

AN EXTREME-VALUE ANALYSIS OF THE LIL FOR BROWNIAN MOTION

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ABSTRACT. We present an extreme-value analysis of the classical law of the iterated logarithm (LIL) for Brownian motion. Our result can be viewed as a new improvement to the LIL.

1. INTRODUCTION

Let $\{B(t)\}_{t \geq 0}$ be a standard Brownian motion. The law of the iterated logarithm (LIL) of Khintchine (1933) states that $\limsup_{t \rightarrow \infty} (2t \ln \ln t)^{-1/2} B(t) = 1$ a.s. Equivalently,

$$(1.1) \quad \text{With probability one, } \sup_{s \geq t} \frac{B(s)}{\sqrt{2s \ln \ln s}} \rightarrow 1 \text{ as } t \rightarrow \infty.$$

The goal of this note is to determine the rate at which this convergence occurs.

We consider the extreme-value distribution function (Resnick, 1987, p. 38),

$$(1.2) \quad \Lambda(x) = \exp(-e^{-x}) \quad \forall x \in \mathbf{R}.$$

Also, we place $\mathcal{L}_k x$ or $\mathcal{L}_k(x)$ in favor of the k -fold, iterated, natural logarithm, $(\ln \cdots \ln)(x)$ (k times). Then, our main result is as follows:

Theorem 1.1. *For all $x \in \mathbf{R}$,*

$$(1.3) \quad \lim_{t \rightarrow \infty} \mathbf{P} \left\{ 2\mathcal{L}_2 t \left(\sup_{s \geq t} \frac{B(s)}{\sqrt{2s \mathcal{L}_2 s}} - 1 \right) - \frac{3}{2} \mathcal{L}_3 t + \mathcal{L}_4 t + \ln \left(\frac{3}{\sqrt{2}} \right) \leq x \right\} = \Lambda(x),$$

$$(1.4) \quad \lim_{t \rightarrow \infty} \mathbf{P} \left\{ 2\mathcal{L}_2 t \left(\sup_{s \geq t} \frac{|B(s)|}{\sqrt{2s \mathcal{L}_2 s}} - 1 \right) - \frac{3}{2} \mathcal{L}_3 t + \mathcal{L}_4 t + \ln \left(\frac{3}{2\sqrt{2}} \right) \leq x \right\} = \Lambda(x).$$

The preceding is accompanied by the following strong law:

Theorem 1.2. *With probability one,*

$$(1.5) \quad \lim_{t \rightarrow \infty} \frac{\mathcal{L}_2 t}{\mathcal{L}_3 t} \left(\sup_{s \geq t} \frac{B(s)}{\sqrt{2s \mathcal{L}_2 s}} - 1 \right) = \frac{3}{4}.$$

This should be compared with the following consequence of the theorem of Erdős (1942):

$$(1.6) \quad \limsup_{t \rightarrow \infty} \frac{\mathcal{L}_2 t}{\mathcal{L}_3 t} \left(\sup_{s \geq t} \frac{B(s)}{\sqrt{2s \mathcal{L}_2 s}} - 1 \right) = \frac{3}{4} \quad \text{a.s.}$$

[Erdős's theorem is stated for Bernoulli walks, but applies equally well—and for the same reasons—to Brownian motion.]

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Theorem 1.1 is derived by analyzing the excursions of the Ornstein–Uhlenbeck process,

$$(1.7) \quad X(t) = e^{-t/2} B(e^t) \quad t \geq 0.$$

Our method is influenced by the ideas of Motoo (1959).

2. PROOF OF THEOREM 1.1

An application of Itô's formula shows us that the process X satisfies the s.d.e,

$$(2.1) \quad X(t) = X(0) + \int_1^{\exp(t)} \frac{1}{\sqrt{s}} dB(s) - \frac{1}{2} \int_0^t X(s) ds.$$

The stochastic integral in (2.1) has quadratic variation $\int_1^{\exp(t)} s^{-1} ds = t$. Therefore, this stochastic integral defines a Brownian motion. Call the said Brownian motion W , to see that X satisfies the s.d.e.,

$$(2.2) \quad dX = dW - \frac{1}{2} X dt.$$

In particular, the quadratic variation of X at time t is t . This means that the semimartingale local times of X are occupation densities (Revuz and Yor, 1999, Chapter VI). In particular, if $\{L_t^0(X)\}_{t \geq 0}$ denotes the local time of X at zero, then

$$(2.3) \quad L_t^0(X) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|X(s)| \leq \varepsilon\}} ds \quad \text{a.s.}$$

See Revuz and Yor (1999, Corollary 1.6, p. 224).

