Eductive Stability in Real Business Cycle Models[*]

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Abstract

Within the standard RBC model we examine issues of expectational coordination on the unique rational expectations equilibrium. We show the sensitivity of agents' plans and decisions to their short-run and long-run expectations is too great to trigger eductive coordination in a world of hyper-rational agents who are endowed with Common Knowledge and contemplate the possibility of small deviations from equilibrium: eductive stability never obtains. We conclude adaptive learning must play a role in real-time dynamics. Our eductive instability theorem has a counterpart under adaptive learning: even with asymptotic stability the transition dynamics can involve large departures from rational expectations.

JEL Classifications: D84, D83, E32

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1 Introduction

The question of expectational coordination has returned to the forefront of intellectual debate in the context of the economic crisis that began in 2007. Some people have argued that the overly optimistic view of the world conveyed by our models arises in particular from the universal adoption of rational expectations (RE). In fact, the axiomatic use of RE has been under critical examination for some time: what is at stake in this debate is the robustness of expectational coordination on RE. This issue, the subject of the present paper, has been discussed using a variety of approaches.

Problems of multiplicity of rational expectations equilibria (REE) have received considerable attention, as in the sunspot literature, e.g. Benhabib and Farmer (1994) and Chiappori and Guesnerie (1991), and in the global games literature, e.g. Morris and Shin (1998). The existence of multiple REE arises also in the New Keynesian model, most dramatically in connection with the zero lower bound, e.g. Benhabib, Schmitt-Grohe and Uribe (2001). The possibility of multiple REE most obviously and forcefully raises the issue of coordination of expectations.[1]

However, the issue of coordination of expectations on RE is crucial also in models with a unique REE. Rational expectations must be viewed as an equilibrium concept. In game theoretic terms the REE is a Nash equilibrium, not a dominant strategy equilibrium. Put differently, it is rational to have RE only if the expectations of other agents are in accordance with RE. This unavoidably leads to the question of how agents would come to coordinate on the REE.

One approach to this question examines the stability of the REE under adaptive learning. See, for example, Marcet and Sargent (1989), Woodford (1990), Brock and Hommes (1997) and Evans and Honkapohja (2001). Most prominently, under the least-squares learning approach, agents form expectations like statisticians or econometricians, updating their forecasting model over time as new data become available. This is a "bounded rationality" approach because the agents do not explicitly take account of the self-referential aspect of the model; and, consequently, agents can learn to have RE only asymptotically as they update their forecast models to observed data over time.

A second approach, called the "eductive" viewpoint, is the main focus of the current paper. See Evans and Guesnerie (1993, 2005) and Guesnerie (2002) for an introductory conceptual assessment in dynamic models. Under

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[1] Also relevant to this discussion is the concept of "rational beliefs" – see Kurz and Motolese (2001). Under their equilibrium concept agents have heterogeneous beliefs consistent with the empirical distribution of aggregate observables.

this approach, agents are assumed to fully understand the structure of the economy and to be fully rational; indeed something stronger than rationality is assumed, what game theorists call Common Knowledge of rationality. An REE is said to be eductively stable if these assumptions guarantee coordination of agents on rational expectations. Eductive stability provides strong support for RE, because RE is then attainable through mental reasoning and does not require the gradual accumulation of data over time.

Consider, for example, a standard exercise, in which the economy at some time is subject to a known but previously unanticipated structural change. Many examples could be given, but a typical application, often seen in textbooks using nonstochastic models with perfect foresight, is a permanent change in fiscal policy. One of the attractions of the RE approach is that, in the event of such a structural change, agents fully incorporate into their decision-making the resulting future changes in the equilibrium paths of wages and interest-rates. If the economy is eductively stable, then coordination on this revised RE path would be achieved by our fully rational agents: the usual textbook RE solutions would be fully vindicated.

In this paper we focus on eductive stability and expectational coordination in models in which agents are long-lived, or more precisely, are far-sighted. The logical framework and the central results for eductive learning, as well as the connections with adaptive (or "evolutive") learning and the issue of sunspot multiplicity, are well understood in many contexts where agents are short-lived, and in particular within simple models of overlapping generations: see, e.g., the assessment of Gauthier and Guesnerie (2004). In these settings, although eductive stability is in general somewhat more demanding, it has close connections with other well-developed coordination criteria, including determinacy, absence of sunspot equilibria, and convergence of adaptive learning rules. With long lived agents, and this is a key message of our findings in this paper, the picture changes significantly.

The assumption that agents are long-lived, or far-sighted, is clearly not economically innocuous. For example, long-lived agents take into account their life-time income rather than income over a short horizon, a fact that of course has a key impact on the understanding and design of macroeconomic policies. This suggests, unless the RE-hypothesis is accepted axiomatically, that the effect of the presence of long-lived agents on eductive stability should receive careful and parallel attention. Do they make expectational coordination more or less robust?

To explore this complex question, we consider in this paper the simplest version of a Real Business Cycle (RBC) model and its standard focal point, the REE, which is presented in Section 2.1. In Section 2.2, we investigate a

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background question that is central for any assessment of the plausibility of expectational coordination on the REE: how do actions and plans of agents react to out-of-equilibrium beliefs? The answer, for small changes of beliefs, is provided by a Preparation Theorem, which appears to be new to the literature. This theorem identifies the changes in consumption and savings decisions associated with a small change in the whole infinite trajectory of beliefs, providing rich and comprehensive information on this question. We obtain a key expectational feedback parameter ξ, depending on the underlying model specification, which governs the response of plans to beliefs.

In Section 3, we develop the eductive approach to expectational coordination in the RBC model. To start, we define eductive stability first in the context of the cobweb model, and then in a general setting.[2] The essence of the approach is to assume full rationality of all agents and to take as given a collection of common knowledge assumptions. This collection includes rationality of other agents, the full structure of the model, and restrictions on the beliefs of agents concerning the path of relevant aggregates. The question is whether this common knowledge is sufficient to trigger a mental reasoning process leading to coordination on the REE. We then describe how to adapt the concept to our specific model, and distinguish between "high-tech" and alternative "low-tech" interpretations.

We find that coordination on the REE in the RBC model is difficult. Indeed, Section 4 establishes the impossibility of the different versions of "strong" eductive stability. Thus, complete coordination on the REE time path cannot be expected. Coordination of expectations of long-lived agents is necessarily weak. There is no collective view of the future – no set of paths close but not identical to the self-fulfilling, equilibrium path – that is able to trigger coordination on the equilibrium, along the eductive lines just described. Taking a real-time perspective, every such collective view is subject, at some stage, to be invalidated by facts. Put more dramatically, in this simple world, a "crisis," here an expectational crisis, is unavoidable, although the time required for the crisis to become manifest, i.e. the extent of the weakness of expectational coordination, depends upon certain system characteristics that we identify.

What should we conclude from the negative eductive stability results of Section 4? The failure of common knowledge to trigger coordination, and the real-time falsification of beliefs in the long run, indicates that in this setting

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[2]The cobweb model is often developed as an infinitely repeated game, as in Bray and Savin (1986). In the cobweb model, hyper-rational agents are able to coordinate on the REE, under natural parametric restrictions, given appropriate common knowledge assumptions. There are close and well-understood connections between the eductive learning viewpoint and the adaptive learning viewpoint.

real-time adaptive learning must play a significant role. The REE, which is unique in our RBC model, remains the natural benchmark. This equilibrium path, and the corresponding expectations, is uniquely the one in which agents would be unable to improve on their forecasts. However, the clear implication of eductive instability is that coordination on the REE can only be attained by an adaptive learning process, a process in which expectations adapt to data over time. Returning to our hypothetical example of an unanticipated novel policy change that disturbs an RE steady state, renewed expectational coordination after the structural break cannot be immediate and can only take place over time.

Section 5 therefore takes up adaptive learning within our setting. Although it is known that the REE in the RBC model can be asymptotically stable under adaptive learning rules, our objective is to determine the links between our eductive instability results and the transitional adaptive-learning dynamics. The Preparation Theorem of Section 2 also plays a central role in our study of adaptive learning. In addition, drawing inspiration from the framework of Section 3, we develop a new stability notion for adaptive learning that emphasizes the extent of deviation from RE.

Specifically, in Section 5 we examine what we call "B-stability," the possibility in this context of maintaining a collective view of the future, i.e. consistency of the realized path with initial beliefs that the path will remain in an initially conjectured neighborhood of the equilibrium (and in this sense avoiding an "expectational crisis"). As is well-known, adaptive learning dynamics depend on the "gain" parameter $0 < \alpha \leq 1$ that governs the extent to which expectations respond to observed data. The equilibrium paths under learning depend both on α and on the key expectational feedback parameter ξ obtained in Section 2.

Section 5 obtains both negative results, reflecting the earlier eductive instability results, and some more positive results that reflect the potential for stability under adaptive learning. The negative result is that the REE is never robustly B-stable under adaptive learning: the consistency of conjectured beliefs paths is not robust in the sense that any collective view of the future will be falsified, at some point in time, for a large set of adaptive learning rules indexed by the gain α. In particular, despite the REE being asymptotically stable for sufficiently small α, the deviation of the learning dynamics from the REE can become arbitrarily large.

The positive results concern partial B-stability (for some gains α) and asymptotic stability. For some values of ξ and α both B-stability and asymptotic stability hold. There are, however, trade-offs that reflect the results of Sections 2 and 3. In particular, for large ξ the REE is never B-stable for any α even

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though it is asymptotically stable for small α. On the other hand, for small ξ the REE is B-stable only when α is sufficiently large.

Sections 6 and 7 discuss the implications of our results and concludes. The REE in our RBC model is the natural benchmark and anchor. However, coordination on the REE cannot be achieved by fully rational agents using eductive reasoning, even under strong common knowledge assumptions. The instability results of Section 4 indicate that full coordination on the REE is implausible and that therefore coordination can only be expected to arise over time through adaptive learning rules. Section 5 shows that while asymptotic stability holds under adaptive learning for a wide range of parameters, B-stability is not robust across our class of adaptive learning rules. Thus, starting from an RE steady state, if there is an unanticipated structural or policy change, then, even if the nature of this structural change is fully understood, it will take time for agents to coordinate on the REE and deviations during the learning transition can be large.

2 Model and Preparation Theorem

We consider a standard RBC model, except that for simplicity we assume a fixed labor supply and omit exogenous productivity shocks.[3] These simplifications, which amount to a focus on a nonstochastic discrete-time Ramsey model, are not critical to our results and are made in order to clarify the central features of our analysis. Elimination of both random shocks and labor-supply response to disequilibrium expectations can be expected to facilitate coordination on the rational expectations equilibrium (REE).[4] Despite eliminating these influences we establish that a strong form of eductive stability fails.

2.1 The model and equilibrium

There is a continuum of identical infinitely-lived households, indexed by $\omega \in [0, 1]$. At time t each household ω holds capital $k_t(\omega)$, resulting from previous decisions, and supplies inelastically one unit of labor. At time $t = 0$, facing interest rate r_0 and prospects of future interest rates r_t and wages q_t, household

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[3] Seminal RBC papers include Kydland and Prescott (1982) and Long and Plosser (1983).

[4] For example, the weak eductive stability conditions, given below, can be shown to be stricter when labor supply is elastic. Also, it is straightforward to extend our instability results to allow for *iid* random productivity shocks with small support.

ω determines his actions today and plans for the future by solving

$$\max \quad E_0(\omega) \sum_{t=0}^{\infty} \beta^t U(c_t(\omega)), \text{ where } 0 < \beta < 1, \qquad (1)$$

$$\text{subject to} \quad k_{t+1}(\omega) = (1 + r_t)k_t(\omega) + q_t - c_t(\omega), \qquad (2)$$

with initial wealth $k_0(\omega)$ given. Here $E_0(\omega)$ captures the expectations of agent ω formed using his subjective distribution.

We will focus on the case in which $k_0(\omega)$ is the same for all agents, but we do not impose this initially. The utility function $U(c)$ is increasing, strictly concave and smooth. We further impose a No Ponzi Game (NPG) condition that the present value of their limiting lifetime wealth be nonnegative. The household's Euler equation is

$$U'(c_t(\omega)) = \beta E_t(\omega) \left((1 + r_{t+1})U'(c_{t+1}(\omega)) \right). \qquad (3)$$

When the future is viewed as deterministic, expectations of future interest rates r_t and wages q_t are point expectations. Iterating forward the household flow budget constraint, and imposing the NPG and transversality conditions, gives the lifetime budget constraint of the household:

$$\sum_{t=0}^{\infty} R_t c_t(\omega) = \sum_{t=0}^{\infty} R_t q_t + (1 + r_0)k_0(\omega), \text{ where } R_t = \prod_{i=1}^{t}(1 + r_i)^{-1}, \ R_0 = 1. \ (4)$$

Goods are produced by firms from capital and labor using a constant returns to scale production function $f(K, L)$, satisfying the usual assumptions, under conditions of perfect competition. Thus r_t and q_t are given by $r_t = f_K(K_t, 1) - \delta$ and $q_t = f_L(K_t, 1)$, where $K_t = \int_0^1 k_t(\omega)d\omega$, $f_K = \partial f/\partial K$, $f_L = \partial f/\partial L$, and $0 \leq \delta \leq 1$ is the depreciation rate. For convenience, below, we also write $f(K)$ in place of $f(K, 1)$ and use the notation $f' = f_K$ and $f'' = f_{KK}$. In addition we have the aggregate capital accumulation equation

$$K_{t+1} = (1 - \delta)K_t + f(K_t) - C_t, \text{ where } C_t = \int_0^1 c_t(\omega)d\omega. \qquad (5)$$

In this model, the definition of an intertemporal perfect foresight equilibrium follows from routine considerations. The model has a unique a (unique) perfect foresight steady state, on which we focus attention.

