2007 Boot Camp, Problem Set 2

1. Linear Algebra

(1) Give an example of a set V which is a finite dimensional vector space over a field F and an infinite dimensional vector space over another field K. Does there exist a set V which has finite dimension n as a vector space over some field F and finite dimension $m \neq n$ as a vector space over some other field K?

(2) Let A be an $n \times n$ matrix. Define the *trace* of A. Is trace a basis invariant property of a linear transformation?

(3) Recall that the dual of a vector space V is $V^* = \{f : V \to F | f \text{ linear}\} = \mathrm{Hom}_F(V, F)$. V^* is the vector space of *linear functionals* on V.

   (a) Let $\alpha : V \to W$ be a linear transformation of F vector spaces. Show that α induces a linear transformation of F vector spaces $\alpha^* : W^* \to V^*$. We also occasionally write α^* as $\mathrm{Hom}_F(\alpha F)$.

   (b) Show that if $\alpha : V \to W$ and $\beta : W \to Y$ are linear transformations of F vector spaces then $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.

   (c) Show that $\mathrm{id} : V \to V$ the identity map becomes $\mathrm{id}^* : V^* \to V^*$, also the identity map. These three properties together show that $\mathrm{Hom}_F(-, F)$ is a *contravariant functor* from the category of F vector spaces to itself.

   (d) Assume V and W are finite dimensional F vector spaces and $\alpha : V \to W$ is linear. Let A be a matrix which represents α. Show that A^T represents α^*.

   (e) Let V be a finite dimensional F vector space. Show that $(V^*)^* \cong V$.

(4) Let F be $\mathbb{R}$ or $\mathbb{C}$ and V a vector space over F.

   (a) Define an *inner product* $< -, - >$ on V and the *standard inner product* on F^n. A vector space V with a specified inner product $< -, - >$ is called an *inner product space.*

   (b) Given an inner product space V, define the *norm* of a vector v, $||v||$.

   (c) Let $|| \cdot ||_1$ and $|| \cdot ||_2$ be two norms on a finite dimensional inner product space V. Show that there are constants c_1 and c_2 such that for all $v \in V$, $||v||_1 \leq c_1 ||v||_2$ and $||v||_2 \leq c_2 ||v||_1$. This shows that the two norms are *isomorphic*. Give an example of an infinite dimensional vector space with two non-isomorphic norms.

   (d) Let V be an inner product space. Define *orthogonal* and *orthonormal* vectors in V and show that a set of vectors which is pairwise orthogonal is linearly independent.

   (e) Let $v_1 = (2, 4 + i, -1 + \frac{i}{3})^T, v_2 = (\frac{1}{3}, -2, \sqrt{2} - i)^T \in \mathbb{C}^3$. Compute $< v_1, v_2 >$ and $||v_1||$ using the standard inner product.

   (f) Let V be the vector space of all continuous complex-valued functions on the unit interval. Let

$$< f, g >= \int_0^1 f(t)\overline{g(t)}dt$$

   Show that this is an inner product on V, compute $< e^t, \cos(\frac{\pi t}{2}) >$ and show that for any $n \neq m \in \mathbb{N}$, $\sqrt{2}\cos(2\pi nt)$ is orthonormal to $\sqrt{2}\cos(2\pi mt)$. (And similarly with sine.)

   (g) Let V be a vector space and $< -, - >$ an inner product on V. True or false: if $< v, w >= 0$ for all $w \in V$ then $v = 0$.

   (h) Let $F = \mathbb{R}$. Prove the *polarization identity*, $< v, w >= \frac{1}{4}||v + w||^2 - \frac{1}{4}||v - w||^2$.

(5) Let V be a finite dimensional $\mathbb{C}$-inner product space.

   (a) Let α be a linear functional on V. Show that there is a unique vector $v_0 \in V$ so that $f(v) = < v, v_0 >$ for all $v \in V$.

   (b) Define *a (sesquilinear) form* on V and a *Hermetian form* on V. All forms are sesquilinear.