Let $\{\tau(t)\}_{t \geq 0}$ denote the right-continuous inverse-process to $L^0(X)$. By the ergodic theorem, $\tau(t)/t$ a.s. converges as t diverges. In fact,

$$(2.4) \quad \lim_{t \rightarrow \infty} \frac{\tau(t)}{t} = \sqrt{2\pi} \quad \text{a.s.}$$

To this, note that $\tau(t)/t \sim t/L_t^0(X)$ a.s. But another application of the ergodic theorem implies that $L_t^0(X) \sim \mathbb{E}[L_t^0(X)]$ a.s. The assertion (2.4) then follows from the fact that $\mathbb{E}[L_t^0(X)] = t/\sqrt{2\pi}$ (Horváth and Khoshnevisan, 1995, Lemma 3.2).

Define

$$(2.5) \quad \varepsilon_t = \sup_{s \geq \tau(t)} \frac{X(s)}{\sqrt{2 \ln s}} \quad \forall t \geq e.$$

Lemma 2.1. *Almost surely,*

$$(2.6) \quad \left| \varepsilon_n - \sup_{j \geq n} \frac{M_j}{\sqrt{2 \ln j}} \right| = O \left(\frac{1}{\ln n} \cdot \sqrt{\frac{\mathcal{L}_2 n}{n}} \right) \quad (n \rightarrow \infty),$$

where

$$(2.7) \quad M_j = \sup_{s \in [\tau(j), \tau(j+1)]} X(s) \quad \forall j \geq 1.$$

Proof. According to (2.4),

$$(2.8) \quad \sup_{s \in [\tau(j), \tau(j+1)]} \left| \frac{1}{\sqrt{\ln \tau(j)}} - \frac{1}{\sqrt{\ln s}} \right| \sim \frac{1}{2 \ln j} \cdot \sqrt{\frac{\mathcal{L}_2 j}{j \ln j}} \quad (j \rightarrow \infty).$$

On the other hand, according to (1.1) and (2.4), almost surely,

$$(2.9) \quad M_j = O \left(\sqrt{\ln \tau(j+1)} \right) = O \left(\sqrt{\ln j} \right) \quad (j \rightarrow \infty).$$

The lemma follows from a little algebra. \square

Lemma 2.1, and monotonicity, together prove that Theorem 1.1 is equivalent to the following: For all $x \in \mathbf{R}$,

$$(2.10) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ 2 \ln n \left(\sup_{j \geq n} \frac{M_j}{\sqrt{2 \ln j}} - 1 \right) - \frac{3}{2} \mathcal{L}_2 n + \mathcal{L}_3 n + \ln \left(\frac{3}{\sqrt{2}} \right) \leq x \right\} = \Lambda(x).$$

We can derive this because: (i) By the strong Markov property of the OU process X , M_1, M_2, \dots is an i.i.d. sequence; and (ii) the distribution of M_1 can be found by a combining a little bit of stochastic calculus with an iota of excursion theory. In fact, one has a slightly more general result for Itô diffusions (i.e., diffusions that solve smooth s.d.e.'s) at no extra cost.

Proposition 2.2. *Let $\{Z_t\}_{t \geq 0}$ denote the regular Itô diffusion on $(-\infty, \infty)$ which solves the s.d.e.*

$$(2.11) \quad dZ_t = \sigma(Z_t) dw_t + a(Z_t) dt,$$

where $\sigma, a \in \mathcal{C}^\infty(\mathbf{R})$, σ is bounded away from zero, and $\{w_t\}_{t \geq 0}$ is a Brownian motion. Write $\{\theta_t\}_{t \geq 0}$ for the inverse local-time of $\{Z_t\}_{t \geq 0}$ at zero. Then for all $\lambda > 0$,

$$(2.12) \quad \mathbb{P} \left\{ \sup_{t \in [0, \theta_1]} Z_t \leq \lambda \mid Z_0 = 0 \right\} = \exp \left(- \frac{f'(0)}{2 \{f(\lambda) - f(0)\}} \right).$$