Definition 1 *The perfect foresight steady state $K_t = k_t(\omega) = \bar{K}$, $C_t = c_t(\omega) = \bar{C}$, $r_t = \bar{r}$ and $q_t = \bar{q}$ is given by $1 = \beta(1 + \bar{r})$, $\bar{r} = f_K(\bar{K}, 1) - \delta$, $\bar{q} = f_L(\bar{K}, 1)$, $\bar{C} = f(\bar{K}, 1) - \delta\bar{K} = \bar{r}\bar{K} + \bar{q}$.*

If $K_0 = \bar{K}$ then under perfect foresight the economy stays in the steady state for all t, and if $K_0 \neq \bar{K}$, then there is a unique perfect foresight path that converges to the steady state as $t \to \infty$. We now assume that the economy is initially in the steady state, with $k_0(\omega) = \bar{K}$ for all ω, and we examine the robustness of expectational coordination on this equilibrium.

2.2 Beliefs, actions, plans and realizations

Consider an individual agent facing the consumption/savings problem (1)-(2). The behavior of the agent is in part determined by his beliefs about the future values of wages and interest rates. In general, an agent's beliefs could be stochastic, summarized by a sequence of joint density functions $\{F_t(q^t, r^t)\}$, where q^t and r^t are the time t wage and interest rate histories, respectively. We restrict attention to deterministic beliefs, i.e. to point expectations. This is not only justified by the fact that our reference solution is nonstochastic, but also, as we will make clear later, because we focus on variations in beliefs that are small deviations from perfect foresight beliefs.

The beliefs of agent ω may therefore be summarized by real sequences of expected wages and interest rates. We choose to assume that the agent understands the relationship between aggregate capital and input prices, that is, the agent knows $r_t = f'(K_t) - \delta$, and $q_t = f(K_t) - f'(K_t)K_t$. Hence his beliefs are completely and consistently captured by a sequence of real numbers identifying his point expectations of future capital stock.[5] We denote these beliefs by $K^e(\omega) = \{K_t^e(\omega)\}_{t\geq 0}$, where $K_0^e(\omega) = K_0 = \bar{K}$ is known to all agents. A beliefs profile is the collection of all agents' beliefs: $K^e = \{K^e(\omega) : \omega \in [0,1]\}$.

The key ingredient of the analysis is the understanding of the effect of changes in individual expectations on changes in individual actions or plans (particularly when these changes occur around the equilibrium). Taking as the reference point the perfect foresight steady-state path $K_0 = \bar{K}, K_t = \bar{K}$, for all t, we examine the solution to the program, which determines the agent's present actions and future plans. This is simply the program (1) subject to (4), where q_t and r_t are replaced by $q_t^e(\omega)$ and $r_t^e(\omega)$, derived as just explained from $K_t^e(\omega)$.

We focus on small changes, around the steady-state values, in the individual agent's initial capital $k_0(\omega)$ and point expectations $K_t^e(\omega)$, and on the effect of these small changes on the agent's initial plans. We measure these small

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[5] Note that a different assumption, that people have beliefs on wages and interest rates inconsistent with the competitive behaviour of firms, would make expectational coordination more difficult and not easier.

changes in capital as deviations from steady state: we set $dk_0(\omega) = k_0(\omega) - \bar{K}$ and $dK_t^e(\omega) = K_t^e(\omega) - \bar{K}$. We also write $dK^e(\omega) = \{dK_t^e(\omega)\}_{t\geq 0}$ for the beliefs path of agent ω, and $dK^e = \{dK^e(\omega) : \omega \in [0, 1]\}$. Similarly, for the agent's corresponding optimal plans $k_t(\omega)$ and $c_t(\omega)$, which as we will see are fully determined by their beliefs and their initial capital holdings, we write $dk_t(\omega) = k_t(\omega) - \bar{K}$ and $dc_t(\omega) = c_t(\omega) - \bar{C}$. Throughout the paper we maintain the assumption that each agent's beliefs are such that their plans are well captured by first-order approximations. Given this assumption, we can identify a particular variable's time path with its first-order approximation, and thus from now on, we use our deviation notation to capture this identification.

2.2.1   Expectations and plans of agents: the preparation theorem

Our objective is, as argued above, to determine the change in the agent's plans as functions of the changes in the agent's expectations and initial savings. We will make the connections between changes in expectations and changes in planned actions fully explicit in Theorem 1, which we call the Preparation Theorem. We begin with several lemmas that will provide intuition for later results. Here, as below, we use the first-order viewpoint, with the just-introduced notation. For the first lemma it is useful to allow for variations in the initial aggregate capital stock, i.e. $dK_0 \neq 0$.

Lemma 1 (Welfare Lemma). *Consider any time path of beliefs $dK_t^e(\omega)$, and any initial saving $dk_0(\omega)$ and aggregate capital stock dK_0. Let $dc_t(\omega)$ be the associated sequence of consumption decisions. Then*

$$\beta^{-1}dk_0(\omega) = \sum_{t\geq 0} \beta^t dc_t(\omega). \tag{6}$$

Lemma 1, which is proven in the Appendix, relies on the fact that, given the individual budget constraint (4), the equilibrium plan $K_t = k_t(\omega) = \bar{K}$, $C_t = c_t(\omega) = \bar{C}$, remains just feasible, to first-order approximation. Indeed, as shown in the proof,

$$\sum_{t\geq 0} \beta^t dc_t(\omega) = \beta^{-1}dk_0(\omega) + \sum_{t\geq 0} \beta^t(dq_t + \bar{K}dr_t),$$

where the income effect is captured by $\sum_{t\geq 0} \beta^t(dq_t + \bar{K}dr_t)$, i.e. the change in the present value of wage and rental income due to a change in current and expected aggregate capital stocks. However, constant returns to scale technology implies $\bar{K}f_{KK}(\bar{K}, 1) + f_{KL}(\bar{K}, 1) = 0$, so that $dq_t + \bar{K}dr_t = 0$, from

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which the result follows. In summary, to first order, changes in current and expected aggregate capital stock have zero income effect.

Lemma 1 implies in particular that, given beliefs $dK_t^e(\omega)$, if $k_0(\omega) = \bar{K}$ then the optimal consumption path satisfies $\sum \beta^t dc_t(\omega) = 0$. Thus a change in the expected path of aggregate capital, and the corresponding expected price changes that it triggers, have no first-order impact on welfare, since it leaves the present value of consumption unchanged. This is the reason we call the above result the Welfare Lemma.[6] The next two lemmas complete the intuitive picture of how expectations affect plans and actions. The household Euler equation (3) leads to (7) and the flow budget constraint (2) leads to (8). See the Appendix for details.

Lemma 2 *The consumption and savings paths for agent ω satisfy*

$$
\begin{aligned}
dc_t(\omega) &= dc_{t-1}(\omega) + \sigma^{-1}\beta\bar{C}f''dK_t^e(\omega) & (7)\\
dk_{t+1}(\omega) &= \beta^{-1}dk_t(\omega) - dc_t(\omega), & (8)
\end{aligned}
$$

where $\sigma = -\bar{C}U''(\bar{C})/U'(\bar{C})$ is the consumption elasticity of marginal utility and where f'' denotes $f''(\bar{K})$.

It is useful to define the "expectations feedback" parameter

$$
\xi = -\sigma^{-1}(1-\beta)^{-1}\beta^2\bar{C}f'' > 0,
$$

which will be of considerable interest, as it plays a central role in determining the responsiveness of consumption and savings plans to changes in expectations of the aggregate capital path.

Lemma 3 *Given beliefs path $dK^e(\omega)$, the optimal plans for agent ω satisfy*

$$
\begin{aligned}
dc_t(\omega) &= \beta^{-1}(1-\beta)dk_t(\omega) + \beta^{-1}(1-\beta)\xi\sum_{n\geq 1}\beta^n dK_{t+n}^e(\omega) & (9)\\
\\
dk_{t+1}(\omega) &= dk_t(\omega) - \beta^{-1}(1-\beta)\xi\sum_{n\geq 1}\beta^n dK_{t+n}^e(\omega). & (10)
\end{aligned}
$$

Equations (9)-(10) can be interpreted as the time t consumption and saving functions of agent ω, since they relate the agent's consumption and saving at t to his wealth $dk_t(\omega)$, and to expected future prices, captured by $dK_{t+n}^e(\omega)$.[7]

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[6]The Appendix provides the same result for time t, i.e. $\beta^{-1}dk_t(\omega) = \sum_{s\geq t}\beta^s dc_s(\omega)$.

[7]Consumption functions, giving the dependence of consumption on wealth and expected future variables, have a long history and are used, for example, in Eusepi and Preston (2011).

These equations motivate our interpretation of ξ as an expectations feedback parameter: ξ measures the impact on consumption and savings of a permanent unit increase in expected future aggregate capital. The following Theorem gives the optimal plans explicitly in terms of the full belief path $dK^e(\omega)$.

Theorem 1 *(Preparation Theorem). Assume $dk_0(\omega) = 0$. Given beliefs path $dK^e(\omega)$, the optimal plans for agent ω are given by*

$$dc_t(\omega) = -\beta^{-1}(1-\beta)\xi \sum_{T \geq 1} \theta^c(t,T)dK_T^e(\omega), \ \textit{for } t \geq 0, \textit{ and}$$

$$dk_t(\omega) = -\xi \sum_{T \geq 1} \theta^k(t,T)dK_T^e(\omega), \ \textit{for } t \geq 1, \textit{where}$$

$$\theta^c(t,T) = \begin{cases} -\beta^T & \textit{if} \quad 0 \leq t < T \\ 1 - \beta^T & \textit{if} \quad t \geq T \end{cases},$$

$$\theta^k(t,T) = \begin{cases} \beta^{T-t} - \beta^T & \textit{if} \quad 1 \leq t < T \\ 1 - \beta^T & \textit{if} \quad t \geq T \end{cases}.$$

We can also write this as

$$dc(\omega) = \Gamma^c\left(dK^e(\omega)\right) \equiv -\xi\beta^{-1}(1-\beta)\Theta^c dK^e(\omega)$$
$$dk(\omega) = \Gamma^k\left(dK^e(\omega)\right) \equiv -\xi\Theta^k dK^e(\omega),$$

where Θ^c and Θ^k are semi-infinite matrices. Note that the first row of Θ^c determines the first coordinate of $dc(\omega)$, i.e. $dc_0(\omega)$, resulting from $dK^e(\omega)$. Thus the (i,j) element of Θ^c corresponds to $\theta^c(i-1,j)$. Analogously the first row of Θ^k determines the first coordinate of $dk(\omega)$, i.e. $dk_1(\omega)$, resulting from $dK^e(\omega)$. Thus the (i,j) element of Θ^k corresponds to $\theta^k(i,j)$. We have

$$\Theta^c = \begin{pmatrix} -\beta & -\beta^2 & -\beta^3 & \cdots \\ 1-\beta & -\beta^2 & -\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & -\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & 1-\beta^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \Theta^k = \begin{pmatrix} 1-\beta & \beta-\beta^2 & \beta^2-\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & \beta-\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & 1-\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & 1-\beta^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A formal proof of Theorem 1 is available in Appendix A. We here give an alternative argument based on the observation that the Nth column of the matrices describes the impact on the paths of consumption and savings plans of a unit increase in $dK_N^e(\omega)$. Thus consider a thought experiment in which $dK_N^e(\omega) > 0$ and $dK_t^e(\omega) = 0, \forall t \neq N$. Our aim is to derive column N of the

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matrix Γ^c. We do so using the Euler equation (7) and equation (6) of the Welfare Lemma. For the path $dK^e(\omega) = \{dK_t^e(\omega)\}_{t \geq 1}$, the expected interest rate remains constant at its steady-state value in all periods, except for period N in which it changes by $f'' dK_N^e(\omega) < 0$.

By the Euler equation (7) planned consumption remains constant before and after the period of expected change; also, because $dK_N^e(\omega) > 0$ implies a lower expected interest rate at $t = N$, the price of time $N - 1$ consumption is low relative to the price of time N consumption. These observations coupled with (7) may be summarized as follows:

$$
\begin{aligned}
d\check{c}(\omega) &\equiv dc_0(\omega) = \cdots = dc_{N-1}(\omega) = dc_N(\omega) - \beta \bar{C} \sigma^{-1} f'' dK_N^e(\omega) \\
&> dc_N(\omega) = dc_{N+1}(\omega) = \cdots \equiv d\hat{c}(\omega).
\end{aligned}
$$

To pin down the values of $d\check{c}(\omega)$ and $d\hat{c}(\omega)$ we use Lemma 1, which states that the present value of consumption is zero. Thus

$$
\begin{aligned}
0 &= \sum_{t=0}^{N-1} \beta^t dc_t(\omega) + \sum_{t=N}^{\infty} \beta^t dc_t(\omega) = (1 - \beta)^{-1} \left((1 - \beta^N) d\check{c}(\omega) + \beta^N d\hat{c}(\omega) \right) \\
&= (1 - \beta)^{-1} \left(d\hat{c}(\omega) - \left(1 - \beta^N \right) \beta \bar{C} \sigma^{-1} f'' dK_N^e(\omega) \right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
d\check{c}(\omega) &= \xi \beta^{-1} (1 - \beta) \beta^N dK_N^e(\omega) \\
d\hat{c}(\omega) &= -\xi \beta^{-1} (1 - \beta) \left(1 - \beta^N \right) dK_N^e(\omega),
\end{aligned}
$$

in accordance with Theorem 1: see $\theta^c(t, N)$.

