1. Linear Algebra

Again, V and W are finite dimensional vector spaces over a field F

(1) Define..
 (a) the *tensor product* $V \otimes W$
 (b) the n^{th} *tensor power* $V^{\otimes n}$ of V.
 (c) the n^{th} *symmetric power* $S^n(V)$ of V
 (d) the n^{th} *exterior power* $\Lambda_n(V)$ of V.
 (e) the *tensor algebra* $T(V)$ over V
 (f) the *symmetric algebra* $S(V)$ over V.
 (g) the *exterior algebra* $\Lambda(V)$ over V
(2) Show that $V^* \otimes W \cong \mathrm{Hom}_F(V, W)$

2. 500 Algebra

(1) Define...
 (a) a *group* G
 (b) the *order* of an element of a group and of a group, written $|g|$ and $|G|$ respectively.
 (c) a *subgroup* $H \leq G$
 (d) the *index* $[G : H]$ of a subgroup $H \leq G$
 (e) an *abelian* group A
 (f) a *normal* subgroup $H \trianglelefteq G$
 (g) the *center* $Z(G)$ of a group G
 (h) the *commutator* $[g, g']$ of two elements $g, g' \in G$
 (i) the *commutator subgroup* $[G, G] \leq G$
 (j) a *homomorphism* $\phi : G \to G'$ of groups
 (k) the *image* and *kernel* of a homomorphism of groups
 (l) an *injective* and a *surjective* homomorphism of groups
 (m) the *direct sum* $G \oplus G'$ of two groups G and G'
 (n) a *(left or right) action* of a group G on a set X – if X has an action by G, S is called a *G-set*.
 (o) Gx, X/G, $\mathrm{Stab}_G(x) = G_x$, S^G (for a G-set X and $x \in X$).
(2) Give at least five different examples of groups, at least two of which are uncountable.
(3) Let G be a group with $|G| = p$, p prime. Show that G is abelian.
(4) Let G be a group with $|G| = p^k$, p prime. For what k is G necessarily abelian?
(5) Give an example of a group G with $|G| = pq$, p, q prime, such that G is not abelian.
(6) Show that for any group homomorphism $\phi : G \to G'$, $\ker(\phi) \trianglelefteq G$ and $\mathrm{im}(\phi) \leq G'$. Is the image of ϕ necessarily a normal subgroup?
(7) Consider a group G with subgroup H
 (a) Show that H acts on G
 (b) Show that each element of G/H is a set of order $|H|$ and that $|G/H| = [G : H]$.
 (c) If $H \trianglelefteq G$, show that G/H is again a group.
 (d) Define the projection map $\pi : G \to G/H$, show that it is a homomorphism and that it is surjective.
 (e) Give an example of a group G with non-normal subgroup H so that G/H is not a group.
(8) Show that $n\mathbb{Z} \trianglelefteq \mathbb{Z}$ for any $n \in \mathbb{N}$. Use this to define the group $\mathbb{Z}/n\mathbb{Z}$ and explain what it is.
(9) Show that $Z(G) \trianglelefteq G$. What can you say about $G/Z(G)$? What are we doing here?
(10) Show that $[G, G] \trianglelefteq G$. What can you say about $G/[G, G]$? What does this do?
(11) Given a group G, a G-set X and $x \in X$, show that $G_x \leq G$. Is it normal in G?
(12) If $H, K \leq G$ are of finite index with $\gcd([G : H], [G : K]) = 1$, show that $HK = G$.
(13) Define a *cyclic* subgroup of a group G. Does every group contain a cyclic subgroup?
(14) Define a *conjugacy class* for a group G. Let G be a finite group of order n. Show that the probability that two elements of G commute is $\frac{k}{n}$ where k is the number of conjugacy classes of G.
(15) Put a group structure on S^1 and T^2. Show that S^1 contains an element of every finite order.

(16) Define $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{R})$. Show that these are groups.