Proof. The scale function of a diffusion is defined only up to an affine transformation. Therefore, we can assume, without loss of generality, that $f'(0) = 1$ and $f(0) = 0$; else, we choose the scale function $x \mapsto \{f(x) - f(0)\}/f'(0)$ instead. Explicitly, $\{Z_t\}_{t \geq 0}$ has the scale function (Revuz and Yor, 1999, Exercise VII.3.20).

$$(2.13) \quad f(x) = \int_0^x \exp \left(-2 \int_0^y \frac{a(u)}{\sigma^2(u)} du \right) dy.$$

Owing to Itô's formula, $N_t := f(Z_t)$ satisfies

$$(2.14) \quad dN_t = f'(Z_t) \sigma(Z_t) dw_t = f'(f^{-1}(N_t)) \sigma(f^{-1}(N_t)) dw_t,$$

and so is a local martingale. According to the Dambis, Dubins–Schwarz representation theorem (Revuz and Yor, 1999, Theorem V.1.6, p. 181), there exists a Brownian motion $\{b(t)\}_{t \geq 0}$ such that

$$(2.15) \quad \begin{aligned} N_t &= b(\alpha_t), \quad \text{where} \\ \alpha_t &= \alpha(t) = \langle N \rangle_t = \int_0^t [f'(f^{-1}(N_r))]^2 \sigma^2(f^{-1}(N_r)) dr \quad \forall t \geq 0. \end{aligned}$$

The process N is manifestly a diffusion; therefore, it has continuous local-time processes $\{L_t^x(N)\}_{t \geq 0, x \in \mathbf{R}}$ which satisfy the occupation density formula (Revuz and Yor, 1999, Corollary VI.1.6, p. 224 and time-change). By (2.13), $f' > 0$, and because σ is bounded away from zero, $\sigma^2 f' > 0$. Therefore, the inverse process $\{\alpha^{-1}(t)\}_{t \geq 0}$ exists a.s., and is uniquely defined by $\alpha(\alpha^{-1}(t)) = t$ for all $t \geq 0$.

Let $\{L_t^x(b)\}_{t \geq 0, x \in \mathbf{R}}$ denote the local-time processes of the Brownian motion b . It is well known (Rogers and Williams, 2000, Theorem V.49.1) that a.s.

$$(2.16) \quad \{L_t^x(b)\}_{t \geq 0, x \in \mathbf{R}} = \{L_{\alpha(t)}^x(N)\}_{t \geq 0, x \in \mathbf{R}};$$

For completeness, we include a brief argument here. Thanks to the occupation density formula,

$$(2.17) \quad \int_{-\infty}^{\infty} h(x) L_t^x(b) dx = \int_0^t h(b(s)) ds = \int_0^t h(N_{\alpha^{-1}(s)}) ds,$$

valid for all Borel-measurable functions $h : \mathbf{R} \rightarrow \mathbf{R}$. See (2.15) for the last equality. We can change variables [$v = \alpha^{-1}(s)$], and use the definition of α_t in (2.15), to note that

$$(2.18) \quad \begin{aligned} \int_{-\infty}^{\infty} h(x) L_t^x(b) dx &= \int_0^{\alpha^{-1}(t)} h(N_v) d\alpha_v \\ &= \int_0^{\alpha^{-1}(t)} h(N_v) d\langle N \rangle_v = \int_{-\infty}^{\infty} h(x) L_{\alpha^{-1}(t)}^x(N) dx. \end{aligned}$$

This establishes (2.16). In particular, with probability one,

$$(2.19) \quad L_t^0(N) = L_{\alpha(t)}^0(b) \quad \forall t \geq 0.$$

By (2.11) and (2.14), $d\langle Z \rangle_t = [f'(f^{-1}(N_t))]^{-2} d\langle N \rangle_t$, so if $L(Z)$ denotes the local times of Z , then almost surely,

$$(2.20) \quad \begin{aligned} \int_{-\infty}^{\infty} h(x) L_t^x(Z) dx &= \int_0^t h(Z_r) d\langle Z \rangle_r \\ &= \int_0^t \frac{h(f^{-1}(N_r))}{[f'(f^{-1}(N_r))]^2} d\langle N \rangle_r \\ &= \int_{-\infty}^{\infty} h(f^{-1}(y)) \frac{L_t^y(N)}{[f'(f^{-1}(y))]^2} d[f^{-1}(y)] \\ &= \int_{-\infty}^{\infty} h(x) \frac{L_t^{f(x)}(N)}{f'(x)} dx \quad [x = f^{-1}(y)]. \end{aligned}$$