Turning now to savings, and continuing with the same thought experiment, we use the flow budget equation (8) to get

$$
dk_1(\omega) = -d\check{c} = -\xi \left(\beta^{N-1} - \beta^N \right) dK_N^e(\omega).
$$

Recursively substituting into the flow budget constraint gives

$$
dk_t(\omega) = -\xi (\beta^{N-t} - \beta^N) dK_N^e(\omega) \text{ for } t \leq N.
$$

See $\theta^k(t, N)$ for $t \leq N$. Since consumption is constant for $t \geq N$, the Welfare Lemma applied to time N (see footnote following Lemma 1) implies that $d\hat{c} = (\beta^{-1} - 1) dk_N(\omega) = \bar{r} dk_N(\omega)$, i.e. agents consume the interest on their savings, and hence their savings is unchanged over time. Thus for $n \geq 1$

$$
dk_{N+n}(\omega) = \beta^{-1} dk_N(\omega) - d\hat{c} = dk_N(\omega) = -\xi (1 - \beta^N) dK_N^e(\omega),
$$

as indicated by Theorem 1: see $\theta^k(t, N)$ for $t > N$. To summarize we have:

$$dk_t(\omega) = \begin{cases} -\xi \left(\beta^{-t} - 1\right) \beta^N dK_N^e(\omega) \text{ for } t \leq N \\ -\xi \left(\beta^{-N} - 1\right) \beta^N dK_N^e(\omega) \text{ for } t > N \end{cases} .$$
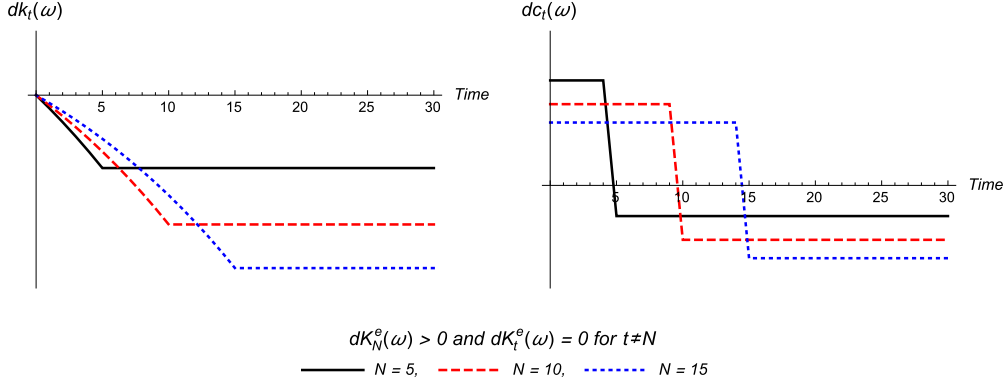

Figure 1: Impact on planned $dk_t(\omega), dc_t(\omega)$ of a future one-period increase of $dK_N^e(\omega)$ for alternative values of N.

Figure 1 illustrates the results for $dK_N^e(\omega) > 0$ and $dK_t^e(\omega) = 0$ for $t \neq N$. Note that the impact on $dk_t(\omega)$ is negative for all $t \geq 1$ while the impact on $dc_t(\omega)$ changes sign over time. We emphasize this point in the next section.

2.2.2   Implications of the Preparation Theorem

The Preparation Theorem, through the expectations feedback parameter ξ and the matrices Θ^c and Θ^k, fully describes the connection between expectations and the planned actions of the individual agents. We will later exploit this knowledge to obtain consequences of these beliefs for describing the possible corresponding paths of aggregate capital. At this stage we provide intuition for later results, limiting attention to discussion of the effects of beliefs on the plans of individual agents.

First we note that the *general sensitivity* of decisions to expectations is governed by the value of ξ. A high ξ means that the individual decisions and plans react strongly to expectations. We discuss later the interpretation of ξ, but it is intuitively clear that a high f'' or a low σ increases expectational sensitivity. In addition to this general expectational sensitivity, our results also stress *the profile effect* of beliefs, which are fully captured by the above

matrices. Taken together, we get rich information – indeed exhaustive to first-order approximation – giving the effects of beliefs on the agent's plans.

This information can be used to provide a more comprehensive intuition about agents' plans. Here we pick out three insights,[8] Corollaries 1 – 3, which are particularly relevant for what follows.

First, as foreshadowed above, a striking and immediate implication from inspecting Θ^k is the following corollary on strategic substitutability.[9]

Corollary 1 *The map Γ^k exhibits local strategic substitutability, i.e.*

$$\frac{\partial \left(dk_t(\omega)\right)}{\partial \left(dK_T^e(\omega)\right)} < 0 \ \text{for all } t, T \geq 1.$$

Second, consider beliefs of a cylinder form, as we shall do later i.e. $-e < dK_t^e(\omega) < e$, where $e > 0$.[10] Implications for savings in period one, $dk_1(\omega)$, which then follow from the first row of Θ^k, are given by Corollary 2.

Corollary 2 *If $-e < dK_t^e(\omega) < e$ for all $t \geq 1$ then $-\xi e < dk_1(\omega) < \xi e$.*

Third, the connection between the infinite path of beliefs and the individual savings decision is further clarified by the following example. Suppose that expected capital is above the steady state by a fixed amount $e > 0$ for $N > 0$ periods before reverting to the steady state. That is, suppose the agent ω's beliefs profile, $dK^e(\omega)$, satisfies

$$dK_T^e(\omega) = \begin{cases} e & \text{for} & T = 1, \ldots, N \\ 0 & \text{for} & T > N \end{cases} . \tag{11}$$

Because of its later importance, we call these "N-period deviation" beliefs. By acting on $dK^e(\omega)$, the operators Γ^k and Γ^c yield the savings and consumption plans of agent ω, respectively: $dk(\omega) = \Gamma^k \left(dK^e(\omega)\right)$ and $dc(\omega) = \Gamma^c \left(dK^e(\omega)\right)$.

Agent ω believes the real interest rate will be at its steady-state value after period N; thus he will hold consumption and savings constant from period N onward. These constant values can be determined using the operators Γ^k and Γ^c. The Extended Appendix shows that we obtain:

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[8] There are many others. For example, the reader is invited to compute the sum of the rows of each matrix to evaluate the effect of extreme beliefs.

[9] This property suggests that we could identify the set of rationalizable trajectories in the RBC model using the findings of Guesnerie-Jara-Moroni (2010), who characterize rationalizable equilibria in finite-dimensional models with strategic substitutabilities. Because our model is infinite dimensional, the results in our setting are not immediate, however, and left for future research. The results of next section suggest that the set under discussion is (very) large.

[10] Here, of course, e is small enough that first-order approximations apply.

Corollary 3 *Suppose an agent has N-period deviation beliefs (11). Then*

$$dk_t(\omega) = \begin{cases} -\xi \left(t - \bar{r}^{-1}(1 - \beta^t)\beta^{N-t}\right) e & \text{for } 1 \leq t < N \\ -\xi \left(N - \bar{r}^{-1}(1 - \beta^N)\right) e & \text{for } t \geq N \end{cases} \tag{12}$$

$$dc_t(\omega) = \begin{cases} \xi \left(1 - \beta^N - \bar{r}t\right) e & \text{for } 0 \leq t < N \\ \bar{r}dk_t(\omega) & \text{for } t \geq N \end{cases}. \tag{13}$$

Note that for $e > 0$ it follows that $dc_t(\omega) > 0$, that $dc_t(\omega)$ declines over time for $t < N$, and that $dc_t(\omega) < 0$ for $t \geq N$. For $t \geq N$ equation (13) is familiar: agent ω consumes the net return to savings, leaving the savings stock unchanged. Thus the expectation of a period of low interest rates associated with $dK_T^e(\omega) = e > 0$ for $T \leq N$ leads agents to shift consumption from the future to the present. However, for our purposes equation (12) is more significant. Most strikingly, for $t \geq N$ the right-hand side implies that $dk_t(\omega)$ becomes large (in magnitude) as N gets large. Thus even if agent ω thinks that capital deviates from steady state by only some small e, and only for some finite period of time, his savings path will move away from, and remain arbitrarily far from, the steady state, provided that period is long enough. The implications of Corollary 3 are shown in Figure 2 in which we set $\beta = 0.95$, $\xi = 0.5$ and $N = 10$.
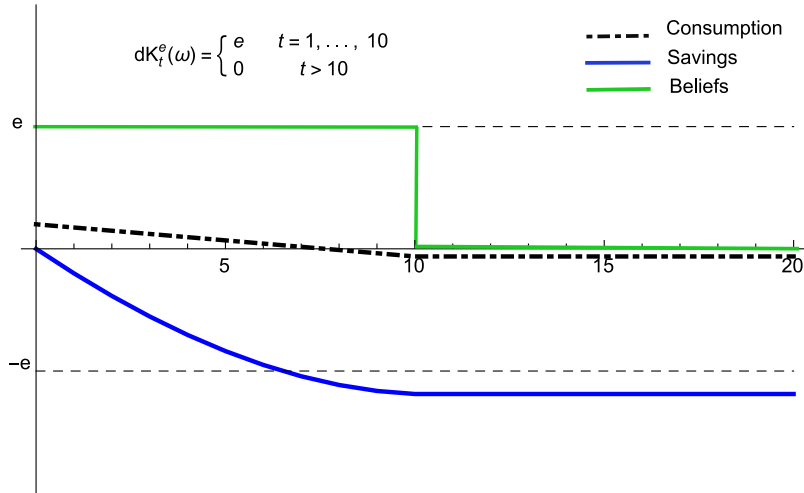

Figure 2: Consumption and savings plan for N-period beliefs when $N = 10$.

With the operators Γ^k and Γ^c in hand we can now turn to our central concern, which is the possibility of rational agents coordinating on the rational expectations steady-state path.

3 Eductive stability

We first provide a definition of eductive stability based on rather abstract game-theoretical considerations. This will be the "high-tech" view on expectational coordination. However we will later see how the sophisticated high-tech viewpoint has a more intuitive counterpart that may be termed "low-tech." Our analysis initially will leave in the shadow the time dimension of the problem. After illustrating the concepts in a simple cobweb framework, we then reintroduce time and adapt the general ideas to our infinite horizon setting.

3.1 Local eductive stability: the high-tech view

We begin by considering an abstract economy populated with rational economic agents (in all the following, we shall assume that these agents are infinitesimal, with the collection modelled as a continuum). The agents know the logic of the collective economic interactions (the underlying model). Both the rationality of the agents and the model are Common Knowledge (CK). The state of the system is denoted E and belongs to some subset $\mathcal{E}$ of some vector space.[11] Emphasizing the expectational aspects of the problem, we view an equilibrium of the system as a state E^* such that if everybody believes that it prevails, it does prevail.[12]

Under eductive learning, as described below, each agent contemplates the possible states of the economy implied by the beliefs and associated actions of the economy's agents. Coordination on a particular equilibrium outcome obtains when this contemplation, together with the knowledge that all agents are engaged in the same contemplation, rules out all potential economic outcomes except the equilibrium. If coordination on an equilibrium is implied by the eductive learning process, then we say that the equilibrium is eductively stable.[13] We call this the "high-tech" view since it relies on higher-order mental reasoning. The argument can be either global or local. We now introduce the local version of eductive stability. As we will see, even local stability is at

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[11] Note that E can be a number (the value of an equilibrium price or a growth rate), a vector (of equilibrium prices), a function (an equilibrium demand function), an infinite trajectory of states, as will be the case in this paper, or a probability distribution.

[12] Note that E^* is such that the assertion "it is CK that $E = E^*$" is meaningful.

[13] The term "strongly rational" is also used. For details on the game theoretic background of our investigation see Guesnerie and Jara-Moroni (2011). A study within a "normal form" framework echoing the preoccupations of the present paper can be found in Matsui and Oyama (2006). We also remark that we here view eductive stability as a zero-one criterion. Less stringent indices of stability could also be developed, e.g. see Desgranges and Ghosal (2010) for one such approach.

issue in our infinite-horizon model.

Formally, we say that E^* is locally eductively stable if and only if one can find some (non-trivial) small neighborhood of E^*, $V(E^*)$, such that Assertion A implies Assertion B:

Assertion A : It is "hypothetically" CK that $E \in V(E^*)$.

Assertion B : It is CK that $E = E^*$.

Assertion A is at this stage hypothetical.[14] In the stable case the mental process that leads from Assertion A to Assertion B is the following:

1. Because everybody knows that $E \in V(E^*)$, everybody knows that everybody limits their responses to actions that are best responses to some probability distributions over $V(E^*)$. It follows that everybody knows that the state of the system will be in a set $\mathcal{E}(1) \subset \mathcal{E}$.

2. If $\mathcal{E}(1)$ is a proper subset of $V(E^*)$, the mental process goes on as in step 1, but based now on $\mathcal{E}(1)$ instead of $V(E^*)$. In this case it follows that everybody knows that the state of the system will be in a set $\mathcal{E}(2) \subset \mathcal{E}$.

3. The process continues inductively provided that at each stage, $\mathcal{E}(n)$ is a proper subset of $\mathcal{E}(n-1)$.

In the stable case, we have a decreasing sequence $V(E^*) \supset \mathcal{E}(1) \supset \cdots \supset \mathcal{E}(n-1) \supset \mathcal{E}(n)$.[15] When the sequence converges to E^*, the equilibrium is locally eductively stable. Here "locally" refers to the fact that the initial neighborhood is small.[16] Note also that intuitively, in the small neighborhood case, whenever the first step conclusion obtains the next step normally follows.[17]

Although inspired by game-theoretic considerations,[18] we view eductive stability as a natural expectational coordination criterion that has independent merit. The connections of the approach with the game theoretical viewpoint, for example the relationship between eductive stability and uniqueness of rationalizable strategies, is assessed in Guesnerie, Jara-Moroni (2011).[19]

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[14] Although it might be sustained by some policy commitment.

[15] A given economic model may be naturally allied with several distinct common knowledge assumptions: these common knowledge assumptions will all impose the recursive reasoning process described above, but will differ in the initial restrictions on agents' beliefs; and because of the central role the initial restriction plays in the eductive learning process, different common knowledge assumptions may produce different stability results.

[16] If the initial neighbourhood is $\mathcal{E}$ then the word global replaces the word local.

[17] At step 1, $\mathcal{E}(1) \subset V(E^*) \subset \mathcal{E}$ is CK and the mental process goes on in step 2, so that the first step contraction still acts with a decreased support. See footnote 22, below.

[18] See the literature on rationalizable beliefs, e.g. Bernheim (1984) and Pearce (1984).

