

MONOIDAL SUPERCATEGORIES

JONATHAN BRUNDAN AND ALEXANDER P. ELLIS

ABSTRACT. This work is a companion to our article “Super Kac-Moody 2-categories,” which introduces super analogs of the Kac-Moody 2-categories of Khovanov-Lauda and Rouquier. In the case of \mathfrak{sl}_2 , the super Kac-Moody 2-category was constructed already in [A. Ellis and A. Lauda, “An odd categorification of $U_q(\mathfrak{sl}_2)$ ”], but we found that the formalism adopted there became too cumbersome in the general case. Instead, it is better to work with *2-supercategories* (roughly, 2-categories enriched in vector superspaces). Then the Ellis-Lauda 2-category, which we call here a Π -2-category (roughly, a 2-category equipped with a distinguished involution in its Drinfeld center), can be recovered by taking the superadditive envelope then passing to the underlying 2-category. The main goal of this article is to develop this language and the related formal constructions, in the hope that these foundations may prove useful in other contexts.

1. INTRODUCTION

1.1. In representation theory, one finds many monoidal categories and 2-categories playing an increasingly prominent role. Examples include the *Brauer category* $\mathcal{B}(\delta)$, the *oriented Brauer category* $\mathcal{OB}(\delta)$, the *Temperley-Lieb category* $\mathcal{TL}(\delta)$, the *web category* $\text{Web}(U_q(\mathfrak{sl}_n))$, the category of *Soergel bimodules* $\mathcal{S}(W)$ associated to a Coxeter group W , and the *Kac-Moody 2-category* $\mathfrak{U}(\mathfrak{g})$ associated to a Kac-Moody algebra \mathfrak{g} . Each of these categories, or perhaps its additive Karoubi envelope, has a definition “in nature,” as well as a diagrammatic description by generators and relations. It is also often instructive after taking additive Karoubi envelope to pass to the *Grothendieck ring*. Let us go through our examples in turn.

- The Brauer category $\mathcal{B}(\delta)$ is the symmetric monoidal category generated by a self-dual object of dimension $\delta \in \mathbb{C}$. By [LZ, Theorem 2.6], it may be presented as the strict monoidal category with one generating object \cdot and three generating morphisms $\times : \cdot \otimes \cdot \rightarrow \cdot \otimes \cdot$, $\cup : \mathbb{1} \rightarrow \cdot \otimes \cdot$ and $\cap : \cdot \otimes \cdot \rightarrow \mathbb{1}$, subject to the following relations:

$$\begin{array}{ccc}
 \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} | \\ | \end{array}, & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array}, & \begin{array}{c} \cup \\ \cap \end{array} = \begin{array}{c} | \\ | \end{array}, & \begin{array}{c} \cup \\ \cap \end{array} = \begin{array}{c} | \\ | \end{array}, \\
 \begin{array}{c} \cup \\ \cap \end{array} = \begin{array}{c} \cap \\ \cup \end{array}, & \begin{array}{c} \cup \\ \cap \end{array} = \begin{array}{c} \cup \\ \cap \end{array}, & \begin{array}{c} \cup \\ \cap \end{array} = \delta. &
 \end{array}$$

Here, we are using the well-known string calculus for morphisms in a strict monoidal category as in [BK]. We remark also that the additive Karoubi envelope of $\mathcal{B}(\delta)$ is Deligne’s interpolating category $\text{REP}(O_\delta)$. There is a similar story for the oriented Brauer category. It is the symmetric monoidal category generated by a dual pair of objects of dimension δ . An explicit

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presentation is recorded in [BCNR, Theorem 1.1]. Its additive Karoubi envelope is Deligne’s interpolating category $\text{REP}(GL_\delta)$.

- For $\delta = -(q + q^{-1}) \in \mathbb{Q}(q)$, the additive Karoubi envelope of $\mathcal{TL}(\delta)$ is monoidally equivalent to the category of finite-dimensional representations of the quantum group $U_q(\mathfrak{sl}_2)$. More generally, for $n \geq 2$, the additive Karoubi envelope of $\text{Web}(U_q(\mathfrak{sl}_n))$ is monoidally equivalent to the category of finite-dimensional representations of $U_q(\mathfrak{sl}_n)$. An explicit diagrammatic presentation was derived by Cautis, Kamnitzer and Morrison [CKM], building on the influential work of Kuperberg [K] which treated the case $n = 3$.
- When W is a Weyl group, Soergel [S] showed that $\mathcal{S}(W)$ is monoidally equivalent to the Hecke category $\mathcal{H}(G/B)$ of Kazhdan-Lusztig (certain B -equivariant sheaves on the associated Lie group G). In general, $\mathcal{S}(W)$ is the additive Karoubi envelope of the category of *Bott-Samelson bimodules*. In almost all cases, a diagrammatic presentation of the latter monoidal category has been derived by Elias and Williamson [EW1]. The Grothendieck ring $K_0(\mathcal{S}(W))$ is isomorphic to the group ring of W ; if one incorporates the natural grading into the picture one actually gets the Iwahori-Hecke algebra $H_q(W)$ associated to W .
- The Kac-Moody 2-category $\mathfrak{U}(\mathfrak{g})$ was defined by generators and relations by Rouquier [R] and Khovanov-Lauda [KL]; see also [B]. The Grothendieck ring of its additive Karoubi envelope is naturally an idempotented ring, with idempotents indexed by the underlying weight lattice, and is isomorphic to the idempotented integral form $\dot{U}(\mathfrak{g})$ of the universal enveloping algebra of \mathfrak{g} ; if one incorporates the grading one gets Lusztig’s idempotented integral form $\dot{U}_q(\mathfrak{g})$ of the associated quantum group. (These statements are still only conjectural outside of finite type.)

1.2. We are interested in this article in *superalgebra*, i.e. $\mathbb{Z}/2$ -graded algebra. Our motivation comes from the belief that there should be interesting super analogs of all of the categories just mentioned. In fact, they are already known in several cases. For example, analogs of the Brauer and oriented Brauer categories are suggested by [KT] and [JK], respectively. Also in [BE], we have defined a super analog of the Kac-Moody 2-category, building on [EL] which treated the case of \mathfrak{sl}_2 . In thinking about such questions, one quickly runs into some basic foundational issues. To start with, already in the literature, there are several competing notions as to what should be called a “super monoidal category.” The goal of the paper is to clarify these notions and the connections between them; see also [U] for further developments.

Let \mathbb{k} be a field of characteristic different from 2. A *superspace* is a $\mathbb{Z}/2$ -graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We use the notation $|v|$ for the parity of a homogeneous vector v in a superspace. Formulae involving this notation for inhomogeneous v should be interpreted by extending additively from the homogeneous case.

Let \mathcal{SVec} (resp. \mathcal{SVec}_{fd}) be the category of all superspaces (resp. finite dimensional superspaces) and (not necessarily homogeneous) linear maps. These categories possess some additional structure:

- A linear map between superspaces V and W is *even* (resp. *odd*) if it preserves (resp. reverses) the parity of vectors. Moreover, any linear map $f : V \rightarrow W$ decomposes uniquely as a sum $f = f_{\bar{0}} + f_{\bar{1}}$ with $f_{\bar{0}}$ even and $f_{\bar{1}}$ odd. This makes each morphism space $\text{Hom}_{\mathcal{SVec}}(V, W)$ into a superspace.
- The usual \mathbb{k} -linear tensor product of two superspaces is again a superspace with $(V \otimes W)_{\bar{0}} = V_{\bar{0}} \otimes W_{\bar{0}} \oplus V_{\bar{1}} \otimes W_{\bar{1}}$ and $(V \otimes W)_{\bar{1}} = V_{\bar{0}} \otimes W_{\bar{1}} \oplus V_{\bar{1}} \otimes W_{\bar{0}}$.

Also the tensor product $f \otimes g$ of two linear maps is the linear map defined from $(f \otimes g)(v \otimes w) := (-1)^{|g||v|} f(v) \otimes g(w)$.

Let $\underline{\mathcal{S}Vec}$ be the subcategory of $\mathcal{S}Vec$ consisting of all superspaces but only the even linear maps. The restriction of the tensor product operation just defined gives a functor $-\otimes- : \underline{\mathcal{S}Vec} \times \underline{\mathcal{S}Vec} \rightarrow \underline{\mathcal{S}Vec}$ making $\underline{\mathcal{S}Vec}$ into a monoidal category. However, $\mathcal{S}Vec$ itself is *not* monoidal in the usual sense, because of the sign in the following formula for composing tensor products of linear maps:

$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|} (f \circ h) \otimes (g \circ k). \quad (1.1)$$

In fact, $\mathcal{S}Vec$ is what we'll call a *monoidal supercategory*. We proceed to the formal definitions.

Definition 1.1. (i) A *supercategory* means a $\underline{\mathcal{S}Vec}$ -enriched category, i.e. each morphism space is a superspace and composition induces an even linear map. (We refer to [K, §1.2] for the basic language of enriched categories.)

(ii) A *superfunctor* $F : \mathcal{A} \rightarrow \mathcal{B}$ between supercategories is a $\underline{\mathcal{S}Vec}$ -enriched functor, i.e. the function $\text{Hom}_{\mathcal{A}}(\lambda, \mu) \rightarrow \text{Hom}_{\mathcal{B}}(F\lambda, F\mu)$, $f \mapsto Ff$ is an even linear map for all $\lambda, \mu \in \text{ob } \mathcal{A}$. (See [K, §1.2] again.)

(iii) Given superfunctors $F, G : \mathcal{A} \rightarrow \mathcal{B}$, a *supernatural transformation* $x : F \Rightarrow G$ is a family of morphisms $x_\lambda = x_{\lambda, \bar{0}} + x_{\lambda, \bar{1}} \in \text{Hom}_{\mathcal{B}}(F\lambda, G\lambda)$ for $\lambda \in \text{ob } \mathcal{A}$, such that $|x_{\lambda, p}| = p$ and $x_{\mu, p} \circ Ff = (-1)^{p|f|} Gf \circ x_{\lambda, p}$ for all $p \in \mathbb{Z}/2$ and $f \in \text{Hom}_{\mathcal{A}}(\lambda, \mu)$. The supernatural transformation x decomposes as a sum of homogeneous supernatural transformations as $x = x_{\bar{0}} + x_{\bar{1}}$ where $(x_p)_\lambda := x_{\lambda, p}$, making the space $\text{Hom}(F, G)$ of all supernatural transformations from F to G into a superspace. (Even supernatural transformations are just the same as the $\underline{\mathcal{S}Vec}$ -enriched natural transformations of [K, §1.2].)

(iv) A superfunctor $F : \mathcal{A} \rightarrow \mathcal{B}$ is a *superequivalence* if there is a superfunctor $G : \mathcal{B} \rightarrow \mathcal{A}$ such that $F \circ G$ and $G \circ F$ are isomorphic to identities via even supernatural transformations. To check that F is a superequivalence, it suffices to show that it is full, faithful, and *evenly dense*, i.e. every object of \mathcal{B} should be isomorphic to an object in the image of F via an even isomorphism.

(v) For any supercategory \mathcal{A} , the *underlying category* $\underline{\mathcal{A}}$ is the category with the same objects as \mathcal{A} but only its even morphisms. If $F : \mathcal{A} \rightarrow \mathcal{B}$ is a superfunctor between supercategories, it restricts to $\underline{F} : \underline{\mathcal{A}} \rightarrow \underline{\mathcal{B}}$. Also an even supernatural transformation $x : F \Rightarrow G$ is the same data as a natural transformation $x : \underline{F} \Rightarrow \underline{G}$. (These definitions are special cases of ones in [K, §1.3].)

Example 1.2. (i) We've already explained how to make $\mathcal{S}Vec$ into a supercategory. The underlying category is $\underline{\mathcal{S}Vec}$.

(ii) A *superalgebra* is a superspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$ equipped with an even linear map $m_A : A \otimes A \rightarrow A$ making A into an associative, unital algebra; we denote the image of $a \otimes b$ under this map simply by ab . Any superalgebra A can be viewed as a supercategory \mathcal{A} with one object whose endomorphism superalgebra is A .

(iii) Suppose we are given superalgebras A and B . Then there is a supercategory $A\text{-SMod-}B$ consisting of all (A, B) -superbimodules and superbimodule homomorphisms. Here, an (A, B) -*superbimodule* is a superspace V plus an even linear map $m_V : A \otimes V \otimes B \rightarrow V$ making V into an (A, B) -bimodule in the usual sense; we denote the image of $a \otimes v \otimes b$ under this map simply by avb . A *superbimodule homomorphism* $f : V \rightarrow W$ is a linear map such that $m_W \circ (1_A \otimes f \otimes 1_B) = f \circ m_V$. In view of the definition of tensor product of linear maps between superspaces, this means explicitly that $f(avb) = (-1)^{|f||a|} af(v)b$.

(iv) For any two supercategories \mathcal{A} and \mathcal{B} , there is a supercategory $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ consisting of all superfunctors and supernatural transformations.

The monoidal category $\mathcal{S}Vec$ is symmetric with braiding $u \otimes v \mapsto (-1)^{|u||v|}v \otimes u$. As in [K, §1.4], this allows us to introduce a product operation $-\boxtimes-$ which makes the category $\mathcal{S}Cat$ of all supercategories and superfunctors into a monoidal category. On objects (i.e. supercategories) \mathcal{A} and \mathcal{B} , this operation is defined by letting $\mathcal{A} \boxtimes \mathcal{B}$ be the supercategory whose objects are ordered pairs (λ, μ) of objects of \mathcal{A} and \mathcal{B} , respectively, and $\text{Hom}_{\mathcal{A} \boxtimes \mathcal{B}}((\lambda, \mu), (\sigma, \tau)) := \text{Hom}_{\mathcal{A}}(\lambda, \sigma) \otimes \text{Hom}_{\mathcal{B}}(\mu, \tau)$. Composition in $\mathcal{A} \boxtimes \mathcal{B}$ is defined using the symmetric braiding in $\mathcal{S}Vec$, so that $(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|}(f \circ h) \otimes (g \circ k)$. The unit object \mathcal{I} is a distinguished supercategory with one object whose endomorphism superalgebra is \mathbb{k} (concentrated in even parity). The definition of $-\boxtimes-$ on morphisms (i.e. superfunctors) is obvious, as are the coherence maps.

Remark 1.3. Example 1.2(iii) is a special case of Example 1.2(iv). Let \mathcal{A} and \mathcal{B} be defined from superalgebras A and B as in Example 1.2(ii). Let \mathcal{B}^{sup} be the supercategory with $\text{ob } \mathcal{B}^{\text{sup}} := \text{ob } \mathcal{B}$, and new composition law $a \bullet b := (-1)^{|a||b|}b \circ a$. Then the supercategory $\mathcal{H}om(\mathcal{A} \boxtimes \mathcal{B}^{\text{sup}}, \mathcal{S}Vec)$ is isomorphic to $A\text{-SM}od\text{-}B$ via the superfunctor which identifies $V : \mathcal{A} \boxtimes \mathcal{B}^{\text{sup}} \rightarrow \mathcal{S}Vec$ with the superspace obtained by evaluating at the only object, viewed as a superbimodule so $avb := (-1)^{|b||v|}V(a \otimes b)(v)$. The data of a supernatural transformation $f : V \rightarrow W$ is exactly the same as the data of a superbimodule homomorphism.

Definition 1.4. (i) A *monoidal supercategory* is a supercategory \mathcal{A} equipped with a superfunctor $-\otimes- : \mathcal{A} \boxtimes \mathcal{A} \rightarrow \mathcal{A}$, a unit object $\mathbb{1}$, and even supernatural isomorphisms¹ $a : (-\otimes-) \otimes - \xrightarrow{\cong} - \otimes (-\otimes-)$, $l : \mathbb{1} \otimes - \xrightarrow{\cong} -$ and $r : - \otimes \mathbb{1} \xrightarrow{\cong} -$ called *coherence maps*, which satisfy axioms analogous to the ones of a monoidal category. In any monoidal supercategory, tensor products of morphisms compose according to the same rule (1.1) that we already observed in $\mathcal{S}Vec$. We call this the *super interchange law*.

(ii) Given monoidal supercategories \mathcal{A} and \mathcal{B} , a *monoidal superfunctor* is a superfunctor $F : \mathcal{A} \rightarrow \mathcal{B}$ plus coherence maps $c : (F-) \otimes (F-) \xrightarrow{\cong} F(-\otimes-)$ and $i : \mathbb{1}_{\mathcal{B}} \xrightarrow{\cong} F\mathbb{1}_{\mathcal{A}}$ satisfying axioms analogous to the ones of a monoidal category; we require that c is an even supernatural isomorphism and that i is an even isomorphism. (Note we implicitly assume all monoidal (super)functors are strong throughout the article.)

(iii) Given monoidal superfunctors $F, G : \mathcal{A} \rightarrow \mathcal{B}$, a *monoidal natural transformation* is an even supernatural transformation $x : F \Rightarrow G$ such that

$$\begin{aligned} x_{\lambda \otimes \mu} \circ (c_F)_{\lambda, \mu} &= (c_G)_{\lambda, \mu} \circ (x_{\lambda} \otimes x_{\mu}), \\ x_{\mathbb{1}_{\mathcal{A}}} \circ i_F &= i_G, \end{aligned}$$

in $\text{Hom}_{\mathcal{B}}((F\lambda) \otimes (F\mu), G(\lambda \otimes \mu))$ and $\text{Hom}_{\mathcal{B}}(\mathbb{1}_{\mathcal{B}}, G\mathbb{1}_{\mathcal{A}})$, respectively. (There is no such thing as a monoidal *supernatural* transformation.)

A monoidal supercategory (resp. superfunctor) is *strict* if its coherence maps are identities. There is a version of Mac Lane's *Coherence Theorem* [Mac] for monoidal

¹By $(-\otimes-)\otimes-$ we mean the superfunctor $(\mathcal{A} \boxtimes \mathcal{A}) \boxtimes \mathcal{A} \rightarrow \mathcal{A}$ obtained by applying \otimes twice in the order indicated. Similarly, $-\otimes(-\otimes-)$ is a superfunctor $\mathcal{A} \boxtimes (\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \mathcal{A}$, but we are viewing it as a superfunctor $(\mathcal{A} \boxtimes \mathcal{A}) \boxtimes \mathcal{A} \rightarrow \mathcal{A}$ by using the canonical isomorphism defined by the associator in $\mathcal{S}Cat$. Also, $\mathbb{1} \otimes - : \mathcal{A} \rightarrow \mathcal{A}$ and $-\otimes \mathbb{1} : \mathcal{A} \rightarrow \mathcal{A}$ denote the superfunctors defined by tensoring on the left and right by the unit object, respectively.

supercategories. It implies that any monoidal supercategory \mathcal{A} is *monoidally super-equivalent* to a strict monoidal supercategory \mathcal{B} , i.e. there are monoidal superfunctors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$ such that $G \circ F$ and $F \circ G$ are isomorphic to identities via monoidal natural transformations; equivalently, there is a monoidal superfunctor $F : \mathcal{A} \rightarrow \mathcal{B}$ which defines a super-equivalence between the underlying supercategories.

With a little care about signs, the string calculus mentioned earlier can be used to represent morphisms in a strict monoidal supercategory \mathcal{A} . Thus, a morphism $f \in \text{Hom}_{\mathcal{A}}(\lambda, \mu)$ is the picture

$$\begin{array}{c} \mu \\ | \\ \textcircled{f} \\ | \\ \lambda \end{array} \cdot \tag{1.2}$$

Often we will omit the object labels λ, μ here. Then the horizontal and vertical compositions $f \otimes g$ and $f \circ g$ are obtained by horizontally and vertically stacking:

$$\begin{array}{c} | \\ \textcircled{f} \\ | \end{array} \otimes \begin{array}{c} | \\ \textcircled{g} \\ | \end{array} = \begin{array}{c} | \\ \textcircled{f} \\ | \end{array} \begin{array}{c} | \\ \textcircled{g} \\ | \end{array}, \quad \begin{array}{c} | \\ \textcircled{f} \\ | \end{array} \circ \begin{array}{c} | \\ \textcircled{g} \\ | \end{array} = \begin{array}{c} | \\ \textcircled{f} \\ | \\ \textcircled{g} \\ | \end{array} \cdot$$

More complicated pictures should be interpreted by *first* composing horizontally *then* composing vertically. For example, the following is $(f \otimes g) \circ (h \otimes k)$:

$$\begin{array}{c} | \\ \textcircled{f} \\ | \\ \textcircled{h} \\ | \end{array} \otimes \begin{array}{c} | \\ \textcircled{g} \\ | \\ \textcircled{k} \\ | \end{array} \cdot$$

Unlike in the purely even setting, this is *not* the same as $(f \circ h) \otimes (g \circ k)$. In fact, in pictures, the super interchange law tells us that

$$\begin{array}{c} | \\ \textcircled{f} \\ | \end{array} \otimes \begin{array}{c} | \\ \textcircled{g} \\ | \end{array} = \begin{array}{c} | \\ \textcircled{f} \\ | \end{array} \begin{array}{c} | \\ \textcircled{g} \\ | \end{array} = (-1)^{|f||g|} \begin{array}{c} | \\ \textcircled{g} \\ | \end{array} \otimes \begin{array}{c} | \\ \textcircled{f} \\ | \end{array} \cdot \tag{1.3}$$

Example 1.5. (i) The supercategory \mathcal{SVec} is a monoidal supercategory with tensor functor as defined above. The unit object is \mathbb{k} . More generally, for a superalgebra A , $A\text{-SMod-}A$ is a monoidal supercategory with tensor functor defined by taking the usual tensor product of superbimodules over A . The unit object is the regular superbimodule A .

(ii) For a supercategory \mathcal{A} , $\mathcal{E}nd(\mathcal{A})$ is a strict monoidal supercategory, with $- \otimes -$ defined on functors $F, G : \mathcal{A} \rightarrow \mathcal{A}$ by $F \otimes G := F \circ G$, and on supernatural transformations $x : F \Rightarrow G$ and $y : H \Rightarrow K$ so that $(x \otimes y)_\lambda := x_{K\lambda} \circ Fy_\lambda$. The unit object is the identity functor $I : \mathcal{A} \rightarrow \mathcal{A}$. Later on, we will denote the horizontal compositions $F \otimes G$ and $x \otimes y$ of two superfunctors or two supernatural transformations simply by FG and xy , respectively. In more complicated horizontal compositions, we often adopt the standard shorthand of writing simply F in place of the identity morphism 1_F , e.g. for $x : F \Rightarrow G$, $y : H \Rightarrow K$, the expressions Fy and xH denote $1_F y : FH \Rightarrow FK$ and $x1_H : FH \Rightarrow GH$, respectively.

(iii) Here is a purely diagrammatic example. The *odd Brauer supercategory* is the strict monoidal supercategory \mathcal{SB} with one generating object \cdot , an *even* generating morphism $\times : \cdot \otimes \cdot \rightarrow \cdot \otimes \cdot$, and two *odd* generating morphisms $\cup : \mathbb{1} \rightarrow \cdot \otimes \cdot$

and $\cap : \cdot \otimes \cdot \rightarrow \mathbb{1}$, subject to the following relations:

$$\begin{array}{ccc} \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \\ \text{X} \end{array} = \text{||}, & \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \\ \text{X} \end{array} = \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \\ \text{X} \end{array}, & \text{N} = \text{I}, & \text{U} = -\text{I}, \\ \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \\ \text{X} \end{array} = \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \\ \text{X} \end{array}, & \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \\ \text{X} \end{array} = \text{U}. & & \end{array}$$

This was introduced in [KT] where it is called the marked Brauer category, motivated by Schur-Weyl duality for the Lie superalgebra $\mathfrak{p}_n(\mathcal{C})$. Unlike the Brauer category defined earlier, there is no parameter δ . Indeed, using the relations and super interchange, one can check that

$$\begin{array}{c} \text{O} \\ \text{O} \\ \text{O} \\ \text{O} \end{array} = -\text{N} \quad \text{hence} \quad \text{O} = \frac{1}{2} \begin{array}{c} \text{O} \\ \text{O} \\ \text{O} \\ \text{O} \end{array} - \frac{1}{2} \begin{array}{c} \text{O} \\ \text{O} \\ \text{O} \\ \text{O} \end{array} = 0.$$

1.3. Now we switch the focus to one of the competing notions. Instead of working with additive categories *enriched* in the monoidal category $\underline{\mathcal{S}Vec}$, one can work with module categories *over* the monoidal category $\underline{\mathcal{S}Vec}_{fd}$ (e.g. see [EGNO, §7.1]), or equivalently, additive \mathbb{k} -linear categories equipped with a strict action of the cyclic group $\mathbb{Z}/2$ (e.g. see [EW2, §1.3]). We adopt the following language for such structures:

Definition 1.6. (i) A Π -category (\mathcal{A}, Π, ξ) is a \mathbb{k} -linear category \mathcal{A} plus a \mathbb{k} -linear endofunctor $\Pi : \mathcal{A} \rightarrow \mathcal{A}$ and a natural isomorphism $\xi : \Pi^2 \xrightarrow{\sim} I$ such that $\xi\Pi = \Pi\xi$ in $\text{Hom}(\Pi^3, \Pi)$. Note then that Π is a self-inverse equivalence.