This proves that a.s., $f'(x) \cdot L_t^x(Z) = L_t^{f(x)}(N)$. In particular, $L_t^0(Z) = L_t^0(N)$ for all $t \geq 0$, a.s. Thanks to (2.19), we have proved the following: Almost surely,

$$(2.21) \quad L_t^0(Z) = L_{\alpha(t)}^0(b) \quad \forall t \geq 0.$$

Define

$$(2.22) \quad \varphi_t = \inf \{s > 0 : L_s^0(b) > t\} \quad \forall t \geq 0.$$

Then thanks to (2.21),

$$(2.23) \quad \varphi_t = \alpha(\theta_t) \quad \forall t \geq 0.$$

Thus,

$$(2.24) \quad \begin{aligned} \mathbb{P} \left\{ \sup_{s \in [0, \theta_1]} Z_s \leq \lambda \mid Z_0 = 0 \right\} &= \mathbb{P} \left\{ \sup_{s \in [0, \theta_1]} N_s \leq f(\lambda) \mid N_0 = 0 \right\} \\ &= \mathbb{P} \left\{ \sup_{s \in [0, \varphi_1]} b_s \leq f(\lambda) \mid b_0 = 0 \right\}. \end{aligned}$$

The last identity follows from (2.15), and the fact that α and α^{-1} are both continuous and strictly increasing a.s.

Define \mathcal{N}_β to be the total number of excursion of the Brownian motion b that exceed β by local-time 1. Then,

$$(2.25) \quad \begin{aligned} \mathbb{P} \left\{ \sup_{s \in [0, \varphi_1]} b_s \leq f(\lambda) \mid b_0 = 0 \right\} &= \mathbb{P} \{ \mathcal{N}_{f(\lambda)} = 0 \mid b_0 = 0 \} \\ &= \exp \{ -\mathbb{E} [\mathcal{N}_{f(\lambda)} \mid b_0 = 0] \}, \end{aligned}$$

because \mathcal{N}_β is a Poisson random variable (Itô, 1970). According to Proposition 3.6 of Revuz and Yor (1999, p. 492), $\mathbb{E}[\mathcal{N}_\beta \mid b_0 = 0] = (2\beta)^{-1}$ for all $\beta > 0$. [See also Revuz and Yor (1999, Exercise XII.4.11).] The result follows. \square

Remark 2.3. Also, the following equality holds:

$$(2.26) \quad \mathbb{P} \left\{ \sup_{t \in [0, \theta_1]} |Z_t| \leq \lambda \mid Z_0 = 0 \right\} = \exp \left(-\frac{f'(0)}{f(\lambda) - f(0)} \right).$$

This follows as above after noting that $f(-x) = -f(x)$, and that $\mathbb{E}[\mathcal{N}'_\beta \mid b_0 = 0] = \beta^{-1}$, where \mathcal{N}'_β denotes the number of excursions of the Brownian motion b that exceed β in absolute value by local-time 1.

Proof of Theorem 1.1. If we apply the preceding computation to the diffusion X itself, then we find that $\mathbb{P}\{M_1 \leq \lambda\} = \exp\{-1/[2S(\lambda)]\}$, where S is the scale function of X which satisfies $S'(0) = 1$ and $S(0) = 0$. According to (2.2) and (2.13), $S(x) = \int_0^x \exp(y^2/2) dy$. Note that $S(x) \sim x^{-1} \exp(x^2/2)$ as $x \rightarrow \infty$.

Let $\{\beta_n(x)\}_{n=1}^\infty$ be a sequence which, for x fixed, satisfies $\beta_n(x) \rightarrow \infty$ as $n \rightarrow \infty$. We assume, in addition, that $\alpha_n(x) := \beta_n(x)/\ln n$ goes to zero as $n \rightarrow \infty$. We will suppress x in the notation and write α_n and β_n for $\alpha_n(x)$ and $\beta_n(x)$, respectively.