[19] Most of their analysis is conducted in a finite-dimensional state space $\mathcal{E}$. Thus their

3.2 Example: Cobweb Model

To illustrate the eductive approach, consider the cobweb model, interpreted as a producers' game in which the strategy of each firm is its output and the optimal choice of output depends on expected price, which then can be viewed as the previous one-dimensional state E. This example is analyzed in detail in Guesnerie (1992). For simplicity, consider a static, nonstochastic model and assume that market demand $d(p)$ is decreasing in price p and that for each firm the cost of producing quantity q is $(2C)^{-1}q^2$, with $C > 0$. There is a continuum of firms $\omega \in [0, 1]$. Each firm ω must make its production decision before the price is known, and chooses its quantity $q(\omega)$ based on expected price $p^e(\omega)$. Thus $q(\omega) = Cp^e(\omega)$ and with market clearing $d(p) = \int q(\omega)d\omega = C \int p^e(\omega)d\omega$. There is a unique perfect foresight equilibrium $\bar{p}$, satisfying $d(\bar{p}) = C\bar{p}$, and $p^e(\omega) = \bar{p}$ for all ω. For small deviations in expected price, the equilibrium price p is given to first order by $p = \mu - \phi \int p^e(\omega)d\omega$, for appropriate μ and $\phi = -C/d'(\bar{p}) > 0$. We now ask whether individually rational agents endowed with common knowledge, as previously discussed, would necessarily coordinate on rational expectations.

The eductive argument works as follows. Let $V(\bar{p})$ denote a small open symmetric interval containing $\bar{p}$. Suppose all agents believe that $p \in V(\bar{p})$. It follows that $p \in \mathcal{E}(1) \equiv \phi V(\bar{p})$, and if $\phi < 1$ then $\mathcal{E}(1)$ is a proper subset of $V(\bar{p})$. Continuing in the stable case, if, in addition, it is hypothetically common knowledge that $p \in V(\bar{p})$, then $p^e(\omega) \in V(\bar{p})$ for all ω and this is also common knowledge. It then follows that it is common knowledge that $p \in \mathcal{E}(1)$. Iterating this argument it follows that $p^e(\omega) \in \mathcal{E}(N) \equiv \phi^N V(\bar{p})$ for all $N = 0, 1, 2, \ldots$, so that high-tech eductive stability is implied. In contrast, eductive stability fails if $\phi > 1$ since, even under the hypothetical common knowledge assumption $p \in V(\bar{p})$, hyper-rational agents are unable to coordinate on $p = \bar{p}$ through mental reasoning.

3.3 Local eductive stability: the low-tech interpretation

The above definition, based on the successful deletion of non-best responses and starting under the assumption that the state of the system is close to the equilibrium state, reflects the local version of a "hyper-rationality" viewpoint. An alternative "low-tech" definition of local expectational stability is the following: there exists a non-trivial neighborhood of the equilibrium such that, if everybody believes that the state of the system is in this neighborhood, it

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results would require an extension to an infinte-dimensional setting to ensure applicability to our RBC framework.

is necessarily the case, whatever the specific form taken by everybody's belief, that the state is in the given neighborhood.[20] With the above formal apparatus, the definition is: one can find some non-trivial small neighborhood of E^*, $V(E^*)$, such that if everybody believes that $E \in V(E^*)$, then the state of the system will be in $\mathcal{E}(1)$, a proper subset of $V(E^*)$. Again, $V(E^*)$ is an initial belief assumption, a universally shared conjecture on the set of possible states, and for low-tech eductive stability we require that this belief cannot be falsified by actual outcomes resulting from individual actions that are best responses to some probability distributions over $V(E^*)$.[21] The argument is here low-tech in the sense that it refers to the rationality of agents, but not to CK of rationality or of the model, and does not even require full knowledge of the model: the criterion focuses on agents' actions which depend only on their beliefs about the state of the system, and not on their beliefs about other agents' beliefs. To put it another way, the criterion appeals only to the results of the first step of the high-tech criterion and emphasizes the one-step elasticity of realizations to expectations. However, we have argued above that it is intuitively plausible that the local high-tech and low-tech criteria turn out to be equivalent in this abstract setting, as previously stressed in the literature.[22] In the cobweb example with $\phi < 1$, low-tech eductive stability follows directly from $\mathcal{E}(1)$ being a proper subset of $V(\bar{p})$.

Finally, some words are in order concerning the connections between the eductive viewpoint and the "evolutive" or adaptive learning viewpoint. At this point let us only say that the failure to find a set $V(E^*)$ for which the equilibrium is locally eductively stable signals a tendency for any near-equilibrium states of beliefs, a priori reachable through some "reasonable" evolutive updating process, to be driven away in some cases, a fact that threatens the convergence of the corresponding learning rule.[23]

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[20]The conjectural equilibrium bounds discussed by Benhabib and Bull (1988), in the context of the overlapping generations model of money, has a similar motivation.

[21]Equivalently, in the absence of such a neighborhood $V(E^*)$, facts may falsify any "collective" conjecture, whatever the proximity of the conjectured set to the equilibrium (unless the conjecture is reduced to the equilibrium E^* itself).

[22]A formal statement of the equivalence requires additional technical assumptions such as the (weak) assumptions stressed of Guesnerie and Jara-Moroni (2011). Their results also allow one to show that the local analysis may concentrate on heterogenous point-expectations.

[23]And certainly forbids a strong form of "monotonic" convergence: see Guesnerie (2002), Guesnerie and Woodford (1991) and Gauthier and Guesnerie (2004).

3.4 Eductive stability within the RBC model

We now return to the RBC model, and in line with the above discussion we begin with the high-tech viewpoint. The time dimension of our problem, and in particular the infinite horizon, as well as the fact that agents are infinitely-lived, brings some additional features to our general framework and then raises some additional issues.

The equilibrium E^* under consideration is given by $dK_t = 0$ for all t, where here and in the sequel we use the deviation notation of Section 2.2. The first issue concerns the notion of a neighborhood $V(E^*)$, which is less straightforward here than in timeless or short-horizon contexts. Consider the economy at time zero with $dk_0(\omega) = 0$ for all ω. We begin with a simple, natural restriction on the initial time paths of beliefs, that they lie within an ε-neighborhood of the steady state, which might be called the "cylinder" assumption. In later sections we consider alternative initial assumptions. The cylinder assumption is:[24]

B: For some $\varepsilon > 0$ sufficiently small, $dK_t \in D(\varepsilon) \equiv [-\varepsilon, \varepsilon]$ for all t.

Under the high-tech eductive approach, we assume that agents are rational, know the model (including the connections between r, w and K), know that all agents are rational, and know that these facts are common knowledge. Finally, we hypothetically assume that B is common knowledge:

CK: It is hypothetically Common Knowledge that B is satisfied.

Given our emphasis on point expectations, which is justified for a small neighborhood of beliefs, a point expectations beliefs profile K^e, as introduced above, satisfies B provided that $dK_t^e(\omega) \in D(\varepsilon)$ for all ω and t. Furthermore, with CK, we assume that agent ω' knows that $dK_t^e(\omega) \in D(\varepsilon)$, and that agent ω'' knows that agent ω' knows that $dK_t^e(\omega) \in D(\varepsilon)$, etc.

Definition 2 *The steady state is CK strongly eductively stable if CK implies that it is common knowledge that the equilibrium path, $dK_t = 0$ for all t, will take place.*[25]

When we need to be explicit we index B by B(ε) and CK by CK(ε).

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[24]Recall that we restrict attention to paths sufficiently close to the steady state that first-order approximations remain valid.

[25]We slightly depart from previous terminology by leaving the local aspects of the analysis implicit (and not referring to local eductive stability) and by using the word "strong" to stress that the simultaneous coordination at time zero concerns the entire time path of aggregate capital.

We denote by $dK_t = \int dk_t(\omega)d\omega$ the time t aggregate planned savings of agents, where we use our first-order notation for convenience. A proof of strong eductive stability would then proceed as follows. The first stage of the mental process places attention on the possible aggregate plans $dK = \{dK_t\}_{t \geq 0}$ generated by the infinite-horizon plans $\{dk_t(\omega)\}_{t \geq 0, \omega \in [0,1]}$ associated with a profile $dK^e = \{dK_t^e(\omega)\}_{t \geq 0, \omega \in [0,1]}$ of initial beliefs satisfying B. If these possible paths dK restrict the initial beliefs, then the process continues given these new belief restrictions, and so on. If this mental process converges, the agents' CK beliefs about the aggregate path coincide with the equilibrium path: they correctly predict the next period's state and all future states resulting from the guessed actions and plans; in our terminology, the steady state is strongly eductively stable. Hence, in our intertemporal context, while the high-tech story is more sophisticated than it is in the abstract timeless or one-period context introduced above, it follows the same logic: rational agents relying on CK are able to deduce at time zero that $dK_t = 0$ for all t.

We now move to a low-tech interpretation for our model consistent with Section 3.3, which is given by the following definition:

Definition 3 *The steady state is B strongly stable under the low-tech viewpoint if beliefs consistent with B imply aggregate plans consistent with B.*

In the timeless environment examined in Sections 3.1-3.3, aggregate plans are fully determined by the beliefs of agents, making the low-tech implementation unambiguous. In our dynamic setting, however, the realizations of the aggregate over time can differ from agents' expectations, leading them to change their plans, and possibly to revise their expectations of the future. We will come to the question of revised expectations later in Section 5.

At this stage, we introduce an alternative low-tech interpretation in our model, which emphasizes the real-time evolution of the economy when initial beliefs $\{K_t^e(\omega)\}_{t=1}^{\infty}$ are strictly maintained as the economy evolves.

Definition 4 *The steady state is B strongly stable, under the low-tech viewpoint, in the alternative sense, if beliefs consistent with B and maintained over time imply aggregate trajectories consistent with B, i.e. $\{dK_s\}_{s=1}^{\infty} \in D^{\infty}(\varepsilon) \equiv D(\varepsilon) \times D(\varepsilon) \times \cdots$, when each agent ω chooses $dc_t(\omega), dk_{t+1}(\omega)$ optimally given his savings $dk_t(\omega), dK_t$ and his expectations $\{dK_s^e(\omega)\}_{s=t+1}^{\infty}$.*

There are close connections between these three notions of stability:[26]

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[26] Indeed, with some additional assumptions the three definitions of strong stabilty are equivalent when ε is small enough. The required additional assumptions for the equivalence would have the same flavor as those referred to in footnote 22.

Proposition 1 *A failure of strong stability under the low-tech viewpoint of Definition 3 implies both a failure of strong stability in the sense of Definition 2 and in the alternative sense of Definition 4.*

In the following section, we will rely on this proposition to stress that a failure of strong stability under the first low-tech definition implies a failure of strong stability in all senses.

4 Impossibility of Strong Eductive Stability

The principle of our investigation is the following: we focus on individual agents ω noting that given their expectations of the aggregate capital stock $\{dK_t^e(\omega)\}_{t=1}^{\infty}$ they formulate optimal dynamic consumption and savings plans, $\{dc_t(\omega)\}_{t=0}^{\infty}$ and $\{dk_t(\omega)\}_{t=1}^{\infty}$. These individual plans generate an aggregate consumption trajectory $dC_t = \int_0^1 dc_t(\omega)d\omega$, for $t \geq 0$, and hence a corresponding trajectory for the aggregate capital stock $\{dK_t\}_{t=1}^{\infty}$. We note that if all agents hold the same expectations then $dC_t = dc_t(\omega)$ and $dK_t = dk_t(\omega)$, and in particular the map Γ^k determines the implied path of aggregate capital. From Proposition 1 a necessary condition for strong stability is that for homogeneous beliefs consistent with B the implied paths lie within $D^{\infty}(\varepsilon)$.

4.1 Preliminaries: weak eductive stability

From the above analysis, a prerequisite to strong stability, whatever its definition, is the following: under the beliefs restrictions B, the optimal plans of agents necessarily lead to consumption and saving decisions $dc_0(\omega), dk_1(\omega)$ that imply $dK_1 \in D(\varepsilon)$. This condition, clearly a necessary condition, will be called *weak eductive stability.*

Definition 5 *The steady state is weakly eductively stable if, given the initial condition $dk_0(\omega) = 0$ for all agents ω, and given that all agents' beliefs satisfy B, then the aggregate capital stock in period $t = 1$ implied by the agents' optimal plans satisfies $dK_1 \in D(\varepsilon)$.*

Using the above remarks and Corollary 1 we immediately conclude:

Corollary 4 *Under B the steady state is weakly eductively stable if $\xi < 1$ (and only if $\xi \leq 1$).*

We have earlier stressed the key role of the parameter ξ in relating real-izations to expectations. The condition $\xi < 1$ implies that high σ and low f'' promote eductive stability. This makes intuitive sense: if f'' is small then a given change in the path $dK^e(\omega)$ results in a small change in the expected interest rate path $dr^e(\omega)$. If σ is large then the substitution effect is small, leading to a small change $dc_0(\omega)$. Thus when f'' is small and σ is large, changes in $dK^e(\omega)$ result in a small change dC_0 in consumption at $t = 0$ and hence a small change in first-period capital dK_1.

For example, suppose production is Cobb-Douglas, so that $f(K) = K^\varphi$, for $0 < \varphi < 1$, and utility takes the form $U(c) = (c^{(1-\sigma)} - 1)/(1 - \sigma)$, for $\sigma > 0$. Then $\bar{K} = \left(\varphi\left(\bar{r} + \delta\right)^{-1}\right)^{1/(1-\varphi)}$ and $f'' = \varphi(\varphi - 1)\bar{K}^{\varphi-2}$. Using these and the steady-state equations $\bar{r} = \beta^{-1} - 1$ and $\bar{C} = \bar{K}^\varphi - \delta\bar{K}$, it can be computed that if, say, $\varphi = 1/3$, $\bar{r} = 0.05$ and $\delta = 0.10$, then we have weak eductive stability if and only if $\sigma > 2/3$. This condition is perhaps plausibly satisfied, though we have obtained the somewhat surprising result that even weak eductive stability of the RBC model cannot be taken for granted.

