$

for all $t,t''\in[0,1]$.

Let $W_i=w_i\otimes \mathrm{id}_{M_{m(A)}}$, $i=0,1,....,n$ and $W_e=w_e\otimes \mathrm{id}_{M_{m(F_1)}}$. Then

$$\|\pi_i(\langle\phi\otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rangle)W_i(\pi_i(\langle\phi\otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle)W_i^*-\omega_j(t_i)\| \quad (e\,0.61)$$

$$<3m(A)^2\epsilon_1+2\gamma_2<1/32. \quad (e\,0.62)$$

We also have

$$\|\langle\phi_e\otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rangle W_e(\langle\phi_e\otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle)W_e^* \quad (e\,0.63)$$

$$-\pi_e(\omega_j)\|<3m(A)^2\epsilon_1+2\gamma_2<1/32. \quad (e\,0.64)$$

It follows from (e 0.61) that there exists selfadjoint elements $b_{i,j} \in M_{m(A)}(F_2)$ such that

$$\exp(\sqrt{-1} b_{i,j}) = \quad \text{(e 0.65)}$$
$$\omega_j(t_i)^*(\pi_i \langle \phi \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rangle) W_i (\pi_i(\langle \phi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle) W_i^*, \quad \text{(e 0.66)}$$

and $b_{e,j} \in M_{m(A)}(F_1)$ such that

$$\exp(\sqrt{-1} b_{e,j}) = \quad \text{(e 0.67)}$$
$$\pi_e(\omega_j)^*(\pi_e \langle \phi \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rangle) W_e (\pi_e(\langle \phi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle) W_e^*, \quad \text{(e 0.68)}$$

and

$$\|b_{i,j}\| < 2 \arcsin(3m(A)^2 \epsilon_1/4 + 2\gamma_2), \ \ j = 1, 2, ..., k(A), \quad \text{(e 0.69)}$$

$i = 0, 1, ..., n, e.$
We write

$$\begin{aligned} b_{i,j} &= (b_{i,j}^{(1)}, b_{i,j}^{(2)}, ..., b_{i,j}^{F(2)}) \in F_2 \ \text{ and} \\ b_{e,j} &= (b_{e,j}^{(1)}, b_{e,j}^{(2)}, ..., b_{e,j}^{(F(1))}) \in F_1. \quad \text{(e 0.70)} \end{aligned}$$

We also have that

$$h_0(b_{e,j}) = b_{0,j} \text{ and } h_1(b_{e,j}) = b_{n,j}. \tag{e 0.71}$$

Note that

$$(\pi_i(\langle \phi \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rangle))W_i(\pi_i(\langle \phi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle)W_i^* \tag{e 0.72}$$

$$= \pi_i(\omega_j)\exp(\sqrt{-1}b_{i,j}), \tag{e 0.73}$$

$j = 1, 2, ..., k(A)$ and $i = 0, 1, ..., n, e$.
Then,

$$\frac{n_s}{2\pi}(t_s \otimes \mathrm{Tr}_{M_{m(A)}})(b_{i,j}^{(s)}) \in \mathbb{Z}, \tag{e 0.74}$$

where t_s is the normalized trace on M_{n_s}, $s = 1, 2, ..., F(2)$, $j = 1, 2, ..., k(A)$, and $i = 0, 1, ..., n$. We also have

$$\frac{m_s}{2\pi}(t_s \otimes \mathrm{Tr}_{M_{m(A)}})(b_{e,j}^{(s)}) \in \mathbb{Z} \tag{e 0.75}$$

where t_s is the normalized trace on M_{m_s}, $s = 1, 2, ..., F(1)$, $j = 1, 2, ..., k(A)$. Let

$$\lambda_{i,j}^{(s)} = \frac{n_s}{2\pi}(t_s \otimes \mathrm{Tr}_{M_{m(A)}})(b_{i,j}^{(s)}) \in \mathbb{Z},$$

where t_s is the normalized trace on M_{n_s}, $s = 1, 2, ..., n$, $j = 1, 2, ..., k(A)$ and $i = 0, 1, 2, ..., n$.
Let

$$\lambda_{e,j}^{(s)} = \frac{m_s}{2\pi}(t_s \otimes \mathrm{Tr}_{M_{m(A)}})(b_{e,j}^{(s)}) \in \mathbb{Z}$$

where t_s is the normalized trace on M_{m_s}, $s = 1, 2, ..., F(1)$ and $j = 1, 2, ..., k(A)$. Let

$$\lambda_{i,j} = (\lambda_{i,j}^{(1)}, \lambda_{i,j}^{(2)}, ..., \lambda_{i,j}^{(F(2)}) \in \mathbb{Z}^{F(2)} \text{ and}$$
$$\lambda_{e,j} = (\lambda_{e,j}^{(1)}, \lambda_{e,j}^{(2)}, ..., \lambda_{e,j}^{(F(1)}) \in \mathbb{Z}^{F(1)}. \qquad (e\,0.76)$$