A little calculus shows that if $\alpha_n(x) > 0$, then

$$(2.27) \quad \begin{aligned} \mathbb{P} \left\{ \sup_{j \geq n} \frac{M_j}{\sqrt{2 \ln j}} \leq 1 + \frac{\alpha_n}{2} \right\} &= \prod_{j=n}^\infty \exp \left(-\frac{1}{2S((1 + \alpha_n/2)\sqrt{2 \ln j})} \right) \\ &= \exp \left(-\frac{(1 + \alpha_n/2)}{\sqrt{2}} \sum_{j=n}^\infty \frac{\sqrt{\ln j}}{j^{(1 + \alpha_n/2)^2}} \right) \\ &= \exp \left(-\frac{[1 + o(1)](1 + \alpha_n/2)}{\sqrt{2}} \int_n^\infty \frac{\sqrt{\ln x}}{x^{(1 + \alpha_n/2)^2}} dx \right) \\ &= \exp \left(-\frac{[1 + o(1)](1 + \alpha_n/2)\sqrt{\ln n}}{\sqrt{2}\alpha_n(1 + \alpha_n/4)\alpha_n n^{\alpha_n(1 + \alpha_n/4)}} \right) \\ &= \exp \left(-[1 + o(1)]q_n(x) \frac{(\ln n)^{3/2} e^{-\beta_n}}{\sqrt{2}\beta_n} \right). \end{aligned}$$

Here,

$$(2.28) \quad q_n(x) = \left(\frac{2 \ln n + \beta_n(x)}{2 \ln n + \beta_n(x)/2} \right) \exp \left(-\frac{\beta_n^2(x)}{4 \ln n} \right).$$

If $\alpha_n(x) \leq 0$, then the probability on the right-hand side of (2.27) is 0. Define

$$(2.29) \quad \varphi_n(x) := \frac{3}{2} \mathcal{L}_2 n - \mathcal{L}_3 n - \ln \left(3/\sqrt{2} \right) + x,$$

and set $\beta_n(x)$ in (2.27) equal to $\varphi_n(x)$. This yields

$$(2.30) \quad \mathbb{P} \left\{ \sup_{j \geq n} \frac{M_j}{\sqrt{2 \ln j}} \leq 1 + \frac{\varphi_n(x)}{2 \ln n} \right\} = \begin{cases} \exp(-[1 + o(1)]c_n(x)e^{-x}) & \text{if } \varphi_n(x) > 0, \\ 0 & \text{if } \varphi_n(x) \leq 0, \end{cases}$$

where

$$(2.31) \quad c_n(x) = \left(\frac{2 \ln n + \varphi_n(x)}{2 \ln n + \varphi_n(x)/2} \right) \exp \left(-\frac{\varphi_n^2}{4 \ln n} \right) \left[1 + \frac{-\mathcal{L}_3 n - \ln(3/\sqrt{2}) + x}{\frac{3}{2} \mathcal{L}_2 n} \right]^{-1}.$$

Note that the little- o in (2.30) is uniform in x . If $x \in \mathbf{R}$ is fixed, letting $n \rightarrow \infty$ in (2.30) shows that

$$(2.32) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ 2 \ln n \left(\sup_{j \geq n} \frac{M_j}{\sqrt{2 \ln j}} - 1 \right) \leq \varphi_n(x) \right\} = \exp(-e^{-x}).$$

This proves (2.10), whence equation (1.3) of Theorem 1.1 follows.

Using (2.26), we obtain also

$$(2.33) \quad \mathbb{P} \left\{ \sup_{j \geq n} \frac{|M_j|}{\sqrt{2 \ln j}} \leq 1 + \frac{\alpha_n}{2} \right\} = \exp \left(-[1 + o(1)]\sqrt{2} \cdot \frac{e^{-\beta_n} (\ln n)^{3/2}}{\beta_n} \right).$$

Let $\beta_n(x) = \frac{3}{2} \mathcal{L}_2 n - \mathcal{L}_3 n - \ln \left(\frac{3}{2\sqrt{2}} \right) + x$ in (2.33) to establish (1.4). \square