One might also naturally ask for what length of time the compatibility of plans and beliefs can be maintained if $\xi < 1$. Using a limiting version of equation (12) one can show that aggregate planned savings dK_t remains within the ε-cylinder for $t < \xi^{-1}$. In this sense, aggregate plans remain compatible with beliefs for a longer period of time the smaller is the expectational feedback parameter ξ.

4.2 Strong eductive stability: impossibility theorems

We now turn to several central results in which we establish the failure of (local) strong eductive stability.

4.2.1 The impossibility of CK strong eductive stability

In Corollary 4 we gave a condition for a minimal consistency requirement: given beliefs B, the initial plans of agents will necessarily be consistent with B *in the first period* $t = 1$ whenever the stated condition $\xi < 1$ is satisfied. However, this is only a weak necessary condition for coordination on the full equilibrium path. Recall that $D(\varepsilon) = [-\varepsilon, \varepsilon]$. For strong eductive stability of the steady state under B(ε), it is necessary[27] that the aggregate planned tra-jectories of aggregate capital lie in $D(\varepsilon)$. Conversely, as stressed in Proposition 1, if for a given beliefs path satisfying B(ε), the corresponding trajectory of

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[27]And in fact, as argued above, this is almost sufficient.

aggregate capital does not remain in $D^\infty(\varepsilon)$, this implies eductive instability, whichever definition is adopted.

Theorem 2 *The steady state is never CK strongly eductively stable.*

We emphasize the ubiquitous instability implied by Theorem 2 and its generalization given by Theorem 3 below: these results hold for all utility and production functions meeting standard assumptions and for all discount factors $0 < \beta < 1$ and depreciation rates $0 \le \delta \le 1$.

We remark that, for the class of homogeneous N-period deviation beliefs given by (11), for ξ sufficiently small the first N such that $|dk_N(\omega)| > \varepsilon$ can be relatively large. This can be viewed as an extension of our weak eductive stability result: if, for all ω, $dK_T^e(\omega) = e$ for $T = 1, \dots N$, where $e = \varepsilon$, and $dK_T^e(\omega) = 0$ for $T > N$, then using equation (12), it can be shown that for $t < \xi^{-1}$ aggregate capital dK_t remains inside $D(\varepsilon)$ in accordance with B.[28]

To summarize the intuition corresponding to Corollary 3, with expectations $dK_s^e(\omega) = e > 0$ for $s \le N$, the optimal plan for agents is to increase consumption today at the expense of future consumption. Assuming expectations homogeneity, if $\xi > 1$ the positive impact on dC_0 is large enough to be immediately destabilizing in the short run in the sense that $dK_1 < -\varepsilon$. However, even for $\xi < 1$, and even if ξ is small, and for homogeneous expectations $dK_s^e(\omega) = e > 0$ for $s \le N$ with $0 < e \le \varepsilon$, even for e small, eventually $dK_t < -\varepsilon$ if N is sufficiently large. The planned future reduction in consumption is not enough to avoid, and is consistent with, dK_t leaving $D(\varepsilon)$. Figure 3 builds on the example from Figure 2, and illustrates the intuition of Theorem 2: by homogeneity the implied path of aggregate capital corresponds to the representative agent's savings plan. Note that ε and e are chosen so that agents' beliefs are consistent with B, while the aggregate planned trajectory of capital escapes $D^\infty(\varepsilon)$.

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[28]For example, the reader will notice that in the previous Cobb-Douglas economy, $\sigma > 2/3$ was enough to get weak eductive stability. Because $\xi = 2/(3\sigma)$ it follows that $\sigma > 3$ implies that the exit time t, just emphasized, is greater than 5.
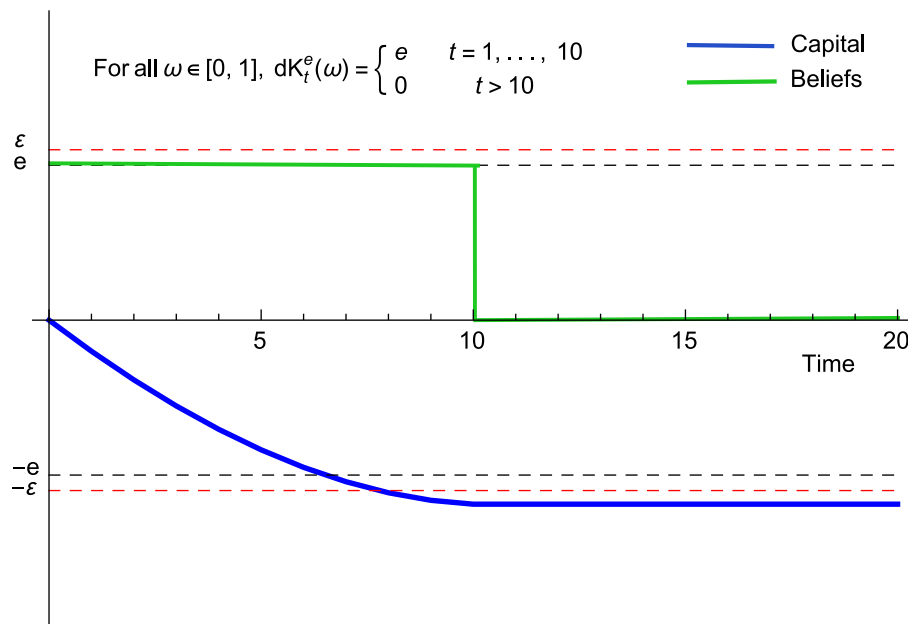
Figure 3: Implied path of aggregate capital for N-period deviation beliefs
with $N = 10$.

Theorem 2 should be contrasted with the fact that, as noted in the Introduction (and as demonstrated in the Appendix), with short-lived agents eductive stability often holds under natural conditions. The long-run horizon dramatically affects expectational coordination, as evaluated from the strong eductive viewpoint.

In the next Section we show that the instability result does not depend on our specific choice of the neighborhood.

4.2.2 The general impossibility of strong eductive stability

We now consider a more general class of beliefs, which nests B:

B′: There exists a specified deterministic sequence $\{\varepsilon_t\}_{t=1}^{\infty}$ with $0 < \varepsilon_t < \bar{\varepsilon}$ and $\bar{\varepsilon}$ sufficiently small, such that $dK_t \in [-\varepsilon_t, \varepsilon_t]$ for all t.

Under B′, all of our concepts and definitions of strong eductive stability are modified in the obvious way. For example we refer to CK′ strong eductive stability when CK is replaced in Definition 2 by CK′ it is hypothetically common knowledge that B′ is satisfied.[29]

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[29]We remark that if we modify CK2 to allow $\varepsilon_t = 0$ for some proper subset of times t,

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Theorem 3 *The steady state is never CK′ strongly eductively stable.*

The negative result of Theorem 3 means in particular that the hyper-rationalistic viewpoint of strong eductive stability is never conclusive.[30] Our hyper-sophisticated agents cannot convince themselves that the rational expectations equilibrium will necessarily prevail.

5 Combining eductive and evolutive learning

We have found that the steady state $\bar{K}$ is not strongly eductively stable according to the various definitions given above: beliefs of the B or B′ type cannot trigger CK of the equilibrium. If common knowledge of full rationality has no predictive power, boundedly rational considerations seem unavoidable and this suggests introducing real-time learning, At the same time, it is known that, for a suitable and natural class of adaptive learning rules, the steady state is locally asymptotically stable. At first sight, this suggests a disconnect between the adaptive and eductive approaches that is much more significant than previously noted in the literature. However, a better way to consider the connection is to adopt the next viewpoint which combines the two approaches, mixing bounded rationality, along the lines of evolutive or adaptive learning, with considerations on the proximity of beliefs to the steady state, associated with the eductive viewpoint.

5.1 The framework

We again endow agents with expectations about the future path of the aggregate capital stock. These expectations are restricted to belong to a set, which for convenience we will take to be the set $D(\varepsilon)$ described by B. This set can here be viewed as describing a collective belief that provides bounds on individual beliefs. Following the eductive approach of this paper, agents' decisions are based on an assessment of the whole future,[31] but in line with

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Theorem 3 still holds.

[30]Instability results also appear in the adaptive learning literature. For example, Howitt (1992) and Evans and Honkapohja (2003) show instability for a class of interest-rate rules in monetary models. However, these models can also suffer from indeterminacy, and stability under adaptive learning can be restored with a suitable choice of interest-rate rule. The generic instability results of the current paper are particularly striking since the RBC model is in general well-behaved.

[31]In the adaptive learning literature, within infinite-horizon models, this approach has been followed, for example, in Sargent (1993, pp. 122-125), Preston (2006), Eusepi and

what we called the alternative sense in Definition 4, we look at the system in real time. Our focus remains on whether the realized path for aggregate capital stays within $D(\varepsilon)$. Now, however, in accordance with the evolutive or adaptive learning viewpoint, the expected trajectory at time t is assumed not only to reflect initial beliefs but also to respond to observed actual capital.

More precisely, we specify a set of adaptive learning rules that determine the way initial expectations change along the real-time trajectory of aggregate capital. Then, coming back to the collective belief interpretation of B, reminiscent of the eductive approach, we ask if the implied path $\{dK_t\}_{t=1}^{\infty}$ will necessarily satisfy B, i.e. if the collective belief, which serves as a frame for the individual beliefs, is subject to real-time falsification. If for some nonempty subset of adaptive learning rules, falsification is impossible, then we say that the steady state is B-stable under evolutive learning, and if this occurs for all adaptive learning rules within the set of rules under consideration, we say the steady state is robustly B-stable under evolutive learning.

5.2   The real-time system

In the real-time system we assume that at each time t each agent ω re-solves their dynamic optimization problem. That is, at each time t agent ω chooses $dk_{t+1}(\omega)$ optimally given their savings, their expectations (which are now revised each period) and the aggregate capital stock. Using Lemma 3, the real-time description of savings behavior is given by

$$dk_{t+1}(\omega) = dk_t(\omega) - \beta^{-1}(1-\beta)\xi \sum_{j \geq 1} \beta^j dK_{t,t+j}^e(\omega), \qquad (14)$$

where now $dK_{t,t+j}^e(\omega)$ is the point expectation of aggregate capital in period $t + j$ held by agent ω in period t, which, as implied by the notation, we now allow to evolve over time.[32]

It can be seen from equation (14) that the agent's decision $dk_{t+1}(\omega)$ depends on a single sufficient statistic for $\left\{dK_{t,t+j}^e(\omega)\right\}_{j=1}^{\infty}$, given by

$$d\hat{K}_t^e(\omega) = \beta^{-1}(1-\beta)\sum_{j \geq 1} \beta^j dK_{t,t+j}^e(\omega).$$

The normalization factor $\beta^{-1}(1-\beta)$ ensures that the sum of the weights on $dK^e_{t,t+j}(\omega)$ is one, so that under B, $d\hat{K}^e_t(\omega)$ must lie in the interval $D(\varepsilon)$. We thus rewrite (14) as $dk_{t+1}(\omega) = dk_t(\omega) - \xi d\hat{K}^e_t(\omega)$, and, following the boundedly rational adaptive learning approach, interpret this equation as providing a behavioral decision rule in which we re-envision the agent's saving choice as depending solely on current wealth and the sufficient statistic $d\hat{K}^e_t(\omega)$. For convenience we refer to $d\hat{K}^e_t(\omega)$ as "expected future capital."

Finally we specify a simple adaptive scheme for the revisions of expectations over time: $d\hat{K}^e_t(\omega) = (1-\alpha)d\hat{K}^e_{t-1}(\omega) + \alpha dK_t$, where $0 < \alpha \leq 1$, called the gain parameter in the literature, describes how expectations reflect current information about the actual capital stock.[33]

We are now in a position to describe the real-time evolution of the system. For the sake of simplicity, we start from an initial situation in which the time-zero belief is the same for everybody:

$$d\hat{K}^e_0(\omega) \equiv \beta^{-1}(1-\beta) \sum_{j \geq 1} \beta^j dK^e_{0,j}(\omega) = d\hat{K}^e_0,$$

and thus the sufficient statistics are initially, and remain, homogeneous across agents. Finally, in line with our earlier analysis, we assume that $dK_0 = 0$.[34]

The homogeneity assumption allows us to calculate the resulting time path, and is also illuminating in the sense that we would hope the system to be stable under learning if we start with small expected future capital $d\hat{K}^e_0$. Under homogeneity the system's dynamics, which depend only on initial $d\hat{K}^e_0$ and on the parameters α and ξ, can be written as

$$dK_{t+1} = dK_t - \xi d\hat{K}^e_t \quad \text{and} \quad d\hat{K}^e_{t+1} = (1-\alpha)d\hat{K}^e_t + \alpha dK_{t+1}. \tag{15}$$

We can now return to the previously suggested concept of B-stability under adaptive learning and give formal definitions.

Definition 6 *The steady state is B-stable under adaptive learning for a given $0 < \alpha \leq 1$ if, for all $\varepsilon > 0$ sufficiently small, $|d\hat{K}^e_0| < \varepsilon$ implies that the trajectory $\{dK_t\}^\infty_{t=1}$, generated by (15) remains in the cylinder $D(\varepsilon)^\infty$.*

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[33]Adaptive learning in nonstochastic models with infinite horizons often assumes that forecasts are the same at all horizons. See, for example, Evans, Honkapohja and Mitra (2009). In the current context this means $dK^e_{t,t+j} = e$ at t for all j. Our formulation in terms of $d\hat{K}^e_t$ allows for greater generality, while retaining a single sufficient statistic that is updated over time. In stochastic models, the time pattern of variables can be estimated and updated using recursive least squares. For technical reasons this procedure cannot be used in nonstochastic systems. Intuitively, in a nonstochastic equilibrium the asymptotic lack of temporal variation makes impossible consistent estimation of time-series parameters. See Evans and Honkapohja (2001, pp. 152-154).

[34]Our results are qualitatively robust to small perturbations of intial aggregate capital.

Definition 7 *The steady state is robustly B-stable under adaptive learning if it is B-stable under adaptive learning for all $0 < \alpha \leq 1$.*

If the steady state is B-stable for some nonempty subset of $0 < \alpha \leq 1$, but is not robustly B-stable, then we will say that it is partially B-stable under adaptive learning.