We have

$$|\frac{\lambda_{i,j}^{(s)}}{n_s}| < d/4, \quad s = 1, 2, ..., F(2), \text{ and} \qquad (e\,0.77)$$
$$|\frac{\lambda_{e,j}^{(s)}}{m_s}| < d/4, \quad s = 1, 2, ..., F(1), \qquad (e\,0.78)$$

$j = 1, 2, ..., k(A)$, $i = 0, 1, 2, ..., n$.

Define $\alpha_i^{(0,1)}: K_1(A) \to \mathbb{Z}^{F(2)}$ by mapping $[g_j]$ to $\lambda_{i,j}$, $j = 1, 2, ..., k(A)$ and $i = 0, 1, 2, ..., n$, and define $\alpha_e^{(0,1)}: K_1(A) \to \mathbb{Z}^{F(1)}$ by mapping $[g_j]$ to $\lambda_{e,j}$, $j = 1, 2, ..., k(A)$. We write $K_0(A \otimes C(\mathbb{T})) = K_0(A) \oplus \boldsymbol{\beta}(K_1(A)))$ (see **??** for the definition of $\boldsymbol{\beta}$). Define $\alpha_i : K_*(A \otimes C(\mathbb{T})) \to K_*(F_2)$ as follows: On $K_0(A \otimes C(\mathbb{T}))$, define

$$\alpha_i|_{K_0(A)} = [\pi_i \circ \phi]|_{K_0(A)}, \;\; \alpha_i|_{\boldsymbol{\beta}(K_1(A))} = \alpha_i \circ \boldsymbol{\beta}|_{K_1(A)} = \alpha_i^{(0,1)} \quad \text{(e 0.79)}$$

and on $K_1(A \otimes C(\mathbb{T}))$,

$$\alpha_i|_{K_1(A \otimes C(\mathbb{T}))} = 0, \quad \text{(e 0.80)}$$

$i = 0, 1, 2, ..., n$, and define $\alpha_e \in \mathrm{Hom}(K_*(A \otimes C(\mathbb{T})), K_*(F_1))$, by

$$\alpha_e|_{K_0(A)} = [\pi_e \circ \phi]|_{K_0(A)}, \;\; \alpha_e|_{\boldsymbol{\beta}(K_1(A))} = \alpha_i \circ \boldsymbol{\beta}|_{K_1(A)} = \alpha_e^{(0,1)} \quad \text{(e 0.81)}$$

on $K_0(A \otimes C(\mathbb{T}))$ and $(\alpha_e)|_{K_1(A \otimes C(\mathbb{T}))} = 0$. Note that

$$(h_0)_* \circ \alpha_e = \alpha_0 \;\text{ and }\; (h_1)_* \circ \alpha_e = \alpha_n. \quad \text{(e 0.82)}$$

Since $A \otimes C(\mathbb{T})$ satisfies the UCT, the map α_e can be lifted to an element of $KK(A \otimes C(\mathbb{T}), F_1)$ which is still denoted by α_e. Then define

$$\alpha_0 = \alpha_e \times [h_0] \text{ and } \alpha_n = \alpha_e \times [h_1] \tag{e 0.83}$$

in $KK(A \otimes C(\mathbb{T}), F_2)$. For $i = 1, ..., n-1$, also pick a lifting of α_i in $KK(A \otimes C(\mathbb{T}), F_2)$, and still denote it by α_i. We estimate that

$$\|(w_i^* w_{i+1})\pi_{t_i} \circ \phi(g) - \pi_{t_i} \circ \phi(g)(w_i^* w_{i+1})\| < \epsilon_1/4 \text{ for all } g \in \mathcal{G}_5,$$

