Basic Homotopy Lemmas Introduction

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$\mathrm{length}(\{u(t)\}) \leq 4\pi + 1.$

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$$\leq \sum_{n=1}^{N} \frac{\epsilon}{6n!} + \epsilon/3 < \epsilon. \tag{e 0.11}$$

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Proof.

The spectrum of u has a gap with the length at least $d = 2\pi/n$. □

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$$\sum_{j=1}^n c_{ij} = a_i, \sum_{i=1}^m c_{ij} = b_j, \text{ for all } i, j \qquad (e\,0.15)$$

and

$$c_{ij} = 0 \text{ unless } (i, j) \in R. \qquad (e\,0.16)$$

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Now suppose that $\phi, \psi : C(X) \to M_n$ such that

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We have, for all $f \in C(X)$,

$$\phi(f) = \sum_{i=1}^{k_1} f(x_i) p_i \text{ and } \psi(f) = \sum_{i=1}^{k_2} f(y_i) q_i, \tag{e 0.22}$$

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$$\tau(\phi(g_A)) \geq \sum_{x_i \in S} a_i/n. \tag{e 0.23}$$

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We may write

$$\phi(f) = \sum_{i,j} f(x_i) p_{ij} \text{ and } \psi(f) = \sum_{i,j} f(y_j) q_{ij}. \tag{e 0.30}$$

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Moreover, $p_{ij} \neq 0$ and $q_{ij} \neq 0$ if and only if $\mathrm{dist}(x_i, y_j) < \eta$. Therefore there exists a unitary $u \in M_n$ such that

$$u^* p_{ij} u = q_{ij} \text{ and } \|\mathrm{Ad}\, u \circ \phi(f) - \psi(f)\| < \epsilon \tag{e 0.31}$$

for all $f \in \mathcal{F}$. Lemma then follows easily.

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$$\phi_{*0} = \psi_{*0}, \ \ |\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \ \text{ for all } \ g \in \mathcal{H}, \qquad \text{(e 0.32)}$$

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for all $f \in \mathcal{F}$. In other words,

$$\|v^*\phi(f)v - \psi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \qquad \text{(e 0.37)}$$

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where $\alpha_i : [0,1] \to X$ is a continuous map, $i = 1, 2, ..., n$.

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$$\|\phi(g)(t) - \phi(g)(t')| < \min\{\epsilon/4, \delta_1/2, \delta/2\} \text{ for all } f \in \mathcal{H} \quad \text{(e 0.40)}$$

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$$\phi(f)(t_{i-1}) = \sum_{j=1}^{n} f(x_{i-1,j}) p_{i-1,j} \text{ for all } f \in C(X), \quad \text{(e 0.41)}$$

where $x_{i-1,j} \in X$ and $\{p_{i-1,1}, p_{i-1,2}, ..., p_{i-1,n}\}$ is a set of mutually orthogonal rank one projections.

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$i = 1, 2, ..., m$. Moreover, we may assume, without loss of generality, that there is a permutation σ_i such that

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$$\|v(t)\phi(f)(t_{i-1}) - \phi(f)(t_{i-1})v(t)\| < \epsilon/4 \text{ for all } f \in \mathcal{F}, \quad \text{(e 0.45)}$$

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$$\psi(f) = \sum_{i=1}^{n} f(\alpha_i) p_i \text{ for all } f \in C(X). \tag{e 0.46}$$

On $t \in [t_{i-1}, t_i]$,

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$$\begin{aligned} \|\phi(f)(t) - \psi(f)(t)\| &= \|\phi(f)(t) - \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j}\| \\ &+ \|\sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} - \sum_{j=1}^{n} f(\alpha_{j,i-1}(t))p_j(t)\| \end{aligned}$$

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On $t \in [t_{i-1}, t_i]$,

$$\begin{aligned}
\|\phi(f)(t) - \psi(f)(t)\| &= \|\phi(f)(t) - \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j}\| \\
&+\|\sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} - \sum_{j=1}^{n} f(\alpha_{j,i-1}(t))p_j(t)\| \\
&< \epsilon/4 + \|\sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} - \sum_{j=1}^{n} f(x_{i-1,j})v^*(t)p_{i-1,j}v(t)\| + \epsilon/4 \\
&= \|\phi(f)(t_{i-1}) - v^*(t)\phi(f)(t_{i-1})v(t)\| + \epsilon/2 < \epsilon/2 + \epsilon/2
\end{aligned}$$

for all $f \in \mathcal{F}$.