   (c) Show that for any form f on V, there is a unique linear operator $T_f : V \to V$ so that $f(v_1, v_2) = < T_f v_1, v_2 >$ for all $v_1, v_2 \in V$. Call T the associated linear operator to f. Given a linear operator T on V, do you necessarily get a form f_T on V?

   (d) Show that if f is a form on V for which $f(v, v) \in \mathbb{R}$ for every $v \in V$, then f is Hermetian.

   (e) Let f be a symmetric form on V (so, $f(v, w) = f(w, v)$ for all $v, w \in V$). What is f?

   (f) A form f is *left non-degenerate* if $f(v, w) = 0$ for all $w \in V$ implies $v = 0$. Prove that T_f is nonsingular if and only if f is left non-degenerate.

(g) Define *right non-degeneracy* of a form and show that f is left non-degenerate if and only if it is right non-degenerate. Thus, we need not use "left" or "right".
(h) Let f be a non-degenerate form and L a linear functional on V. Show that there exists a unique vector $v_0 \in V$ so that $L(v) = f(v, v_0)$ for all $v \in V$.

2. 500 Analysis

(1) Find $\lim_{n\to\infty}(1+\frac{1}{2n})^n$.
(2) State...
  (a) Bolzano-Weierstrass Theorem
  (b) Heine-Borel Theorem
  (c) Intermediate Value Theorem
  (d) Mean Value Theorem
  (e) Extreme Value Theorem
  (f) Weierstrass Approximation Theorem or the Stone-Weierstrass Theorem
(3) Prove the Extreme Value Theoerem
(4) Let S be any infinite bounded subset of $\mathbb{R}$. True or False: There exists a convergent sequence $(s_n) \subseteq S$.
(5) Give at least two (non-trivially different) examples of sequences of irrational numbers converging to a rational number and vice-versa.
(6) Define *uniform continuity* and prove that $x^{17}\sin(x) - e^x\cos(3x)$ is unifomly continuous on $[0, \pi]$.
(7) Define *uniform convergence* for a sequence (f_n) of real valued functions on an interval $[a, b]$. Prove that a uniformly convergent sequence of uniformly continuous functions is continuous. Is this the strongest version of the statement you can prove?
(8) Define *uniform boundedness* for a sequence (f_n) of real valued functions. Let (f_n) be a sequence of continuous functions on $[0, 1]$ which are uniformly bounded and which converge pointwise to 0. Is it true that they converge uniformly to 0?
(9) Define *algebraic numbers* over $\mathbb{C}$, prove that the set of complex algebraic numbers is countable and thus that there exist complex numbers which are not algebraic.
(10) Use the following outline to prove the Banach Fixed Point Theorem.
  (a) Let $q \in [0, 1)$ and $m, n \in \mathbb{N}$ with $m \geq n$. Prove:
    (i) $q^n + \cdots + q^m = \frac{q^n - q^{m+1}}{1-q}$
    (ii) $q^n + \cdots + q^m < \frac{q^n}{1-q}$
    (iii) If $n > N$, then $q^n + \cdots + q^m < \frac{q^N}{1-q}$
  (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for some fixed $q \in [0, 1)$
$$|f(x) - f(y)| \leq q|x - y| \text{ for all } x, y \in \mathbb{R}$$
    (i) Let $x_1 = 1$ and $x_{n+1} = f(x_n)$ for $n > 1$. Show that the sequence (x_n) converges.
    (ii) Prove that there is a solution to the equation $f(x) = x$.

3. 600 Analysis

(1) State the definition of a *σ-algebra*, a *measure space* and a *measurable function.* Define the *measure* of a subset of a measure space.
(2) Let $\epsilon \in (0, 1)$. Construct an open dense subset of $\mathbb{R}$ with measure ϵ.
(3) Show that every infinite σ-algebra is uncountable.
(4) Let Y and Z be topological spaces and X a measure space. Suppose $g : Y \to Z$ is continuous. Prove that $g \circ f$ is measurable if $f : X \to Y$ is a measurable function.
(5) Let g be a real function on a measure space X such that $\{x \in X | g(x) \geq r\}$ is measurable for every $r \in \mathbb{Q}$. Prove that g is measurable.