$i = 0, 1, ..., n-1$. Let $\Lambda_{i,i+1} : C(\mathbb{T}) \otimes A \to F_2$ be a unital contractive completely positive linear map given by the pair $w_i^* w_{i+1}$ and $\pi_{t_i} \circ \phi$ (by **??**, see 2.8 of [**?**]). Denote $V_{i,j} = \langle \pi_{t_i} \circ \phi \otimes \mathrm{id}_{M_{m(A)}}(g_j) \rangle$, $j = 1, 2, ..., k(A)$ and $i = 0, 1, 2, ..., n-1$.
Write

$$V_{i,j} = (V_{i,j,1}, V_{i,j,2}, ..., V_{i,j,F(2)}) \in F_2, \ \ j = 1, 2, ..., k(A), \ \ i = 0, 1, 2, ..., n.$$

Similarly, write

$$W_i = (W_{i,1}, W_{i,2}, ..., W_{i,F(2)}) \in F_2, \ \ i = 0, 1, 2, ..., n.$$

We have

$$\|W_i V_{i,j}^* W_i^* V_{i,j} V_{i,j}^* W_{i+1} V_{i,j} W_{i+1}^* - 1\| < 1/16 \qquad \text{(e 0.84)}$$

$$\|W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^* - 1\| < 1/16 \qquad \text{(e 0.85)}$$

and there is a continuous path $Z(t)$ of unitaries such that $Z(0) = V_{i,j}$ and $Z(1) = V_{i+1,j}$. Since

$$\|V_{i,j} - V_{i+1,j}\| < \delta_1/12, \;\; j = 1, 2, ..., k(A),$$

we may assume that $\|Z(t) - Z(1)\| < \delta_1/6$ for all $t \in [0, 1]$. We also write

$$Z(t) = (Z_1(t), Z_2(t), ..., Z_{F(2)}(t)) \in F_2 \;\text{ and }\; t \in [0, 1].$$

We obtain a continuous path

$$W_i V_{i,j}^* W_i^* V_{i,j} Z(t)^* W_{i+1} Z(t) W_{i+1}^*$$

which is in $CU(M_{nm(A)})$ for all $t \in [0, 1]$ and

$$\|W_i V_{i,j}^* W_i^* V_{i,j} Z(t)^* W_{i+1} Z(t) W_{i+1}^* - 1\| < 1/8 \;\text{ for all }\; t \in [0, 1].$$

It follows that

$$(1/2\pi\sqrt{-1})(t_s\otimes \mathrm{Tr}_{M_{m(A)}})[\log(W_{i,s}V_{i,j,s}^*W_{i,s}^*V_{i,j,s}Z_s(t)^*W_{i+1,s}Z_s(t)W_{i+1,s}^*)]$$

is a constant, where t_s is the normalized trace on M_{n_s}. In particular,

$$\begin{aligned}&(1/2\pi\sqrt{-1})(t_s\otimes \mathrm{Tr}_{M_{m(A)}})(\log(W_{i,s}V_{i,j,s}^*W_{i,s}^*W_{i+1,s}V_{i,j,s}W_{i+1}^*))\\ &=(1/2\pi\sqrt{-1})(t_s\otimes \mathrm{Tr}_{M_{m(A)}})(\log(W_{i,s}V_{i,j,s}^*W_{i,s}^*V_{i,j}V_{i+1,j,s}^*W_{i+1}V_{i,j,s}W_{i+1}^*)).\end{aligned}$$

Also

$$\begin{aligned}&W_iV_{i,j}^*W_i^*V_{i,j}V_{i+1,j}^*W_{i+1}V_{i+1,j}W_{i+1}^* && (e\,0.86)\\ &=(\omega_j(t_i)\exp(\sqrt{-1}b_{i,j}))^*\omega_j(t_i)\exp(\sqrt{-1}b_{i+1,j}) && (e\,0.87)\\ &=\exp(-\sqrt{-1}b_{i,j})\omega_j(t_i)^*\omega_j(t_{i+1})\exp(\sqrt{-1}b_{i+1,j}). && (e\,0.88)\end{aligned}$$

Note that, by (**??**) and (e 0.56), for $t\in[t_i,t_{i+1}]$,

$$\|\omega_j(t_i)^*\omega_j(t)-1\|<3(3\epsilon_1'+2\gamma_2)<3/32, \qquad (e\,0.89)$$

$j=1,2,...,k(A)$, $i=0,1,...,n-1$.

