Research Statement

Chad Giusti

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I am currently engaged in three distinct research projects. The areas of study are diverse but are all centered around the interplay between geometric and algebraic topology. I approach knot theory through the algebraic topology of the space of knots, and geometric data is essential in my work in in group and equivariant cohomology.

My dissertation work extends and refines the program of Vassiliev's which initiated the subject of finite-type knot invariants. In order to do so, I introduce and classify *plumbers' knots* [7]. These allow me, for example, to extend the notion of the derivative of knot invariant.

In [10], Paolo Salvatore, Dev Sinha and I explore the Hopf ring structue on $H^*(B\Sigma_\bullet; \mathbb{Z}/2)$. This structure allows us to produce a component-wise additive basis with explicit cup product multiplication rule and to understand the action of the Steenrod algebra on this basis.

Lastly, Bill Kronholm and I are utilizing his $RO(\mathbb{Z}/2)$-graded Serre spectral sequence to classify the possible geometries of $Rep(\mathbb{Z}/2)$-equivariant cell complexes. In [9], we compute the equivariant cohomology of Moore spaces with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$. In this context, we obtain a version of his $\underline{\mathbb{Z}/2}$-freeness theorem [12].

Contents

1 Unstable Vassiliev theory

In his seminal work [16], Vassiliev considers a directed system of spaces of knots parameterized by polynomials whose limit is a suitable space of long knots. Through analysis of this directed system by applications of duality and resolution of singularities he initiated the theory of finite-type, or Vassiliev, invariants. These invariants include the Jones polynomial (after a change of variables) and other quantum knot invariants [3], which at the time were the among most powerful knot invariants studied.

My work takes a similar approach to knot theory, but with a new set of finite-dimensional approximations, namely the "plumbers' knots" (see Figure 1 below). These approximations have considerable differences from Vassiliev's polynomial approximations, which have generally made them easier to analyze. For example, while the question of isotopy through finite polynomial parameterization is intractable, I have managed to find an explicit algorithm to determine the isotopy classes of finite-length plumbers' knots [7]. Indeed, plumbers' knots seem uniquely attuned to questions of components, since their direct limit only recovers the components

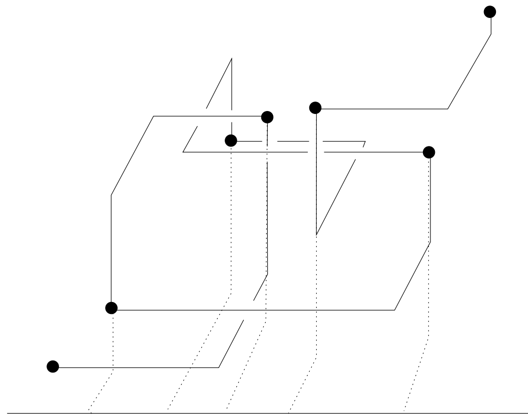

Figure 1: A plumbers' curve of 6 moves whose associated x-permutation is 24153.

of the knot space, in contrast to polynomials which recover the weak homotopy type. And, while Vassiliev's blowups of the finite polynomial knot spaces are abstractly decomposible in a stable range, my model for the blowup of each plumbers' knot space carries a geometric cellular structure.

This choice of settings allows me to make considerable progress in pushing the Vassilev program further. In the English translations of Vassiliev's work, only a sequence of spectral sequences is written down, and each of those only converges to some subset of cohomology of its particular finite-dimensional approximation. In my work, I get an honest inverse system of spectral sequences, each of which converges to the full cohomology of the space in question. In Vassiliev's setting, there are derivatives (a formalism popularized by Bar-Natan [2]) only for collections of double-points. In the plumbers' setting, surprisingly, there are derivatives defined for all singularity types. Vassiliev's combinatorics lead to chord diagrams. Mine are leading to a combination of chord diagrams and cube diagrams [1], cousins to the grid diagrams which allow combinatorial computation of knot Floer homology. Vassiliev has conjectured a "homotopical splitting" which would imply collapse of his spectral sequence, a result proved by Kontsevich along the diagonal with rational coefficients. I have been able to prove collapse of the spectral sequences with integer coefficients for each finite space of plumbers' knots.

Indeed, while the theory of finite-type invariants has recently taken a back seat to theories such as Khovanov homology and knot Floer homology, extension of Vassiliev's techniques could help bring them to the fore again and unify them with these approaches, in particular the Floer theory. To that end, I am currently working on understanding what the analogues of weight systems are in plumbers' Vassiliev theory, as well as trying to do calculations of derivatives for both all knot invariants in the first plumbers' spaces as well as some known invariants across the entire directed system. Success so far combined with the ability to continually compare with the complex, but completely combinatorial, classification for finite plumbers' knot theory makes me optimistic about the prospects for further developments.