3. 500 Topology

(1) Define...
  (a) a *toplogy*
  (b) a *base* for a topology
  (c) a *sub-base* for a topology
  (d) a *topological space*
  (e) a *subspace* of a topological space
  (f) a *closed subset* of a toplogical space X
  (g) a *compact subspet* of a toplogical space X
  (h) a *continuous function* $f : X \to Y$, X and Y both topological spaces – f is often called a *map*
  (i) a *homeomorphism* $f : X \to Y$
  (j) a *Hausdorff* space – sometimes, a T_2 space
  (k) a *surface*
  (l) a *closed* surface
(2) For any map of sets $f : X \to Y$, determine which of the following are true or false:
  (a) $f(f^{-1}(Y)) \subseteq Y$
  (b) $Y \subseteq f(f^{-1}(Y))$
  (c) $f^{-1}(f(X)) \subseteq X$
  (d) $X \subseteq f^{-1}f(X))$
(3) Can you give necessary and/or sufficient conditions such that:
  (a) $f(f^{-1}(Y)) = Y$
  (b) $f^{-1}(f(X)) = X$
(4) Give at least three examples of surfaces, at least one of which is not closed.
(5) Give at least three examples of topologies on the real line so that none of the resultant topological spaces are homeomorphic
(6) Give as many examples of topological spaces as you can, at least two of which are not Hausdorff
(7) For a metric space X, define $B_\epsilon(x) = \{y \in X | d(y, x) < \epsilon\}$. Show that the set of open balls $\{B_\epsilon(x) | x \in X, \epsilon > 0\}$ forms a base for a topology of X and that the notion of continuous functions between metric spaces $f : X \to Y$ under this *induced* topology agrees with the usual $\epsilon - \delta$ definition.
(8) Let Y be a subspace of X. If A is open in Y and Y is open in X, show that A is open in X.
(9) Prove that the function $h(x) = \frac{e^x}{1+e^x}$ is a homeomorphism, $h : \mathbb{R} \cong (0, 1)$. Modify this to produce a homeomorphism $\hat{h} : \mathbb{R} \cong (a, b)$ for any $a < b$.
(10) Show that there exists no homeomorphism $h : \mathbb{R} \to S^1$
(11) Show that the continuous image of a compact set is compact.
(12) Show that a compact subset of a Hausdorff space is closed.
(13) Define the following topological spaces: S^n, T^2, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$
(14) Define a topology for $\mathrm{GL}_n(\mathbb{R})$. This makes it a *topological group*. Show that the canonical left action of $\mathrm{GL}_n(\mathbb{R})$ on itself is continuous.

4. 600 Algebra

(1) Define...
  (a) a *ring* R (note – *all* rings in Algebra have identity)
  (b) a *commutative* ring
  (c) a *right ideal* $I \trianglelefteq_r R$
  (d) a *left ideal* $I \trianglelefteq_l R$
  (e) a *two sided ideal* $I \trianglelefteq R$
  (f) a *principal* ideal
  (g) a *prinicipal ideal domain* or *PID*
  (h) a *euclidean domain* or *ED*
  (i) a *unique factorization domain* or *UFD*
(2) List some rings and ideals (left, right and two sided) in those rings.

(3) Prove that any right ideal in a commutative ring is also a left ideal and vice-versa.
(4) Give an example of a ring R which is not a PID and a non-principal ideal I in R
(5) Show that $\mathbb{Z}$ is a PID.
(6) Define $\mathrm{Mat}_n(\mathbb{R})$ (resp. $\mathrm{Mat}_n(F)$ for any field F) and give an example that shows that it is not a commutative ring. Give an example of a left ideal of $\mathrm{Mat}_3(\mathbb{R})$ which is *not* a right ideal. Can you find any two sided ideals?
(7) $\mathrm{GL}_n(\mathbb{R})$ is not a ring. Why not?
(8) Using each of PID, ED and UFD once, fill in the following: *blank* $\subseteq$ *blank* $\subseteq$ *blank*. Find an example of each each of the larger classes which is not a memeber of the smaller one.
(9) Let G, H, K be abelian groups. Show that...
  (a) $\mathrm{Hom}_{\mathbb{Z}}(G \otimes H, K)$ and $\mathrm{Hom}_{\mathbb{Z}}(G, \mathrm{Hom}_{\mathbb{Z}}(H, K))$ are abelian groups
  (b) $\mathrm{Hom}_{\mathbb{Z}}(G \otimes H, K) \cong \mathrm{Hom}_{\mathbb{Z}}(G, \mathrm{Hom}_{\mathbb{Z}}(H, K))$.

5. 600 Topology

(1) Let $f : X \to Y$ and $g : X \to Y$ be maps. Define a *homotopy* H between f and g.
(2) Define a *deformation retraction* of a topological space X onto a subspace A. Construct an explicit deformation retraction from $\mathbb{R}^2 \setminus \{0\}$ to S^1
(3) Let $(X, *)$ be a based topological space. Define $\pi_1(X, *)$.
(4) Explain what it means for a topological space to be connected and to be path connected. Show that one implies the other but that the ideas are not the same.
(5) Prove that $\pi_1(S^1, *) \cong \mathbb{Z}$.
(6) Define a *n-dimensional topological manifold M*, the n-dimensional topological manifolds S^n and $\mathbb{R}P^n$, the $2n$-dimensional topological manifold $\mathbb{C}P^n$ and the real *Grassman* manifold $G(n, k)$.
(7) Define a *fiber bundle* and a *fibration* over a topological space.

6. 600 Geometry

Here, let $U, V \subseteq \mathbb{R}^n$ for some n.

(1) Define...
  (a) a *smooth* map $\phi : U \to V$
  (b) a *diffeomorphism* $\phi : U \to V$
  (c) a *n-dimensional smooth real manifold M*
  (d) a *chart*, an *atlas* and a *smooth structure* for M
(2) Demonstrate two incompatable smooth structures for $\mathbb{R}$, but construct a diffeomorphism between the two.
(3) Explain what it means for a smooth manifold to be *closed*, *compact*, *with boundary* and *with corners*
(4) Classify all compact smooth 1-manifolds up to diffeomorphism.