(ii) Given Π -categories $(\mathcal{A}, \Pi_{\mathcal{A}}, \xi_{\mathcal{A}})$ and $(\mathcal{B}, \Pi_{\mathcal{B}}, \xi_{\mathcal{B}})$, a Π -functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is a \mathbb{k} -linear functor plus the data of a natural isomorphism $\beta_F : \Pi_{\mathcal{B}}F \xrightarrow{\sim} F\Pi_{\mathcal{A}}$ such that $\xi_{\mathcal{B}}F(\xi_{\mathcal{A}})^{-1} = \beta_F\Pi_{\mathcal{A}} \circ \Pi_{\mathcal{B}}\beta_F$ in $\text{Hom}((\Pi_{\mathcal{B}})^2F, F(\Pi_{\mathcal{A}})^2)$. For example, the identity functor I is a Π -functor with $\beta_I = 1_{\Pi}$, and Π is a Π -functor with $\beta_{\Pi} := -1_{\Pi^2}$. Note also that the composition of two Π -functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ is itself a Π -functor with $\beta_{GF} := G\beta_F \circ \beta_GF$.

(iii) Given Π -functors $F, G : \mathcal{A} \rightarrow \mathcal{B}$, a Π -natural transformation is a natural transformation $x : F \Rightarrow G$ such that $x\Pi_{\mathcal{A}} \circ \beta_F = \beta_G \circ \Pi_{\mathcal{B}}x$ in $\text{Hom}(\Pi_{\mathcal{B}}F, G\Pi_{\mathcal{A}})$.

There is a close relationship between supercategories and Π -categories. To explain this formally, we need the following intermediate notion. Actually, our experience suggests this is often the most convenient place to work in practice.

Definition 1.7. A Π -supercategory $(\mathcal{A}, \Pi, \zeta)$ is a supercategory \mathcal{A} plus the extra data of a superfunctor $\Pi : \mathcal{A} \rightarrow \mathcal{A}$ and an odd supernatural isomorphism $\zeta : \Pi \xrightarrow{\sim} I$. Note then that $\xi := \zeta\zeta : \Pi^2 \xrightarrow{\sim} I$ is an even supernatural isomorphism, i.e. \mathcal{A} is equipped with canonical even isomorphisms $\xi_{\lambda} : \Pi^2\lambda \xrightarrow{\sim} \lambda$ satisfying

$$\xi_{\lambda} = \zeta_{\lambda} \circ \Pi\zeta_{\lambda} = -\zeta_{\lambda} \circ \zeta_{\Pi\lambda} \tag{1.4}$$

for all $\lambda \in \text{ob } \mathcal{A}$. Moreover, we have that $\xi\Pi = \Pi\xi$ in $\text{Hom}(\Pi^3, \Pi)$.

To specify the extra data needed to make a supercategory into a Π -supercategory, one just needs to give objects $\Pi\lambda$ and odd isomorphisms $\zeta_{\lambda} : \Pi\lambda \xrightarrow{\sim} \lambda$ for each $\lambda \in \text{ob } \mathcal{A}$. The effect of Π on a morphism $f : \lambda \rightarrow \mu$ is uniquely determined by the requirement that $\zeta_{\mu} \circ \Pi f = (-1)^{|f|} f \circ \zeta_{\lambda}$. It is then automatic that $\zeta = (\zeta_{\lambda}) : \Pi \xrightarrow{\sim} I$ is an odd supernatural isomorphism.

Example 1.8. Given superalgebras A and B , $A\text{-SMod-}B$ is a Π -supercategory; hence, taking $A = B = \mathbb{k}$, so is \mathcal{SVec} . To specify Π and ζ , we just need to define an (A, B) -supermodule ΠV and an odd isomorphism $\zeta_V : \Pi V \xrightarrow{\sim} V$ for each (A, B) -superbimodule V . We take ΠV to be the same underlying vector space as V viewed as a superspace with the opposite $\mathbb{Z}/2$ -grading $(\Pi V)_{\bar{0}} := V_{\bar{1}}$ and $(\Pi V)_{\bar{1}} := V_{\bar{0}}$. The superbimodule structure on ΠV is defined in terms of the original action by $a \cdot v \cdot b := (-1)^{|a|} avb$. This ensures that the identity function on the underlying vector space defines an odd superbimodule isomorphism $\zeta_V : \Pi V \xrightarrow{\sim} V$. Everything else is forced; for example, for a morphism $f : V \rightarrow W$ we must have that $\Pi f : \Pi V \rightarrow \Pi W$ is the function $(-1)^{|f|} f$; also, $\xi_V : \Pi^2 V \xrightarrow{\sim} V$ is minus the identity.

Now we can explain the connection between supercategories and Π -categories. Let \mathcal{SCat} be the category of supercategories and superfunctors as above. Also let $\Pi\text{-}\mathcal{SCat}$ be the category of Π -supercategories and superfunctors, and $\Pi\text{-}\mathcal{Cat}$ be the category of Π -categories and Π -functors. There are functors

$$\mathcal{SCat} \xrightarrow{(1)} \Pi\text{-}\mathcal{SCat} \xrightarrow{(2)} \Pi\text{-}\mathcal{Cat}. \quad (1.5)$$

The functor (1) is defined in Definition 1.10 below; it sends supercategory \mathcal{A} to its Π -envelope \mathcal{A}_π . The functor (2) sends Π -supercategory $(\mathcal{A}, \Pi, \zeta)$ to the *underlying* Π -category $(\underline{\mathcal{A}}, \underline{\Pi}, \xi)$, where $\underline{\mathcal{A}}$ and $\underline{\Pi}$ are as in Definition 1.1(v), and $\xi := \zeta\zeta$; it sends superfunctor $F : (\mathcal{A}, \Pi_{\mathcal{A}}, \zeta_{\mathcal{A}}) \rightarrow (\mathcal{B}, \Pi_{\mathcal{B}}, \zeta_{\mathcal{B}})$ to the Π -functor (\underline{F}, β_F) , where $\beta_F := -\zeta_{\mathcal{B}} F(\zeta_{\mathcal{A}})^{-1} : \Pi_{\mathcal{B}} F \xrightarrow{\sim} F \Pi_{\mathcal{A}}$.

Theorem 1.9. *The functors just defined have the following properties:*

- *The functor (1) is left 2-adjoint to the forgetful functor $\nu : \Pi\text{-}\mathcal{SCat} \rightarrow \mathcal{SCat}$ in the sense that there is a superequivalence $\mathcal{H}om(\mathcal{A}, \nu\mathcal{B}) \rightarrow \mathcal{H}om(\mathcal{A}_\pi, \mathcal{B})$ for every supercategory \mathcal{A} and Π -supercategory \mathcal{B} .*
- *The functor (2) is an equivalence of categories.*

Definition 1.10. The Π -envelope \mathcal{A}_π of supercategory \mathcal{A} is the Π -supercategory with objects $\{\Pi^a \lambda \mid \lambda \in \text{ob } \mathcal{A}, a \in \mathbb{Z}/2\}$, i.e. we double the objects in \mathcal{A} . Morphisms are defined from

$$\text{Hom}_{\mathcal{A}_\pi}(\Pi^a \lambda, \Pi^b \mu) := \Pi^{a+b} \text{Hom}_{\mathcal{A}}(\lambda, \mu),$$

where the Π on the right hand side is the parity-switching functor on \mathcal{SVec} from Example 1.8. We denote the morphism $\Pi^a \lambda \rightarrow \Pi^b \mu$ in \mathcal{A}_π coming from a homogeneous morphism $f : \lambda \rightarrow \mu$ in \mathcal{A} under this identification by f_a^b . Thus, if $|f|$ denotes the parity of f in \mathcal{A} , then $f_a^b : \Pi^a \lambda \rightarrow \Pi^b \mu$ is of parity $a+b+|f|$ in \mathcal{A}_π . Composition in \mathcal{A}_π is induced by composition in \mathcal{A} , so $f_b^c \circ g_a^b := (f \circ g)_a^c$. To make \mathcal{A}_π into a Π -supercategory, we set $\Pi(\Pi^a \lambda) := \Pi^{a+\bar{1}} \lambda$ and $\zeta_{\Pi^a \lambda} := (1_\lambda)_{a+\bar{1}}^a : \Pi^{a+\bar{1}} \lambda \rightarrow \Pi^a \lambda$. If $F : \mathcal{A} \rightarrow \mathcal{B}$ is a superfunctor, it extends to $F_\pi : \mathcal{A}_\pi \rightarrow \mathcal{B}_\pi$ sending $\Pi^a \lambda \mapsto \Pi^a (F\lambda)$ and $f_a^b \mapsto (Ff)_a^b$.

Remark 1.11. In [Man], one finds already the notion of a *superadditive category*. In our language, this is an additive Π -supercategory. The *superadditive envelope* of a supercategory \mathcal{A} may be constructed by first taking the Π -envelope, then taking the usual additive envelope after that.

1.4. We can now introduce *monoidal Π -categories* and *monoidal Π -supercategories*. It is best to start with monoidal Π -supercategories, since this definition is on the surface. Then we'll recover the correct definition of monoidal Π -category on passing to the underlying category.

Definition 1.12. A *monoidal Π -supercategory* $(\mathcal{A}, \pi, \zeta)$ is a monoidal supercategory \mathcal{A} with the additional data of a distinguished object π and an odd isomorphism $\zeta : \pi \xrightarrow{\sim} \mathbb{1}$ from π to the unit object $\mathbb{1}$.

Any monoidal Π -supercategory $(\mathcal{A}, \pi, \zeta)$ is a Π -supercategory in the sense of Definition 1.7 with parity-switching functor $\Pi := \pi \otimes - : \mathcal{A} \rightarrow \mathcal{A}$ and $\zeta_\lambda := l_\lambda \circ \zeta \otimes 1_\lambda : \Pi \lambda \xrightarrow{\sim} \lambda$. One could also choose to define Π to be the functor $- \otimes \pi$, but that is isomorphic to our choice because there is an even supernatural isomorphism $\beta : \pi \otimes - \xrightarrow{\sim} - \otimes \pi$ with β_λ defined as the composite

$$\pi \otimes \lambda \xrightarrow{\zeta \otimes 1_\lambda} \mathbb{1} \otimes \lambda \xrightarrow{l_\lambda} \lambda \xrightarrow{r_\lambda^{-1}} \lambda \otimes \mathbb{1} \xrightarrow{1_\lambda \otimes \zeta^{-1}} \lambda \otimes \pi.$$

We observe moreover that the pair (π, β) is an object in the *Drinfeld center* of \mathcal{A} , i.e. we have that

$$l_\pi \circ \beta_{\mathbb{1}} = r_\pi, \quad (1.6)$$

$$a_{\lambda, \mu, \pi} \circ \beta_{\lambda \otimes \mu} \circ a_{\pi, \lambda, \mu} = (1_\lambda \otimes \beta_\mu) \circ a_{\lambda, \pi, \mu} \circ (\beta_\lambda \otimes 1_\mu), \quad (1.7)$$

for all objects $\lambda, \mu \in \text{ob } \mathcal{A}$. Moreover, $\beta_\pi = -1_{\pi \otimes \pi}$. There is also an even isomorphism $\xi := (l_{\mathbb{1}} = r_{\mathbb{1}}) \circ \zeta \otimes \zeta : \pi \otimes \pi \rightarrow \mathbb{1}$ such that

$$(1_\lambda \otimes \xi^{-1}) \circ r_\lambda^{-1} \circ l_\lambda \circ (\xi \otimes 1_\lambda) = a_{\lambda, \pi, \pi} \circ (\beta_\lambda \otimes 1_\pi) \circ a_{\pi, \lambda, \pi}^{-1} \circ (1_\pi \otimes \beta_\lambda) \circ a_{\pi, \pi, \lambda} \quad (1.8)$$

in $\text{Hom}_{\mathcal{A}}((\pi \otimes \pi) \otimes \lambda, \lambda \otimes (\pi \otimes \pi))$.

Example 1.13. (i) We've already explained how $A\text{-SMod-}A$ is both a monoidal supercategory and a Π -supercategory. In fact, it is a monoidal Π -supercategory with $\pi := \Pi A$ and $\zeta : \Pi A \xrightarrow{\sim} A$ being the identity function. In particular, this makes \mathcal{SVec} into a monoidal Π -supercategory.

(ii) If $(\mathcal{A}, \Pi, \zeta)$ is any Π -supercategory, then $(\mathcal{E}nd(\mathcal{A}), \Pi, \zeta)$ is a strict monoidal Π -supercategory.

Definition 1.14. (i) A *monoidal Π -category* $(\mathcal{A}, \pi, \beta, \xi)$ is a \mathbb{k} -linear monoidal category \mathcal{A} plus the extra data of an object (π, β) in its Drinfeld center with $\beta_\pi = -1_{\pi \otimes \pi}$, and an isomorphism $\xi : \pi \otimes \pi \xrightarrow{\sim} \mathbb{1}$ satisfying (1.8).

(ii) A *monoidal Π -functor* between monoidal Π -categories $(\mathcal{A}, \pi_{\mathcal{A}}, \beta_{\mathcal{A}}, \xi_{\mathcal{A}})$ and $(\mathcal{B}, \pi_{\mathcal{B}}, \beta_{\mathcal{B}}, \xi_{\mathcal{B}})$ is a \mathbb{k} -linear monoidal functor $F : \mathcal{A} \rightarrow \mathcal{B}$ with its usual coherence maps c and i , plus an additional coherence map $j : \pi_{\mathcal{B}} \xrightarrow{\sim} F\pi_{\mathcal{A}}$ which is an isomorphism compatible with the β 's and the ξ 's in the sense that

$$\begin{aligned} F(\beta_{\mathcal{A}})_\lambda \circ c_{\pi_{\mathcal{A}}, \lambda} \circ (j \otimes 1_{F\lambda}) &= c_{\lambda, \pi_{\mathcal{A}}} \circ (1_{F\lambda} \otimes j) \circ (\beta_{\mathcal{B}})_{F\lambda}, \\ i \circ \xi_{\mathcal{B}} &= F\xi_{\mathcal{A}} \circ c_{\pi_{\mathcal{A}}, \pi_{\mathcal{A}}} \circ (j \otimes j), \end{aligned}$$

in $\text{Hom}(\pi_{\mathcal{B}} \otimes F\lambda, F(\lambda \otimes \pi_{\mathcal{A}}))$ and $\text{Hom}(\pi_{\mathcal{B}} \otimes \pi_{\mathcal{B}}, F\mathbb{1}_{\mathcal{A}})$, respectively.

(iii) A *monoidal Π -natural transformation* $x : F \Rightarrow G$ between monoidal Π -functors $F, G : \mathcal{A} \rightarrow \mathcal{B}$ is a monoidal natural transformation as usual, such that $x_{\pi_{\mathcal{A}}} \circ j_F = j_G$ in $\text{Hom}_{\mathcal{B}}(\pi_{\mathcal{B}}, G\pi_{\mathcal{A}})$.

There are categories \mathcal{SMon} , $\Pi\text{-}\mathcal{SMon}$ and $\Pi\text{-}\mathcal{Mon}$ consisting of all monoidal supercategories, monoidal Π -supercategories and monoidal Π -categories, respectively. Morphisms in \mathcal{SMon} and $\Pi\text{-}\mathcal{SMon}$ are monoidal superfunctors as in Definition 1.4(ii). Morphisms in $\Pi\text{-}\mathcal{Mon}$ are monoidal Π -functors in the sense of Definition 1.14(ii). Now, just like in (1.5), there are functors

$$\mathcal{SMon} \xrightarrow{(1)} \Pi\text{-}\mathcal{SMon} \xrightarrow{(2)} \Pi\text{-}\mathcal{Mon}. \quad (1.9)$$

The functor (1) is defined by the Π -envelope construction explained in Definition 1.16 below. The functor (2) sends monoidal Π -supercategory $(\mathcal{A}, \pi, \zeta)$ to the underlying category $\underline{\mathcal{A}}$ with the obvious monoidal structure, made into a monoidal Π -category $(\underline{\mathcal{A}}, \pi, \beta, \xi)$ as explained before Definition 1.14. It sends a monoidal superfunctor F between monoidal Π -supercategories \mathcal{A} and \mathcal{B} to $\underline{F} : \underline{\mathcal{A}} \rightarrow \underline{\mathcal{B}}$, made into a monoidal Π -functor by setting $j := (F\zeta_{\mathcal{A}})^{-1} \circ i \circ \zeta_{\mathcal{B}} : \pi_{\mathcal{B}} \xrightarrow{\sim} F\pi_{\mathcal{A}}$.

Theorem 1.15. *The functors just defined satisfy analogous properties to Theorem 1.9: (1) is left 2-adjoint to the forgetful functor and (2) is an equivalence.*

Definition 1.16. The Π -envelope of a monoidal supercategory \mathcal{A} is the monoidal Π -supercategory $(\mathcal{A}_{\pi}, \pi, \zeta)$ where \mathcal{A}_{π} is as in Definition 1.10, $\pi := \Pi^{\bar{1}}\mathbb{1}$, $\zeta := (\mathbb{1}_{\mathbb{1}})^{\bar{0}}$, and tensor products of objects and morphisms are defined from

$$\begin{aligned} (\Pi^a \lambda) \otimes (\Pi^b \mu) &:= \Pi^{a+b}(\lambda \otimes \mu), \\ f_a^b \otimes g_c^d &:= (-1)^{a|g|+|f|d+ad+ac} (f \otimes g)_{a+c}^{b+d}. \end{aligned}$$

The unit object of \mathcal{A}_{π} is $\Pi^{\bar{0}}\mathbb{1}$. The coherence maps a, l and r extend to \mathcal{A}_{π} in an obvious way. Also if $F : \mathcal{A} \rightarrow \mathcal{B}$ is a monoidal superfunctor then the superfunctor $F_{\pi} : \mathcal{A}_{\pi} \rightarrow \mathcal{B}_{\pi}$ from Definition 1.10 is naturally monoidal too.

In the strict case, one can work with \mathcal{A}_{π} diagrammatically as follows. For f as in (1.2), we represent $f_a^b \in \text{Hom}_{\mathcal{A}_{\pi}}(\Pi^a \lambda, \Pi^b \mu)$ by the diagram

Then the rules for horizontal and vertical composition become:

In order to appreciate the need for the sign in this definition of horizontal composition, the reader might want to verify the super interchange law in \mathcal{A}_{π} .

1.5. Let us make a few remarks about Grothendieck groups/rings. Recall for a category \mathcal{A} that its *additive Karoubi envelope* $\text{Kar}(\mathcal{A})$ is the idempotent completion of the additive envelope of \mathcal{A} . The *Grothendieck group* $K_0(\text{Kar}(\mathcal{A}))$ is the Abelian group generated by isomorphism classes of objects of $\text{Kar}(\mathcal{A})$, subject to the relations $[V] + [W] = [V \oplus W]$. In case \mathcal{A} is a monoidal category, the monoidal structure on \mathcal{A} extends canonically to $\text{Kar}(\mathcal{A})$, hence we get a ring structure on $K_0(\text{Kar}(\mathcal{A}))$ with $[V] \cdot [W] = [V \otimes W]$.

For a supercategory \mathcal{A} , we propose that the role of additive Karoubi envelope should be played by the Π -category $\text{SKar}(\mathcal{A}) := \text{Kar}(\underline{\mathcal{A}}_{\pi})$, i.e. one first passes to the Π -envelope, then to the underlying category, and then one takes additive Karoubi envelope as usual. The Grothendieck group $K_0(\text{SKar}(\mathcal{A}))$ comes equipped with a distinguished involution π defined from $\pi([V]) := [\Pi V]$, making it into a module over the ring

$$\mathbb{Z}^{\pi} := \mathbb{Z}[\pi]/(\pi^2 - 1).$$

In case \mathcal{A} is a monoidal supercategory, $\text{SKar}(\mathcal{A})$ is a monoidal Π -category. The tensor product induces a multiplication on $K_0(\text{SKar}(\mathcal{A}))$, making it into a \mathbb{Z}^{π} -algebra.

Example 1.17. (i) Suppose A is a superalgebra viewed as a supercategory \mathcal{A} with one object. Then $\mathrm{SKar}(\mathcal{A})$ is equivalent to the category of finitely generated projective A -supermodules and even A -supermodule homomorphisms. Hence, $K_0(\mathrm{SKar}(\mathcal{A}))$ is the usual split Grothendieck group of the superalgebra A .

(ii) Recall that \mathcal{I} , the unit object of the monoidal category \mathcal{SCat} , is a supercategory with one object whose endomorphism superalgebra is \mathbb{k} . There is a unique way to define a tensor product making \mathcal{I} into a strict monoidal supercategory. Its super Karoubi envelope $\mathrm{SKar}(\mathcal{I})$ is monoidally equivalent to $\underline{\mathcal{SVec}}_{fd}$. Hence, it is a semisimple Abelian category with just two isomorphism classes of irreducible objects represented by \mathbb{k} and $\Pi\mathbb{k}$, and $K_0(\mathrm{SKar}(\mathcal{I})) \cong K_0(\underline{\mathcal{SVec}}_{fd}) \cong \mathbb{Z}^\pi$.

(iii) Here is an example which may be of independent interest. For $\delta \in \mathbb{k}$, the *odd Temperley-Lieb supercategory* is the strict monoidal supercategory $\mathcal{STL}(\delta)$ with one generating object \cdot and two *odd* generating morphisms $\cup : \mathbb{1} \rightarrow \cdot \otimes \cdot$ and $\cap : \cdot \otimes \cdot \rightarrow \mathbb{1}$, subject to the following relations:

$$\begin{array}{c} \cup \\ \cap \end{array} = |, \quad \begin{array}{c} \cap \\ \cup \end{array} = -|, \quad \bigcirc = \delta.$$

The following theorem will be proved in the appendix.

Theorem 1.18. *Assume that $\delta = -(q - q^{-1})$ for $q \in \mathbb{k}^\times$ that is not a root of unity. Then $\mathrm{SKar}(\mathcal{STL}(\delta))$ is a semisimple Abelian category. Moreover, as a based ring with canonical basis coming from the isomorphism classes of irreducible objects, $K_0(\mathrm{SKar}(\mathcal{STL}(\delta)))$ is isomorphic to the subring of $\mathbb{Z}^\pi[x, x^{-1}]$ spanned over \mathbb{Z} by $\{[n+1]_{x,\pi}, \pi[n+1]_{x,\pi} \mid n \in \mathbb{N}\}$, where*

$$[n+1]_{x,\pi} := x^n + \pi x^{n-2} + \cdots + \pi^n x^{-n}. \quad (1.10)$$

When \mathbb{k} is of characteristic zero, we will explain this result by constructing a monoidal equivalence between $\mathrm{SKar}(\mathcal{STL}(\delta))$ and the category of finite-dimensional representations of the quantum superalgebra $U_q(\mathfrak{osp}_{1|2})$ as defined by Clark and Wang [CW]. We note that

$$[n+1]_{x,\pi}[m+1]_{x,\pi} = \sum_{r=0}^{\min(m,n)} \pi^r [n+m-2r+1]_{x,\pi}, \quad (1.11)$$

which may be interpreted as the analog of Clebsch-Gordan for $U_q(\mathfrak{osp}_{1|2})$. Also

$$\sum_{n=0}^{\infty} [n]_{x,\pi} t^n = \frac{1}{1 - [2]_{x,\pi} t + \pi t^2}, \quad (1.12)$$

which is a π -deformed version of the generating function for Chebyshev polynomials of the second kind. It follows that $K_0(\mathrm{SKar}(\mathcal{STL}(\delta)))$ is a polynomial algebra over \mathbb{Z}^π generated by $[2]_{x,\pi}$, which is the isomorphism class of the generating object \cdot .