1.1 Plumbers' knots

An m-move plumbers' map $\phi_{\mathbf{v}} : I \to [0,1]^3$ is uniquely determined by a collection $\mathbf{v}$ of $(m-1)$ vertices in $(0,1)^3$ as the map which moves from vertex to vertex along segments parallel to the coordinate axes x, y and z, in this order. As such, space P_m of all such is homeomorphic to $\left((0,1)^3\right)^{m-1}$. Each segment parallel to an axis is called a *pipe*, and pipes which are separated by three or more intervening pipes are called *distant*. $K_m \subseteq P_m$ is the space of *m-move plumbers' knots*, consisting of those plumbers' maps for which distant pipes do not intersect. (One feature that distinguishes this theory from that of PL knots is that our definition of knot allows for up to two adjacent zero-length pipes.)

The space of plumbers' maps of m moves admits a cellular decomposition generated by open cells homeomorphic to $\left(\Delta^{m-1}\right)^{\times 3}$. Each such cell is indexed by a triple of permutations of $(m-1)$ elements which describe the order in which the vertices appear when projected onto the x, y or z-axes respectively (see Figure 1 above). Write $\textsc{Cell}_*(P_m)$ for the cellular complex generated by all such.

The principal object of interest here, $S_m = P_m \setminus K_m$, is the *discriminant*, consisting of all singular plumbers' maps. S_m inherits a cellular structure from P_m in the form of a closed subcomplex $\textsc{Cell}_*(S_m) \subseteq \textsc{Cell}_*(P_m)$ generated by some of the cells in dimension $3m-4$.

Additionally, there are stabilization maps $\iota_m : P_m \hookrightarrow P_{m+1}$, under which $\varinjlim K_m$ has the components of the space of long knots (but not its weak homotopy type), so this system is a model for classical knot theory.

The combinatorial nature of the cell structure has allowed me to produce an algorithm to search the spaces K_m to determine "unstable isotopy classes". Given two topological knots, one can choose plumbers' knot representatives and search for unstable isotopies between them at each level. This search terminates if two knots are isotopic, behavior which lies in contrast to the application of computable knot invariants.

Several authors have worked on the problem of enumerating components of finite-complexity knot spaces. Notably, Calvo [4] has enumerated the components of the first few spaces of piecewise-linear (stick) knots, but the complexity of the geometry has forced his approach to be largely ad hoc. The algorithm above has allowed me to enumerate the components of several K_m and, in theory, provides a deterministic method to do so for any such space. Results of these computations show that the spaces are trivial up through K_4, that K_5 has 7 components (an unknot and three unstable representatives of each trefoil), K_6 has 49 components (including multiple representatives of each nontrivial knot up through 5 crossings) and K_7 has 1008 components. While complete enumerations beyond this point would require substantial optimization, searching particular components remains viable for much larger spaces. This data on which cells lie in the same component makes explicit computations of unstable Vassiliev derivatives, as described below, much easier.

Boundaries of cells in $\textsc{Cell}_\bullet(S_m)$ are indexed by collections $\mathbf{C}$ of coordinate equalities on the vertices which define their elements. One says that the plumbers maps in this cell *respect* $\mathbf{C}$. For example, $\mathbf{e}(3142_x, 4132_y, 1324_z)$ has a codimension 4 boundary cell indexed by the set $\{(3\,1)_x, (4\,2)_x, (4\,1)_y, (1\,3)_y\}$. Note, however, that this cell also respects the larger set $\mathbf{C}' = \mathbf{C} \cup \{(3\,4)_y\}$.

This leads to another convenient description of the space of singular maps S_m. Write $[\mathbf{m}] = \{1, \ldots, m\}$, let $\binom{I}{2}$ be the collection of two element subsets of I and let $\mathcal{P}(I)$ be the power set.

Definition 1.1. *Define $\mathcal{S}_m$ be the mth* singularity category *whose objects are non-empty elements of $\mathcal{P}\left(\binom{[\mathbf{m-1}]}{2} \times \{x, y, z\}\right)$ and whose morphisms are inclusions.*

Definition 1.2. *Let $B_m : \mathcal{S}_m \to$ **Top** be the contravariant functor given by $B_m(\mathbf{C}) = \{\phi \in S_m : \phi \text{ respects } \mathbf{C}\}$.*

Proposition 1.3. *$S_m = \operatorname{colim} B_m$.*

It is in this sense that the space of singular maps is combinatorial. Spaces of knots do not seem to admit such decompositions.

1.2 Vassiliev theory in the plumbers' knot setting

Vassiliev theory proceeds from two insights. First, one can understand a knot invariant, viewed as an element of cohomology in degree zero, through Alexander duality for finite-dimensional approximations to the space of all knots. Secondly, as alluded to above, the advantage of this is that while the space of knots is uniform, the space of singular knots has combinatorial structure. Indeed, one may assemble (or decompose, depending on your point of view) the discriminant according to singularity data. In the plumbers' knots case, I use the singularity category described above.