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Let X *be a compact metric space, let* $P \in M_r(C(X))$ *be a projection and let* $C = PM_r(C(X))P$. *Suppose that* $\phi : C \to C([0,1], M_n)$, *where* $n \geq 1$ *is an integer. For any* $\epsilon > 0$ *and any finite subset* $\mathcal{F} \subset C$, *there exists a set of mutually orthogonal rank projections* $p_1, p_2, ..., p_n \in C([0,1], M_n)$ *such that*

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Proof.

$C(X) = \lim_{n\to\infty}(C(X_n), \imath_n)$, where X_n is a polygon and $\imath_n$ is an injective homomorphism. □

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$$u_n = \begin{pmatrix} e^{2\pi i/n} & 0 & 0 \cdots & \\ 0 & e^{4\pi i/n} & 0 \cdots & \\ & & \ddots & \\ & & & e^{2n\pi i/n} \end{pmatrix}$$

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$$f(e^{2\pi it}) = \begin{cases} 1-2t, & \text{if } 0 \le t \le 1/2; \\ -1+2t, & \text{if } 1/2 < t \le 1, \end{cases}$$

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The proof is just a modification of that of Theorem 1.4. □

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Remark: $\mathcal{P}$ can be chosen to be a set of mutually orthogonal projections which corresponds to a set of disjoint clopen subsets with union X.

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$$\phi_1=\phi_{1,0}\oplus\phi_{1,1},\ \ \phi_2=\phi_{2,0}\oplus\phi_{2,1}, \tag{e 10.53}$$

$$\tau\circ\phi_{1,1}(p_i)=\tau\circ\phi_{1,2}(p_i),\ \ i=1,2,...,k_1. \tag{e 10.54}$$

By replacing ϕ_1 by $\mathrm{Ad}\, v \circ \phi_1$, simplifying the notation, without loss of generality, we may assume that $P_{0,1} = P_{0,2}$. It follows (see **??**) that

$$[\phi_{1,1}]|_{\mathcal{P}} = [\phi_{2,1}]|_{\mathcal{P}}. \tag{e 10.55}$$

By (e 10.52) and choice of σ_0, we also have

$$\tau \circ \phi_{1,1}(g) \geq \Delta_1(\hat{g}) \text{ for all } g \in \mathcal{H}_1' \text{ and} \tag{e 10.56}$$

$$|\tau \circ \phi_{1,1}(g) - \tau \circ \phi_{1,2}(g)| < \sigma_0 \cdot \delta_1 \text{ for all } g \in \mathcal{H}_2'. \tag{e 10.57}$$

Therefore

$$t \circ \phi_{1,1}(g) \geq \Delta_1(\hat{g}) \text{ for all } g \in \mathcal{H}_1' \text{ and} \tag{e 10.58}$$

$$|t \circ \phi_{1,1}(g) - t \circ \phi_{1,2}(g)| < \delta_1 \text{ for all } g \in \mathcal{H}_2', \tag{e 10.59}$$

where t is the tracial state on $(1 - P_{1,0})M_n(1 - P_{1,0})$. By applying **??**, there exists a unitary $v_1 \in (1 - P_{1,0})M_n(1 - P_{1,0})$ such that

$$\|\mathrm{Ad}\, v_1 \circ \phi_{1,1}(f) - \phi_{2,1}(f)\| < \epsilon/16 \text{ for all } f \in \mathcal{F}. \tag{e 10.60}$$

Put $H = \phi_{2,1}$ and $p = P_{1,0}$. The lemma for the case that $A = M_r(C(X))$ follows.