1.7. In the remainder of the article, we will work in the more general setting of 2-categories. Recalling that a monoidal category is essentially the same as a 2-category with one object, the reader should have no trouble recovering the definitions made in this introduction from the more general ones formulated later on.

In Section 2, we will discuss *2-supercategories*, which (in the strict case) are categories enriched in \mathcal{SCat} ; the basic example is the 2-supercategory of supercategories, superfunctors and supernatural transformations. Then in Section 3, we

introduce Π -2-supercategories; the basic example is the Π -2-supercategory of Π -supercategories, superfunctors and supernatural transformations. Section 4 develops the appropriate generalization of the notion of Π -envelope to 2-supercategories, in particular establishing the properties of the functors (1) above. In Section 5, we discuss Π -2-categories; the basic example is the Π -2-category of Π -categories, Π -functors and Π -natural transformations. Then we prove that the functors (2) above are equivalences; more generally, we show that the categories of Π -2-categories and Π -2-supercategories are equivalent.

The approach to $\mathbb{Z}/2$ -graded categories developed by this point can also be applied in almost exactly the same way to \mathbb{Z} -graded categories. We give a brief account of this in the final Section 6. Actually, we will combine the two gradings into a single $\mathbb{Z} \oplus \mathbb{Z}/2$ -grading, and develop a theory of *graded supercategories*. Although we won't discuss it further here, there are two natural ways to suppress the $\mathbb{Z}/2$ -grading (thereby leaving the domain of superalgebra): one can either view \mathbb{Z} -gradings as $\mathbb{Z} \oplus \mathbb{Z}/2$ -gradings with the $\mathbb{Z}/2$ -grading being trivial, i.e. concentrated in parity $\bar{0}$; or one can view \mathbb{Z} -gradings as $\mathbb{Z} \oplus \mathbb{Z}/2$ -gradings with the $\mathbb{Z}/2$ -grading being induced by the \mathbb{Z} -grading, i.e. all elements of degree $n \in \mathbb{Z}$ are of parity $n \pmod{2}$. The first of these variations is already extensively used in representation theory, e.g. see the last paragraph of [R, §2.2.1] or [BLW, §5.2].

We would like to say finally that many of the general definitions in this article can be found in some equivalent form in many places in the literature. We were influenced especially by the work of Kang, Kashiwara and Oh in [KKO, Section 7]; see also [EL, Section 2]. Our choice of terminology is different. We include here a brief dictionary for readers familiar with [KKO] and [EL]; note also that in [KKO] additivity is assumed from the outset.

Our language	Language of [KKO, EL]
Supercategory	1-supercategory [KKO, Def. 7.7]
Superfunctor	Superfunctor [KKO, Def. 7.7]
Supernatural transformation	Even and odd morphisms [KKO, Def. 7.8]
2-supercategory	2-supercategory [KKO, Def. 7.12]
Π -category	Supercategory [KKO, Def. 7.1], [EL, Def. 2.13]
Π -functor	Superfunctor [KKO, Def. 7.1], [EL, Def. 2.13]
Π -natural transformation	Supernatural transformation [EL, Def. 2.16]
Π -2-category	Super-2-category [EL, Def. 2.17]

There is a similar linguistic clash in our development of the graded theory in Section 6: by a *graded category*, we mean a category enriched in graded vector spaces. It is more common in the literature for a graded category to mean a category equipped with a distinguished autoequivalence. When working with the latter structure, we will denote this autoequivalence by Q , and call it a *Q-category*.

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2. SUPERCATEGORIES

In the main body of the article, \mathbb{k} will denote some fixed commutative ground ring. By *superspace*, we mean now a $\mathbb{Z}/2$ -graded \mathbb{k} -module $V = V_{\bar{0}} \oplus V_{\bar{1}}$; as usual when working over a commutative ring, we make no distinction between left modules and right modules, indeed, we'll often view \mathbb{k} -modules as (\mathbb{k}, \mathbb{k}) -bimodules whose left and right actions are related by $cv = vc$. By a linear map, we mean a \mathbb{k} -module

homomorphism. Viewing \mathbb{k} as a superalgebra concentrated in even parity, these are the same as \mathbb{k} -supermodules and \mathbb{k} -supermodule homomorphisms².

We have the Π -supercategory \mathcal{SVec} of all superspaces³ and linear maps defined just like in the introduction. The underlying category $\underline{\mathcal{SVec}}$ consisting of superspaces and even linear maps is a symmetric monoidal category with braiding defined as in the introduction.

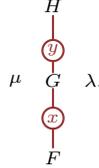
Recall also the definitions of *superfunctor* and *supernatural transformation* from Definition 1.1. Let \mathcal{SCat} be the category of all supercategories and superfunctors. We make it into a monoidal category with tensor functor denoted \boxtimes as explained after Example 1.2.

Definition 2.1. A *strict 2-supercategory* is a category enriched in the monoidal category \mathcal{SCat} just defined. Thus, for objects λ, μ in a strict 2-supercategory \mathfrak{A} , there is given a supercategory $\mathcal{H}om_{\mathfrak{A}}(\lambda, \mu)$ of morphisms from λ to μ , whose objects F, G are the *1-morphisms* of \mathfrak{A} , and whose morphisms $x : F \rightarrow G$ are the *2-morphisms* of \mathfrak{A} . We use the shorthand $\mathcal{H}om_{\mathfrak{A}}(F, G)$ for the superspace $\mathcal{H}om_{\mathcal{H}om_{\mathfrak{A}}(\lambda, \mu)}(F, G)$ of all such 2-morphisms.

The string calculus explained for monoidal supercategories in the introduction can also be used for strict 2-supercategories: given 1-morphisms $F, G : \lambda \rightarrow \mu$, one represents a 2-morphism $x : F \Rightarrow G$ by the picture



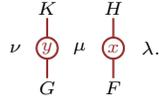
The composition $y \circ x$ of x with another 2-morphism $y \in \mathcal{H}om_{\mathfrak{A}}(G, H)$ is obtained by vertically stacking pictures:



The composition law in \mathfrak{A} gives a coherent family of superfunctors

$$T_{\nu, \mu, \lambda} : \mathcal{H}om_{\mathfrak{A}}(\mu, \nu) \boxtimes \mathcal{H}om_{\mathfrak{A}}(\lambda, \mu) \rightarrow \mathcal{H}om_{\mathfrak{A}}(\lambda, \nu)$$

for all objects $\lambda, \mu, \nu \in \mathfrak{A}$. Given 2-morphisms $x : F \rightarrow H, y : G \rightarrow K$ between 1-morphisms $F, H : \lambda \rightarrow \mu, G, K : \mu \rightarrow \nu$, we denote $T_{\nu, \mu, \lambda}(y \otimes x) : T_{\nu, \mu, \lambda}(G, F) \rightarrow T_{\nu, \mu, \lambda}(K, H)$ simply by $yx : GF \rightarrow KH$, and represent it by horizontally stacking pictures:



²In Sections 2–4, one can actually work even more generally over any commutative superalgebra $\mathbb{k} = \mathbb{k}_{\bar{0}} \oplus \mathbb{k}_{\bar{1}}$, interpreting a superspace as a (\mathbb{k}, \mathbb{k}) -superbimodule whose left and right actions are related by $cv = (-1)^{|c||v|}vc$.

³One should be careful about set-theoretic issues here by fixing a Grothendieck universe and taking only *small* superspaces. We won't be doing anything high enough for this to cause difficulties, so will ignore issues of this nature.

When confusion seems unlikely, we will use the same notation for a 1-morphism F as for its identity 2-morphism. With this convention, we have that $yH \circ Gx = yx = (-1)^{|x||y|}Kx \circ yF$, or in pictures:

$$\begin{array}{c} K \\ | \\ \nu \text{---} \textcircled{y} \text{---} \mu \\ | \\ G \end{array} \quad \begin{array}{c} H \\ | \\ \textcircled{x} \text{---} \lambda \\ | \\ F \end{array} = \begin{array}{c} K \\ | \\ \nu \text{---} \textcircled{y} \text{---} \mu \\ | \\ G \end{array} \quad \begin{array}{c} H \\ | \\ \textcircled{x} \text{---} \lambda \\ | \\ F \end{array} = (-1)^{|x||y|} \begin{array}{c} K \\ | \\ \nu \text{---} \textcircled{y} \text{---} \mu \\ | \\ G \end{array} \quad \begin{array}{c} H \\ | \\ \textcircled{x} \text{---} \lambda \\ | \\ F \end{array}.$$

This identity is a special case of the *super interchange law* in a strict 2-supercategory, which is proved by the following calculation:

$$\begin{aligned} (vu) \circ (yx) &= T_{\nu,\mu,\lambda}(v \otimes u) \circ T_{\nu,\mu,\lambda}(y \otimes x) = T_{\nu,\mu,\lambda}((v \otimes u) \circ (y \otimes x)) \\ &= (-1)^{|u||y|} T_{\nu,\mu,\lambda}((v \circ y) \otimes (u \circ x)) = (-1)^{|u||y|} (v \circ y)(u \circ x). \end{aligned}$$

The presence of the sign here means that a strict 2-supercategory is *not* a 2-category in the usual sense.

For example, we can make \mathcal{SCat} into a strict 2-supercategory \mathfrak{SCat} by declaring that its morphism categories are the supercategories $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ consisting of all superfunctors from \mathcal{A} to \mathcal{B} , with morphisms being all supernatural transformations. The horizontal composition GF of two superfunctors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ is defined by $GF := G \circ F$. The horizontal composition $yx : GF \Rightarrow KH$ of supernatural transformations $x : F \Rightarrow H$ and $y : G \Rightarrow K$ is given by $(yx)_\lambda := y_{H\lambda} \circ Gx_\lambda$ for each object λ of \mathcal{A} . We leave it to the reader to verify that the super interchange law holds; this works because of the signs built into the definition of supernatural transformation.

So far, we have only defined the notion of *strict* 2-supercategory. There is also a “weak” notion, which we call simply *2-supercategory*, in which the horizontal composition is only assumed to be associative and unital up to some even supernatural isomorphisms. The following are the superizations of the definitions in the purely even setting (e.g. see the definition of bicategory in [L], or [R, §2.2.2]), replacing the usual Cartesian product \times of categories with the product \boxtimes .

Definition 2.2. (i) A *2-supercategory* \mathfrak{A} consists of:

- A set of objects $\text{ob } \mathfrak{A}$.
- A supercategory $\mathcal{H}om_{\mathfrak{A}}(\mu, \lambda)$ for each $\lambda, \mu \in \text{ob } \mathfrak{A}$, whose objects and morphisms are called 1-morphisms and 2-morphisms, respectively. We refer to the composition of 2-morphisms in these supercategories as *vertical composition*.
- A family of 1-morphisms $\mathbb{1}_\lambda : \lambda \rightarrow \lambda$ for each $\lambda \in \text{ob } \mathfrak{A}$.
- Superfunctors $T_{\nu,\mu,\lambda} : \mathcal{H}om_{\mathfrak{A}}(\mu, \nu) \boxtimes \mathcal{H}om_{\mathfrak{A}}(\lambda, \mu) \rightarrow \mathcal{H}om_{\mathfrak{A}}(\lambda, \nu)$ for all $\lambda, \mu, \nu \in \text{ob } \mathfrak{A}$. We usually denote $T_{\nu,\mu,\lambda}$ simply by $--$, and call it *horizontal composition*.
- Even supernatural isomorphisms $a : (--)- \xrightarrow{\sim} -(--)$, $l : \mathbb{1}_\lambda - \xrightarrow{\sim} -$ and $r : - \mathbb{1}_\lambda \xrightarrow{\sim} -$ in all situations that such horizontal compositions makes sense.

Then we require that the following diagrams of supernatural transformations commute:

$$\begin{array}{ccc}
 & ((- -) -) - & \\
 & \swarrow a - & \searrow a \\
 (- (- -)) - & & (- -) (- -) \\
 & \searrow a & \swarrow a \\
 - ((- -) -) & \xrightarrow{-a} & - (- (- -))
 \end{array}
 , \quad
 \begin{array}{ccc}
 (- \mathbb{1}_\lambda) - & & \\
 & \searrow r - & \\
 & & - - - \\
 & \swarrow -l & \\
 - (\mathbb{1}_\lambda -) & &
 \end{array}$$

A 1-morphism $F : \lambda \rightarrow \mu$ in a 2-supercategory is called a *superequivalence* if there is a 1-morphism G in the other direction such that $GF \cong \mathbb{1}_\lambda$ and $FG \cong \mathbb{1}_\mu$ via even 2-isomorphisms.

(ii) A 2-supercategory $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$ between 2-supercategories is the following data:

- A function $\mathbb{R} : \text{ob } \mathfrak{A} \rightarrow \text{ob } \mathfrak{B}$.
- Superfunctors $\mathbb{R} : \mathcal{H}om_{\mathfrak{A}}(\lambda, \mu) \rightarrow \mathcal{H}om_{\mathfrak{B}}(\mathbb{R}\lambda, \mathbb{R}\mu)$ for $\lambda, \mu \in \text{ob } \mathfrak{A}$.
- Even supernatural isomorphisms $c : (\mathbb{R} -) (\mathbb{R} -) \xrightarrow{\cong} \mathbb{R}(- -)$.
- Even 2-isomorphisms $i : \mathbb{1}_{\mathbb{R}\lambda} \xrightarrow{\cong} \mathbb{R}\mathbb{1}_\lambda$ for all $\lambda \in \text{ob } \mathfrak{A}$.

Then we require that the following diagrams commute:

$$\begin{array}{ccc}
 & (\mathbb{R}(- -)) (\mathbb{R} -) & \\
 & \swarrow c(\mathbb{R} -) & \searrow c \\
 ((\mathbb{R} -) (\mathbb{R} -)) (\mathbb{R} -) & & \mathbb{R}((- -) -) \\
 & \searrow a & \swarrow \mathbb{R}a \\
 (\mathbb{R} -) ((\mathbb{R} -) (\mathbb{R} -)) & & \mathbb{R}(- (- -)) \\
 & \swarrow (\mathbb{R} -) c & \searrow c \\
 & (\mathbb{R} -) (\mathbb{R}(- -)) &
 \end{array}
 , \quad
 \begin{array}{ccc}
 (\mathbb{R} -) \mathbb{1}_{\mathbb{R}\lambda} & \xrightarrow{(\mathbb{R} -) i} & (\mathbb{R} -) (\mathbb{R}\mathbb{1}_\lambda) & \mathbb{1}_{\mathbb{R}\mu} (\mathbb{R} -) & \xrightarrow{i(\mathbb{R} -)} & (\mathbb{R}\mathbb{1}_\mu) (\mathbb{R} -) \\
 r \downarrow & & \downarrow c & l \downarrow & & \downarrow c \\
 \mathbb{R} - & \xleftarrow{\mathbb{R} r} & \mathbb{R}(- \mathbb{1}_\lambda) & \mathbb{R} - & \xleftarrow{\mathbb{R} l} & \mathbb{R}(\mathbb{1}_\mu -)
 \end{array}$$

There is a natural way to compose two 2-supercategories. Also each 2-supercategory \mathfrak{A} possesses an identity 2-supercategory, which will be denoted \mathbb{I} . Hence, we get a category 2-SCat consisting of 2-supercategories and 2-supercategories.

(iii) Given 2-supercategories $\mathbb{R}, \mathbb{S} : \mathfrak{A} \rightarrow \mathfrak{B}$ for 2-supercategories \mathfrak{A} and \mathfrak{B} , a 2-natural transformation⁴ $(X, x) : \mathbb{R} \Rightarrow \mathbb{S}$ is the following data:

- 1-morphisms $X_\lambda : \mathbb{R}\lambda \rightarrow \mathbb{S}\lambda$ in \mathfrak{B} for each $\lambda \in \text{ob } \mathfrak{A}$.
- Even supernatural transformations $x_{\mu, \lambda} : X_\mu(\mathbb{R} -) \Rightarrow (\mathbb{S} -)X_\lambda$ (which are superfunctors $\mathcal{H}om_{\mathfrak{A}}(\lambda, \mu) \rightarrow \mathcal{H}om_{\mathfrak{B}}(\mathbb{R}\lambda, \mathbb{S}\mu)$) for all $\lambda, \mu \in \text{ob } \mathfrak{A}$

⁴In *nLab* this is an *oplax natural transformation*.

We require that the following diagrams commute for all $F : \lambda \rightarrow \mu$ and $G : \mu \rightarrow \nu$:

$$\begin{array}{ccc}
X_\nu((\mathbb{R}G)(\mathbb{R}F)) & \xrightarrow{a^{-1}} & (X_\nu(\mathbb{R}G))(\mathbb{R}F) \xrightarrow{(x_{\nu,\mu})_G(\mathbb{R}F)} ((\mathbb{S}G)X_\mu)(\mathbb{R}F) \\
\downarrow X_\nu c & & \downarrow a \\
X_\nu \mathbb{R}(GF) & & (\mathbb{S}G)(X_\mu(\mathbb{R}F)) \\
\downarrow (x_{\nu,\lambda})_{GF} & & \downarrow (\mathbb{S}G)(x_{\mu,\lambda})_F \\
\mathbb{S}(GF)X_\lambda & \xleftarrow{c_{X_\lambda}} & ((\mathbb{S}G)(\mathbb{S}F))X_\lambda \xleftarrow{a^{-1}} (\mathbb{S}G)((\mathbb{S}F)X_\lambda) \\
& & \\
& & \mathbb{1}_{\mathbb{S}\lambda}X_\lambda \xrightarrow{l} X_\lambda \xrightarrow{r^{-1}} X_\lambda \mathbb{1}_{\mathbb{R}\lambda} \\
& & \downarrow i_{X_\lambda} \qquad \qquad \qquad \downarrow X_\lambda i \\
& & (\mathbb{S}\mathbb{1}_\lambda)X_\lambda \xleftarrow{(x_{\lambda,\lambda})_{\mathbb{1}_\lambda}} X_\lambda(\mathbb{R}\mathbb{1}_\lambda)
\end{array}$$

A 2-natural transformation (X, x) is *strong*⁵ if each $x_{\mu,\lambda}$ is an isomorphism. There is a 2-category $2\text{-}\mathfrak{SCat}$ consisting of all 2-supercategories, 2-superfunctors and 2-natural transformations.

(iv) Suppose that $(X, x), (Y, y) : \mathbb{R} \rightarrow \mathbb{S}$ are 2-natural transformations for 2-superfunctors $\mathbb{R}, \mathbb{S} : \mathfrak{A} \rightarrow \mathfrak{B}$. A *supermodification* $\alpha : (X, x) \Rightarrow (Y, y)$ is a family of 2-morphisms $\alpha_\lambda = \alpha_{\lambda, \bar{0}} + \alpha_{\lambda, \bar{1}} : X_\lambda \Rightarrow Y_\lambda$ for all $\lambda \in \text{ob } \mathfrak{A}$, such that the diagram

$$\begin{array}{ccc}
X_\mu(\mathbb{R}F) & \xrightarrow{(x_{\mu,\lambda})_F} & (\mathbb{S}F)X_\lambda \\
\downarrow \alpha_\mu(\mathbb{R}F) & & \downarrow (\mathbb{S}F)\alpha_\lambda \\
Y_\mu(\mathbb{R}F) & \xrightarrow{(y_{\mu,\lambda})_F} & (\mathbb{S}F)Y_\lambda
\end{array}$$

commutes for all 1-morphisms $F : \lambda \rightarrow \mu$ in \mathfrak{A} . We have that $\alpha = \alpha_{\bar{0}} + \alpha_{\bar{1}}$ where $(\alpha_p)_\lambda := \alpha_{\lambda,p}$. This makes the space $\text{Hom}((X, x), (Y, y))$ of supermodifications $\alpha : (X, x) \Rightarrow (Y, y)$ into a superspace. There is a supercategory $\mathcal{H}om(\mathbb{R}, \mathbb{S})$ consisting of all 2-natural transformations and supermodifications. There is a 2-supercategory $\mathfrak{H}om(\mathfrak{A}, \mathfrak{B})$ consisting of 2-superfunctors, 2-natural transformations and supermodifications; it is strict if \mathfrak{B} is strict. These are the morphism 2-supercategories in the strict 3-supercategory of 2-supercategories. Since we won't do anything with this here, we omit the details.

We note that a strict 2-supercategory in the sense of Definition 2.1 is the same thing as a 2-supercategory whose coherence maps a, l and r are identities. In the strict case, the unit objects $\mathbb{1}_\lambda$ are uniquely determined, so do not need to be given as part of the data. A *strict 2-superfunctor* is a 2-superfunctor whose coherence maps c and i are identities. There exist 2-superfunctors between strict 2-supercategories which are themselves not strict.

Recall for superalgebras A and B that $B\text{-}\mathcal{S}Mod\text{-}A$ denotes the supercategory of (B, A) -superbimodules; see Example 1.2(iii). Given another superalgebra C , the usual tensor product over B gives a superfunctor

$$- \otimes_B - : C\text{-}\mathcal{S}Mod\text{-}B \boxtimes B\text{-}\mathcal{S}Mod\text{-}A \rightarrow C\text{-}\mathcal{S}Mod\text{-}A.$$

The 2-supercategory \mathfrak{SBim} of *superbimodules* has objects that are superalgebras, the morphism supercategories are defined from $\mathcal{H}om_{\mathfrak{SBim}}(A, B) := B\text{-}\mathcal{S}Mod\text{-}A$, and horizontal composition comes from the tensor product operation just mentioned. It gives a basic example of a 2-supercategory which is not strict.

⁵Or a *pseudonatural transformation* in $n\text{Lab}$.

Two 2-supercategories \mathfrak{A} and \mathfrak{B} are *2-superequivalent* if there are 2-superfunctors $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$ and $\mathbb{S} : \mathfrak{B} \rightarrow \mathfrak{A}$ such that $\mathbb{S} \circ \mathbb{R}$ and $\mathbb{R} \circ \mathbb{S}$ are superequivalent to the identities in $\mathfrak{H}om(\mathfrak{A}, \mathfrak{A})$ and $\mathfrak{H}om(\mathfrak{B}, \mathfrak{B})$, respectively. Equivalently, there is a 2-superfunctor $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$ that induces a superequivalence $\mathfrak{H}om_{\mathfrak{A}}(\lambda, \mu) \rightarrow \mathfrak{H}om_{\mathfrak{B}}(\mathbb{R}\lambda, \mathbb{R}\mu)$ for all $\lambda, \mu \in \text{ob } \mathfrak{A}$, and every $\nu \in \text{ob } \mathfrak{B}$ is superequivalent to an object of the form $\mathbb{R}\lambda$ for some $\lambda \in \text{ob } \mathfrak{A}$.

The *Coherence Theorem* for 2-supercategories implies that any 2-supercategory is 2-superequivalent to a strict 2-supercategory. The proof can be obtained by mimicking the argument in the purely even case from [L]. In view of this result, we will sometimes assume for simplicity that we are working in the strict case.