3. PROOF OF THEOREM 1.2

In light of (1.6) it suffices to prove that

$$(3.1) \quad \liminf_{t \rightarrow \infty} \frac{\mathcal{L}_2 t}{\mathcal{L}_3 t} \left(\sup_{s \geq t} \frac{B(s)}{\sqrt{2s\mathcal{L}_2 s}} - 1 \right) \geq \frac{3}{4} \quad \text{a.s.}$$

We aim to prove that almost surely,

$$(3.2) \quad \sup_{j \geq n} \frac{M_j}{\sqrt{2 \ln j}} > \sqrt{1 + c \frac{\mathcal{L}_2 n}{\ln n}} \quad \text{eventually a.s. if } c < \frac{3}{2}$$

Theorem 1.2 follows from this by the similar reasons that yielded Theorem 1.1 from (2.10). But (3.2) follows from (2.27):

$$(3.3) \quad \mathbb{P} \left\{ \sup_{j \geq n} \frac{M_j}{\sqrt{2 \ln j}} \leq \sqrt{1 + c \frac{\mathcal{L}_2 n}{\ln n}} \right\} = \exp \left\{ -\frac{1 + o(1)}{2c\sqrt{2}} \cdot \frac{(\ln n)^{(3/2)-c}}{\mathcal{L}_2 n} \right\}.$$

Replace n by ρ^n where $\rho > 1$ is fixed. We find that if $c < (3/2)$ then the probabilities sum in n . Thus, by the Borel–Cantelli lemma, for all $\rho > 1$ and $c < (3/2)$ fixed,

$$(3.4) \quad \sup_{j \geq \rho^n} \frac{M_j}{\sqrt{2 \ln j}} > \sqrt{1 + c \frac{\mathcal{L}_2(\rho^n)}{\ln(\rho^n)}} \quad \text{eventually a.s.}$$

Equation (3.2) follows from this and monotonicity.

4. AN EXPECTATION BOUND

We can use our results to improve on the bounds of Dobric and Marano (2003) for the rate of convergence of $E[\sup_{s \geq t} B_s(2s\mathcal{L}_2s)^{-1/2}]$ to 1.

Proposition 4.1. *As $t \rightarrow \infty$,*

$$(4.1) \quad E \left[\sup_{s \geq t} \frac{B(s)}{\sqrt{2s\mathcal{L}_2s}} \right] = 1 + \frac{3\mathcal{L}_3t}{4\mathcal{L}_2t} - \frac{1\mathcal{L}_4t}{2\mathcal{L}_2t} + \frac{1}{2} \frac{\gamma - \ln(3/\sqrt{2})}{\mathcal{L}_2t} + o\left(\frac{1}{\mathcal{L}_2t}\right),$$

where $\gamma \approx 0.5772$ denotes Euler's constant.

Proof. Define

$$(4.2) \quad U_n := 2 \ln n \left(\sup_{j \geq n} \frac{M_j}{\sqrt{2 \ln j}} - 1 \right) - \frac{3}{2}\mathcal{L}_2n + \mathcal{L}_3n + \ln(3/\sqrt{2}).$$

We have shown that U_n converges weakly to Λ . We now establish that $\sup_n E[U_n^2] < \infty$. This implies uniform integrability, whence we can deduce that $E[U_n] \rightarrow \int x d\Lambda(x)$.

Let $\varphi_n(x)$ be as defined in (2.29), and note that $x_n^* := (3/2)\mathcal{L}_2n - \mathcal{L}_3n - \ln(3/\sqrt{2})$ solves $\varphi_n(-x_n^*) = 0$.

Recalling the definition of $c_n(x)$ in (2.31) and rewriting (2.30) shows that for $x > -x_n^*$

$$(4.3) \quad P\{U_n \leq x\} = \exp(-[1 + o(1)]c_n(x)e^{-x}).$$

Consequently, for n large enough,

$$(4.4) \quad \int_0^\infty xP\{U_n \leq -x\} \leq \int_0^{x_n^*} x \exp\left(-\frac{1}{2}c_n(-x)e^x\right) dx.$$

For n sufficiently large and $0 \leq x < x_n^*$, $c_n(-x) \geq e^{-2}/(3/2) \geq (1/12)$. Thus, for n sufficiently large,

$$(4.5) \quad \int_0^\infty xP\{U_n \leq -x\} \leq \int_0^{x_n^*} x e^{-\frac{1}{24}e^x} dx \leq \int_0^\infty x e^{-\frac{1}{24}e^x} dx < \infty.$$