By Lemma 3.5 of [?],

$$(t_s \otimes \mathrm{Tr}_{m(A)})(\log(\omega_{j,s}(t_i)^* \omega_{j,s}(t_{i+1}))) = 0. \qquad (e\,0.90)$$

It follows that (by the Exel formula, using (??), (e 0.88) and (e 0.90))

$$t \otimes \mathrm{Tr}_{m(A)})(\mathrm{bott}_1(V_{i,j}, W_i^* W_{i+1})) \qquad (e\,0.91$$

$$= (\frac{1}{2\pi\sqrt{-1}})(t \otimes \mathrm{Tr}_{m(A)})(\log(V_{i,j}^* W_i^* W_{i+1} V_{i,j} W_{i+1}^* W_i)) \qquad (e\,0.92$$

$$= (\frac{1}{2\pi\sqrt{-1}})(t \otimes \mathrm{Tr}_{m(A)})(\log(W_i V_{i,j}^* W_i^* W_{i+1} V_{i,j} W_{i+1}^*))$$

$$= (\frac{1}{2\pi\sqrt{-1}})(t \otimes \mathrm{Tr}_{m(A)})(\log(W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^*))$$

$$= (\frac{1}{2\pi\sqrt{-1}})(t \otimes \mathrm{Tr}_{m(A)})(\log(\exp(-\sqrt{-1} b_{i,j}) \omega_j(t_i)^* \omega_j(t_{i+1}) \exp(\sqrt{-1} b_{i+1,j}$$

$$= (\frac{1}{2\pi\sqrt{-1}})[(t \otimes \mathrm{Tr}_{k(n)})(-\sqrt{-1} b_{i,j}) + (t \otimes \mathrm{Tr}_{k(n)})(\log(\omega_j(t_i)^* \omega_j(t_{i+1}))$$

$$+(t \otimes \mathrm{Tr}_{k(n)})(\sqrt{-1} b_{i,j})] \qquad (e\,0.93$$

$$= \frac{1}{2\pi}(t \otimes \mathrm{Tr}_{k(n)})(-b_{i,j} + b_{i+1,j}) \qquad (e\,0.94$$

for all $t \in T(F_2)$. In other words,

$$\mathrm{bott}_1(V_{i,j}, W_i^* W_{i+1})) = -\lambda_{i,j} + \lambda_{i+1,j} \tag{e 0.95}$$

$j = 1, 2, ..., m(A)$, $i = 0, 1, ..., n-1$.
Consider $\alpha_0, ..., \alpha_n \in KK(A \otimes C(\mathbb{T}), F_2)$ and $\alpha_e \in KK(A \otimes C(\mathbb{T}), F_1)$.
Note that

$$|\alpha_i(g_j)| = |\lambda_{i,j}|,$$

and by (e 0.77), one has

$$m_s, n_j \geq N_2(8/d + 1).$$

By applying **4.4** (using (e 0.78), among other items), there are unitaries $z_i \in F_2$, $i = 1, 2, ..., n-1$, and $z_e \in F_1$ such that

$$\|[z_i, \pi_{t_i} \circ \phi(g)]\| < \delta_u \text{ for all } g \in \mathcal{G}_u \tag{e 0.96}$$

$$\mathrm{Bott}(z_i, \pi_{t_i} \circ \phi) = \alpha_i \text{ and } \mathrm{Bott}(z_e, \pi_e \circ \phi) = \alpha_e. \tag{e 0.97}$$

Put

$$z_0 = h_0(z_e) \quad \textit{and} \quad z_n = h_1(z_e).$$

One verifies (by (e 0.83)) that

$$\mathrm{Bott}(z_0, \pi_{t_0} \circ \phi) = \alpha_0 \text{ and } \mathrm{Bott}(z_n, \pi_{t_n} \circ \phi) = \alpha_n. \tag{e 0.98}$$

Let $U_{i,i+1} = z_i (w_i)^* w_{i+1} (z_{i+1})^*$, $i = 0, 1, 2, ..., n-1$. Then

$$\|[U_{i,i+1}, \pi_{t_i} \circ \phi(g)]\| < \min\{\delta_1, \delta_2\}, \quad g \in \mathcal{G}_u, \ i = 0, 1, 2, ..., n-1 \tag{e 0.99}$$

Moreover, for $i = 0, 1, 2, ..., n-1$,

$$\begin{aligned} \mathrm{bott}_1(U_{i,i+1}, \pi_{t_i} \circ \phi) &= \mathrm{bott}_1(z_i, \pi_{t_i} \circ \phi)) + \mathrm{bott}_1((w_i^* w_{i+1}, \pi_{t_i} \circ \phi)) \\ &\quad + \mathrm{bott}_1((z_{i+1}^*, \pi_{t_i} \circ \phi) \\ &= (\lambda_{i,j}) + (-\lambda_{i,j} + \lambda_{i+1,j}) + (-\lambda_{i+1,j}) \\ &= 0. \end{aligned}$$