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then, there exists a unitary $u \in M_n$ such that

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Remark: If X has infinitely many points, then there is no need to mention the integer N.

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Let $\epsilon_1 = \min\{\epsilon/16, \sigma_0\}$ and let $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{H}_0$. Choose $d_0 > 0$ such that

$$|f(x) - f(x')| < \epsilon_1 \text{ for all } f \in \mathcal{F}_1, \tag{e 10.63}$$

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if $i \neq j$. There is, for each j, a function $h_j \in C(X)$ with $0 \le h_j \le 1$, $h_j(x) = 1$ if $x \in B(\xi_j, d_1/2)$ and $h_j(x) = 0$ if $x \notin B(\xi_j, d_1)$. Define $\mathcal{H}_1 = \mathcal{H}_0 \cup \{h_j : 1 \le j \le m\}$ and put

$$\sigma_1 = \min\{\Delta(\hat{g}) : g \in \mathcal{H}_1\}. \qquad \text{(e 10.65)}$$

Choose an integer $N_0 \ge 1$ such that $1/N_0 < \sigma_1 \cdot (1-\alpha)/4$ and $N = 4m(N_0+1)^2(K+1)^2$.
Now let $H : C(X) \to M_n$ be a unital homomorphism with $n \ge N$

Let $Y_1 = \overline{B(\xi_1, d_0/2)} \setminus \cup_{i=2}^m B(\xi_i, d_1)$,
$Y_2 = \overline{B(\xi_2, d_0/2)} \setminus (Y_1 \cup \cup_{i=3}^m B(\xi_i, d_1)$,
$Y_j = \overline{B(\xi_j, d_0/2)} \setminus (\cup_{i=1}^{j-1} Y_i \cup \cup_{i=j+1}^m B(\xi_i, d_1))$, $j = 1, 2, ..., m$. Note that $Y_j \cap Y_i = \emptyset$ if $i \neq j$ and $B(\xi_j, d_1) \subset Y_j$. We write that

$$H(f) = \sum_{i=1}^n f(x_i)p_i = \sum_{j=1}^m (\sum_{x_i \in Y_j} f(x_i)p_i) \text{ for all } f \in C(X), \quad \text{(e 10.66)}$$

where $\{p_1, p_2, ..., p_n\}$ is a set of mutually orthogonal rank one projections in M_n, $\{x_1, x_2, ..., x_n\} \subset X$. Let R_j be the cardinality of $\{x_i : x_i \in Y_j\}$. Then, by (e 10.61),

$$R_j \geq N\tau \circ H(h_j) \geq N\Delta(\hat{h}_j) \geq (N_0 + 1)^2 K\sigma_1 \geq (N_0 + 1)K^2, \quad \text{(e 10.67)}$$

$j = 1, 2, ..., m$. Write $R_j = S_j K + r_j$, where $S_j \geq N_0 Km$ and $0 \leq r_j < K$, $j = 1, 2, ..., m$. Choose $x_{j,1}, x_{j,2}, ..., x_{j,r_j} \subset \{x_i \in Y_j\}$ and denote $Z_j = \{x_{j,1}, x_{j,2}, ..., x_{j,r_j}\}$, $j = 1, 2, ..., m$.

Therefore we may write

$$H(f) = \sum_{j=1}^{m} \Big(\sum_{x_i \in Y_j \setminus Z_j} f(x_i) p_i \Big) + \sum_{j=1}^{m} \Big(\sum_{i=1}^{r_j} f(x_{j,i}) p_{j,i} \Big) \qquad \text{(e 10.68)}$$

for $f \in C(X)$. Note that the cardinality of $\{x_i \in Y_j \setminus Z_j\}$ is KS_j, $j = 1, 2, ..., m$. Define

$$\Psi(f) = \sum_{j=1}^{m} f(\xi_j) P_j = \sum_{k=1}^{K} \Big(\sum_{j=1}^{m} f(\xi_j) Q_{j,k} \Big) \text{ for all } f \in C(X), \qquad \text{(e 10.69)}$$