5.3   The results

The first result is again an impossibility theorem.

Theorem 4 *Under adaptive learning the steady state is not robustly B-stable.*

The proof is given in the Appendix. In fact the result should be expected in view of Proposition 1 and Theorem 2. The trajectory under adaptive learning is continuous in α (for small $\alpha > 0$ the α-trajectory is close to the $\alpha = 0$ trajectory under an appropriate metric). The path of dK_t under adaptive learning and when $\alpha = 0$ corresponds to the path that would be realized by the actions of eductive agents with the constantly held beliefs $dK_t^e(\omega) = d\hat{K}_0^e$. Failure of robust B-stability would then follow by continuity as a result of the impossibility of strong eductive stability in the alternative sense.

A striking feature of Theorem 4 is that instability arises for small α, which in the adaptive and least-squares learning literature is usually viewed as a stabilizing case (the "small gain" limit).[35] In our approach, the problem is that in this case the initial collective belief will be falsified, which we view as a fragility of expectational coordination.

The instability result is stronger than stated in the following sense, which can be verified from the proof of Theorem 4. Given $\varepsilon > 0$, consider any initial beliefs $0 < |d\hat{K}_0^e| < \varepsilon$. Then there exists $\alpha > 0$ such that the corresponding trajectory under adaptive learning will have $|dK_t| > \varepsilon$ for some $t \geq 1$. That is, starting from the steady state, all initial expectations, for appropriate α, lead to paths dK_t that leave $D(\varepsilon)$.

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[35]The connection between eductive stability and stability under evolutive (or adaptive) learning rules has been discussed in Evans and Guesnerie (1993), Guesnerie (2002) and Hommes and Wagener (2010). In short-horizon set-ups, eductive instability is usually reflected in adaptive instability for large gains (here $\alpha < 1$ large). This is seen for the overlapping generations model with money in Guesnerie and Woodford (1991) and Evans and Honkapohja (1995). We also note that in experiments the complex expectational dynamics found in Hommes (2011) for the cobweb model arise in the eductively unstable case we identify in Section 3.2. For related work see Bao and Duffy (2013).

We next turn to partial B-stability under adaptive learning. We will see that a natural necessary condition is given by asymptotic stability of the system (15). Asymptotic stability is the classical stability criterion used for adaptive learning. We therefore start with the following result:

Lemma 4 *System (15) is asymptotically stable if and only if $\xi < 4\alpha^{-1} - 2$.*

Lemma 4 implies asymptotic stability for all α if $\xi < 2$ and for $\alpha < 4(2 + \xi)^{-1}$ if $\xi \geq 2$. For given $\varepsilon > 0$, asymptotic stability implies that $|dK_t|, |d\hat{K}_t^e| \leq \varepsilon$ for t sufficiently large. In fact asymptotic stability is a necessary condition for partial B-stability under adaptive learning.[36] However this is only one necessary condition for partial B-stability under adaptive learning. It is immediate from (15) that $\xi \leq 1$ is another necessary condition: indeed, since $dK_0 = 0$ we have $dK_1 = -\xi d\hat{K}_0^e$. A more complete picture is provided by the following Theorem.

Theorem 5 *We have the following results on partial B-stability and asymptotic stability for the adaptive learning system (15) with $\xi > 0$ and $0 < \alpha \leq 1$:*

1. *Given α, the steady state is B-stable for sufficiently small ξ.*

2. *Given ξ satisfying $\xi \leq 1$:*

   (a) *the steady state is not B-stable for sufficiently small α;*

   (b) *the steady state is B-stable for sufficiently large α;*

   (c) *the steady state is asymptotically stable for all α.*

3. *Given ξ satisfying $\xi > 1$:*

   (a) *the steady state is not B-stable for any α;*

   (b) *for $1 < \xi < 2$, the steady state is asymptotically stable for all α;*

   (c) *for $\xi \geq 2$, it is asymptotically stable for $\alpha < 4(2 + \xi)^{-1}$.*

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[36] To see this, assume the system is not asymptotically stable. It can be shown that the eigenvalues in this case are real. (The proof of Lemma 4 implies that in the complex case the modulus of the eigenvalues is $1 - \alpha$.) We must show that $dK_0 = 0$, $d\hat{K}^e \neq 0$ leads to a divergent path. Noting that $(0, 1)$ is not an eigenvector it follows that all initial conditions $(0, d\tilde{K}^e)$ lead to divergent paths provided at least one of the eigenvalues has magnitude larger than one.

This theorem emphasizes the relevance of the expectations feedback parameter for understanding real-time learning. Indeed, the coefficient ξ, stressed in Sections 2 and 3, plays a key role in both partial B-stability and asymptotic stability under adaptive learning. Clearly small ξ is stabilizing: for any α, B-stability obtains for sufficiently small ξ; and whenever $\xi < 1$ B-stability holds for α sufficiently large. Similarly, large ξ is destabilizing. If $\xi > 1$ then partial B-stability fails: for some initial beliefs satisfying B, the path will falsify these beliefs, *whatever* the specific (asymptotically stable or not) adaptive learning rule used by the agents. Furthermore for $\xi > 2$ asymptotic stability obtains only for α sufficiently small, while for $\xi < 2$ asymptotic stability holds for all α.[37] In the top panel of Figure 4, α is fixed and chosen so that asymptotic stability holds for all examined values of ξ. However, B-stability obtains only for $\xi = 0.1$.

On the other hand, for given ξ it is small values of α that generate the failure of robust B-stability under adaptive learning, even though the asymptotic stability condition $\xi < 4\alpha^{-1} - 2$ is easier to satisfy when $\alpha > 0$ is small. The reason for this is that small α under adaptive learning leads to a cumulative movement of aggregate capital away from the steady-state value, which over finite time periods, as $\alpha \to 0$, track the possible dK_t paths deduced by agents in our eductive setting.[38] This behavior is illustrated in the bottom panel of Figure 4.

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[37]Partial B-stability results for $\xi < 1$ exhibit a trade-off between ξ and α: as $\xi \to 1$ from below, the partial B-stability region tends to $\alpha > 1/2$. Numerical results indicate partial B-stability for $\alpha \in (\gamma(\xi), 1]$ where $\gamma(\xi)$ is continuous and monotonically increasing in ξ with $\gamma(\xi) \to 0$ as $\xi \to 0$. When $\xi = 1$ we have partial B-stability for $\alpha \in [1/2, 1]$. We note a discontinuity in B-stability for given α, which is caused by first-period behavior: $dK_1 = -\xi dK_0^e$ implies B-instability for any $\xi > 1$.

[38]An interesting feature of the lack of robust B-stability under adaptive learning, which can be seen in the proof of Theorem 4 given in the Appendix, is that the instability is associated with long cyclical movements in dK_t.
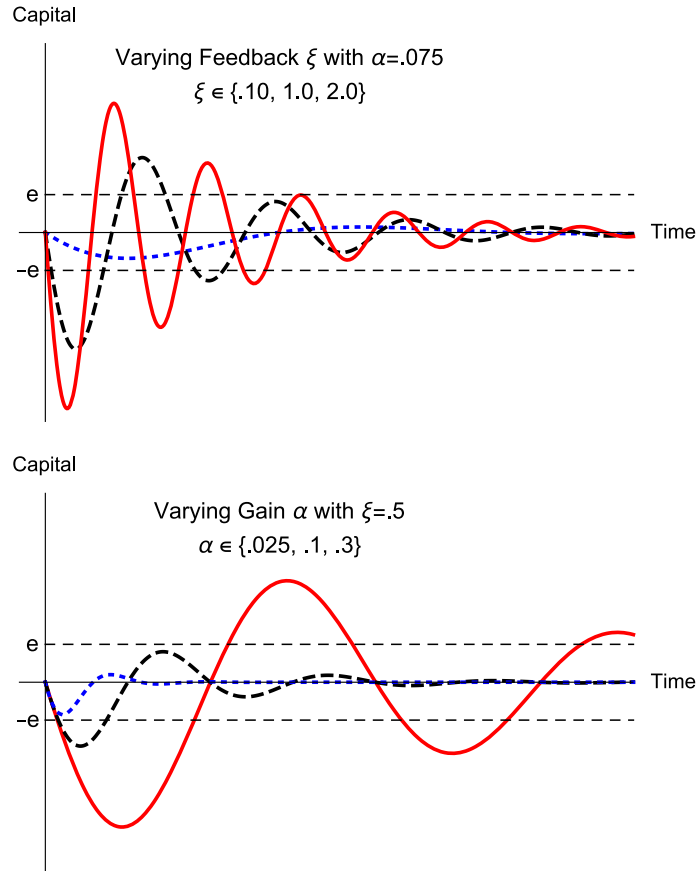
Capital

Varying Feedback ξ with α=.075

$\xi \in \{.10, 1.0, 2.0\}$

e

−e

Time

Capital

Varying Gain α with ξ=.5

$\alpha \in \{.025, .1, .3\}$

e

−e

Time

Figure 4: Aggregate capital paths under adaptive learning. Upper figure: maximum amplitude of paths increasing in feedback parameter ξ. Lower figure: maximum amplitude of paths decreasing in gain parameter α.

6 Discussion

This paper examines the impact of the presence of long-lived agents on the coordination of expectations in a simple RBC model. Expectational coordination is assessed from two competing yet complementary viewpoints: eductive and adaptive.

In both cases we found coordination difficult to achieve. This difficulty relates to the elasticities of economic outcomes to agents' expectations. A basic contribution of the paper is to show that, within the model under considera-

tion, the presence of far-sighted agents implies that these elasticities are high: this is the lesson of the Preparation Theorem. Our analysis emphasizes the key roles of the expectations feedback parameter ξ and the matrices Θ^c and Θ^k, which govern the magnitude of the elasticities and reveal the cumulative effects of small but persistent deviations of expectations from the equilibrium.

The Preparation Theorem implies a generic failure of strong eductive stability in the RBC model: hypothetical common knowledge assumptions cannot trigger common knowledge of the equilibrium.[39] Thus full coordination on the REE, as of time zero, cannot be expected based on full rationality and on common knowledge of the model. Strikingly, this negative result holds independently of the system characteristics, in sharp contrast to previous eductive learning studies that emphasize significant classes of "good cases" in which CK considerations provide support for the possibility of expectational coordination on the REE – see Guesnerie (2005).

There is a sense, however, in which small ξ still favours expectational coordination: $\xi < 1$ is the condition for weak eductive stability. Furthermore, as noted in Section 4.1, the length of time for which plans will necessarily be compatible with beliefs is inversely proportional to ξ. However, even with small ξ the incompatibility of aggregate plans with beliefs will eventually emerge: given an expectation that aggregate K will fall below $\bar{K}$ by a small amount for a long enough stretch of time, the optimal dynamic plans of agents call for them to eventually accumulate capital in excess of $\bar{K}$ by a large amount.

Our instability result, as stressed in the Introduction, is in striking contrast to results for dynamic models with short-lived agents. In Appendix C we study eductive stability in the standard overlapping generations (OLG) model with capital in which agents live for two periods. This OLG model could also be interpreted as a variant of the RBC model in which agents are myopic in the sense that their planning horizon extends only to the following period. In line with the findings stressed in the Introduction, we find that the short planning horizon makes expectational coordination easier. In particular we obtain a strong eductive stability condition that is satisfied for frequently employed parametric specifications of preferences and technology.

Returning to our RBC framework, a logical consequence of the failure of strong eductive stability is that the evolution of the economy requires bounded

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[39]We do not claim however that this difficulty occurs in every model with infinitely-lived agents. For example, it would not occur in the world of Lucas (1978) for the reasons, which have an eductive flavor, that are stressed in Section 6 of his paper. The difference in conclusions reflects the fact that in the Lucas tree model the dividend of a tree is exogenous, while in the RBC setting beliefs of agents affect rental rates though the endogeneity of aggregate capital.

rationality considerations that incorporate adaptive learning rules. We therefore considered paths generated by adaptive learning, and asked whether their dynamics reflect the eductive instability results. We have presented a number of adaptive learning results that link real-time adaptive learning dynamics to the expectational feedback parameter ξ that plays a central role in Sections 2 and 4. However, most strikingly, we provide a strong adaptive learning result that is closely linked to strong eductive instability: Theorem 4 shows that while real-time adaptive learning under small gain is asymptotically stable, the resulting time paths will include periods of large deviations from the REE, even if initial expectations are close to the steady state.

These results suggest that non-negligible deviations from the REE are likely. For example, suppose, starting from a steady state, that there is a small shock to expectations created by some news event. Then not only will coordination on the REE be impossible using eductive learning, but also some plausible adaptive learning rules, even if they return the economy to the steady state asymptotically, will necessarily first lead the economy further *away* from the steady state.

Our eductive and adaptive results thus go hand-in-hand. The lack of strong eductive stability in the RBC model is reflected in a lack of robust adaptive B stability. In addition, the factors governing weaker forms of eductive stability are reflected in asymptotic and partial B stability under adaptive learning. Thus the eductive viewpoint opens another door to the new territories being investigated in adaptive learning studies with long-horizon agents and provides a powerful tool for their exploration.[40]

7 Conclusions

In this paper we have examined the issue of coordination of expectations on the benchmark rational expectations solution in the benchmark RBC model. The difficulties of expectational coordination can be ascertained from two sides, the eductive one and the adaptive one. In both cases, far-sighted agents are sensitive to the whole path of expectations and long-run expectations matter significantly.

It is not surprising that long-run concerns influence present decisions and future plans. However, the sensitivity to expectations of long-run plans envisaged today is extreme, and this is at the heart of the impossibility of strong eductive stability: fully rational agents with strong common knowledge as-

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[40] For RBC models, important adaptive learning dynamics have been noted in Eusepi and Preston (2011), Branch and McGough (2011) and Mitra, Evans and Honkapohja (2013).

sumptions are unable to coordinate on RE. Although rational expectations is clearly the benchmark solution, the failure of strong eductive stability in RBC models indicates that real-time adaptive learning dynamics must play a role.