Definition 1.4. *The homotopical blowup of the discriminant is $\tilde{S}_m = hocolim B_m$.*

As is standard, the projection map $\pi : \tilde{S}_m \to S_m$ is a homotopy equivalence. But my success in analyzing the unstable theory yields more than one can usually say about such a map: the cell structure $\text{Cell}_*(S_m)$ lifts to a cell structure $C_*(\tilde{S}_m)$.

This cell structure contains both the singularity data from the original discriminant and combinatorial data analagous to that in Vassiliev's auxillary spectral sequences from [16]. This wealth of combinatorial data helps me do detailed analysis at the chain level. Indeed, there is a canonical choice of chain representative for a plumbers' knot invariant.

Leveraging geometric information, certain codimension one cells in the blowup "separate" pairs of cells from P_m containing plumbers' curves. Define the derivative of an invariant $[\alpha]$ across such a cell $\tilde{e}$ to be the difference of the value of the invariant on those two cells, written $d_{\tilde{\mathbf{e}}}([\alpha])$.

Theorem 1.5. *Let $[\alpha] \in H^0(K_m)$. The lift of its Alexander dual cycle $[\alpha^\vee]$ to $H_{3m-4}(\tilde{S}_m)$ has a chain representative given by $\alpha^\vee = \sum_{\tilde{\mathbf{e}} \in C_{3m-4}(\tilde{S}_m)} (-1)^{|\tilde{e}|} d_{\tilde{\mathbf{e}}}([\alpha])\tilde{e}$.*

This theorem is in stark contrast to Vassiliev's acyclicity results, which allowed him to focus on restricted singularity types (among transversal singularities) but not yield canonical representatives. Quickly following from this theorem is the following.

Corollary 1.6. *There exists a canonical Vassiliev derivative for plumbers' knot invariants associated to each given singularity type for plumbers' knots.*

Vassiliev's original techniques and subsequent work have assumed that derivatives only exist for isolated unions of double points. Here, because all singularity types come into play, we deduce the following "Taylor's theorem."

Corollary 1.7. *Each $\alpha \in \bar{H}^0(K_m)$ is completely determined by its collection of Vassilev derivatives.*

Finally, I introduce a filtration on discriminant, roughly by counting the number of distant pipes which intersect and correcting for certain unstable artifacts such as "sliding around corners". (A combinatorial decription of the filtration exists, but I have yet to locate a concise statement of the rule set.) This filtration agrees, on "stable" singularity data, with that in the Vassiliev spectral sequence.

Define $E^r_{p',q'}(m)$ to be the homology spectral sequence of the filtration on $\hat{S}_m$. Identically to Vassiliev, reindexing by $p = -p', q = 3n - 4 + p' - q'$ obtains a second-quadrant cohomological spectral sequence $E_r^{p,q}(m)$. By Alexander duality, this sequence converges to $\bar{H}^{p+q}(K_m)$. However, in this setting, each such spectral sequence is computable from the cell structure on its underlying space. That is, I have an inverse system of spectral sequences which can be analyzed directly. In fact, using my computation of the chain representatives of an invariant, I can now arrive at the following unstable collapse result.

Theorem 1.8. *$E_r^{p,q}(m)$ collapses at the E_2 page.*

I suspect that this result can be improved to show collapse at the E_1 page. Such a result would be a first step toward resolution of Vassiliev's "homotopical splitting" conjecture, that the wedge of complexity filtration quotients of the discriminant of the space of long knots is "homotopy equivalent" to the original space.

1.3 Current directions

There are several directions in which I am currently directing my efforts for this project. These are certainly not all of the questions available in this context, but best represent what I hope to accomplish in the near future.

1. Given a knot invariant, particularly a finite-type invariant, I would like to understand the derivatives of its restriction to K_m. It is important to emphasize that, in this setting, all singularity types occur. I hope that by looking at the evolution of these derivatives in the inverse system I will be able to determine whether integer valued weight systems are "integrable".

This is also an area which could intersect other approaches to knot theory, in particular knot Floer homology. Baldridge and Lowrance [1] describe *cube diagrams* which compute knot Floer homology and which can be identified as a natural subspace of the spaces of plumbers' knots, so it seems likely that some information about their Vassiliev derivatives in this setting can be gained.

2. "Taylor's theorem" leads naturally to the question of which collections of derivatives produce invariants of plumbers' knots. Such a classification would be analagous to the construction of weight systems for Vassiliev's classical spectral sequence, providing new combinatorial tools for understanding these unstable sequences.