where $P_j = \sum_{x_i \in Y_j \setminus Z_j} p_i = \sum_{k=1}^{K} Q_{j,k}$ and $\mathrm{rank} Q_{j,k} = S_j$, $j = 1, 2, ..., m$. Put $e_0 = \sum_{i=1}^{m} (\sum_{i=1}^{r_j} p_{j,i})$, $e_k = \sum_{j=1}^{m} Q_{j,k}$, $k = 1, 2, ..., K$. Note that

$$\mathrm{rank}(e_0) = \sum_{j=1}^{m} r_j < mK \text{ and } \mathrm{rank}(e_k) = S_j \qquad \text{(e 10.70)}$$

$$S_j \geq N_0 mK > mK, \;\; j = 1, 2, .., K. \qquad \text{(e 10.71)}$$

It follows that $e_0 \lesssim e_1$ and e_i is equivalent to e_1.

Moreover, we may write

$$\Psi(f) = \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K}) \text{ for all } f \in A, \tag{e 10.72}$$

where $\psi(f) = \sum_{j=1}^{m} f(\xi_j) Q_{j,1}$ for all $f \in A$. We also estimate that

$$\|H(f) - (\phi(f) \oplus \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K}))\| < \epsilon_1 \text{ for all } f \in \mathcal{F}_1. \tag{e 10.73}$$

We also compute that

$$\tau \circ \psi(g) \geq (1/K)(\Delta(\hat{g}) : g \in \mathcal{H}_0\} - \epsilon_1 - \frac{mK}{N_0 K m}) \geq \alpha \frac{\Delta(\hat{g})}{K} \tag{e 10.74}$$

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Suppose that $\mathcal{H}_1 \subset (A_0)_+^{\mathbf{1}} \setminus \{0\}$ is a finite subset, $\sigma > 0$ is positive number and $n \geq 1$ is an integer. There exists a finite subset $\mathcal{H}_2 \subset (A_0)_+^{\mathbf{1}} \setminus \{0\}$ satisfying the following: Suppose that $\phi : A = A_0 \otimes C(\mathbb{T}) \to M_k$ (for some integer $k \geq 1$) is a unital homomorphism and

$$tr \circ \phi(h \otimes 1) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_2. \tag{e 10.75}$$

Then there exist mutually orthogonal projections $e_0, e_1, e_2, ..., e_n \in M_k$ such that $e_1, e_2, ..., e_n$ are equivalent and $\sum_{i=0}^{n} e_i = 1$, and there exists a unital homomorphisms $\psi_0 : A = A_0 \otimes C(\mathbb{T}) \to e_0 M_k e_0$ and $\psi : A = A_0 \otimes C(\mathbb{T}) \to e_1 M_k e_1$ such that one may write that

$$\|\phi(f) - \mathrm{diag}(\psi_0(f), \overbrace{\psi(f), \psi(f), ..., \psi(f)}^{n})\| < \epsilon \tag{e 10.76}$$

$$\text{and } \mathrm{tr}(e_0) < \sigma \tag{e 10.77}$$

for all $f \in \mathcal{F}$, where tr is the tracial state on M_k.

Moreover,

$$tr(\psi(g \otimes 1)) \geq \frac{\Delta(\hat{g})}{2n} \text{ for all } g \in \mathcal{H}_1. \tag{e 10.78}$$

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there exists a unitary $U \in M_{Km+n}$ such that

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In particular, if $h_1(1_A) = h_2(1_A)$, $W \in U(pM_{n+1}(B)p)$.

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$$\pi_0(A) = \pi_0(C) \cong C \text{ and } \ker\pi_0 \supset J, \qquad (e\,10.82)$$

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$1 - q = \lim_{n\to\infty}(f_{1/16}(a))^{1/n}$. So q is closed and $pq = 0$.

It follows Brown interpolation lemma that there is a projection $e \in D$ such that

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Let $a \in (I_0)_+$ be a strictly positive element and let $J = \overline{\Psi(a)Q\Psi(a)}$ be the hereditary C^*-subalgebra of Q generated by $\Psi(a)$. Put $C_2 = \Psi(C_1) + J$.