Definition 2.3. Let \mathfrak{A} be a 2-supercategory. The *Drinfeld center* of \mathfrak{A} is the monoidal supercategory of all strong 2-natural transformations $\mathbb{I} \Rightarrow \mathbb{I}$ and super-modifications. Thus, an object (X, x) of the Drinfeld center is a coherent family of 1-morphisms $X_\lambda : \lambda \rightarrow \lambda$ and even supernatural isomorphisms $x_{\mu, \lambda} : X_\mu - \overset{\sim}{\cong} - X_\lambda$ for $\lambda, \mu \in \text{ob } \mathfrak{A}$; a morphism $\alpha : (X, x) \Rightarrow (Y, y)$ is coherent family of 2-morphisms $\alpha_\lambda : X_\lambda \Rightarrow Y_\lambda$. The tensor product $(X \otimes Y, x \otimes y)$ of objects (X, x) and (Y, y) is defined from $(X \otimes Y)_\lambda := X_\lambda Y_\lambda$, $(x \otimes y)_{\mu, \lambda} := x_{\mu, \lambda} y_{\mu, \lambda}$; the tensor product $\alpha \otimes \beta$ of morphisms $\alpha : (X, x) \rightarrow (U, u)$ and $\beta : (Y, y) \rightarrow (V, v)$ is defined from $(\alpha \otimes \beta)_\lambda := \alpha_\lambda \beta_\lambda$. If \mathfrak{A} is strict then its Drinfeld center is strict too.

We remark that the Drinfeld center of a 2-supercategory is a *braided* monoidal supercategory, although we omit the definition of such a structure. (See [MS] for more about Drinfeld center in the purely even setting.)

3. Π -SUPERCATEGORIES

According to Definition 1.7, a *Π -supercategory* is a supercategory with the additional data of a parity-switching functor Π and an odd supernatural isomorphism $\zeta : \Pi \overset{\sim}{\cong} I$. It is an easy structure to work with as there are no additional axioms, unlike the situation for the Π -categories of Definition 1.6. The same goes for superfunctors and supernatural transformations between Π -supercategories: there are no additional compatibility constraints with respect to Π .

Definition 3.1. A *Π -2-supercategory* $(\mathfrak{A}, \pi, \zeta)$ is a 2-supercategory \mathfrak{A} plus families $\pi = (\pi_\lambda)$ and $\zeta = (\zeta_\lambda)$ of 1-morphisms $\pi_\lambda : \lambda \rightarrow \lambda$ and odd 2-isomorphisms $\zeta_\lambda \in \text{Hom}_{\mathfrak{A}}(\pi_\lambda, \mathbb{1}_\lambda)$ for each object $\lambda \in \mathfrak{A}$. It is *strict* if \mathfrak{A} is strict.

Let $\Pi\text{-SCat}$ be the category of all Π -supercategories and superfunctors. Let $\Pi\text{-SCat}$ be the strict 2-supercategory of all Π -supercategories, superfunctors and supernatural transformations. The latter gives the archetypal example of a strict Π -2-supercategory: the additional data of $\pi = (\pi_{\mathcal{A}})$ and $\zeta = (\zeta_{\mathcal{A}})$ are defined by letting $\pi_{\mathcal{A}}$ be the parity-switching functor $\Pi_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$ on the Π -supercategory \mathcal{A} , and taking $\zeta_{\mathcal{A}} : \pi_{\mathcal{A}} \overset{\sim}{\cong} \mathbb{1}_{\mathcal{A}}$ to be the given odd supernatural isomorphism $\Pi_{\mathcal{A}} \overset{\sim}{\cong} I_{\mathcal{A}}$.

The basic example of a Π -2-supercategory that is not strict is the 2-supercategory \mathfrak{SBim} defined at the end of the previous section. Recall the objects are superalgebras, the 1-morphisms are superbimodules, the 2-morphisms are superbimodule homomorphisms, and horizontal composition is given by tensor product. Also, for each object (i.e. superalgebra) A , the unit 1-morphism $\mathbb{1}_A$ is the regular superbimodule A . The extra data π and ζ needed to make \mathfrak{SBim} into a Π -2-supercategory are given by declaring that $\pi_A := \Pi A$ (i.e. we apply the parity-switching functor to the regular superbimodule), and each $\zeta_A : \pi_A \overset{\sim}{\cong} \mathbb{1}_A$ comes from the superbimodule homomorphism $\Pi A \rightarrow A$ that is the identity function on the underlying set.

Each morphism supercategory $\mathcal{H}om_{\mathfrak{A}}(\lambda, \mu)$ in a Π -2-supercategory \mathfrak{A} admits a parity-switching functor Π making it into a Π -supercategory, namely, the endofunctor π_{μ} arising by horizontally composing on the left by π_{μ} . Alternatively, one could take the endofunctor $-\pi_{\lambda}$ defined by horizontally composing on the right by π_{λ} . These two choices are isomorphic according to our first lemma.

Lemma 3.2. *Let $(\mathfrak{A}, \pi, \zeta)$ be a Π -2-supercategory. For objects λ, μ , there is an even supernatural isomorphism*

$$\beta_{\mu, \lambda} : \pi_{\mu} - \xrightarrow{\cong} -\pi_{\lambda}.$$

Assuming \mathfrak{A} is strict for simplicity, this is defined by $(\beta_{\mu, \lambda})_F := -\zeta_{\mu} F \zeta_{\lambda}^{-1}$ for each 1-morphism $F : \lambda \rightarrow \mu$. Setting $\beta := (\beta_{\mu, \lambda})$, the pair (π, β) is an object in the Drinfeld center of \mathfrak{A} as in Definition 2.3, i.e. the following hold (still assuming strictness):

- (i) $(\beta_{\nu, \lambda})_{GF} = G(\beta_{\mu, \lambda})_F \circ (\beta_{\nu, \mu})_G F$ for 1-morphisms $F : \lambda \rightarrow \mu$ and $G : \mu \rightarrow \nu$;
- (ii) $(\beta_{\lambda, \lambda})_{1_{\lambda}} = 1_{\pi_{\lambda}}$.

Moreover:

- (iii) $\pi_{\lambda} \zeta_{\lambda} = -\zeta_{\lambda} \pi_{\lambda}$ hence $(\beta_{\lambda, \lambda})_{\pi_{\lambda}} = -1_{\pi_{\lambda}^2}$;
- (iv) $\xi_{\lambda} := \zeta_{\lambda} \zeta_{\lambda} : \pi_{\lambda}^2 \Rightarrow 1_{\lambda}$ is an even 2-isomorphism such that $\xi_{\mu} F \zeta_{\lambda}^{-1} = (\beta_{\mu, \lambda})_F \pi_{\lambda} \circ \pi_{\mu} (\beta_{\mu, \lambda})_F$ in $\text{Hom}_{\mathcal{C}}(\pi_{\mu}^2 F, F \pi_{\lambda}^2)$ for all $F : \lambda \rightarrow \mu$.

Proof. To show that $\beta_{\mu, \lambda}$ is an even supernatural isomorphism, we need to show for any 2-morphism $x : F \Rightarrow G$ between 1-morphisms $F, G : \lambda \rightarrow \mu$ that

$$x \pi_{\lambda} \circ (\beta_{\mu, \lambda})_F = (\beta_{\mu, \lambda})_G \circ \pi_{\mu} x. \quad (3.1)$$

This follows from the following calculation with the super interchange law:

$$x \pi_{\lambda} \circ \zeta_{\mu} G \zeta_{\lambda}^{-1} = (-1)^{|x|} \zeta_{\mu} x \zeta_{\lambda}^{-1} = \zeta_{\mu} F \zeta_{\lambda}^{-1} \circ \pi_{\mu} x.$$

For (i), we must show that $G \zeta_{\mu} F \zeta_{\lambda}^{-1} \circ \zeta_{\nu} G \zeta_{\mu}^{-1} F = -\zeta_{\nu} G F \zeta_{\lambda}^{-1}$, which is clear by the super interchange law again. For (ii), we have that $-\zeta_{\lambda} \zeta_{\lambda}^{-1} = \zeta_{\lambda}^{-1} \circ \zeta_{\lambda} = 1_{\pi_{\lambda}}$. For (iii), $\zeta_{\lambda} \zeta_{\lambda} = \zeta_{\lambda} \circ \pi_{\lambda} \zeta_{\lambda} = -\zeta_{\lambda} \circ \zeta_{\lambda} \pi_{\lambda}$. Cancelling ζ_{λ} on the left, we deduce that $\pi_{\lambda} \zeta_{\lambda} = -\zeta_{\lambda} \pi_{\lambda}$, hence $-\zeta_{\lambda} \pi_{\lambda} \zeta_{\lambda}^{-1} = \pi_{\lambda} \zeta_{\lambda}^{-1} \circ \zeta_{\lambda} \pi_{\lambda} = -1_{\pi_{\lambda}^2}$. Finally (iv) follows from the calculation:

$$\begin{aligned} F \xi_{\lambda} \circ (\beta_{\mu, \lambda})_F \pi_{\lambda} \circ \pi_{\mu} (\beta_{\mu, \lambda})_F &= F \zeta_{\lambda} \zeta_{\lambda} \circ \zeta_{\mu} F \zeta_{\lambda}^{-1} \pi_{\lambda} \circ \pi_{\mu} \zeta_{\mu} F \zeta_{\lambda}^{-1} \\ &= -\zeta_{\mu} F \pi_{\lambda} \zeta_{\lambda} \circ \pi_{\mu} \zeta_{\mu} F \zeta_{\lambda}^{-1} = \zeta_{\mu} \zeta_{\mu} F = \xi_{\mu} F. \end{aligned}$$

□

Applying Lemma 3.2 to the strict Π -2-supercategory $\Pi\text{-}\mathfrak{S}\mathfrak{C}\mathfrak{a}\mathfrak{t}$, we obtain the following.

Corollary 3.3. *Let $(\mathcal{A}, \Pi_{\mathcal{A}}, \zeta_{\mathcal{A}})$ and $(\mathcal{B}, \Pi_{\mathcal{B}}, \zeta_{\mathcal{B}})$ be Π -supercategories. As in Definition 1.7, there are even supernatural isomorphisms $\xi_{\mathcal{A}} : \Pi_{\mathcal{A}}^2 \xrightarrow{\cong} I_{\mathcal{A}}$ and $\xi_{\mathcal{B}} : \Pi_{\mathcal{B}}^2 \xrightarrow{\cong} I_{\mathcal{B}}$ both defined by setting $\xi := \zeta \zeta$.*

- (i) *We have that $\Pi \zeta = -\zeta \Pi$ in $\text{Hom}(\Pi^2, \Pi)$, hence $\Pi \xi = \xi \Pi$ in $\text{Hom}(\Pi^3, \Pi)$.*
- (ii) *For a superfunctor $F : \mathcal{A} \rightarrow \mathcal{B}$, define $\beta_F := -\zeta_{\mathcal{B}} F (\zeta_{\mathcal{A}})^{-1} : \Pi_{\mathcal{B}} F \Rightarrow F \Pi_{\mathcal{A}}$. This is an even supernatural isomorphism such that $\xi_{\mathcal{B}} F (\xi_{\mathcal{A}})^{-1} = \beta_F \Pi_{\mathcal{A}} \circ \Pi_{\mathcal{B}} \beta_F$ in $\text{Hom}((\Pi_{\mathcal{B}})^2 F, F (\Pi_{\mathcal{A}})^2)$.*
- (iii) *If $x : F \Rightarrow G$ is a supernatural transformation between superfunctors $F, G : \mathcal{A} \rightarrow \mathcal{B}$ then $\beta_G \circ \Pi_{\mathcal{B}} x = x \Pi_{\mathcal{A}} \circ \beta_F$ in $\text{Hom}(\Pi_{\mathcal{B}} F, G \Pi_{\mathcal{A}})$.*
- (iv) *For superfunctors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$, we have that $\beta_{GF} = G \beta_F \circ \beta_G F$. Also $\beta_I = 1_{\Pi}$ and $\beta_{\Pi} = -1_{\Pi^2}$.*

When working with Π -2-supercategories, notions of 2-superfunctors, 2-natural transformations and supermodifications are just as defined for 2-supercategories in Definition 2.2: there are no additional compatibility constraints. Let Π -2- \mathcal{SCat} be the category of all Π -2-supercategories and 2-superfunctors, and Π -2- \mathfrak{SCat} be the strict 2-category of all Π -2-supercategories, 2-superfunctors and 2-natural transformations.

4. ENVELOPES

In this subsection, we prove the statements about the functors (1) in Theorems 1.9 and 1.15. We will also construct Π -envelopes of 2-supercategories. We start at the level of supercategories. Recall the functor $-\pi : \mathcal{SCat} \rightarrow \Pi\text{-}\mathcal{SCat}$ from Definition 1.10, which sends supercategory \mathcal{A} to its Π -envelope $(\mathcal{A}_\pi, \Pi, \zeta)$, and superfunctor F to F_π . In fact, this is part of the data of a strict 2-superfunctor

$$-\pi : \mathfrak{SCat} \rightarrow \Pi\text{-}\mathfrak{SCat}, \quad (4.1)$$

sending a supernatural transformation $x : F \Rightarrow G$ to $x_\pi : F_\pi \Rightarrow G_\pi$ defined from $(x_\pi)_{\Pi^a \lambda} := (-1)^{|x|^a} (x_\lambda)_a^a$.

For any supercategory \mathcal{A} , there is a canonical superfunctor $J : \mathcal{A} \rightarrow \mathcal{A}_\pi$ which sends $\lambda \mapsto \Pi^{\bar{0}} \lambda$ and $f \mapsto f_{\bar{0}}^{\bar{0}}$. This is full and faithful. It is also dense: each object $\Pi^{\bar{0}} \lambda$ of \mathcal{A}_π is obviously in the image, while $\Pi^{\bar{1}} \lambda$ is isomorphic to $\Pi^{\bar{0}} \lambda$ via the odd isomorphism $(1_\lambda)_{\bar{1}}^{\bar{0}}$. This means that \mathcal{A} and \mathcal{A}_π are equivalent as abstract categories. However they need not be superequivalent as J need not be evenly dense:

Lemma 4.1. *The canonical superfunctor $J : \mathcal{A} \rightarrow \mathcal{A}_\pi$ is a superequivalence if and only if \mathcal{A} is Π -complete, meaning that every object of \mathcal{A} is the target of an odd isomorphism.*

Proof. The ‘‘only if’’ direction is clear as every object $\Pi^a \lambda$ of \mathcal{A}_π is the target of the odd isomorphism $(1_\lambda)_{a+\bar{1}}^a : \Pi^{a+\bar{1}} \lambda \rightarrow \Pi^a \lambda$. Conversely, assume that \mathcal{A} is Π -complete. To show that J is a superequivalence, it suffices to check that it is evenly dense. Let λ be an object of \mathcal{A} and $f : \mu \rightarrow \lambda$ be an odd isomorphism in \mathcal{A} . Then $f_{\bar{0}}^{\bar{1}} : \Pi^{\bar{0}} \mu \rightarrow \Pi^{\bar{1}} \lambda$ is an even isomorphism in \mathcal{A}_π . Hence, $\Pi^{\bar{1}} \lambda$ is isomorphic via an even isomorphism to something in the image of J , as of course is $\Pi^{\bar{0}} \lambda$. \square

Here is the universal property of Π -envelopes.

Lemma 4.2. *Suppose \mathcal{A} is a supercategory and $(\mathcal{B}, \Pi, \zeta)$ is a Π -supercategory.*

- (i) *Given a superfunctor $F : \mathcal{A} \rightarrow \mathcal{B}$, there is a canonical superfunctor $\tilde{F} : \mathcal{A}_\pi \rightarrow \mathcal{B}$ such that $F = \tilde{F}J$.*
- (ii) *Given a supernatural transformation $x : F \Rightarrow G$ between superfunctors $F, G : \mathcal{A} \rightarrow \mathcal{B}$, there is a unique supernatural transformation $\tilde{x} : \tilde{F} \Rightarrow \tilde{G}$ such that $x = \tilde{x}J$.*

Proof. (i) For $\lambda \in \text{ob } \mathcal{A}$, we set $\tilde{F}(\Pi^a \lambda) := F\lambda$ if $a = \bar{0}$ or $\Pi(F\lambda)$ if $a = \bar{1}$. For a morphism $f : \lambda \rightarrow \mu$ in \mathcal{A} , let $\tilde{F}(f_a^b) : \tilde{F}(\Pi^a \lambda) \rightarrow \tilde{F}(\Pi^b \mu)$ be $(\zeta_{F\mu}^b)^{-1} \circ Ff \circ \zeta_{F\lambda}^a$, where $\zeta_{F\lambda}^a$ denotes $1_{F\lambda}$ if $a = \bar{0}$ or $\zeta_{F\lambda}$ if $a = \bar{1}$, and $\zeta_{F\mu}^b$ is interpreted similarly.

(ii) We are given that $\tilde{x}_{\Pi^{\bar{0}} \lambda} = x_\lambda$ for each $\lambda \in \text{ob } \mathcal{A}$. Also, by the definition in (i), we have that $\tilde{F}\zeta_{\Pi^{\bar{0}} \lambda} = \zeta_{F\lambda}$ for each $\lambda \in \text{ob } \mathcal{A}$. Hence, to ensure the supernaturality property on the morphism $\zeta_{\Pi^{\bar{0}} \lambda} : \Pi^{\bar{1}} \lambda \rightarrow \Pi^{\bar{0}} \lambda$, we must have that $\tilde{x}_{\Pi^{\bar{1}} \lambda} = (-1)^{|x|} (\zeta_{G\lambda}^a)^{-1} \circ x_\lambda \circ \zeta_{F\lambda}$. Thus, in general, we have that

$$\tilde{x}_{\Pi^a \lambda} = (-1)^{a|x|} (\zeta_{G\lambda}^a)^{-1} \circ x_\lambda \circ (\zeta_{F\lambda}^b). \quad (4.2)$$

It just remains to check that this is indeed a supernatural transformation, i.e. it satisfies supernaturality on all other morphisms in \mathcal{A}_π . Take a homogeneous $f : \lambda \rightarrow \mu$ in \mathcal{A} and consider $f_a^b : \Pi^a \lambda \rightarrow \Pi^b \mu$. We must show that

$$(\zeta_{G\mu}^b)^{-1} \circ Gf \circ (\zeta_{G\lambda}^a) \circ \tilde{x}_{\Pi^a \lambda} = (-1)^{|x|(|f|+a+b)} \tilde{x}_{\Pi^b \mu} \circ (\zeta_{F\mu}^b)^{-1} \circ Ff \circ (\zeta_{F\lambda}^a).$$

This follows on substituting in the definitions of the \tilde{x} 's from (4.2) and using that $Gf \circ x_\lambda = (-1)^{|x||f|} x_\mu \circ Ff$. \square

Most of the time, Lemmas 4.1–4.2 are all that one needs when working with Π -envelopes in practice. The following gives a more formal statement, enough to establish the claim made about the functor (1) in Theorem 1.9 from the introduction. To state it, we let $\nu : \Pi\text{-}\mathfrak{SCat} \rightarrow \mathfrak{SCat}$ be the obvious forgetful 2-superfunctor.

Theorem 4.3. *For all supercategories \mathcal{A} and Π -supercategories \mathcal{B} , there is a functorial superequivalence $\mathcal{H}om(\mathcal{A}, \nu\mathcal{B}) \rightarrow \mathcal{H}om(\mathcal{A}_\pi, \mathcal{B})$, sending superfunctor F to \tilde{F} and supernatural transformation x to \tilde{x} , both as defined in Lemma 4.2. Hence, the strict 2-superfunctor $-\pi$ is left 2-adjoint to ν .*

Proof. We must show that the given superfunctor is fully faithful and evenly dense. The fully faithfulness follows from Lemma 4.2(ii). To see that it is evenly dense, take a superfunctor $F : \mathcal{A}_\pi \rightarrow \mathcal{B}$. Consider the composite functor $FJ : \mathcal{A} \rightarrow \nu\mathcal{B}$. Then there is an even supernatural isomorphism $\tilde{F}J \xrightarrow{\sim} F$, which is defined by the following even isomorphisms $\tilde{F}J(\Pi^a \lambda) \xrightarrow{\sim} F(\Pi^a \lambda)$ for each $\lambda \in \text{ob } \mathcal{A}$ and $a \in \mathbb{Z}/2$: if $a = \bar{0}$, then $\tilde{F}J(\Pi^{\bar{0}} \lambda) = F(\Pi^{\bar{0}} \lambda)$, and we just take the identity map; if $a = \bar{1}$, then $\tilde{F}J(\Pi^{\bar{1}} \lambda) = \Pi F(\Pi^{\bar{0}} \lambda)$, so we need to produce an isomorphism $\Pi F(\Pi^{\bar{0}} \lambda) \xrightarrow{\sim} F(\Pi^{\bar{1}} \lambda)$, which we get from Corollary 3.3(ii). We leave it to the reader to check the naturality. \square

We turn our attention to 2-supercategories.

Definition 4.4. The Π -envelope of a 2-supercategory \mathfrak{A} is the Π -2-supercategory $(\mathfrak{A}_\pi, \pi, \zeta)$ with morphism supercategories that are the Π -envelopes of the morphism supercategories in \mathfrak{A} :

- The object set for \mathfrak{A}_π is the same as for \mathfrak{A} .
- The set of 1-morphisms $\lambda \rightarrow \mu$ in \mathfrak{A}_π is

$$\{\Pi^a F \mid \text{for all 1-morphisms } F : \lambda \rightarrow \mu \text{ in } \mathfrak{A} \text{ and } a \in \mathbb{Z}/2\}.$$

- The horizontal composition of 1-morphisms $\Pi^a F : \lambda \rightarrow \mu$ and $\Pi^b G : \mu \rightarrow \nu$ is defined by $(\Pi^b G)(\Pi^a F) := \Pi^{a+b}(GF)$.
- The superspace of 2-morphisms $\Pi^a F \Rightarrow \Pi^b G$ in \mathfrak{A}_π is defined from

$$\text{Hom}_{\mathfrak{A}_\pi}(\Pi^a F, \Pi^b G) := \Pi^{a+b} \text{Hom}_{\mathfrak{A}}(F, G).$$

We denote the 2-morphism $\Pi^a F \Rightarrow \Pi^b G$ coming from $x : F \Rightarrow G$ under this identification by x_a^b . If x is homogeneous of parity $|x|$ then x_a^b is homogeneous of parity $|x| + a + b$.