   In order to better understand how the derivatives can fit together, I am constructing a piece of software for use in analyzing the filtered plumbers' spaces. For a fixed $\hat{S}_m$, one can enumerate the components of $F_p \setminus F_{p+1}$ and construct a graph which describes the cellular boundary connections between these various filtration levels. By studying these graphs for small spaces of plumbers' knots, I hope to gain intuition about the geometry of the filtration quotients which can be extended to the general case.

3. Working with rational coefficients, one observes a natural splitting of $H^0(K_m;\mathbb{Q})$ into *stable* and *unstable* pieces. Notice that the stable part of a plumbers' knot invariant must take the same value on any two components of K_m which will eventually be identified. For each topological knot type κ which has representatives in K_m, define the stable part of a plumbers' knot invariant α to be the average of its value over all such components.

   The rational splitting described above allows one to ask about the fundamental differences between deriviatves of stable and unstable invariants of plumbers' knots. The expectation is that this will help to understand the behavior, if any, of unstable elements in the limit. I am investigating whether any such elements survive to to the limit. The assumption is that there are none, which would imply that finite-type invariants distinguish knots. My techniques have the potential to yield either a proof or a counterexample to this famous conjecture.

2 Cohomology of symmetric groups as a Hopf ring

The cohomology of symmetric groups is a subject of classical interest. Several authors have contributed to our understanding of these rings: Nakaoka [14] computed the mod two homology of the groups (and their cup coproduct structure), a computation on which much of our work relies, while Hu'ng [11] and later Feshbach [6] apply invariant theoretic techniques to study generators and relations.

In [10], we consider a Hopf ring structure on the cohomology of $B\Sigma_\bullet = \coprod_n B\Sigma_n$. The products in the ring are the transfer product, $\odot$, introduced by Strickland and Turner [15] and the component-wise cup product, while the coproduct Δ is dual to the standard Pontrjagin product on homology.

Theorem 2.1 ([10], Theorem 1.2)**.** *As a Hopf ring $H^*(B\Sigma_\bullet;\mathbb{Z}/2)$ is generated by a collection of classes $\gamma_{\ell,n} \in H^{n(2^\ell-1)}(B\Sigma_{n2^\ell};\mathbb{Z}/2)$ along with unit classes on each component. The coproduct of $\gamma_{\ell,n}$ is given by*

$$\Delta\gamma_{\ell,n} = \sum_{i+j=n} \gamma_{\ell,i} \otimes \gamma_{\ell,j}$$

Relations between transfer products of these generators are given by

$$\gamma_{\ell,n} \odot \gamma\ell, m = \binom{n+m}{n} \gamma_{\ell,n+m}$$

Relations between cup products of generators are that cup products of generators on different components are zero.

This data is sufficient to completely determine the mod two cohomology ring structure on each component. In particular, it allows us to describe an additive basis for each and to explicitly describe the cup product structure for this basis. As there are Cartan formulae for both products, the action of the Steenrod algebra on this ring is completely determined by its action on the additive basis, which we also compute.

Preliminary computations suggest that we will be able to perform a similar analysis of the cohomology of symmetric groups all primes. Further, we are pleased to note that this Hopf ring structure appears to have broad potential application. There are fundamental geometric connections with configuration spaces, through which are accessible much broader classes of groups. In particular, we have outlined a program for the study of cohomology of families of finite Coxeter groups along these lines.

3 $RO(\mathbb{Z}/2)$-graded cohomology of Moore spaces

In the study of study of motivic homotopy theory, there is a forgetful map from Voevodsky's motivic cohomology to $RO(\mathbb{Z}/2)$-graded cohomology [13]. This implies that one can utilize the more computationally accessible world of $Rep(\mathbb{Z}/2)$-complexes to answer questions in the study if motivic cohomology. In his dissertation, Kronholm constructs a Serre spectral sequence in this context and applies it to show that $RO(\mathbb{Z}/2)$-graded cohomology with coefficients in the constant Mackey functor $\underline{\mathbb{Z}/2}$ is always free as a module over the cohomology of a point [12].

In [5], Dugger constructs a spectral sequence $H^{*,*}(X;\underline{\mathbb{Z}}) \Rightarrow KR^*(X)$, where $KR^*(X)$ is Atyiah's KR-theory. Computations in this spectral sequence would be greatly simplifed by an analogue of Kronholm's freeness theorem for cohomology with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$.

In this spirit, Kronholm and I [9] have begun an analysis of the modules over $H^{*,*}(pt,\underline{\mathbb{Z}})$ which can arise from $Rep(\mathbb{Z}/2)$-complexes. The results fall into two broad classes: one in which only free modules arise and one which the spaces are the expected families of "equivariant Moore spaces". However, there are boundary cases between the two classes in which extension problems occur in the spectral sequence. At this time, we are working to understand the geometry of these equivariant cohomology classes in order to resolve these extension problems.

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