Also for n sufficiently large,

$$(4.6) \quad \int_0^\infty xP\{U_n > x\} \leq \int_0^\infty x \left(1 - e^{-\frac{3}{2}c_n(x)e^{-x}}\right) dx \leq \int_0^\infty \frac{3}{2}xc_n(x)e^{-x} dx.$$

We can get the easy bound for $x > 0$, $c_n(x) \leq (3/2 + x)/(1/4) = (6 + 4x)$, yielding

$$(4.7) \quad \int_0^\infty xP\{U_n > x\} dx \leq \int_0^\infty \frac{3}{2}x(6 + 4x)e^{-x} dx < \infty.$$

Now we can write

$$(4.8) \quad E[U_n^2] = 2 \int_0^\infty xP\{|U_n| > x\} dx = 2 \int_0^\infty xP\{U_n > x\} dx + 2 \int_0^\infty xP\{U_n < -x\} dx.$$

We have just show that the two terms on the right-hand side are bounded uniformly in n , which establishes uniform integrability.

From Lemma 2.1, it follows that

$$(4.9) \quad 2\mathcal{L}_2t \left(E \left[\sup_{s \geq t} \frac{B_s}{\sqrt{2s\mathcal{L}_2s}} \right] - 1 \right) - \frac{3}{2}\mathcal{L}_3t + \mathcal{L}_4t + \ln\left(\frac{3}{\sqrt{2}}\right) \rightarrow \int_{-\infty}^\infty x d\Lambda(x).$$

It suffices to prove that $\int x d\Lambda(x) = \gamma$. But this follows because

$$(4.10) \quad \int_{-\infty}^{\infty} x d[e^{-e^{-x}}] = - \int_0^{\infty} \ln(t) e^{-t} dt = - \frac{d}{dz} \int_0^{\infty} t^{z-1} e^{-t} dt \Big|_{z=1} = - \frac{\Gamma'(z)}{\Gamma(z)} \Big|_{z=1}.$$

Cf. Davis (1965, Equation 6.3.2). \square

5. AN APPLICATION TO RANDOM WALKS

Let X_1, X_2, \dots be i.i.d. random variables with

$$(5.1) \quad \mathbb{E}[X_1] = 0, \quad \text{Var}(X_1) = 1, \quad \text{and} \quad \mathbb{E}[X_1^2 \mathcal{L}_2(X_1 \vee e^e)] < \infty.$$

Let $S_n = X_1 + \dots + X_n$ ($n \geq 1$) denote the corresponding random walk. Then, according to Theorem 2 of Einmahl (1987), there exists a probability space on which one can construct $\{S_n\}_{n=1}^{\infty}$ together with a Brownian motion B such that $|S_n - B(n)|^2 = o(n/\mathcal{L}_2 n)$ a.s. On the other hand, by the Borel–Cantelli lemma, $|B(t) - B(n)|^2 = o(n/\mathcal{L}_2 n)$ uniformly for all $t \in [n, n+1]$ a.s. These remarks, and a few more lines of elementary computations, together yield the following.

Proposition 5.1. *If (5.1) holds, then for all $x \in \mathbf{R}$,*

$$(5.2) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ 2\mathcal{L}_2 n \left(\sup_{k \geq n} \frac{S_k}{\sqrt{2k\mathcal{L}_2 k}} - 1 \right) - \frac{3}{2}\mathcal{L}_3 n + \mathcal{L}_4 n + \ln \left(\frac{3}{\sqrt{2}} \right) \leq x \right\} = \Lambda(x),$$

$$(5.3) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ 2\mathcal{L}_2 n \left(\sup_{k \geq n} \frac{|S_k|}{\sqrt{2k\mathcal{L}_2 k}} - 1 \right) - \frac{3}{2}\mathcal{L}_3 n + \mathcal{L}_4 n + \ln \left(\frac{3}{2\sqrt{2}} \right) \leq x \right\} = \Lambda(x).$$

Remark 5.2. It would be interesting to know if the preceding remains valid if only $\mathbb{E}[X_1] = 0$ and $\text{Var}(X_1) = 1$. We believe the answer to be, “No.”

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