- The vertical composition of $x_a^b : \Pi^a F \Rightarrow \Pi^b G$ and $y_b^c : \Pi^b G \Rightarrow \Pi^c H$ is defined from

$$y_b^c \circ x_a^b := (y \circ x)_a^c : \Pi^a F \Rightarrow \Pi^c H. \quad (4.3)$$

- The horizontal composition of $x_a^c : \Pi^a F \Rightarrow \Pi^c H$ and $y_b^d : \Pi^b G \Rightarrow \Pi^d K$ is defined by

$$y_b^d x_a^c := (-1)^{b|x|+|y|c+bc+ab} (yx)_{a+b}^{c+d} : \Pi^{a+b}(GF) \Rightarrow \Pi^{c+d}(KH). \quad (4.4)$$

- The units $\mathbb{1}_\lambda$ in \mathfrak{A}_π are the 1-morphisms $\Pi^{\bar{0}}\mathbb{1}_\lambda$. Also define $\pi = (\pi_\lambda)$ by $\pi_\lambda := \Pi^{\bar{1}}\mathbb{1}_\lambda$ and $\zeta = (\zeta_\lambda)$ by $\zeta_\lambda := (1_{\mathbb{1}_\lambda})_{\bar{1}}^{\bar{0}} : \pi_\lambda \xrightarrow{\cong} \mathbb{1}_\lambda$; in particular, $\xi_\lambda := \zeta_\lambda \zeta_\lambda : \pi_\lambda^2 \xrightarrow{\cong} \mathbb{1}_\lambda$ is minus the identity.
- The structure maps a, l and r in \mathfrak{A}_π are induced by the ones in \mathfrak{A} in the obvious way, but there are some signs to be checked to see that this makes sense. For example, for the associator, one needs to note that the signs in the following two expressions agree:

$$\begin{aligned} (z_c^f y_b^e) x_a^d &= (-1)^{c|y|+|z|e+ce+bc+(b+c)|x|+(|y|+|z|)d+(b+c)d+a(b+c)} ((zy)x)_{a+b+c}^{d+e+f}, \\ z_c^f (y_b^e x_a^d) &= (-1)^{b|x|+|y|d+bd+ab+c(|x|+|y|)+|z|(d+e)+c(d+e)+(a+b)c} (z(yx))_{a+b+c}^{d+e+f}. \end{aligned}$$

The main check needed to verify that this is indeed a Π -2-supercategory is made in the following lemma:

Lemma 4.5. *The horizontal and vertical compositions from (4.3)–(4.4) satisfy the super interchange law.*

Proof. We need to show that

$$(v_d^f u_c^e) \circ (y_b^d x_a^c) = (-1)^{(c+e+|u|)(b+d+|y|)} (v_d^f \circ y_b^d) (u_c^e \circ x_a^c).$$

The left hand side equals

$$\begin{aligned} (-1)^{d|u|+|v|e+de+cd+b|x|+|y|c+bc+ab} (vu)_{c+d}^{e+f} \circ (yx)_{a+b}^{c+d} &= \\ (-1)^{d|u|+|v|e+de+cd+b|x|+|y|c+bc+ab} ((vu) \circ (yx))_{a+b}^{e+f} &= \\ (-1)^{d|u|+|v|e+de+cd+b|x|+|y|c+bc+ab+|u||y|} ((v \circ y)(u \circ x))_{a+b}^{e+f}, \end{aligned}$$

using the super interchange law in \mathfrak{A} for the last equality. The right hand side equals

$$\begin{aligned} (-1)^{(c+e+|u|)(b+d+|y|)} (v \circ y)_b^f (u \circ x)_a^e &= \\ (-1)^{(c+e+|u|)(b+d+|y|)+b(|u|+|x|)+(|v|+|y|)e+be+ab} ((v \circ y)(u \circ x))_{a+b}^{e+f}. \end{aligned}$$

We leave it to the reader to check that the signs here are indeed equal. \square

For any 2-supercategory \mathfrak{A} , there is a canonical strict 2-superfunctor $\mathbb{J} : \mathfrak{A} \rightarrow \mathfrak{A}_\pi$; it is the identity on objects, it sends the 1-morphism $F : \lambda \rightarrow \mu$ to $\Pi^{\bar{0}}F$, and the 2-morphism $x : F \Rightarrow G$ to $x_{\bar{0}}^{\bar{0}} : \Pi^{\bar{0}}F \Rightarrow \Pi^{\bar{0}}G$. The analog of Lemma 4.1 is as follows:

Lemma 4.6. *For a 2-supercategory \mathfrak{A} , the canonical 2-superfunctor $\mathbb{J} : \mathfrak{A} \rightarrow \mathfrak{A}_\pi$ is a 2-superequivalence if and only if \mathfrak{A} is Π -complete, meaning that it possesses 1-morphisms $\pi_\lambda : \lambda \rightarrow \lambda$ and odd 2-isomorphisms $\pi_\lambda \cong \mathbb{1}_\lambda$ for every $\lambda \in \text{ob } \mathfrak{A}$.*

Proof. Applying Lemma 4.1 to the morphism supercategories, we get that \mathbb{J} is a 2-superequivalence if and only if every 1-morphism in \mathfrak{A} is the target of an odd 2-isomorphism. It is clearly sufficient to verify this condition just for the unit 1-morphisms $\mathbb{1}_\lambda$ in \mathfrak{A} . \square

Taking Π -envelopes actually defines a strict 2-functor

$$-\pi : 2\text{-}\mathfrak{SCat} \rightarrow \Pi\text{-}2\text{-}\mathfrak{SCat}. \quad (4.5)$$

We still need to specify this on 2-superfunctors and 2-natural transformations:

- Suppose that $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$ is a 2-superfunctor with coherence maps $c : (\mathbb{R}-)(\mathbb{R}-) \xrightarrow{\cong} \mathbb{R}(-)$ and $i : \mathbb{1}_{\mathbb{R}\lambda} \xrightarrow{\cong} \mathbb{R}\mathbb{1}_\lambda$ for each $\lambda \in \text{ob } \mathfrak{A}$. Then we let $\mathbb{R}_\pi : \mathfrak{A}_\pi \rightarrow \mathfrak{B}_\pi$ be the 2-superfunctor equal to \mathbb{R} on objects, and given by the rules $\Pi^a F \mapsto \Pi^a(\mathbb{R}F)$ on 1-morphisms and $x_a^b \mapsto (\mathbb{R}x)_a^b$ on 2-morphisms.

Its coherence maps $c_\pi : (\mathbb{R}_\pi -)(\mathbb{R}_\pi -) \xrightarrow{\sim} \mathbb{R}_\pi(-)$ and $i_\pi : \mathbb{1}_{\mathbb{R}_\pi \lambda} \xrightarrow{\sim} \mathbb{R}_\pi \mathbb{1}_\lambda$ for \mathbb{R}_π are defined by $(c_\pi)_{\Pi^a F, \Pi^b G} := (c_{F, G})_{a+b}^{a+b}$ and $i_\pi := i_0^{\bar{0}}$.

- If $(X, x) : \mathbb{R} \Rightarrow \mathbb{S}$ is a 2-natural transformation, we let $(X_\pi, x_\pi) : \mathbb{R}_\pi \Rightarrow \mathbb{S}_\pi$ be the 2-natural transformation defined from $(X_\pi)_\lambda := \Pi^{\bar{0}} X_\lambda$ and $((x_\pi)_{\mu, \lambda})_{\Pi^a F} := ((x_{\mu, \lambda})_F)_a^a$.

Lemma 4.7. *Suppose \mathfrak{A} is a 2-supercategory and $(\mathfrak{B}, \pi, \zeta)$ is a Π -2-supercategory.*

- (i) *Given a graded 2-superfunctor $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$, there is a canonical graded 2-superfunctor $\tilde{\mathbb{R}} : \mathfrak{A}_\pi \rightarrow \mathfrak{B}$ such that $\mathbb{R} = \tilde{\mathbb{R}}\mathbb{J}$.*
- (ii) *Given a 2-natural transformation $(X, x) : \mathbb{R} \Rightarrow \mathbb{S}$ between 2-superfunctors $\mathbb{R}, \mathbb{S} : \mathfrak{A} \rightarrow \mathfrak{B}$, there is a unique 2-natural transformation $(\tilde{X}, \tilde{x}) : \tilde{\mathbb{R}} \Rightarrow \tilde{\mathbb{S}}$ such that $\tilde{X}_\lambda = X_\lambda$ and $x_{\mu, \lambda} = \tilde{x}_{\mu, \lambda}\mathbb{J}$ for all $\lambda, \mu \in \text{ob } \mathfrak{A}$.*

Proof. To simplify notation throughout this proof, we will assume that \mathfrak{B} is strict.

(i) On objects, we take $\tilde{\mathbb{R}}\lambda := \mathbb{R}\lambda$. To specify its effect on 1- and 2-morphisms, we first introduce some notation: for $a \in \mathbb{Z}/2$ and $\lambda \in \text{ob } \mathfrak{B}$, let $\zeta_\lambda^a : \pi_\lambda^a \Rightarrow \mathbb{1}_\lambda$ denote the 2-morphism $1_{\mathbb{1}_\lambda} \in \text{Hom}_{\mathfrak{B}}(\mathbb{1}_\lambda, \mathbb{1}_\lambda)$ if $a = \bar{0}$ or the 2-morphism $\zeta_\lambda \in \text{Hom}_{\mathfrak{B}}(\pi_\lambda, \mathbb{1}_\lambda)$ if $a = \bar{1}$. Then, for a 1-morphism $F : \lambda \rightarrow \mu$ in \mathfrak{A} and $a \in \mathbb{Z}/2$, we set $\tilde{\mathbb{R}}(\Pi^a F) := \pi_{\mathbb{R}\mu}^a(\mathbb{R}F)$. Also, if $x : F \Rightarrow G$ is a 2-morphism in \mathfrak{A} between 1-morphisms $F, G : \lambda \rightarrow \mu$, we define $\tilde{\mathbb{R}}(x_a^b) : \tilde{\mathbb{R}}(\Pi^a F) \Rightarrow \tilde{\mathbb{R}}(\Pi^b G)$ to be the following composition:

$$\pi_{\mathbb{R}\mu}^a(\mathbb{R}F) \xrightarrow{\zeta_{\mathbb{R}\mu}^a(\mathbb{R}F)} \mathbb{R}F \xrightarrow{\mathbb{R}x} \mathbb{R}G \xrightarrow{(\zeta_{\mathbb{R}\mu}^b)^{-1}(\mathbb{R}G)} \pi_{\mathbb{R}\mu}^b(\mathbb{R}G).$$

In other words, by the super interchange law, we have that

$$\tilde{\mathbb{R}}(x_a^b) = (-1)^{a|x|} (\zeta_{\mathbb{R}\mu}^b)^{-1} \zeta_{\mathbb{R}\mu}^a(\mathbb{R}x). \quad (4.6)$$

Recalling (4.3), it is easy to see from this definition that $\tilde{\mathbb{R}}(y_b^c \circ x_a^b) = \tilde{\mathbb{R}}(y_b^c) \circ \tilde{\mathbb{R}}(x_a^b)$. Thus, we have specified the first two pieces of data from Definition 2.2(ii) that are required to define the 2-superfunctor $\tilde{\mathbb{R}}$.

For the other two pieces of required data, let $c : (\mathbb{R}-)(\mathbb{R}-) \xrightarrow{\sim} \mathbb{R}(-)$ and $i : \mathbb{1}_{\mathbb{R}\lambda} \xrightarrow{\sim} \mathbb{R}\mathbb{1}_\lambda$ be the coherence maps for \mathbb{R} . The coherence map \tilde{i} for $\tilde{\mathbb{R}}$ is just the same as i . We define the other coherence map \tilde{c} for $\tilde{\mathbb{R}}$ by letting $\tilde{c}_{\Pi^b G, \Pi^a F} : \tilde{\mathbb{R}}(\Pi^b G) \tilde{\mathbb{R}}(\Pi^a F) \Rightarrow \tilde{\mathbb{R}}(\Pi^{a+b}(GF))$ be the following composition (for $F : \lambda \rightarrow \mu$ and $G : \mu \rightarrow \nu$):

$$\pi_{\mathbb{R}\nu}^b(\mathbb{R}G) \pi_{\mathbb{R}\mu}^a(\mathbb{R}F) \xrightarrow{\pi_{\mathbb{R}\nu}^b(\beta_{\mathbb{R}\nu, \mathbb{R}\mu}^a)^{-1}(\mathbb{R}G)} \pi_{\mathbb{R}\nu}^b \pi_{\mathbb{R}\nu}^a(\mathbb{R}G)(\mathbb{R}F) \xrightarrow{m_{b, a}^{cG, F}} \pi_{\mathbb{R}\nu}^{a+b} \mathbb{R}(GF).$$

Here, $\beta_{\mathbb{R}\nu, \mathbb{R}\mu}^a : \pi_{\mathbb{R}\nu}^a - \xrightarrow{\sim} -\pi_{\mathbb{R}\mu}^a$ is the identity if $a = \bar{0}$ or the even supernatural isomorphism $\beta_{\mathbb{R}\nu, \mathbb{R}\mu}^a$ from Lemma 3.2 if $a = \bar{1}$, and $m_{b, a} : \pi_{\mathbb{R}\nu}^b \pi_{\mathbb{R}\nu}^a \xrightarrow{\sim} \pi_{\mathbb{R}\nu}^{a+b}$ is the identity if $ab = \bar{0}$, or the 2-isomorphism $-\zeta_{\mathbb{R}\nu} = -\zeta_{\mathbb{R}\nu} \zeta_{\mathbb{R}\nu}$ from Lemma 3.2(iv) if $ab = \bar{1}$.

The key point now is to check the naturality of \tilde{c} . Take $x : F \Rightarrow H$ and $y : G \Rightarrow K$. We must show that the following diagram commutes for all $a, b, c, d \in \mathbb{Z}/2$:

$$\begin{array}{ccc} \pi_{\mathbb{R}\nu}^b(\mathbb{R}G) \pi_{\mathbb{R}\mu}^a(\mathbb{R}F) & \xrightarrow{\tilde{c}_{\Pi^b G, \Pi^a F}} & \pi_{\mathbb{R}\nu}^{a+b} \mathbb{R}(GF) \\ \tilde{\mathbb{R}}(y_b^d) \tilde{\mathbb{R}}(x_a^c) \downarrow & & \downarrow \tilde{\mathbb{R}}(y_b^d x_a^c) \\ \pi_{\mathbb{R}\nu}^d(\mathbb{R}K) \pi_{\mathbb{R}\mu}^c(\mathbb{R}H) & \xrightarrow{\tilde{c}_{\Pi^d K, \Pi^c H}} & \pi_{\mathbb{R}\nu}^{c+d} \mathbb{R}(KH). \end{array}$$

Recalling (4.4) and (4.6), the composite of the top and right hand maps is equal to $(-1)^{a|x|+(a+b+c)|y|+ab+bc} ((\zeta_{\mathbb{R}\nu}^{c+d})^{-1} \zeta_{\mathbb{R}\nu}^{a+b} \mathbb{R}(yx)) \circ (m_{b,a} c_{G,F}) \circ (\pi_{\mathbb{R}\nu}^b (\beta_{\mathbb{R}\nu, \mathbb{R}\mu}^a)_{\mathbb{R}G}^{-1} (\mathbb{R}F))$.

Also the composite of the bottom and left hand maps is

$$(-1)^{a|x|+b|y|} (m_{d,c} c_{K,H}) \circ (\pi_{\mathbb{R}\nu}^d (\beta_{\mathbb{R}\nu, \mathbb{R}\mu}^c)_{\mathbb{R}K}^{-1} (\mathbb{R}H)) \circ ((\zeta_{\mathbb{R}\nu}^d)^{-1} \zeta_{\mathbb{R}\nu}^b (\mathbb{R}y) (\zeta_{\mathbb{R}\mu}^c)^{-1} \zeta_{\mathbb{R}\mu}^a (\mathbb{R}x)).$$

To see that these two are indeed equal, use the following commutative diagrams:

$$\begin{array}{ccc} \pi_{\mathbb{R}\nu}^b \pi_{\mathbb{R}\nu}^a & \xrightarrow{m_{b,a}} & \pi_{\mathbb{R}\nu}^{a+b} \\ (\zeta_{\mathbb{R}\nu}^d)^{-1} \zeta_{\mathbb{R}\nu}^b (\zeta_{\mathbb{R}\nu}^c)^{-1} \zeta_{\mathbb{R}\nu}^a \downarrow & & \downarrow (-1)^{ab+bc} (\zeta_{\mathbb{R}\nu}^{c+d})^{-1} \zeta_{\mathbb{R}\nu}^{a+b} \\ \pi_{\mathbb{R}\nu}^d \pi_{\mathbb{R}\nu}^c & \xrightarrow{m_{d,c}} & \pi_{\mathbb{R}\nu}^{c+d}, \\ \pi_{\mathbb{R}\nu}^a (\mathbb{R}G) & \xrightarrow{(\beta_{\mathbb{R}\nu, \mathbb{R}\mu}^a)_{\mathbb{R}G}} & (\mathbb{R}G) \pi_{\mathbb{R}\mu}^a \\ (\zeta_{\mathbb{R}\nu}^c)^{-1} \zeta_{\mathbb{R}\nu}^a (\mathbb{R}y) \downarrow & & \downarrow (-1)^{(a+c)|y|} (\mathbb{R}y) (\zeta_{\mathbb{R}\mu}^c)^{-1} \zeta_{\mathbb{R}\mu}^a \\ \pi_{\mathbb{R}\nu}^c (\mathbb{R}K) & \xrightarrow{(\beta_{\mathbb{R}\nu, \mathbb{R}\mu}^c)_{\mathbb{R}K}} & (\mathbb{R}K) \pi_{\mathbb{R}\mu}^c. \end{array}$$

To establish the latter two diagrams, note by the definitions of $m_{b,a}$ and $\beta_{\mathbb{R}\nu, \mathbb{R}\mu}^a$ that $\zeta_{\mathbb{R}\nu}^b \zeta_{\mathbb{R}\nu}^a = (-1)^{ab} \zeta_{\mathbb{R}\nu}^{a+b} \circ m_{b,a}$ and $((\mathbb{R}y) \zeta_{\mathbb{R}\mu}^a) \circ (\beta_{\mathbb{R}\nu, \mathbb{R}\mu}^a)_{\mathbb{R}G} = (-1)^{a|y|} \zeta_{\mathbb{R}\nu}^a (\mathbb{R}y)$, then use the super interchange law.

We leave it to the reader to verify that the coherence axioms hold, i.e. the two diagrams of Definition 2.2(ii) commute. This depends crucially on Lemma 3.2.

(ii) Take a 1-morphism $F : \lambda \rightarrow \mu$ in \mathfrak{A} . We are given that $(\tilde{x}_{\mu, \lambda})_{\Pi^0 F} = (x_{\mu, \lambda})_F$. In order for $\tilde{x}_{\mu, \lambda}$ to satisfy naturality on the 2-morphism $(1_F)_{\Pi^1}^0 : \Pi^1 F \Rightarrow \Pi^0 F$, we are also forced to have $(\tilde{x}_{\mu, \lambda})_{\Pi^1 F} = (\zeta_{\mathbb{S}\mu}(\mathbb{S}F) X_\lambda)^{-1} \circ (x_{\mu, \lambda})_F \circ (X_\mu \zeta_{\mathbb{R}\mu}(\mathbb{R}F))$. Thus, in general, we have that

$$\begin{aligned} (\tilde{x}_{\mu, \lambda})_{\Pi^1 F} &= (\zeta_{\mathbb{S}\mu}^a(\mathbb{S}F) X_\lambda)^{-1} \circ (x_{\mu, \lambda})_F \circ (X_\mu \zeta_{\mathbb{R}\mu}^a(\mathbb{R}F)) \\ &= (\pi_{\mathbb{S}\mu}^a(x_{\mu, \lambda})_F) \circ ((\beta_{\mathbb{S}\mu, \mathbb{R}\mu}^a)^{-1}(\mathbb{R}F)). \end{aligned} \quad (4.7)$$

To check naturality in general, take some homogeneous $x : F \Rightarrow G$, and consider $x_a^b : \Pi^a F \Rightarrow \Pi^b G$. We know that $(x_{\mu, \lambda})_G \circ (X_\mu(\mathbb{R}x)) = ((\mathbb{S}x) X_\lambda) \circ (x_{\mu, \lambda})_F$, and need to prove that $(\tilde{x}_{\mu, \lambda})_{\Pi^b G} \circ (\tilde{X}_\mu(\mathbb{R}x_a^b)) = ((\tilde{\mathbb{S}}x_a^b) \tilde{X}_\lambda) \circ (\tilde{x}_{\mu, \lambda})_{\Pi^a F}$. On expanding all the definitions, this reduces to checking the following identity:

$$\begin{aligned} (\zeta_{\mathbb{S}\mu}^b(\mathbb{S}G) X_\lambda)^{-1} \circ (x_{\mu, \lambda})_G \circ (X_\mu \zeta_{\mathbb{R}\mu}^b(\mathbb{R}G)) \circ (X_\mu (\zeta_{\mathbb{R}\mu}^b)^{-1} \zeta_{\mathbb{R}\mu}^a(\mathbb{R}x)) = \\ ((\zeta_{\mathbb{S}\mu}^b)^{-1} \zeta_{\mathbb{S}\mu}^a(\mathbb{S}x) X_\lambda) \circ (\zeta_{\mathbb{S}\mu}^a(\mathbb{S}F) X_\lambda)^{-1} \circ (x_{\mu, \lambda})_F \circ (X_\mu \zeta_{\mathbb{R}\mu}^a(\mathbb{R}F)), \end{aligned}$$

which is quite straightforward.

It remains to verify that (\tilde{X}, \tilde{x}) satisfies the two axioms for 2-natural transformations from Definition 2.2(iii). We leave this to the reader again; one needs to use Lemma 3.2 repeatedly. \square

Example 4.8. Assume that \mathbb{k} is a field, and recall the monoidal supercategory \mathcal{I} with one object from Example 1.17(ii). Its Π -envelope \mathcal{I}_π is a monoidal Π -supercategory with two objects Π^0 and Π^1 . Each morphism space $\text{Hom}_{\mathcal{I}_\pi}(\Pi^a, \Pi^b)$ is one-dimensional with basis 1_a^b . The tensor product satisfies $\Pi^b \otimes \Pi^a = \Pi^{a+b}$ and $1_b^d \otimes 1_a^c = (-1)^{(a+c)b} 1_{a+b}^{c+d}$. We also have the monoidal Π -supercategory \mathcal{SVec} from Example 1.13(i). By Lemma 4.7(i), the canonical superfunctor $F : \mathcal{I} \rightarrow \mathcal{SVec}$ sending the only object to \mathbb{k} extends to a monoidal superfunctor $\tilde{F} : \mathcal{I}_\pi \rightarrow \mathcal{SVec}$.

This sends $\Pi^a \mapsto \Pi^a \mathbb{k}$ and $1_a^b \mapsto (\text{id}_a^b : \Pi^a \mathbb{k} \rightarrow \Pi^b \mathbb{k}, 1 \mapsto 1)$; its coherence maps are $\Pi^b \mathbb{k} \otimes \Pi^a \mathbb{k} \rightarrow \Pi^{a+b} \mathbb{k}, 1 \otimes 1 \mapsto 1$. The signs are consistent because the linear map $\text{id}_b^d \otimes \text{id}_a^c : \Pi^b \mathbb{k} \otimes \Pi^a \mathbb{k} \rightarrow \Pi^d \mathbb{k} \otimes \Pi^c \mathbb{k}$ sends $1 \otimes 1 \mapsto (-1)^{(a+c)b} 1 \otimes 1$.

Using Lemma 4.7, one can prove the following. In the statement, ν denotes the obvious forgetful functor (actually, here it is a 2-functor).

Theorem 4.9. *For all 2-supercategories \mathfrak{A} and Π -2-supercategories \mathfrak{B} , there is a functorial equivalence $\mathcal{H}om(\mathfrak{A}, \nu \mathfrak{B}) \rightarrow \mathcal{H}om(\mathfrak{A}_\pi, \mathfrak{B})$, sending 2-superfunctor \mathbb{R} to $\tilde{\mathbb{R}}$ and 2-natural transformation (X, x) to (\tilde{X}, \tilde{x}) , both as defined in Lemma 4.7. Hence, the strict 2-functor $-\pi$ is left 2-adjoint to ν .*

On specializing to 2-supercategories with one object, this implies the result about the functor (1) made in the statement of Theorem 1.15 from the introduction.

Remark 4.10. In fact, we should really go one level higher here, viewing $-\pi$ as a strict 3-superfunctor from the strict 3-supercategory of 2-supercategories to the strict 3-supercategory of Π -2-supercategories, by associating a supermodification $\alpha_\pi : (X_\pi, x_\pi) \Rightarrow (Y_\pi, y_\pi)$ to each supermodification $\alpha : (X, x) \Rightarrow (Y, y)$ defined from $(\alpha_\pi)_\lambda := (\alpha_\lambda)_0^0$. We leave it to the reader to formulate an appropriate part (iii) to Lemma 4.7 explaining how to extend $\alpha : (X, x) \Rightarrow (Y, y)$ to $\tilde{\alpha} : (\tilde{X}, \tilde{x}) \Rightarrow (\tilde{Y}, \tilde{y})$. Then Theorem 4.9 becomes a 2-superequivalence

$$\mathfrak{H}om(\mathfrak{A}, \nu \mathfrak{B}) \rightarrow \mathfrak{H}om(\mathfrak{A}_\pi, \mathfrak{B}).$$

In particular, it follows that there is an induced monoidal superfunctor from the Drinfeld center of a 2-supercategory \mathfrak{A} to the Drinfeld center of its Π -envelope \mathfrak{A}_π ; the latter is a monoidal Π -supercategory in the sense of Definition 1.12.