Note that for any $x \in \bigoplus_{*=0,1} \bigoplus_{k=1}^{\infty} K_*(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z})$, one has $Nx = 0$. Therefore

$$\mathrm{Bott}((\underbrace{U_{i,i+1}, ..., U_{i,i+1}}_{N}), (\underbrace{\pi_{t_i} \circ \phi, ..., \pi_{t_i} \circ \phi}_{N}))|_{\mathcal{P}} = N\mathrm{Bott}(U_{i,i+1}, \pi_{t_i} \circ \phi)|_{\mathcal{P}} = 0 \tag{e 0.100}$$

$i = 0, 1, 2, ..., n-1.$
Note that, by the assumption (e 0.39),

$$t_s \circ \pi_t \circ \phi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1', \tag{e 0.101}$$

where t_s is the normalized trace on M_{n_s}, $1 \leq s \leq F(2)$.
By applying **??**, using (e 0.101), (e 0.99) and (e 0.100), there exists a continuous path of unitaries, $\{\tilde{U}_{i,i+1}(t) : t \in [t_i, t_{i+1}]\} \subset F_2 \otimes M_N(\mathbb{C})$ such that

$$\tilde{U}_{i,i+1}(t_i) = \mathrm{id}_{F_2 \otimes M_N(\mathbb{C})}, \ \ \tilde{U}_{i,i+1}(t_{i+1}) = (z_i w_i^* w_{i+1} z_{i+1}^*) \otimes 1_{M_N(\mathbb{C})}, \tag{e 0.102}$$

and

$$\|\tilde{U}_{i,i+1}(t)(\underbrace{\pi_{t_i} \circ \phi(f), ..., \phi_{t_i} \circ \phi(f)}_{N})\tilde{U}_{i,i+1}(t)^* - (\underbrace{\pi_{t_i} \circ \phi(f), ..., \phi_{t_i} \circ \phi(f)}_{N})\| < \epsilon \tag{e 0.103}$$

for all $f \in \mathcal{F}$ and for all $t \in [t_i, t_{i+1}]$. Define $W \in C \otimes M_N$ by

$$W(t) = (w_i z_i^* \otimes 1_{M_N})\tilde{U}_{i,i+1}(t) \text{ for all } t \in [t_i, t_{i+1}], \tag{e 0.104}$$

$i = 0, 1, ..., n-1$. Note that $W(t_i) = w_i z_i^* \otimes 1_{M_N}$, $i = 0, 1, ..., n$. Note also that

$$W(0) = w_0 z_0^* \otimes 1_{M_N} = h_0(w_e z_e^*) \otimes 1_{M_N}$$

and

$$W(1) = w_n z_n^* \otimes 1_{M_N} = h_1(w_e z_e^*) \otimes 1_{M_N}.$$

So $W \in C \otimes M_N$. One then checks that, by (e 0.56), (e 0.103) , (e 0.96) and (e 0.59), for $t \in [t_i, t_{i+1}]$,

$$\begin{aligned} & \|W(t)((\pi_t \circ \phi)(f) \otimes 1_{M_N})W(t)^* - (\pi_t \circ \psi)(f) \otimes 1_{M_N}\| \qquad \text{(e 0.105)} \\ < & \|W(t)((\pi_t \circ \phi)(f) \otimes 1_{M_N})W(t)^* - W(t)((\pi_{t_i} \circ \phi)(f) \otimes 1_{M_N})W^*(t)\| \\ & + \|W(t)(\pi_{t_i} \circ \phi)(f)W(t)^* - W(t_i)\pi_{t_i} \circ \phi)(f)W(t_i)^*\| \\ & + \|W(t_i)((\pi_{t_i} \circ \phi)(f) \otimes 1_{M_N})W(t_i)^* - (w_i(\pi_{t_i} \circ \phi)(f)w_i^*) \otimes 1_{M_N}\| \\ & + \|w_i(\pi_{t_i} \circ \phi)(f)w_i^* - \pi_{t_i} \circ \psi(f)\| \\ & + \|\pi_{t_i} \circ \psi(f) - \pi_t \circ \phi(f)\| \\ < & \epsilon_1/16 + \epsilon/32 + \delta_u + \epsilon_1/16 + \epsilon_1/16 < \epsilon \end{aligned}$$

for all $f \in \mathcal{F}$.