The RE steady state is asymptotically stable under adaptive learning dynamics, for at least a range of adaptive gain parameters, but the sources of instability obtained in the eductive framework remain pivotal. If the adaptation parameter is large then unstable overshooting can arise in the short run, while if the adaptation rate is small then low-frequency swings over the medium run will necessarily generate instability during the learning transition. Because occasional structural shifts, policy changes or news events will in practice disrupt the equilibrium, transitional learning dynamics will continue to be important.

The analysis of this paper presents a useful starting point, and provides a necessary anchor, for an overdue more general reflection on the appropriate incorporation of expectational coordination in long-horizon economic models. Our findings indicate several directions for future research. First, our generic eductive instability in the RBC model contrasts strikingly with earlier research in short-horizon dynamic models, and with eductive stability in a parametric class of overlapping generations growth models. The standard assumption of infinite-lived agents in the RBC model clearly plays a major role in the analysis of the RBC model, but of course the infinite-lifetime assumption is made for theoretical convenience rather than empirical realism. In future research we therefore plan to study the implications for eductive stability in models with long but finite lifetimes.

The other line of research suggested by our analysis is the incorporation into adaptive learning of considerations that are central to eductive analysis. Adaptive learning is a bounded rationality approach in which agents make forecasts based on statistical or econometric models that are updated over time. An advantage of this approach is that the agents require minimal knowledge of the economic structure and do not need to think explicitly about strategic interaction, factors that are central to the eductive viewpoint. This is an advantage but it may also be a disadvantage. If agents do understand the economic structure and how the expectations of others help to determine key economic aggregates, then it may be possible for them to use this information within an adaptive econometric forecasting framework. Particularly for cases in which eductive stability fails to hold, integration of eductive considerations into adaptive approaches is a natural avenue for future research.

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Appendix A: Proofs of Results

Proof of Lemma 1. The agent's lifetime budget constraint, dated here at time t and with his transversality condition incorporated, is given by

$$\sum_{n=0}^{\infty} R_t^n c_{t+n}(\omega) = \sum_{n=0}^{\infty} R_t^n q_{t+n} + (1+r_t)k_t(\omega), \text{ where } R_t^n = \prod_{i=1}^{n}(1+r_{t+i})^{-1}$$

and $R_t^0 = 1$. We compute total derivatives at the steady state. Noting from $dR_t^n = -\beta^{n+1}\sum_{i=1}^{n} dr_{t+i}$ that $\sum_{n\geq 0} dR_t^n = -\frac{1}{r}\sum_{n\geq 1}\beta^n dr_{t+n}$, we obtain

$$\sum_{n\geq 0}\beta^n dc_{t+n}(\omega) = \beta^{-1}dk_t(\omega) + dq_t + \bar{K}dr_t + \sum_{n\geq 1}\beta^n dq_{t+n} + \bar{r}^{-1}\left(\bar{C}-\bar{q}\right)\sum_{n\geq 1}\beta^n dr_{t+n}.$$

At the steady state, we have $\bar{C} = \bar{q} + \bar{r}\bar{K}$. Also, by constant returns to scale, $q = f(K) - Kf'(K)$, so that $dq + \bar{K}dr = 0$. It follows that

$$\sum_{n\geq 0}\beta^n dc_{t+n}(\omega) = \beta^{-1}dk_t(\omega) + \sum_{n\geq 1}\beta^n(dq_{t+n} + \bar{K}dr_{t+n}) = \beta^{-1}dk_t(\omega).$$

Setting $t = 0$ gives the result claimed in Lemma 1. ∎

Proof of Lemma 2. From the agent's flow budget constraint (2) we have, to first order, that $dk_{t+1}(\omega) = (1+\bar{r})dk_t(\omega) + \bar{K}dr_t + dq_t - dc_t(\omega)$ from which (8) follows using the zero income effect $dq + \bar{K}dr = 0$. Under point expectations equation (3) is $U'(c_t(\omega)) = \beta(1 + r_{t+1}^e(\omega))U'(c_{t+1}(\omega))$. Thus to first order $U''(\bar{C})dc_t(\omega) = \beta(1 + \bar{r})U''(\bar{C})dc_{t+1}(\omega) + \beta U'(\bar{C})dr_{t+1}^e(\omega)$. Using $dr_{t+1}^e(\omega) = f''(\bar{K})dK_{t+1}^e(\omega)$, $\beta(1+\bar{r}) = 1$ and the definition of σ gives (7). ∎

Proof of Lemma 3. By recursive substitution the linearized Euler equation (7) taken at time t gives $dc_{t+n}(\omega) = dc_t(\omega) + \frac{\beta\bar{C}f''}{\sigma}\sum_{s=1}^{n} dK_{t+s}^e(\omega)$. Applying the Welfare Lemma at time t, i.e. $\sum_{n\geq 0}\beta^n dc_{t+n}(\omega) = \beta^{-1}dk_t(\omega)$, gives

$$\beta^{-1}dk_t(\omega) = (1-\beta)^{-1}dc_t(\omega) + \frac{\beta\bar{C}f''}{\sigma}\sum_{n\geq 1}\beta^n\sum_{s=1}^{n} dK_{t+s}^e(\omega).$$

Using $\sum_{n\geq 1}\beta^n\sum_{s=1}^{n} dK_{t+s}^e(\omega) = (1-\beta)^{-1}\sum_{n\geq 1}\beta^n dK_{t+n}^e(\omega)$ yields the optimal plan for consumption (9). Combining this with (8) yields $dk_{t+1}(\omega) = dk_t(\omega) + \frac{\beta\bar{C}f''}{\sigma}\sum_{n\geq 1}\beta^n dK_{t+n}^e(\omega)$, which is equivalent to (10). ∎

Proof of Theorem 1. We start with the savings plan. It suffices to show that

$$dk_t(\omega) = -\xi\sum_{n\geq 1}\theta^k(t,n)dK_n^e(\omega) \tag{16}$$

satisfies (10), where we recall that

$$\xi = -\frac{\beta^2 \bar{C}}{\sigma(1-\beta)}f'' \quad \text{and} \quad \theta^k(t,n) = \begin{cases} 1-\beta^n & \text{if} \quad n \leq t \\ \beta^{n-t}(1-\beta^t) & \text{if} \quad n > t \end{cases}.$$

We compute that $dk_t(\omega) + \frac{\beta \bar{C} f''}{\sigma}\sum_{n\geq 1}\beta^n dK^e_{t+n}(\omega)$

$$
\begin{aligned}
&= -\xi\sum_{n\geq 1}\theta^k(t,n)dK^e_n(\omega) - \xi\left(\frac{1-\beta}{\beta}\right)\sum_{m\geq 1}\beta^m dK^e_{t+m}(\omega) \\
&= -\xi\sum_{n=1}^{t}\theta^k(t,n)dK^e_n(\omega) - \xi\sum_{m\geq 1}\left(\beta^{m-1}(1-\beta) + \beta^m(1-\beta^t)\right)dK^e_{t+m}(\omega) \\
&= -\xi\sum_{n=1}^{t}\theta^k(t,n)dK^e_n(\omega) - \xi\sum_{m\geq 1}\left(\beta^{m-1} - \beta^{t+m}\right)dK^e_{t+m}(\omega) \\
&= -\xi\sum_{n=1}^{t}\theta^k(t+1,n)dK^e_n(\omega) - \xi\sum_{m\geq 1}\beta^{m-1}\left(1 - \beta^{t+1}\right)dK^e_{t+m}(\omega) \\
&= -\xi\sum_{n=1}^{t}\theta^k(t+1,n)dK^e_n(\omega) - \xi\sum_{m\geq 1}\theta(t+1,t+m)dK^e_{t+m}(\omega) \\
&= -\xi\sum_{n\geq 1}\theta^k(t+1,n)dK^e_n(\omega) = dk_{t+1}(\omega).
\end{aligned}
$$

This establishes the first part of Theorem 1.

To prove the second part concerning the consumption plan, we combine (9) from Lemma 3 with (16) to obtain

$$
\begin{aligned}
dc_0(\omega) &= \left(\frac{1-\beta}{\beta}\right)\xi\sum_{n\geq 1}\beta^n dK^e_n(\omega) \\
dc_t(\omega) &= -\left(\frac{1-\beta}{\beta}\right)\xi\left(\sum_{n\geq 1}\theta^k(t,n)dK^e_n(\omega) - \sum_{n\geq 1}\beta^n dK^e_{t+n}(\omega)\right), \quad \text{for } t \geq 1.
\end{aligned}
$$

It follows that

$$\theta^c(t,T) = \begin{cases} \theta^k(t,T) & \text{if} \quad T \leq t \\ \theta^k(t,T) - \beta^{T-t} & \text{if} \quad T > t \end{cases},$$

which completes the proof. ∎

Proof of Corollary 3. We apply Lemma 1. For $1 \leq t < N$ we have $dk_t(\omega)$ equals $-\xi e$ times

$$
\begin{aligned}
\sum_{T=1}^{t} \theta^k(t,T) + \sum_{T=t+1}^{N} \theta^k(t,T) &= \sum_{T=1}^{t}(1-\beta^T) + \sum_{T=t+1}^{N}(\beta^{T-t}-\beta^T) \\
&= \sum_{T=1}^{t}(1-\beta^T) + (1-\beta^t)\sum_{T=1}^{N-t}\beta^T \\
&= t - \beta\frac{1-\beta^t}{1-\beta} + (1-\beta^t)\frac{\beta}{1-\beta}(1-\beta^{N-t}) \\
&= t - \bar{r}^{-1}(1-\beta^t)\beta^{N-t}.
\end{aligned}
$$

Similarly for $0 \leq t < N$ we have $dc_t(\omega)$ equals $-\xi e$ times

$$
\begin{aligned}
\bar{r}\sum_{T=1}^{t}\theta^c(t,T) + \bar{r}\sum_{T=t+1}^{N}\theta^c(t,T) &= \bar{r}\sum_{T=1}^{t}(1-\beta^T) - \bar{r}\sum_{T=t+1}^{N}\beta^T \\
&= \bar{r}\left(t - \beta\frac{1-\beta^t}{1-\beta}\right) - \bar{r}\beta^{t+1}\frac{1-\beta^{N-t}}{1-\beta} \\
&= \bar{r}t - 1 + \beta^N.
\end{aligned}
$$

For $t \geq N$ we have $dk_t(\omega)$ equals $-\xi e$ times

$$
\sum_{T=1}^{N}\theta^k(t,T) = \sum_{T=1}^{N}(1-\beta^T) = N - \frac{\beta}{1-\beta}(1-\beta^N) = N - \bar{r}^{-1}(1-\beta^N)
$$

and $dc_t(\omega)$ equals $-\xi e$ times

$$
\begin{aligned}
\bar{r}\sum_{T=1}^{N}\theta^c(t,T) &= \bar{r}\sum_{T=1}^{N}(1-\beta^T) \\
&= \bar{r}\left(N - \frac{\beta}{1-\beta}(1-\beta^N)\right) \\
&= \bar{r}\left(N - \bar{r}^{-1}(1-\beta^N)\right),
\end{aligned}
$$

which implies $dc_t(\omega) = \bar{r}dk_t(\omega)$. ∎

Proof of Proposition 1. If stability in the sense of Definition 3 does not hold, agents are thus unable to deduce from CK that the path for capital must remain in $D(\varepsilon)$. The hypothetical common knowledge assumption cannot be

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confirmed, the mental process cannot go further and eductive stability fails in the sense of 2. To sketch the argument for instability in the alternative sense of Definition 4, note that because there are no income effects associated with changes in aggregate capital, it follows from the Preparation Lemma that the plans of agent ω are to first order independent of aggregate capital. The same is true of the actual decisions for unchanged beliefs. Hence, if the aggregate plans considered in Definition 3 went outside $D(\varepsilon)$, the same would be true for the actual trajectory.■

Proof of Theorem 2. To prove that strong stability fails in the sense of Definition 3, we consider the homogeneous belief class in which all agents have the N-period deviation beliefs as given by (11). Setting $0 < e \le \varepsilon$, it can be seen from (12) that $|dk_N(\omega)| > \varepsilon$ for N large. ■

Proof of Theorem 3. As above, it is enough to prove that stability fails in the low-tech sense of Definition 3. Letting $s_m = \sum_{i=1}^{m} \varepsilon_i$, we observe that for any $\eta > 0$, there is some m so that $\eta s_m > \varepsilon_m$. Indeed, if s_m diverges the result is immediate; otherwise, ηs_m is increasing and $\varepsilon_m \to 0$. Next, fix some horizon $N > 0$, and let agent ω have beliefs $dK_N^e(\omega) = \varepsilon_t$ for $1 \le t \le N$, and zero otherwise. His savings decision is $dk^N(\omega) = -\xi \Theta^k \cdot dK_N^e(\omega)$. Noting that the entries of Θ^k are all positive, that the first m entries in row m are increasing, and that $\theta^k(m,1) = 1 - \beta$ for all m, we have that

$$\left| dk_N^N(\omega) \right| = \xi \sum_{i=1}^{N} \theta^k(N,i)\varepsilon_i \ge \xi(1 - \beta) \sum_{i=1}^{N} \varepsilon_i.$$

For $\eta = \xi(1-\beta)$ and N large, the proof is completed by the initial observation and the sufficiency of considering individual behavior. ■

While the given proof exploits the functional form of the matrix Θ^k, in Theorem A1 below we show that this fundamental instability result follows from strategic substitutability of the map taking expectations to actions, which the map Γ^k exhibits.