5. Π -CATEGORIES

We continue to assume that \mathbb{k} is a commutative ground ring. Let $\Pi\text{-Cat}$ be the category of all Π -categories and Π -functors in the sense of Definition 1.6. Recall also that we denote the underlying category of a supercategory \mathcal{A} by $\underline{\mathcal{A}}$; see Definition 1.1(v). If $(\mathcal{A}, \Pi_{\mathcal{A}}, \zeta_{\mathcal{A}})$ is a Π -supercategory and we set $\xi_{\mathcal{A}} := \zeta_{\mathcal{A}} \zeta_{\mathcal{A}}$, then $(\underline{\mathcal{A}}, \underline{\Pi}_{\mathcal{A}}, \xi_{\mathcal{A}})$ is a Π -category thanks to Corollary 3.3(i). Given another Π -supercategory $(\mathcal{B}, \Pi_{\mathcal{B}}, \zeta_{\mathcal{B}})$ and a superfunctor $F : \mathcal{A} \rightarrow \mathcal{B}$, Corollary 3.3(ii) explains how to construct the additional natural isomorphism β_F needed to make the underlying functor \underline{F} into a Π -functor from $(\underline{\mathcal{A}}, \underline{\Pi}_{\mathcal{A}}, \xi_{\mathcal{A}})$ to $(\underline{\mathcal{B}}, \underline{\Pi}_{\mathcal{B}}, \xi_{\mathcal{B}})$. Using also Corollary 3.3(iv), this shows that there is a functor

$$E_1 : \Pi\text{-SCat} \rightarrow \Pi\text{-Cat} \quad (\mathcal{A}, \Pi, \zeta) \mapsto (\underline{\mathcal{A}}, \underline{\Pi}, \xi), \quad F \mapsto (\underline{F}, \beta_F). \quad (5.1)$$

This is the functor (2) in (1.5).

In order to complete the proof of Theorem 1.9, we must show that the functor E_1 is an equivalence, so that a Π -supercategory $(\mathcal{A}, \Pi, \zeta)$ can be recovered up to superequivalence from its underlying category $(\underline{\mathcal{A}}, \underline{\Pi}, \xi)$. To establish this, we define a functor in the other direction:

$$D_1 : \Pi\text{-Cat} \rightarrow \Pi\text{-SCat} \quad (\mathcal{A}, \Pi, \xi) \mapsto (\widehat{\mathcal{A}}, \widehat{\Pi}, \zeta), \quad (F, \beta_F) \mapsto \widehat{F}. \quad (5.2)$$

This sends Π -category (\mathcal{A}, Π, ξ) to the associated Π -supercategory $(\widehat{\mathcal{A}}, \widehat{\Pi}, \zeta)$, which is the supercategory with the same objects as \mathcal{A} and morphisms $\text{Hom}_{\widehat{\mathcal{A}}}(\lambda, \mu)_{\bar{0}} := \text{Hom}_{\mathcal{A}}(\lambda, \mu)$, $\text{Hom}_{\widehat{\mathcal{A}}}(\lambda, \mu)_{\bar{1}} := \text{Hom}_{\mathcal{A}}(\lambda, \Pi\mu)$. Composition in $\widehat{\mathcal{A}}$ is induced by the composition in \mathcal{A} : if $\hat{f} : \lambda \rightarrow \mu$ and $\hat{g} : \mu \rightarrow \nu$ are homogeneous morphisms in $\widehat{\mathcal{A}}$ then

- if \hat{f} and \hat{g} are both even, so $\hat{f} = f$ and $\hat{g} = g$ for morphisms $f : \lambda \rightarrow \mu$ and $g : \mu \rightarrow \nu$ in \mathcal{A} , then we set $\hat{g} \circ \hat{f} := g \circ f$;
- if \hat{f} is even and \hat{g} is odd, so $\hat{f} = f$ and $\hat{g} = g$ for $f : \lambda \rightarrow \mu$ and $g : \mu \rightarrow \Pi\nu$ in \mathcal{A} , then we again set $\hat{g} \circ \hat{f} := g \circ f$;
- if \hat{f} is odd and \hat{g} is even, so $\hat{f} = f$ and $\hat{g} = g$ for $f : \lambda \rightarrow \Pi\mu$ and $g : \mu \rightarrow \nu$, we set $\hat{g} \circ \hat{f} := (\Pi g) \circ f$;
- if \hat{f} and \hat{g} are both odd, so $\hat{f} = f$ and $\hat{g} = g$ for $f : \lambda \rightarrow \Pi\mu$ and $g : \mu \rightarrow \Pi\nu$, we set $\hat{g} \circ \hat{f} := \xi_\nu \circ (\Pi g) \circ f$.

The check that $(\hat{h} \circ \hat{g}) \circ \hat{f} = \hat{h} \circ (\hat{g} \circ \hat{f})$ for odd $\hat{f}, \hat{g}, \hat{h}$ depends on the axiom $\xi \Pi = \Pi \xi$. To make $\widehat{\mathcal{A}}$ into a Π -supercategory, we define $\widehat{\Pi} : \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{A}}$ to be the superfunctor that is equal to Π on objects, while $\widehat{\Pi} \hat{f} := \Pi f$ if \hat{f} is even coming from $f : \lambda \rightarrow \mu$ in \mathcal{A} , and $\widehat{\Pi} \hat{f} := -\Pi f$ if \hat{f} is odd coming from $f : \lambda \rightarrow \Pi\mu$ in \mathcal{A} . The odd natural isomorphism $\zeta : \widehat{\Pi} \rightarrow I$ is defined on object λ by $\zeta_\lambda := 1_{\Pi\lambda}$, i.e. it is the identity morphism $\Pi\lambda \rightarrow \Pi\lambda$ in \mathcal{A} viewed as an odd morphism $\Pi\lambda \rightarrow \lambda$ in $\widehat{\mathcal{A}}$. Finally, if $(F, \beta_F) : \mathcal{A} \rightarrow \mathcal{B}$ is a Π -functor, we get induced a superfunctor $\widehat{F} : \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{B}}$ between the associated supercategories as follows: it is the same as F on objects; on a homogeneous morphism $\hat{f} : \lambda \rightarrow \mu$ in $\widehat{\mathcal{A}}$ we have that $\widehat{F} \hat{f} := Ff$ if \hat{f} is even coming from $f : \lambda \rightarrow \mu$ in \mathcal{A} , or $\widehat{F} \hat{f} := (\beta_F)_\mu^{-1} \circ Ff$ if \hat{f} is odd coming from $f : \lambda \rightarrow \Pi\mu$ in \mathcal{A} . The check that $\widehat{F}(\hat{g} \circ \hat{f}) = (\widehat{F} \hat{g}) \circ (\widehat{F} \hat{f})$ for odd \hat{f}, \hat{g} depends on the axiom $F\xi_{\mathcal{A}} = \xi_{\mathcal{B}}F \circ \Pi_{\mathcal{B}}\beta_F^{-1} \circ \beta_F^{-1}\Pi_{\mathcal{A}}$.

Lemma 5.1. *The functors $D_1 : \Pi\text{-Cat} \rightarrow \Pi\text{-SCat}$ and $E_1 : \Pi\text{-SCat} \rightarrow \Pi\text{-Cat}$ are mutually inverse equivalences of categories.*

Proof. We have simply that $E_1 \circ D_1 = I_{\Pi\text{-Cat}}$. It remains to show that $D_1 \circ E_1 \cong I_{\Pi\text{-SCat}}$. To see this, we have to define a natural isomorphism $T : D_1 \circ E_1 \xrightarrow{\cong} I_{\Pi\text{-SCat}}$. So for each Π -supercategory $(\mathcal{A}, \Pi, \zeta)$, we need to produce an isomorphism of supercategories $T_{\mathcal{A}} : \widehat{(\mathcal{A})} \xrightarrow{\cong} \mathcal{A}$. We take $T_{\mathcal{A}}$ to be the identity on objects (which are the same in $\widehat{(\mathcal{A})}$ as in \mathcal{A}). On a morphism $\hat{f} : \lambda \rightarrow \mu$ in $\widehat{(\mathcal{A})}$, we let $T_{\mathcal{A}}(\hat{f}) := f$ if \hat{f} is even coming from an even morphism $f : \lambda \rightarrow \mu$ in \mathcal{A} , or $\zeta_\mu \circ f$ if \hat{f} is odd coming from an even morphism $f : \lambda \rightarrow \Pi\mu$ in \mathcal{A} .

To check that $T_{\mathcal{A}}$ is a functor, we need to show that $T_{\mathcal{A}}(\hat{g} \circ \hat{f}) = T_{\mathcal{A}}(\hat{g}) \circ T_{\mathcal{A}}(\hat{f})$ for $\hat{f} : \lambda \rightarrow \mu$ and $\hat{g} : \mu \rightarrow \nu$:

- This is clear if both \hat{f} and \hat{g} are even.
- If \hat{f} is even and \hat{g} is odd, so $\hat{f} = f$ and $\hat{g} = g$ for even $f : \lambda \rightarrow \mu$ and $g : \mu \rightarrow \Pi\nu$ in \mathcal{A} , we have that $T_{\mathcal{A}}(\hat{g} \circ \hat{f}) = \zeta_\nu \circ g \circ f = T_{\mathcal{A}}(\hat{g}) \circ T_{\mathcal{A}}(\hat{f})$.
- If f is odd and g is even, so $\hat{f} = f$ and $\hat{g} = g$ for $f : \lambda \rightarrow \Pi\mu$ and $g : \mu \rightarrow \nu$, then $T_{\mathcal{A}}(\hat{g} \circ \hat{f}) = \zeta_\nu \circ (\Pi g) \circ f$, while $T_{\mathcal{A}}(\hat{g}) \circ T_{\mathcal{A}}(\hat{f}) = g \circ \zeta_\mu \circ f$. These are equal as $\zeta_\nu \circ \Pi g = g \circ \zeta_\mu$ by the supernaturality of ζ .
- If both are odd, so $\hat{f} = f$ and $\hat{g} = g$ for $f : \lambda \rightarrow \Pi\mu$ and $g : \mu \rightarrow \Pi\nu$, then $T_{\mathcal{A}}(\hat{g} \circ \hat{f}) = \xi_\nu \circ (\Pi g) \circ f$. By the super interchange law, $\xi_\nu = -\zeta_\nu \circ \zeta_{\Pi\nu}$, while supernaturality of ζ gives that $\zeta_{\Pi\nu} \circ \Pi g = -g \circ \zeta_\mu$. Hence, $T_{\mathcal{A}}(\hat{g} \circ \hat{f})$ equals $\zeta_\nu \circ g \circ \zeta_\mu \circ f = T_{\mathcal{A}}(\hat{g}) \circ T_{\mathcal{A}}(\hat{f})$.

To see that $T_{\mathcal{A}}$ is an isomorphism, we just need to see that it is bijective on morphisms. This is clear on even morphisms, and follows on odd morphisms because the function $\text{Hom}_{\widehat{(\mathcal{A})}}(\lambda, \mu)_{\bar{1}} = \text{Hom}_{\mathcal{A}}(\lambda, \Pi\mu)_{\bar{0}} \rightarrow \text{Hom}_{\mathcal{A}}(\lambda, \mu)_{\bar{1}}, f \mapsto \zeta_\mu \circ f$ is invertible with inverse $f \mapsto \zeta_\mu^{-1} \circ f$.

Finally we must check the naturality of T : there is an equality of superfunctors $FT_{\mathcal{A}} = T_{\mathcal{B}}(\widehat{F}) : (\widehat{\mathcal{A}}) \rightarrow \mathcal{B}$ for any superfunctor $F : \mathcal{A} \rightarrow \mathcal{B}$ between Π -supercategories \mathcal{A} and \mathcal{B} . This is clear on objects and even morphisms. Consider an odd morphism $\hat{f} : \lambda \rightarrow \mu$ in $(\widehat{\mathcal{A}})$ coming from an even morphism $f : \lambda \rightarrow \Pi_{\mathcal{A}}\mu$ in \mathcal{A} . We have that $F(T_{\mathcal{A}}\hat{f}) = F(\zeta_{\mathcal{A}})_{\mu} \circ Ff$ and $T_{\mathcal{B}}(\widehat{F}\hat{f}) = (\zeta_{\mathcal{B}})_{F\mu} \circ (\beta_F)_{\mu}^{-1} \circ Ff$. These are equal because $\zeta_{\mathcal{B}}F = F\zeta_{\mathcal{A}} \circ \beta_F$ by the definition in Corollary 3.3(ii) and the super interchange law. \square

We need one more general notion, which we spell out below under the simplifying assumption that our 2-categories are strict. The reader should have no trouble interpreting this in the non-strict case; see Definition 1.14 from the introduction where this is done when there is only one object. Our general conventions regarding 2-categories are analogous to the ones for 2-supercategories in Definition 2.2.

Definition 5.2. (i) A Π -2-category $(\mathfrak{A}, \pi, \beta, \xi)$ is a \mathbb{k} -linear 2-category \mathfrak{A} plus a family $\pi = (\pi_{\lambda})$ of 1-morphisms $\pi_{\lambda} : \lambda \rightarrow \lambda$, a family $\beta = (\beta_{\mu, \lambda})$ of natural isomorphisms $\beta_{\mu, \lambda} : \pi_{\mu} - \xrightarrow{\sim} -\pi_{\lambda}$, and a family $\xi = (\xi_{\lambda})$ of 2-isomorphisms $\xi_{\lambda} : \pi_{\lambda}^2 \xrightarrow{\sim} \mathbb{1}_{\lambda}$, such that (assuming \mathfrak{A} is strict):

- the pair (π, β) is an object in the Drinfeld center of \mathfrak{A} , i.e. the properties from Lemma 3.2(i)–(ii) hold;
- $(\beta_{\lambda, \lambda})_{\pi_{\lambda}} = -1_{\pi_{\lambda}^2}$;
- $\xi_{\mu} F \xi_{\lambda}^{-1} = (\beta_{\mu, \lambda})_F \pi_{\lambda} \circ \pi_{\mu} (\beta_{\mu, \lambda})_F$ in $\text{Hom}_{\mathfrak{A}}(\pi_{\mu}^2 F, F \pi_{\lambda}^2)$ for all 1-morphisms $F : \lambda \rightarrow \mu$.

Using the second two of these properties, we get that $\xi_{\mu} \pi_{\mu} = \pi_{\mu} \xi_{\mu}$ in $\text{Hom}_{\mathfrak{A}}(\pi_{\mu}^3, \pi_{\mu})$. Hence, each of the morphism categories $\mathcal{H}om_{\mathfrak{A}}(\lambda, \mu)$ in a Π -2-category is itself a Π -category, with $\Pi := \pi_{\mu} -$ and $\xi := \xi_{\mu} -$.

(ii) A Π -2-functor between two Π -2-categories \mathfrak{A} and \mathfrak{B} is a \mathbb{k} -linear 2-functor $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$ with its usual coherence maps c and i , plus an additional family of 2-isomorphisms $j : \pi_{\mathbb{R}\lambda} \xrightarrow{\sim} \mathbb{R}\pi_{\lambda}$ for each $\lambda \in \text{ob } \mathfrak{A}$, such that the following commute (assuming \mathfrak{A} and \mathfrak{B} are strict):

$$\begin{array}{ccc}
 & (\mathbb{R}\pi_{\mu})(\mathbb{R}-) & \\
 j(\mathbb{R}-) \nearrow & & \searrow c \\
 \pi_{\mathbb{R}\mu}(\mathbb{R}-) & & \mathbb{R}(\pi_{\mu}-) \\
 \beta_{\mathbb{R}\mu, \mathbb{R}\lambda} \downarrow & & \downarrow \mathbb{R}\beta_{\mu, \lambda} \\
 (\mathbb{R}-)\pi_{\mathbb{R}\lambda} & & \mathbb{R}(-\pi_{\lambda}) \\
 (\mathbb{R}-)j \searrow & & \nearrow c \\
 & (\mathbb{R}-)\mathbb{R}\pi_{\lambda} & \\
 \pi_{\mathbb{R}\lambda}^2 \xrightarrow{jj} & (\mathbb{R}\pi_{\lambda})^2 & \xrightarrow{c} \mathbb{R}(\pi_{\lambda}^2) \\
 \xi_{\mathbb{R}\lambda} \downarrow & & \downarrow \mathbb{R}\xi_{\lambda} \\
 \mathbb{1}_{\mathbb{R}\lambda} & \xrightarrow{i} & \mathbb{R}\mathbb{1}_{\lambda}
 \end{array}$$

A Π -2-functor is *strict* if its coherence maps c, i and j are identities.

(iii) A Π -2-natural transformation $(X, x) : \mathbb{R} \Rightarrow \mathbb{S}$ between two Π -2-functors $\mathbb{R}, \mathbb{S} : \mathfrak{A} \rightarrow \mathfrak{B}$ is a 2-natural transformation as usual, with one additional coherence

axiom:

$$\begin{array}{ccc} \pi_{\mathbb{S}\lambda} X_\lambda & \xrightarrow{(\beta_{\mathbb{S}\lambda, \mathbb{R}\lambda})_{X_\lambda}} & X_\lambda \pi_{\mathbb{R}\lambda} \\ j X_\lambda \downarrow & & \downarrow X_\lambda j \\ (\mathbb{S}\pi_\lambda) X_\lambda & \xleftarrow{(x_{\lambda, \lambda})_{\pi_\lambda}} & X_\lambda (\mathbb{R}\pi_\lambda) \end{array} .$$

The basic example of a strict Π -2-category is $\Pi\text{-}\mathcal{C}\text{at}$: objects are Π -categories, 1-morphisms are Π -functors, and 2-morphisms are Π -natural transformations. We define the additional data π, β and ξ so that $\pi_{\mathcal{A}} := \Pi_{\mathcal{A}}$ for each Π -category \mathcal{A} , and ξ and β come from the natural transformations of Definition 1.6(i)–(ii).

For a 2-supercategory \mathfrak{A} , the *underlying 2-category* $\underline{\mathfrak{A}}$ is the 2-category with the same objects as \mathfrak{A} , morphism categories that are the underlying categories of the morphism supercategories in \mathfrak{A} , and horizontal composition that is the restriction of the one in \mathfrak{A} . If $(\mathfrak{A}, \pi, \zeta)$ is a (strict) Π -2-supercategory, Lemma 3.2 shows how to define β and ξ making $(\underline{\mathfrak{A}}, \pi, \beta, \xi)$ into a Π -2-category. In particular, starting from the Π -2-supercategory $\Pi\text{-}\mathcal{S}\mathcal{C}\text{at}$, we see that $\Pi\text{-}\mathcal{S}\mathcal{C}\text{at}$ is a Π -2-category.

Now we upgrade the functors E_1 and D_1 from (5.1)–(5.2) to strict Π -2-functors

$$\mathbb{E}_1 : \Pi\text{-}\mathcal{S}\mathcal{C}\text{at} \rightarrow \Pi\text{-}\mathcal{C}\text{at}, \quad \mathbb{D}_1 : \Pi\text{-}\mathcal{C}\text{at} \rightarrow \Pi\text{-}\mathcal{S}\mathcal{C}\text{at}. \quad (5.3)$$

These agree with E_1 and D_1 on objects and 1-morphisms. On 2-morphisms, \mathbb{E}_1 sends an even supernatural transformation $x : F \Rightarrow G$ to $\underline{x} : \underline{F} \Rightarrow \underline{G}$ defined from $\underline{x}_\lambda := x_\lambda$, which is a Π -natural transformation thanks to Corollary 3.3(iii). In the other direction, \mathbb{D}_1 sends a Π -natural transformation $y : F \Rightarrow G$ to $\hat{y} : \hat{F} \Rightarrow \hat{G}$ defined from $\hat{y}_\lambda := y_\lambda$. In order to check that \hat{y} is an even supernatural transformation, the subtle point is to show that $\hat{y}_\mu \circ \hat{F}\hat{f} = \hat{G}\hat{f} \circ \hat{y}_\lambda$ for an odd morphism $\hat{f} : \lambda \rightarrow \mu$ coming from $f : \lambda \rightarrow \Pi_{\mathcal{A}}\mu$ in \mathcal{A} , i.e. $\Pi_{\mathcal{B}}y_\mu \circ (\beta_F)_\mu^{-1} \circ Ff = (\beta_G)_\mu^{-1} \circ Gf \circ y_\lambda$. This follows from the property $\beta_G \circ \Pi_{\mathcal{B}}y = y\Pi_{\mathcal{A}} \circ \beta_F$ from Definition 1.6(iii), plus the fact that $y_{\Pi_{\mathcal{A}}\mu} \circ Ff = Gf \circ y_\lambda$ by the naturality of y . The following strengthens Lemma 5.1 by taking natural transformations into account:

Theorem 5.3. *The strict Π -2-functors \mathbb{D}_1 and \mathbb{E}_1 from (5.3) give mutually inverse Π -2-equivalences between $\Pi\text{-}\mathcal{C}\text{at}$ and $\Pi\text{-}\mathcal{S}\mathcal{C}\text{at}$.*

Proof. We have that $\mathbb{E}_1 \circ \mathbb{D}_1 = \mathbb{I}_{\Pi\text{-}\mathcal{C}\text{at}}$. Conversely, we show that $\mathbb{D}_1 \circ \mathbb{E}_1$ is isomorphic (not merely equivalent!) to $\mathbb{I}_{\Pi\text{-}\mathcal{S}\mathcal{C}\text{at}}$ in the 2-category $\Pi\text{-}\mathcal{S}\mathcal{C}\text{at}$ by producing a Π -2-natural isomorphism

$$(T, t) : \mathbb{D}_1 \circ \mathbb{E}_1 \xrightarrow{\cong} \mathbb{I}_{\Pi\text{-}\mathcal{S}\mathcal{C}\text{at}}.$$

Thus, we need to supply supercategory isomorphisms $T_{\mathcal{A}} : \widehat{\mathcal{A}} \xrightarrow{\cong} \mathcal{A}$ and even supernatural isomorphisms $(t_{\mathcal{B}, \mathcal{A}})_F : T_{\mathcal{B}}(\widehat{F}) \xrightarrow{\cong} FT_{\mathcal{A}}$ for all Π -supercategories and superfunctors $F : \mathcal{A} \rightarrow \mathcal{B}$. The isomorphisms $T_{\mathcal{A}}$ have already been defined in the proof of Lemma 5.1. Also, in the last paragraph of that proof, we observed that $T_{\mathcal{B}}(\widehat{F}) = FT_{\mathcal{A}}$. So we can simply take each $(t_{\mathcal{B}, \mathcal{A}})_F$ to be the identity. To see that $t_{\mathcal{B}, \mathcal{A}}$ is natural, one needs to observe that $xT_{\mathcal{A}} = T_{\mathcal{B}}(\underline{x})$ for all even supernatural isomorphisms $x : F \Rightarrow G$. The only other non-trivial check required is for the coherence axiom of Definition 5.2(iii). For this, we must show that $(\beta_{\mathcal{A}, \widehat{\mathcal{A}}})_{T_{\mathcal{A}}}$ is the identity for each Π -supercategory \mathcal{A} . This amounts to checking that the natural transformations $\zeta_{\mathcal{A}}T_{\mathcal{A}}$ and $T_{\mathcal{A}}\zeta_{\widehat{\mathcal{A}}}$ are equal. By definition, on an object λ , $\zeta_{\widehat{\mathcal{A}}}$ is the odd morphism $\hat{1}_{\Pi_{\mathcal{A}}\lambda} : \Pi_{\mathcal{A}}\lambda \rightarrow \lambda$ in $\widehat{\mathcal{A}}$ associated to the identity morphism

$1_{\Pi_{\mathcal{A}}\lambda}$. Hence, according to the definition from the first paragraph of the proof of Lemma 5.1, $(T_{\mathcal{A}}\zeta_{\widehat{\mathcal{A}}})_{\lambda} = T_{\mathcal{A}}\hat{1}_{\Pi_{\mathcal{A}}\lambda} = (\zeta_{\mathcal{A}})_{\lambda} = (\zeta_{\mathcal{A}}T_{\mathcal{A}})_{\lambda}$, as required. \square

Recall that $\Pi\text{-}2\text{-}SCat$ is the category of Π -2-supercategories and 2-superfunctors. Also let $\Pi\text{-}2\text{-}Cat$ denote the category of Π -2-categories and Π -2-functors. There is a functor

$$E_2 : \Pi\text{-}2\text{-}SCat \rightarrow \Pi\text{-}2\text{-}Cat, \quad (\mathfrak{A}, \pi, \zeta) \mapsto (\mathfrak{A}, \pi, \beta, \xi), \quad \mathbb{R} \mapsto \underline{\mathbb{R}}. \quad (5.4)$$

We've already defined the effect of this on Π -2-supercategories. On a 2-superfunctor $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$, we define $\underline{\mathbb{R}}$ to be the same function as \mathbb{R} on objects and the underlying functor to \mathbb{R} on morphism categories. The coherence maps c and i restrict in an obvious way to give coherence maps for $\underline{\mathbb{R}}$. We also need the additional coherence map $j : \pi_{\mathbb{R}\lambda} \xrightarrow{\cong} \mathbb{R}\pi_{\lambda}$, which is defined so that the following diagram commutes:

$$\begin{array}{ccc} \pi_{\mathbb{R}\lambda} & \xrightarrow{j} & \mathbb{R}\pi_{\lambda} \\ \zeta_{\mathbb{R}\lambda} \downarrow & & \downarrow \mathbb{R}\zeta_{\lambda} \\ \mathbb{1}_{\mathbb{R}\lambda} & \xrightarrow{i} & \mathbb{R}\mathbb{1}_{\lambda} \end{array}.$$

Now one has to check that the two axioms from Definition 5.2(ii) are satisfied. The first of these is a consequence of the second two diagrams from Definition 2.2(ii) plus the definition of β . For the second one, we have by the super interchange law that

$$((\mathbb{R}\zeta_{\lambda})(\mathbb{R}\zeta_{\lambda})) \circ jj = ((\mathbb{R}\zeta_{\lambda}) \circ j)((\mathbb{R}\zeta_{\lambda}) \circ j) = (i \circ \zeta_{\mathbb{R}\lambda})(i \circ \zeta_{\mathbb{R}\lambda}) = ii \circ (\zeta_{\mathbb{R}\lambda}\zeta_{\mathbb{R}\lambda}).$$

Also, by the naturality of c , we have that $\mathbb{R}(\zeta_{\lambda}\zeta_{\lambda}) \circ c = c \circ ((\mathbb{R}\zeta_{\lambda})(\mathbb{R}\zeta_{\lambda}))$. Putting these together gives $\mathbb{R}(\zeta_{\lambda}\zeta_{\lambda}) \circ c \circ jj = c \circ ii \circ (\zeta_{\mathbb{R}\lambda}\zeta_{\mathbb{R}\lambda})$, and the conclusion follows easily.