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$$0 \to J \to C_2 \to B \to 0$$

is a quasidiagonal extension.

One then concludes there is a projection $P \in J$ and a unital homomorphism $\psi_0 : B \to (1-P)C_2(1-P)$ such that

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Then, for sufficiently large n, by (e 10.94) and (e 10.95),

$$\|(1-p_n)\phi_n(f)-\phi_n(f)(1-p_n)\|<\epsilon_0, \qquad \text{(e 10.96)}$$

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$$0 < t_0(\Psi(b)) < \sigma_0/2 \text{ for all } b \in I_0 \text{ with } 0 \le n \le 1. \qquad (e\,10.98)$$

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This contradicts with (e 10.86).

Corollary **2.13.**

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$$\text{and } \ tr(1-p) = tr(1-q) < \eta_0,$$

Corollary 2.13.

Let A be a unital C^*-algebra whose irreducible representations have bounded dimensions. Let $\eta > 0$, let $\mathcal{E} \subset A$ be a finite subset and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: Suppose that $\phi,\ \psi : A \to M_n$ (for some integer $n \geq 1$) are two $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps. Then, there exist projections $p, q \in M_n$ with $\mathrm{rank}(p) = \mathrm{rank}(q)$ and unital homomorphisms $\phi_0 : A \to pM_np$ and $\psi_0 : A \to qM_nq$ such that

$$\begin{aligned} &\|p\phi(a) - \phi(a)p\| < \eta, \quad \|q\psi(a) - \psi(a)q\| < \eta, \quad a \in \mathcal{E}, \\ &\|\phi(a) - [(1-p)\phi(a)(1-p) + \phi_0(a)]\| < \eta, \\ &\|\psi(a) - [(1-q)\psi(a)(1-q) + \psi_0(a)]\| < \eta, \quad a \in \mathcal{E} \\ &\ \text{and}\ \ tr(1-p) = tr(1-q) < \eta_0, \end{aligned}$$

where tr is the normalized trace on M_n.

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$$tr \circ L_1(h) \geq \Delta(\hat{h}) \text{ and } tr \circ L_2(h) \geq \Delta(\hat{h}) \quad \text{for all } h \in \mathcal{H}_2, \text{ and}$$

$$|tr \circ L_1(h) - tr \circ L_2(h)| < \sigma \quad \text{for all } h \in \mathcal{H}_2. \tag{e 10.99}$$

Then there exist mutually orthogonal projections $e_0, e_1, e_2, ..., e_K \in M_n$ *such that* $e_1, e_2, ..., e_K$ *are equivalent,* $e_0 \lesssim e_1$, $tr(e_0) < \epsilon_1$ *and* $e_0 + \sum_{i=1}^K e_i = 1$, *and there exist a unital* ϵ_0-$\mathcal{G}_0$*-multiplicative contractive completely positive linear maps* $\psi_1, \psi_2 : A \to e_0 M_k e_0$, *a unital homomorphism* $\psi : A \to e_1 M_k e_1$, *and unitary* $u \in M_n$ *such that one may write that*

$$\|L_1(f) - \mathrm{diag}(\psi_1(f), \overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K})\| < \epsilon \text{ and} \tag{e 10.100}$$

$$\|uL_2(f)u^* - \mathrm{diag}(\psi_2(f), \overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K})\| < \epsilon \tag{e 10.101}$$

for all $f \in \mathcal{F}$, *where tr is the tracial state on* M_n. *Moreover,*

$$tr(\psi(g)) \geq \frac{\Delta(\hat{g})}{3K} \text{ for all } g \in \mathcal{H}_1. \tag{e 10.102}$$

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$$\|\mathrm{Ad}\, u \circ L_1(f) - L_2(f)\| < \epsilon \;\; \textit{for all} \;\; f \in \mathcal{F}. \qquad (\mathrm{e}\,10.103)$$

It follows from a combination of Lemma 2.14 and Lemma 2.6.