Proof of Theorem 4. To examine B-stability we consider the system (15) with the assumed initial conditions $dK_0 = 0$ and $dK_0^e(\omega) = e$, where $e = \pm\varepsilon$ for all ω. The dynamics of dK_t can equivalently be written as

$$dK_{t+2} = (2 - \alpha(1 + \xi))dK_{t+1} - (1 - \alpha)dK_t,$$

with $dK_0 = 0$ and $dK_1 = -\xi e$. The eigenvalues of the dynamic system are

$$\lambda_1, \lambda_2 = \frac{1}{2} \left\{ 2 - \alpha(1 + \xi) \pm \sqrt{\alpha}\sqrt{\alpha(1 + \xi)^2 - 4\xi} \right\}.$$

If $\alpha(1+\xi)^2 < 4\xi$ these eigenvalues are complex conjugates, and the solution is given by $dK_t = -\frac{\xi e}{r \sin \psi} r^t \sin(\psi t)$, where $r^2 = 1 - \alpha$ and

$$\psi = \sin^{-1}\left(\frac{1}{2}\sqrt{(1-\alpha)^{-1}\left(4\xi\alpha - \alpha^2(1+\xi)^2\right)}\right).$$

If $\alpha(1+\xi)^2 > 4\xi$ the eigenvalues are real and distinct (we omit the non-generic case of repeated roots) and $dK_t = -(\lambda_1 - \lambda_2)^{-1}\xi e(\lambda_1^t - \lambda_2^t)$.

To show lack of B-stability for small $\alpha > 0$, note that for sufficiently small α the roots are complex and at $t = T = \frac{\pi}{2\psi}$

$$dK_T = -\frac{2\xi e}{\sqrt{\alpha}\sqrt{4\xi - \alpha(1+\xi)^2}}(1-\alpha)^{\pi/(4\psi)}.$$

It can be verified that $\lim_{\alpha\downarrow 0} dK_{T(\psi(\alpha))} = \pm\infty$, where the sign is opposite to the sign of e. It follows that for $\alpha > 0$ sufficiently small we have $|dK_t| > \varepsilon$ for values of t near T. ∎

Proof of Lemma 4. The system can be written as

$$\begin{pmatrix} dK_{t+1} \\ d\hat{K}_{t+1}^e \end{pmatrix} = \begin{pmatrix} 1 & -\xi \\ \alpha & 1 - \alpha(1+\xi) \end{pmatrix} \begin{pmatrix} dK_t \\ d\hat{K}_t^e \end{pmatrix}.$$

Let A denote the 2×2 matrix that governs the dynamics. For asymptotic stability we need both eigenvalues within the unit circle. Equivalently (see LaSalle (1986), p. 28) we require $|\det(A)| < 1$ and $|\mathrm{tr}(A)| < 1 + \det(A)$. Since $\det(A) = 1 - \alpha$ the first condition is satisfied for all $0 < \alpha < 1$. Using $\mathrm{tr}(A) = 2 - \alpha(1+\xi)$ leads to the stated condition. ∎

Proof of Theorem 5. 1) (Sketch) For ξ sufficiently small, the eigenvalues of the system (15) are real and positive. The proof of Theorem 4 shows that, in this case, the path dK_t is proportional to $(\lambda_1 - \lambda_2)(\lambda_1^t - \lambda_2^t)$, which, as a continuously differentiable function of t, has at most one critical point for $t > 0$. This, together with the fact that the maximum magnitude of dK_t is continuous in ξ, and approaches zero as $\xi \to 0$, yields the result. 2) Asymptotic stability is immediate from Lemma 4. Failure of B-stability for small $\alpha > 0$ follows from the proof of Theorem 4. To show B-stability for large $\alpha < 1$, note that for $\alpha = 1$ the solution is $dK_t = -(1-\xi)^t\xi e$. By continuity $\{dK_t\}_{t=0}^\infty \in D(\varepsilon)^\infty$ for all $\alpha < 1$ sufficiently large. 3) Lack of partial B-stability is immediate since $dK_1 = -\xi e$ for $dK_0 = 0$ and $dK_0^e(\omega) = e$, where $e = \pm\varepsilon$ for all ω. The asymptotic stability results follow from Lemma 4. ∎

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Appendix B: Strategic Substitutability

In this Appendix we establish the important connection between our instability results and a natural notion of strategic substitutability in our dynamic infinite-horizon setting. To establish this notion, we let $dK^e(\omega) = \{dK_t^e(\omega)\}$ be a beliefs path of a given agent. His corresponding savings plan is then given by $dk(\omega) = \{dk_t(\omega)\} = \Gamma^k(dK^e(\omega))$. As noted before, the map Γ^k can be represented as a semi-infinite matrix $\{\Gamma_{tm}^k\}_{t,m \geq 1}$. Using this notation, we may write

$$dk_t(\omega) = \sum_{m \geq 1} \Gamma_{tm}^k dK_m^e(\omega). \tag{17}$$

The map Γ^k exhibits *local strategic substitutability* if $\Gamma_{tm}^k < 0$ for all $t, m \in \mathbb{N}$.

Theorem A1 *If Γ^k exhibits local strategic substitutability, and if*

$$\eta \equiv \inf_{n > 0} |\Gamma_{nn}^k| > 0,$$

then the perfect foresight steady state is never CK' strongly eductively stable.

Proof. For notational simplicity, we forgo identifying the dependence of beliefs and actions on ω. Let $\varepsilon = \{\varepsilon_n\}$ be any uniformly bounded sequence of non-negative real numbers, with at least one strictly positive entry. Define sequences of expectations $dK^e(n)$ as follows:

$$dK_t^e(n) = \left\{ \begin{array}{ll} \varepsilon_t & \text{if } t \leq n \\ 0 & \text{if } t > n \end{array} \right. ,$$

and let $dk(n) = \Gamma^k (dK^e(n))$.

By strategic substitutability, $dk_t(\varepsilon) < dk_t(n) < 0$; thus it suffices to find some n and t so that $|dk_t(n)| > \varepsilon_t$. Next, notice that since $dK_{n+s}^e(n) = 0$ for $s > 0$, it follows that $dk_{n+s}(n) = dk_n(n)$. This allows us to compute

$$
\begin{aligned}
dk_{n+1}(n+1) &= dk_{n+1}(n) + \Gamma_{n+1,n+1}^k \cdot \varepsilon_{n+1} \\
&= dk_n(n) + \Gamma_{n+1,n+1}^k \cdot \varepsilon_{n+1} \\
&\leq dk_n(n) - \eta \cdot \varepsilon_{n+1}.
\end{aligned}
$$

Thus, substituting in recursively and using $dk_0 = 0$, we have

$$dk_{n+1}(n+1) \leq -\eta \sum_{m=1}^{n+1} \varepsilon_m.$$

The proof is completed by applying the initial observation in the proof of Theorem 3. ∎

We note that by Corollary 1, Γ^k exhibits local strategic substitutability. Since $\inf_{n>0} |\Gamma^k_{nn}| = \xi(1 - \beta) > 0$, Theorem A1 yields Theorem 3 as a corollary. Also, Corollary 1 and Theorem A1 focus on the implications of local strategic substitutability precisely because our map Γ^k exhibits this characteristic; however, analogous results with analogous proofs hold for maps that exhibit local strategic complementarity.

Appendix C: A Finite-horizon Model with Capital

Here we show that strong eductive stability, even under the high-tech interpretation, can obtain in a finite horizon model with capital. To show this we examine a dynamic model with the shortest possible horizon: the two-period OLG model with capital.

Population is constant and normalized to one, and all markets are competitive. Let ω_t be an agent born at time t. He is endowed with one unit of labor, which he supplies inelastically for real wage q_t. He then allocates his income between savings $s(\omega_t) = k(\omega_t)$ and consumption $c_1(\omega_t)$. In period $t + 1$, this agent is now old: he rents his savings for net real return r_{t+1}, consumes the gross return, plus his share of profits from ownership of firms, and dies. Thus agent ω_t solves the following problem

$$\max E(\omega_t) \left\{ u \left(c_1(\omega_t), c_2(\omega_t) \right) \right\}$$
$$\text{s.t.} \quad c_1(\omega_t) + s(\omega_t) = q_t \tag{18}$$
$$c_2(\omega_t) = (1 + r^e_{t+1}(\omega_t))s(\omega_t) + \pi(\omega_t) \tag{19}$$

Notice that when agent ω_t makes his savings decision, he does not know the value of r_{t+1}. Below we assume constant returns to scale production so that $\pi = 0$.

The agent ω_t's first-order condition is given by

$$u_{c_1} \left(c_1(\omega_t), c_2(\omega_t) \right) = \beta(1 + r^e_{t+1}(\omega_t))u_{c_2} \left(c_1(\omega_t), c_2(\omega_t) \right). \tag{20}$$

Equations (18)–(20) may be used to compute the savings decision of agent ω_t based on current and expected future factor prices:

$$s(\omega_t) = s(q_t, r^e_{t+1}(\omega_t)).$$

Firms hire workers and rent capital in competitive factor markets, and employ constant returns to scale technology to manufacture goods: $Y = f(K, L)$; thus profits are zero and factors prices are given by the respective marginal products. Capital is inelastically supplied "in the morning" by the old and

depreciation is zero: the capital accumulation equation is given accordingly by

$$K_{t+1} = \int s(\omega_t) d\omega_t = \int s(q_t, r^e_{t+1}(\omega_t)) d\omega_t.$$

Assuming agents know the relationship between real interest rates and marginal products, and so form expectations of aggregate capital instead of real interest rates, we have

$$K_{t+1} = \int s \left(f_L(K_t, 1), f_K(K^e_{t+1}(\omega_t), 1) \right) d\omega_t, \tag{21}$$

where $K^e_{t+1}(\omega_t)$ is agent ω_t's forecast of aggregate capital tomorrow. Equation (21) captures the dynamics of the economy: given aggregate capital today and forecasts of aggregate capital tomorrow, the actual value of aggregate capital tomorrow can be determined. It also highlights a key difference between the OLG model and the RBC model: in the OLG model aggregate capital depends only on one-period-ahead forecasts; in the RBC model, aggregate capital depends on forecasts at all horizons.

To motivate the definition of eductive stability, we consider the following thought experiment at time $t = 0$: Let $\bar{K}$ be a steady state of (21): $\bar{K} = s \left(f_L(\bar{K}, 1), f_K(\bar{K}, 1) \right)$. Assume that at time $t = 0$ every old household has capital $k_0(\omega) = \bar{K}$. This determines the wage, and therefore the income, of the young, as given by $\bar{q}$. Each young agent ω_t forecasts capital stock tomorrow, $K^e(\omega_t)$, and determines his savings decision $s(\bar{q}, f_K(K^e(\omega_t), 1))$. He then contemplates the savings decisions of other agents. We again make the common knowledge CK:

CK: It is common knowledge that for some $\varepsilon > 0$ sufficiently small, $K_t \in D(\varepsilon) \equiv [\bar{K} - \varepsilon, \bar{K} + \varepsilon]$ for all t.

These CK beliefs are assumed held by all agents at all times, i.e. for all ω_t for all t.

The definitions of weak and strong eductive stability under CK are identical to the definitions given in Sections 3 and 4. We have the following result:

Theorem A2 *The steady state $\bar{K}$ is strongly eductively stable if and only if*

$$\left| \partial \mathbf{s} / \partial \mathbf{r} \left(f_L(\bar{K}, 1), f_K(\bar{K}, 1) \right) \cdot f_{KK} \left(\bar{K}, 1 \right) \right| < 1. \tag{22}$$

Proof. To see this, first note that (22) holds if and only if there is $\zeta \in (0, 1)$ such that for small $\varepsilon > 0$, whenever $|K^e(\omega_t) - \bar{K}| \leq \varepsilon$ it follows that

$$\left| s \left(f_L(\bar{K}, 1), f_K(K^e(\omega_t), 1) \right) - \bar{K} \right| < \zeta \varepsilon.$$

Under CK, if (22) holds then $s(\hat{\omega}_0) \in D(\zeta\varepsilon)$ for all $\hat{\omega}_0$. Because each agent ω_0 knows that $K^e(\hat{\omega}_0) \in D(\varepsilon)$, he concludes that $s(\hat{\omega}_0) \in D(\zeta\varepsilon)$ for all $\hat{\omega}_0$ and hence that $K_1 \in D(\zeta\varepsilon)$. Thus it is common knowledge that $K_1 \in D(\zeta\varepsilon)$. Iterating the argument it follows that $K_1 \in \bigcap_{n=1}^{\infty} D(\zeta^n\varepsilon) = \{\bar{K}\}$. In contrast if (22) fails then for some beliefs compatible with CK, the aggregate capital stock at $t = 1$ implied by the agents' optimal plans fails to satisfy $K_1 \in D(\varepsilon)$.

It follows that it is common knowledge that $K_1 = \bar{K}$ if and only if (22) holds. For agents at time $t = 1$ the situation is identical to the situation at time $t = 0$. Thus at $t = 1$ agents will conclude that $K_2 = \bar{K}$ and this implies that it will be the case that $K_2 = \bar{K}$ and this will also be common knowledge for agents at $t = 0$. By induction it follows that the fact that the equilibrium path will be $K_t = \bar{K}$, for all t, is common knowledge. ∎

We remark that this stability result is local and can provide a refinement criterion in the case of multiple steady states.

Note the crucial role played by the short planning horizon in going from weak to strong eductive stability. Because agents at time $t = 0$ care only about outcomes at $t = 0, 1$, the weak eductive stability result $K_1 \in D(\zeta\varepsilon)$, can be iterated to show common knowledge that $K_1 = \bar{K}$, and hence common knowledge that the next generation of agents will be in an identical position.

As an exercise for illustrating our results, we specify particular functional forms and conduct numerical analysis. Assume utility is time separable and takes the constant relative risk-aversion form

$$u(c_1, c_2) = \frac{1}{1-\sigma}\left(c_1^{1-\sigma} + c_2^{1-\sigma} - 2\right),$$

for $\sigma > 0$, and assume that production is Cobb-Douglas, $f(K, L) = K^\theta L^{1-\theta}$. In this case there is a unique positive steady-state level of capital, and parameter values for θ and σ completely characterize the model. For all parameters examined – $\theta \in (0, 1)$ and $\sigma \in (0, 100)$ – the steady state is strongly eductively stable.

This example provides a striking contrast to the coordination problems we have demonstrated for the RBC model with infinitely-lived agents.

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