In the other direction, we define a functor

$$D_2 : \Pi\text{-}2\text{-}Cat \rightarrow \Pi\text{-}2\text{-}SCat, \quad (\mathfrak{A}, \pi, \beta, \xi) \mapsto (\widehat{\mathfrak{A}}, \pi, \zeta), \quad \mathbb{R} \mapsto \widehat{\mathbb{R}} \quad (5.5)$$

as follows. The 2-supercategory $\widehat{\mathfrak{A}}$ has the same objects as \mathfrak{A} . Its morphism supercategories $\mathcal{H}om_{\widehat{\mathfrak{A}}}(\lambda, \mu)$ arise as associated Π -supercategories to the morphism categories $\mathcal{H}om_{\mathfrak{A}}(\lambda, \mu)$. Thus the 1-morphisms in $\widehat{\mathfrak{A}}$ are the same as in \mathfrak{A} , while for 1-morphisms $F, G : \lambda \rightarrow \mu$ we have that $\mathcal{H}om_{\widehat{\mathfrak{A}}}(F, G)_{\bar{0}} := \mathcal{H}om_{\mathfrak{A}}(F, G)$ and $\mathcal{H}om_{\widehat{\mathfrak{A}}}(F, G)_{\bar{1}} := \mathcal{H}om_{\mathfrak{A}}(F, \pi_{\mu}G)$. To describe horizontal and vertical composition in $\widehat{\mathfrak{A}}$, we assume to simplify the exposition that \mathfrak{A} is strict. Then vertical composition in $\widehat{\mathfrak{A}}$ is induced by that of \mathfrak{A} (using ξ when composing two odd 2-morphisms). Horizontal composition of 1-morphisms in $\widehat{\mathfrak{A}}$ is the same as in \mathfrak{A} ; the horizontal composition $\hat{y}\hat{x}$ of homogeneous 2-morphisms $\hat{x} : F \Rightarrow H$ and $\hat{y} : G \Rightarrow K$ for $F, H : \lambda \rightarrow \mu$ and $G, K : \mu \rightarrow \nu$ in $\widehat{\mathfrak{A}}$ is defined as follows:

- if they are both even, so $\hat{x} = x$ and $\hat{y} = y$ for morphisms $x : F \Rightarrow H$ and $y : G \Rightarrow K$ in \mathfrak{A} , we define $\hat{y}\hat{x}$ to be the horizontal composition $yx : GF \Rightarrow KH$ in \mathfrak{A} ;
- if \hat{x} is even and \hat{y} is odd, so $\hat{x} = x$ and $\hat{y} = y$ for $x : F \Rightarrow H$ and $y : G \Rightarrow \pi_{\nu}K$ in \mathfrak{A} , we let $\hat{y}\hat{x}$ be the horizontal composition $yx : GF \Rightarrow \pi_{\nu}KH$ in \mathfrak{A} viewed as an odd 2-morphism $GF \Rightarrow KH$ in $\widehat{\mathfrak{A}}$;
- if \hat{y} is even and \hat{x} is odd, so $\hat{y} = y$ and $\hat{x} = x$ for $x : F \Rightarrow \pi_{\mu}H$ and $y : G \Rightarrow K$, we let $\hat{y}\hat{x}$ be $(\beta_{\nu, \mu})_{\bar{K}}^{-1}H \circ yx : GF \Rightarrow K\pi_{\mu}H \Rightarrow \pi_{\nu}KH$;
- if both are odd, so $\hat{x} = x$ and $\hat{y} = y$ for $x : F \Rightarrow \pi_{\mu}H$ and $y : G \Rightarrow \pi_{\nu}K$, we let $\hat{y}\hat{x}$ be $-\xi_{\nu}KH \circ \pi_{\nu}(\beta_{\nu, \mu})_{\bar{K}}^{-1}H \circ yx : GF \Rightarrow \pi_{\nu}K\pi_{\mu}H \Rightarrow \pi_{\nu}^2KH \Rightarrow KH$.

We leave it as an instructive exercise for the reader to check the super interchange law using the axioms from Definition 5.2(i); see also [EL, (2.44)–(2.45)] for helpful pictures. To make $\widehat{\mathfrak{A}}$ into a Π -2-supercategory, we already have the required data $\pi = (\pi_\lambda)$, and we get $\zeta = (\zeta_\lambda)$ by defining $\zeta_\lambda : \pi_\lambda \Rightarrow \mathbb{1}_\lambda$ to be 1_{π_λ} viewed as an odd 2-isomorphism in $\widehat{\mathfrak{A}}$.

To complete the definition of D_2 , we still need to define the 2-superfunctor $\widehat{\mathbb{R}} : \widehat{\mathfrak{A}} \rightarrow \widehat{\mathfrak{B}}$ given a Π -2-functor $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$. For simplicity, we assume that \mathfrak{A} and \mathfrak{B} are strict. Then $\widehat{\mathbb{R}}$ is the same as \mathbb{R} on objects and 1-morphisms. On an even 2-morphism $\hat{x} : F \Rightarrow G$, coming from $x : F \Rightarrow G$ in \mathfrak{A} , we let $\widehat{\mathbb{R}}\hat{x}$ be the even 2-morphism associated to $\mathbb{R}x : \mathbb{R}F \Rightarrow \mathbb{R}G$. On an odd 2-morphism $\hat{x} : F \Rightarrow G$, coming from $x : F \Rightarrow \pi_\mu G$, we let $\widehat{\mathbb{R}}\hat{x}$ be the odd 2-morphism associated to the composition $j^{-1}(\mathbb{R}G) \circ c^{-1} \circ \mathbb{R}x : \mathbb{R}F \Rightarrow \mathbb{R}(\pi_\mu G) \Rightarrow (\mathbb{R}\pi_\mu)(\mathbb{R}G) \Rightarrow \pi_{\mathbb{R}\mu}(\mathbb{R}G)$. We take the coherence maps c and i for $\widehat{\mathbb{R}}$ that are defined by the same data as c and i for \mathbb{R} . As usual, there are various checks to be made:

- To see that $\widehat{\mathbb{R}}$ is a well-defined functor on morphism supercategories, one needs to check that $\widehat{\mathbb{R}}(\hat{y} \circ \hat{x}) = \widehat{\mathbb{R}}\hat{y} \circ \widehat{\mathbb{R}}\hat{x}$ for $\hat{x} : F \Rightarrow G, \hat{y} : G \Rightarrow H$ and $F, G, H : \lambda \rightarrow \mu$. This is immediate if \hat{x} is even. If \hat{x} is odd, it comes from some 2-morphism $x : F \Rightarrow \pi_\mu G$ in \mathfrak{A} . Suppose \hat{y} is even, coming from $y : G \Rightarrow H$ in \mathfrak{A} . Then we need to show that

$$j^{-1}(\mathbb{R}H) \circ c^{-1} \circ \mathbb{R}(\pi_\mu y) \circ \mathbb{R}x = \pi_{\mathbb{R}\mu}(\mathbb{R}y) \circ j^{-1}(\mathbb{R}G) \circ c^{-1} \circ \mathbb{R}x.$$

This follows by the commutativity of the following hexagon of 2-morphisms in \mathfrak{B} :

$$\begin{array}{ccccc}
 & & (\mathbb{R}\pi_\mu)(\mathbb{R}G) & & \\
 & j(\mathbb{R}G) \nearrow & \downarrow & \searrow c & \\
 \pi_{\mathbb{R}\mu}(\mathbb{R}G) & & & & \mathbb{R}(\pi_\mu G) \\
 \pi_{\mathbb{R}\mu}(\mathbb{R}y) \downarrow & & (\mathbb{R}\pi_\mu)(\mathbb{R}y) & & \downarrow \mathbb{R}(\pi_\mu y) \\
 \pi_{\mathbb{R}\mu}(\mathbb{R}H) & & & & \mathbb{R}(\pi_\mu H) \\
 & j(\mathbb{R}H) \searrow & \downarrow & \nearrow c & \\
 & & (\mathbb{R}\pi_\mu)(\mathbb{R}H) & &
 \end{array}$$

To see this, note the left hand square commutes by the interchange law, and the right hand square commutes by naturality of c . The case that \hat{y} is odd, coming from $y : G \Rightarrow \pi_\mu H$, is similar but a little more complicated; ultimately, it depends on the second coherence axiom from Definition 5.2(ii).

- To see that c is a supernatural transformation, one needs to check that

$$\begin{array}{ccc}
 (\widehat{\mathbb{R}}G)(\widehat{\mathbb{R}}F) & \xrightarrow{c} & \widehat{\mathbb{R}}(GF) \\
 (\widehat{\mathbb{R}}\hat{y})(\widehat{\mathbb{R}}\hat{x}) \downarrow & & \downarrow \widehat{\mathbb{R}}(\hat{y}\hat{x}) \\
 (\widehat{\mathbb{R}}K)(\widehat{\mathbb{R}}H) & \xrightarrow{c} & \widehat{\mathbb{R}}(KH)
 \end{array}$$

commutes. We leave this lengthy calculation to the reader, just noting when \hat{x} is odd that it depends also on the first coherence axiom from Definition 5.2(ii).

The proof of the next lemma is similar to the proof of Lemma 5.1. Note also that the remaining part of Theorem 1.15 from the introduction follows from this result (on restricting to 2-(super)categories with one object).

Lemma 5.4. *The functors D_2 and E_2 are mutually inverse equivalences between the categories $\Pi\text{-}2\text{-Cat}$ and $\Pi\text{-}2\text{-SCat}$.*

Proof. We first observe that $E_2 \circ D_2 = I_{\Pi\text{-}2\text{-Cat}}$. To show that $D_2 \circ E_2 \cong I_{\Pi\text{-}2\text{-SCat}}$, we have to define a natural isomorphism $\mathbb{T} : D_2 \circ E_2 \xrightarrow{\cong} I_{\Pi\text{-}2\text{-SCat}}$. So for each $\Pi\text{-}2\text{-supercategory}$ $(\mathfrak{A}, \pi, \zeta)$, we need to produce an isomorphism of 2-supercategories $\mathbb{T}_{\mathfrak{A}} : \widehat{(\mathfrak{A})} \xrightarrow{\cong} \mathfrak{A}$. This is the identity on objects and 1-morphisms. On a homogeneous 2-morphism $\hat{x} : F \Rightarrow G$ between 1-morphisms $F, G : \lambda \rightarrow \mu$ in $\widehat{(\mathfrak{A})}$, we let $\mathbb{T}_{\mathfrak{A}}\hat{x} := x$ if \hat{x} is even coming from $x : F \Rightarrow G$ in \mathfrak{A} , or $\mathbb{T}_{\mathfrak{A}}\hat{x} := \zeta_{\mu}G \circ x$ if \hat{x} is odd coming from $x : F \Rightarrow \pi_{\mu}G$ in \mathfrak{A} . Since $\mathbb{T}_{\mathfrak{A}}$ is clearly bijective on 2-morphisms, it will certainly be a 2-isomorphism, but we still need to verify that it is indeed a well-defined 2-superfunctor, i.e. we need to show that it respects horizontal and vertical composition of 2-morphisms. In the next paragraph, we go through the details of this in the most interesting situation when both 2-morphisms are odd (also assuming \mathfrak{A} is strict to simplify notation).

For vertical composition, take $F, G, H : \lambda \rightarrow \mu$ and odd 2-morphisms $\hat{x} : F \Rightarrow G$, $\hat{y} : G \Rightarrow H$ in $\widehat{(\mathfrak{A})}$ coming from $x : F \Rightarrow \pi_{\mu}G$, $y : G \Rightarrow \pi_{\mu}H$ in \mathfrak{A} . The vertical composition $\hat{y} \circ \hat{x}$ in $\widehat{(\mathfrak{A})}$ is by definition the composition $\xi_{\mu}H \circ \pi_{\mu}y \circ x$ in \mathfrak{A} . We need to show that this is equal to $\zeta_{\mu}H \circ y \circ \zeta_{\mu}G \circ x$:

$$\zeta_{\mu}H \circ y \circ \zeta_{\mu}G \circ x = -\zeta_{\mu}H \circ \zeta_{\mu}\pi_{\mu}H \circ \pi_{\mu}y \circ x = \xi_{\mu}H \circ \pi_{\mu}y \circ x.$$

For horizontal composition, take $F, H : \lambda \rightarrow \mu$, $G, K : \mu \rightarrow \nu$ and odd 2-morphisms $\hat{x} : F \Rightarrow H$, $\hat{y} : G \Rightarrow K$ coming from $x : F \Rightarrow \pi_{\mu}H$, $y : G \Rightarrow \pi_{\nu}K$. Recalling that $(\beta_{\nu, \mu})_K^{-1} = \zeta_{\nu}^{-1}K\zeta_{\mu}$, we have that $\zeta_{\nu}K \circ (\beta_{\nu, \mu})_K^{-1} = K\zeta_{\mu}$, hence $\xi_{\nu}K \circ \Pi_{\nu}(\beta_{\nu, \mu})_K^{-1} = \zeta_{\nu}K\zeta_{\mu}$. We deduce that

$$-\xi_{\nu}KH \circ \Pi_{\nu}(\beta_{\nu, \mu})_K^{-1}H \circ yx = -\zeta_{\nu}K\zeta_{\mu}H \circ yx = (\zeta_{\nu}K \circ y)(\zeta_{\mu}H \circ x),$$

establishing that $\Phi(\hat{y}\hat{x}) = \Phi(\hat{y})\Phi(\hat{x})$.

To complete the proof we need to check naturality: we have that $\mathbb{RT}_{\mathfrak{A}} = \mathbb{T}_{\mathfrak{B}}(\widehat{\mathbb{R}})$ for each 2-superfunctor $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$ between $\Pi\text{-}2\text{-supercategories}$ \mathfrak{A} and \mathfrak{B} . The only tricky point is to see that they are equal on an odd 2-morphism $\hat{x} : F \Rightarrow G$ in $\widehat{(\mathfrak{A})}$ coming from $x : F \Rightarrow \pi_{\mu}G$ in \mathfrak{A} . For this, one needs to use the last of the unit axioms from Definition 2.2(ii) plus the definition of j . \square

Finally, we upgrade E_2 and D_2 to strict 2-functors

$$\mathbb{E}_2 : \Pi\text{-}2\text{-SCat} \rightarrow \Pi\text{-}2\text{-Cat}, \quad \mathbb{D}_2 : \Pi\text{-}2\text{-Cat} \rightarrow \Pi\text{-}2\text{-SCat}. \quad (5.6)$$

We take \mathbb{E}_2 to be equal to E_2 on objects and 1-morphisms. On 2-morphisms, \mathbb{E}_2 sends 2-natural transformation $(X, x) : \mathbb{R} \Rightarrow \mathbb{S}$ to $(\underline{X}, \underline{x}) : \underline{\mathbb{R}} \Rightarrow \underline{\mathbb{S}}$ defined by $\underline{X}_{\lambda} := X_{\lambda}$ and $\underline{x}_{\mu, \lambda} := x_{\mu, \lambda}$. To check the coherence axiom from Definition 5.2(iii),

3-equivalent to the strict 3-category of Π -2-supercategories, 2-superfunctors, 2-natural transformations and even supermodifications. In particular, this assertion implies that the monoidal category underlying the Drinfeld center of a Π -2-supercategory \mathfrak{A} is monoidally equivalent to the Drinfeld center of $\underline{\mathfrak{A}}$.

6. GRADINGS

In the final section, we explain how to incorporate an additional \mathbb{Z} -grading. Since this is all is very similar to the theory so far (and there are no additional issues with signs!), we will be quite brief, introducing suitable language but leaving detailed proofs to the reader. We continue to assume that \mathbb{k} is a commutative ground ring⁶, so a superspace means a $\mathbb{Z}/2$ -graded \mathbb{k} -module as before.

By a *graded superspace* we mean a \mathbb{Z} -graded superspace

$$V = \bigoplus_{n \in \mathbb{Z}} V_n = \bigoplus_{n \in \mathbb{Z}} V_{n, \bar{0}} \oplus V_{n, \bar{1}}.$$

We stress that the \mathbb{Z} - and $\mathbb{Z}/2$ -gradings on a graded superspace are independent of each other. We denote the degree n of $v \in V_n$ also by $\deg(v)$. Let $\underline{\mathcal{G}}\mathcal{S}\mathcal{V}ec$ be the category of graded superspaces and degree-preserving even linear maps, i.e. \mathbb{k} -module homomorphisms $f : V \rightarrow W$ such that $f(V_{n,p}) \subseteq W_{n,p}$ for each $n \in \mathbb{Z}$ and $p \in \mathbb{Z}/2$. This is a symmetric monoidal category with $(V \otimes W)_n = \bigoplus_{r+s=n} V_r \otimes W_s$, and the same braiding as in $\underline{\mathcal{S}}\mathcal{V}ec$.

Definition 6.1. By a *graded supercategory* we mean a category enriched in $\underline{\mathcal{G}}\mathcal{S}\mathcal{V}ec$. A *graded superfunctor* between graded supercategories is a superfunctor that preserves degrees of morphisms. A supernatural transformation $x : F \Rightarrow G$ between graded superfunctors F and G is said to be *homogeneous of degree n* if $x_\lambda : F\lambda \rightarrow G\lambda$ is of degree n for all objects λ . Let $\text{Hom}(F, G)_n$ denote the superspace of all homogeneous supernatural transformations of degree n . Then a *graded supernatural transformation* from F to G is an element of the graded superspace $\text{Hom}(F, G) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}(F, G)_n$.

If \mathcal{A} is a graded supercategory, the *underlying category* $\underline{\mathcal{A}}$ is the \mathbb{k} -linear category with the same objects as \mathcal{A} but only the even morphisms of degree zero. Here are some basic examples of graded supercategories:

- Any graded superalgebra $A = \bigoplus_{n \in \mathbb{Z}} A_n = \bigoplus_{n \in \mathbb{Z}} A_{n, \bar{0}} \oplus A_{n, \bar{1}}$ can be viewed as a graded supercategory with one object.
- For graded superalgebras A and B , let $A\text{-}\underline{\mathcal{G}}\mathcal{S}\mathcal{M}od\text{-}B$ denote the graded supercategory of *graded (A, B) -superbimodules* $V = \bigoplus_{n \in \mathbb{Z}} V_n = \bigoplus_{n \in \mathbb{Z}} V_{n, \bar{0}} \oplus V_{n, \bar{1}}$. Morphisms are defined from $\text{Hom}(V, W) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}(V, W)_n$ where $\text{Hom}(V, W)_n$ consists of all (A, B) -superbimodule homomorphisms $f : V \rightarrow W$ that are *homogeneous of degree n* , i.e. $f(V_m) \subseteq W_{m+n}$ for all $m \in \mathbb{Z}$.
- Taking $A = B = \mathbb{k}$ in (ii), we get the graded supercategory $\underline{\mathcal{G}}\mathcal{S}\mathcal{V}ec$ of graded superspaces. The underlying category is $\underline{\mathcal{G}}\mathcal{S}\mathcal{V}ec$ as defined above.
- For graded supercategories \mathcal{A}, \mathcal{B} , the graded supercategory $\underline{\mathcal{H}}om(\mathcal{A}, \mathcal{B})$ consists of all graded superfunctors and graded supernatural transformations.

Let $\underline{\mathcal{G}}\mathcal{S}\mathcal{C}at$ be the category of all graded supercategories and graded superfunctors. We make $\underline{\mathcal{G}}\mathcal{S}\mathcal{C}at$ into a monoidal category with tensor product operation $- \boxtimes -$ defined in just the same way as was explained after Example 1.2 in the introduction.

⁶Actually, everything prior to Definition 6.12 makes sense more generally working over a graded commutative superalgebra $\mathbb{k} = \bigoplus_{n \in \mathbb{Z}} \mathbb{k}_n = \bigoplus_{n \in \mathbb{Z}} \mathbb{k}_{n, \bar{0}} \oplus \mathbb{k}_{n, \bar{1}}$.

Definition 6.2. A *strict graded 2-supercategory* is a category enriched in \mathcal{GSCat} , i.e. it is a 2-supercategory with an additional grading on 2-morphisms which is respected by both horizontal and vertical composition.

The basic example of a strict graded 2-supercategory is \mathcal{GSCat} : graded supercategories, graded superfunctors and graded supernatural transformations. There is also the “weak” notion of *graded 2-supercategory*, which is the obvious graded analog of Definition 2.2(i). For example, there is a graded 2-supercategory \mathcal{GSBim} of *graded superbimodules*, which has objects that are graded superalgebras, the morphism supercategories are defined from $\mathcal{H}om_{\mathcal{GSBim}}(A, B) := B\text{-}\mathcal{GSM}od\text{-}A$, and horizontal composition is defined by tensor product.

Here is the graded analog of Definition 2.2(ii):

Definition 6.3. For graded 2-supercategories \mathfrak{A} and \mathfrak{B} , a *graded 2-superfunctor* $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$ consists of:

- A function $\mathbb{R} : \text{ob } \mathfrak{A} \rightarrow \text{ob } \mathfrak{B}$.
- Graded superfunctors $\mathbb{R} : \mathcal{H}om_{\mathfrak{A}}(\lambda, \mu) \rightarrow \mathcal{H}om_{\mathfrak{B}}(\mathbb{R}\lambda, \mathbb{R}\mu)$ for $\lambda, \mu \in \text{ob } \mathfrak{A}$.
- Homogeneous graded supernatural isomorphisms $c : (\mathbb{R} -) (\mathbb{R} -) \xrightarrow{\cong} \mathbb{R}(- -)$ that are even of degree zero.
- Homogeneous 2-isomorphisms $i : \mathbb{1}_{\mathbb{R}\lambda} \xrightarrow{\cong} \mathbb{R}\mathbb{1}_{\lambda}$ that are even of degree zero for all $\lambda \in \text{ob } \mathfrak{A}$.

This data should satisfy the same axioms as in Definition 2.2(ii).

We leave it to the reader to formulate the graded versions of Definition 2.2(iii) (2-natural transformations between graded 2-superfunctors) and Definition 2.2(iv) (graded supermodifications).

The next two definitions give the graded analogs of Definitions 1.7 and 3.1.

Definition 6.4. A *graded (Q, Π) -supercategory* is a graded supercategory \mathcal{A} plus the extra data of graded superfunctors $Q, Q^{-1}, \Pi : \mathcal{A} \rightarrow \mathcal{A}$, an odd supernatural isomorphism $\zeta : \Pi \xrightarrow{\cong} I$ that is homogeneous of degree 0, and even supernatural isomorphisms $\sigma : Q \xrightarrow{\cong} I$ and $\bar{\sigma} : Q^{-1} \xrightarrow{\cong} I$ that are homogeneous of degrees -1 and 1 , respectively. Note that $\mathfrak{i} := \bar{\sigma}\sigma : Q^{-1}Q \xrightarrow{\cong} I$, $\mathfrak{j} := \sigma\bar{\sigma} : QQ^{-1} \xrightarrow{\cong} I$ and $\xi := \zeta\zeta : \Pi^2 \xrightarrow{\cong} I$ are even isomorphisms of degree zero, so that Q and Q^{-1} are mutually inverse graded superequivalences, and Π is a self-inverse graded superequivalence.

For example, for graded superalgebras A and B , we can view $A\text{-}\mathcal{GSM}od\text{-}B$ as a graded (Q, Π) -supercategory by defining Π and ζ as in Example 1.8, and letting $Q, Q^{-1} : A\text{-}\mathcal{GSM}od\text{-}B \rightarrow A\text{-}\mathcal{GSM}od\text{-}B$ be the upward and downward grading shift functors, i.e. $(QV)_n := V_{n-1}$, $(Q^{-1}V)_n := V_{n+1}$. We take $\sigma, \bar{\sigma}$ to be induced by the identity function on the underlying sets.

Definition 6.5. A *graded (Q, Π) -2-supercategory* is a graded 2-supercategory \mathfrak{A} plus families $q = (q_{\lambda} : \lambda \rightarrow \lambda), q^{-1} = (q_{\lambda}^{-1} : \lambda \rightarrow \lambda)$ and $\pi = (\pi_{\lambda} : \lambda \rightarrow \lambda)$ of 1-morphisms, and families $\sigma = (\sigma_{\lambda} : q_{\lambda} \xrightarrow{\cong} \mathbb{1}_{\lambda}), \bar{\sigma} = (\bar{\sigma}_{\lambda} : q_{\lambda}^{-1} \xrightarrow{\cong} \mathbb{1}_{\lambda})$ and $\zeta = (\zeta_{\lambda} : \pi_{\lambda} \xrightarrow{\cong} \mathbb{1}_{\lambda})$ of 2-isomorphisms which are even, even and odd of degrees -1 , 1 and 0 , respectively.

For example, there is a graded (Q, Π) -2-supercategory $(Q, \Pi)\text{-}\mathcal{GSCat}$ consisting of all graded (Q, Π) -supercategories, graded superfunctors and graded supernatural transformations.

Lemma 6.6. *Let \mathfrak{A} be a graded (Q, Π) -2-supercategory, which we assume is strict for simplicity.*

- (i) There are families $\beta = (\beta_{\mu,\lambda} : \pi_\mu - \xrightarrow{\sim} -\pi_\lambda)$ of even supernatural isomorphisms of degree zero and $\xi = (\xi_\lambda : \pi_\lambda^2 \xrightarrow{\sim} \mathbb{1}_\lambda)$ of even 2-isomorphisms of degree zero defined as in Lemma 3.2. They satisfy the properties from Definition 5.2(i).
- (ii) There is a family $\gamma = (\gamma_{\mu,\lambda} : q_\mu - \xrightarrow{\sim} -q_\lambda)$ of even supernatural isomorphisms of degree zero defined from $(\gamma_{\mu,\lambda})_F := \sigma_\mu F \sigma_\lambda^{-1}$ for a 1-morphism $F : \lambda \rightarrow \mu$. The pair (q, γ) is an invertible object of the Drinfeld center of \mathfrak{A} with $(\gamma_{\lambda,\lambda})_{q_\lambda} = 1_{q_\lambda^2}$ and $(\gamma_{\lambda,\lambda})_{\pi_\lambda} = (\beta_{\lambda,\lambda})_{q_\lambda}^{-1}$.
- (iii) There are even 2-isomorphisms of degree zero $\iota_\lambda := \bar{\sigma}_\lambda \sigma_\lambda : q_\lambda^{-1} q_\lambda \xrightarrow{\sim} \mathbb{1}_\lambda$ and $\jmath_\lambda := \sigma_\lambda \bar{\sigma}_\lambda : q_\lambda q_\lambda^{-1} \xrightarrow{\sim} \mathbb{1}_\lambda$. Moreover $q_\lambda \iota_\lambda = \jmath_\lambda q_\lambda$ and $\iota_\lambda q_\lambda^{-1} = q_\lambda^{-1} \jmath_\lambda$ in $\text{Hom}_{\mathfrak{A}}(q_\lambda q_\lambda^{-1} q_\lambda, q_\lambda)$ and $\text{Hom}_{\mathfrak{A}}(q_\lambda^{-1} q_\lambda q_\lambda^{-1}, q_\lambda^{-1})$, respectively.

Proof. Similar arguments to those in the proof of Lemma 3.2. \square

Corollary 6.7. *Let \mathcal{A} and \mathcal{A}' be graded (Q, Π) -supercategories.*

- (i) There are even supernatural isomorphisms of degree zero $\xi := \zeta \zeta : \Pi^2 \xrightarrow{\sim} I$, $\iota := \bar{\sigma} \sigma : Q^{-1} Q \Rightarrow I$ and $\jmath := \sigma \bar{\sigma} : Q Q^{-1} \Rightarrow I$. Moreover, we have that $\xi \Pi = \Pi \xi$, and ι^{-1} and \jmath define the unit and counit of an adjunction making (Q, Q^{-1}) into an adjoint pair of auto-equivalences of \mathcal{A} .
- (ii) Suppose that $F : \mathcal{A} \rightarrow \mathcal{A}'$ is a graded superfunctor. There are even supernatural isomorphisms of degree zero $\beta_F := -\zeta' F \zeta^{-1} : \Pi' F \xrightarrow{\sim} F \Pi$ and $\gamma_F := \sigma' F \sigma^{-1} : Q' F \xrightarrow{\sim} F Q$, with $\xi' F \xi^{-1} = \beta_F \Pi \circ \Pi' \beta_F$ as in Corollary 3.3(ii). Also $\gamma_\Pi = \beta_Q^{-1}$.
- (iii) Suppose that $x : F \Rightarrow G$ is a graded supernatural transformation. Then $\beta_G \circ \Pi' x = x \Pi \circ \beta_F$ as in Corollary 3.3(iii). Similarly, $\gamma_G \circ Q' x = x Q \circ \gamma_F$.
- (iv) We have that $\beta_{GF} = G \beta_F \circ \beta_G F$, $\beta_I = 1_\Pi$ and $\beta_\Pi = -1_{\Pi^2}$ as in Corollary 3.3(iv). Similarly, $\gamma_{GF} = G \gamma_F \circ \gamma_G F$, $\gamma_I = 1_Q$ and $\gamma_Q = 1_{Q^2}$.

Proof. Everything follows by applying Lemma 6.6 to the (Q, Π) -2-supercategory (Q, Π) - \mathfrak{GSCat} . In particular, the assertion in (i) that ι^{-1} and \jmath are the unit and counit of an adjunction means that $Q \iota^{-1} \circ \jmath Q : Q \Rightarrow Q Q^{-1} Q \Rightarrow Q$ and $\iota^{-1} Q^{-1} \circ Q^{-1} \jmath : Q^{-1} \Rightarrow Q^{-1} Q Q^{-1} \Rightarrow Q^{-1}$ are identities; this follows because $Q \iota = \jmath Q$ and $\iota Q^{-1} = Q^{-1} \jmath$. \square

The analog of Definition 1.10 in the presence of a grading is as follows.

Definition 6.8. The (Q, Π) -envelope of a graded supercategory \mathcal{A} is the graded (Q, Π) -supercategory $\mathcal{A}_{q,\pi}$ with objects $\{Q^m \Pi^a \lambda \mid \lambda \in \text{ob } \mathcal{A}, m \in \mathbb{Z}, a \in \mathbb{Z}/2\}$ and

$$\text{Hom}_{\mathcal{A}_{q,\pi}}(Q^m \Pi^a \lambda, Q^n \Pi^b \mu) := Q^{n-m} \Pi^{a+b} \text{Hom}_{\mathcal{A}}(\lambda, \mu),$$

where Q and Π on the right hand side are the (invertible) grading and parity shift functors on \mathcal{GVec} . We denote the morphism $Q^m \Pi^a \lambda \rightarrow Q^n \Pi^b \mu$ coming from a homogeneous $f : \lambda \rightarrow \mu$ under this identification by $f_{m,a}^{n,b}$. Composition in $\mathcal{A}_{q,\pi}$ is defined by $g_{m,b}^{n,c} \circ f_{l,a}^{m,b} := (g \circ f)_{l,a}^{n,c}$. The parity-switching functor Π and ζ are defined as in Definition 1.10. The degree shift functors Q, Q^{-1} are given by $Q(Q^m \Pi^a \lambda) := Q^{m+1} \Pi^a \lambda$, $Q^{-1}(Q^m \Pi^a \lambda) := Q^{m-1} \Pi^a \lambda$, and $\sigma, \bar{\sigma}$ are induced by the identity morphism in \mathcal{A} .

In an analogous way to (4.1), Definition 6.8 may be extended to produce a strict graded 2-superfunctor

$$-_{q,\pi} : \mathfrak{GSCat} \rightarrow (Q, \Pi)\text{-}\mathfrak{GSCat}. \quad (6.1)$$

There is a canonical graded superfunctor $J : \mathcal{A} \rightarrow \mathcal{A}_{q,\pi}$ which satisfies a universal property similar to Lemma 4.2. Also J is a graded superequivalence if and only if \mathcal{A} is (Q, Π) -complete, meaning that every object λ of \mathcal{A} is the target of even isomorphisms of degrees ± 1 and the target of an odd isomorphism of degree 0; cf. Lemma 4.1. The analog of Theorem 4.3 is as follows:

Theorem 6.9. *For all graded supercategories \mathcal{A} and graded (Q, Π) -supercategories \mathcal{B} , there is a functorial graded superequivalence $\mathcal{H}om(\mathcal{A}, \nu\mathcal{B}) \rightarrow \mathcal{H}om(\mathcal{A}_{q,\pi}, \mathcal{B})$, where $\nu : (Q, \Pi)\text{-}\mathfrak{S}\mathfrak{C}\mathfrak{a}\mathfrak{t} \rightarrow \mathfrak{S}\mathfrak{C}\mathfrak{a}\mathfrak{t}$ denotes the obvious forgetful 2-superfunctor. Hence, the strict graded 2-superfunctor $-_{q,\pi}$ is left 2-adjoint to ν .*

Moving on to 2-categories, here is the graded analog of Definition 4.4:

Definition 6.10. The (Q, Π) -envelope of a graded 2-supercategory \mathfrak{A} is the (Q, Π) -2-supercategory $\mathfrak{A}_{q,\pi}$ with the same object set as \mathfrak{A} and morphism supercategories that are the (Q, Π) -envelopes of the graded morphism supercategories in \mathfrak{A} . Thus, the set of 1-morphisms $\lambda \rightarrow \mu$ in $\mathfrak{A}_{q,\pi}$ is

$$\{Q^m \Pi^a F \mid \text{for all 1-morphisms } F : \lambda \rightarrow \mu \text{ in } \mathfrak{A}, m \in \mathbb{Z} \text{ and } a \in \mathbb{Z}/2\}.$$

The graded superspace of 2-morphisms $Q^m \Pi^a F \Rightarrow Q^n \Pi^b G$ in $\mathfrak{A}_{q,\pi}$ is defined from

$$\text{Hom}_{\mathfrak{A}_{q,\pi}}(Q^m \Pi^a F, Q^n \Pi^b G) := Q^{n-m} \Pi^{a+b} \text{Hom}_{\mathfrak{A}}(F, G).$$

We denote the 2-morphism $Q^m \Pi^a F \Rightarrow Q^n \Pi^b G$ coming from a homogeneous 2-morphism $x : F \Rightarrow G$ in \mathfrak{A} under this identification by $x_{m,a}^{n,b}$. In the strict case, one might represent $x_{m,a}^{n,b}$ diagrammatically by

$$\begin{array}{c} G \\ \text{---} n \text{---} b \text{---} \\ | \\ \mu \text{---} \textcircled{x} \text{---} \lambda \text{---} \\ | \\ m \text{---} F \text{---} a \end{array}$$

This is of parity $|x| + a + b$ and degree $\deg(x) + n - m$ (where $|x|$ and $\deg(x)$ denote the parity and degree of x in \mathfrak{A}). Vertical composition is defined from

$$y_{m,b}^{n,c} \circ x_{l,a}^{m,b} := (y \circ x)_{l,a}^{n,c}.$$

Horizontal composition of 1-morphisms is defined by

$$(Q^n \Pi^b G)(Q^m \Pi^a F) := Q^{m+n} \Pi^{a+b}(GF)$$

and 2-morphisms by

$$y_{n,b}^{l,d} x_{m,a}^{k,c} := (-1)^{b|x|+|y|c+bc+ab} (yx)_{m+n,a+b}^{k+l,c+d}.$$

Finally, q, q^{-1} and π are given by $q_\lambda := Q^1 \Pi^0 \mathbb{1}_\lambda, q_\lambda^{-1} := Q^{-1} \Pi^0 \mathbb{1}_\lambda$ and $\pi_\lambda := Q^0 \Pi^1 \mathbb{1}_\lambda$; the 2-morphisms $\sigma_\lambda, \bar{\sigma}_\lambda$ and ξ_λ are induced by $1_{\mathbb{1}_\lambda}$.

Again, there is a canonical strict 2-superfunctor $\mathbb{J} : \mathfrak{A} \rightarrow \mathfrak{A}_{q,\pi}$, which is a graded 2-superequivalence if and only if \mathfrak{A} is (Q, Π) -complete, meaning that for each $\lambda \in \text{ob } \mathfrak{A}$ it possesses 1-morphisms $q_\pi^\pm : \lambda \rightarrow \lambda$ and $\pi_\lambda : \lambda \rightarrow \lambda$, and homogeneous 2-isomorphisms $q_\pi^\pm \xrightarrow{\cong} \mathbb{1}_\lambda$ that are even of degrees ∓ 1 , and $\pi_\lambda \xrightarrow{\cong} \mathbb{1}_\lambda$ that is odd of degree 0. Like in (4.5), one can extend Definition 6.10 to obtain a strict 2-functor

$$-_{q,\pi} : 2\text{-}\mathfrak{S}\mathfrak{C}\mathfrak{a}\mathfrak{t} \rightarrow (Q, \Pi)\text{-}2\text{-}\mathfrak{S}\mathfrak{C}\mathfrak{a}\mathfrak{t}. \quad (6.2)$$

The analog of Lemma 4.7 is as follows.

Lemma 6.11. *Suppose \mathfrak{A} is a graded 2-supercategory and \mathfrak{B} is a graded (Q, Π) -2-supercategory.*

- (i) Given a graded 2-superfunctor $\mathbb{R} : \mathfrak{A} \rightarrow \mathfrak{B}$, there is a canonical graded 2-superfunctor $\tilde{\mathbb{R}} : \mathfrak{A}_{q,\pi} \rightarrow \mathfrak{B}$ such that $\mathbb{R} = \tilde{\mathbb{R}}\mathbb{J}$.
- (ii) Given a 2-natural transformation $(X, x) : \mathbb{R} \Rightarrow \mathbb{S}$ for graded 2-superfunctors $\mathbb{R}, \mathbb{S} : \mathfrak{A} \rightarrow \mathfrak{B}$, there is a unique 2-natural transformation $(\tilde{X}, \tilde{x}) : \tilde{\mathbb{R}} \Rightarrow \tilde{\mathbb{S}}$ such that $\tilde{X}_\lambda = X_\lambda$ and $x_{\mu,\lambda} = \tilde{x}_{\mu,\lambda}\mathbb{J}$ for all $\lambda, \mu \in \text{ob } \mathfrak{A}$.

Proof. Since this is similar to the proof of Lemma 4.7, we just go briefly through the definition of $\tilde{\mathbb{R}}$ in (i) (assuming that \mathfrak{B} is strict). On objects, we take $\tilde{\mathbb{R}}\lambda := \mathbb{R}\lambda$. For $\lambda \in \text{ob } \mathfrak{B}$, let $\zeta_\lambda^a : \pi_\lambda^a \xrightarrow{\sim} \mathbb{1}_\lambda$ be defined as in the proof of Lemma 4.7. Also for $m \in \mathbb{Z}$ let $\sigma_\lambda^m : q_\lambda^m \xrightarrow{\sim} \mathbb{1}_\lambda$ be $(\sigma_\lambda)^m : (q_\lambda)^m \xrightarrow{\sim} \mathbb{1}_\lambda$ if $m \geq 0$ or $(\bar{\sigma}_\lambda)^{-m} : (q_\lambda^{-1})^{-m} \xrightarrow{\sim} \mathbb{1}_\lambda$ if $m \leq 0$. Then, for a 1-morphism $F : \lambda \rightarrow \mu$ in \mathfrak{A} , $m \in \mathbb{Z}$ and $a \in \mathbb{Z}/2$, we set $\tilde{\mathbb{R}}(Q^m \Pi^a F) := q_{\mathbb{R}\mu}^m \pi_{\mathbb{R}\mu}^a(\mathbb{R}F)$. Also, if $x : F \rightarrow G$ is a 2-morphism in \mathfrak{A} for $F, G : \lambda \rightarrow \mu$, we define $\tilde{\mathbb{R}}(x_{m,a}^n) : \tilde{\mathbb{R}}(\Pi^a F) \Rightarrow \tilde{\mathbb{R}}(\Pi^a G)$ to be the following composition:

$$q_{\mathbb{R}\mu}^m \pi_{\mathbb{R}\mu}^a(\mathbb{R}F) \xrightarrow{\sigma_{\mathbb{R}\mu}^m \zeta_{\mathbb{R}\mu}^a(\mathbb{R}F)} \mathbb{R}F \xrightarrow{\mathbb{R}x} \mathbb{R}G \xrightarrow{(\sigma_{\mathbb{R}\mu}^n \zeta_{\mathbb{R}\mu}^b)^{-1}(\mathbb{R}G)} q_{\mathbb{R}\mu}^n \pi_{\mathbb{R}\mu}^b(\mathbb{R}G).$$

We also need coherence maps \tilde{i} and \tilde{c} for $\tilde{\mathbb{R}}$, which are defined like in the proof of Lemma 4.7. In particular, $\tilde{c}_{Q^n \Pi^b G, Q^m \Pi^a F}$ is the following composition:

$$q_{\mathbb{R}\nu}^n \pi_{\mathbb{R}\nu}^b(\mathbb{R}G) q_{\mathbb{R}\mu}^m \pi_{\mathbb{R}\mu}^a(\mathbb{R}F) \longrightarrow q_{\mathbb{R}\nu}^n q_{\mathbb{R}\nu}^m \pi_{\mathbb{R}\nu}^b \pi_{\mathbb{R}\nu}^a(\mathbb{R}G)(\mathbb{R}F) \longrightarrow q_{\mathbb{R}\nu}^{m+n} \pi_{\mathbb{R}\nu}^{a+b} \mathbb{R}(GF),$$

where the first map is defined using the supernatural isomorphisms β and γ from Lemma 6.6(i)–(ii), and the second map is defined by collapsing powers of $q_{\mathbb{R}\nu}$ using the 2-isomorphisms from Lemma 6.6(iii), collapsing $\pi_{\mathbb{R}\nu} \pi_{\mathbb{R}\nu}$ using $-\xi$, and also using the given coherence map $c_{G,F} : (\mathbb{R}G)(\mathbb{R}F) \xrightarrow{\sim} \mathbb{R}(GF)$. \square

Using Lemma 6.11, one gets also the analog of Theorem 4.9: the functor $-_{q,\pi}$ from (6.2) is left 2-adjoint to the forgetful functor.

Next, we explain the graded analogs of Definitions 1.6 and 5.2, and extend the results of Section 5. The following is an efficient formulation of the general notion of a strict action of the group $\mathbb{Z} \oplus \mathbb{Z}/2$ on a \mathbb{k} -linear category.

Definition 6.12. (i) A (Q, Π) -category is a \mathbb{k} -linear category \mathcal{A} equipped with the following additional data: an endofunctor $\Pi : \mathcal{A} \rightarrow \mathcal{A}$ and a natural isomorphism $\xi : \Pi^2 \xrightarrow{\sim} I$ such that $\xi\Pi = \Pi\xi$ in $\text{Hom}(\Pi^3, \Pi)$; endofunctors $Q, Q^{-1} : \mathcal{A} \rightarrow \mathcal{A}$ and natural isomorphisms $\imath : Q^{-1}Q \xrightarrow{\sim} I, \jmath : QQ^{-1} \xrightarrow{\sim} I$ so that \imath^{-1} and \jmath define a unit and a counit making (Q, Q^{-1}) into an adjoint pair of auto-equivalences; a natural isomorphism $\beta_Q : \Pi Q \xrightarrow{\sim} Q\Pi$ such that $\xi Q\xi^{-1} = \beta_Q \Pi \circ \Pi \beta_Q$ in $\text{Hom}(\Pi^2 Q, Q\Pi^2)$.

(ii) Given (Q, Π) -categories \mathcal{A} and \mathcal{A}' , a (Q, Π) -functor $F : \mathcal{A} \rightarrow \mathcal{A}'$ is a \mathbb{k} -linear functor with the additional data of natural isomorphisms $\beta_F : \Pi' F \xrightarrow{\sim} F\Pi$ and $\gamma_F : Q' F \xrightarrow{\sim} FQ$ such that $\xi' F\xi^{-1} = \beta_F \Pi \circ \Pi' \beta_F$ in $\text{Hom}((\Pi')^2 F, F\Pi^2)$. For example, I, Π and Q are (Q, Π) -functors with $\beta_I := 1_\Pi, \beta_\Pi := -1_{\Pi^2}, \beta_Q$ as specified in (i), $\gamma_I := 1_Q, \gamma_\Pi := \beta_Q^{-1}$ and $\gamma_Q := 1_{Q^2}$.

(iii) Given (Q, Π) -functors $F, G : \mathcal{A} \rightarrow \mathcal{A}'$, a (Q, Π) -natural transformation is a natural transformation $x : F \Rightarrow G$ such that $x\Pi \circ \beta_F = \beta_G \circ \Pi' x$ and $xQ \circ \gamma_F = \gamma_G \circ Q' x$.

There is a 2-category $(Q, \Pi)\text{-}\mathfrak{Cat}$ consisting of (Q, Π) -categories, (Q, Π) -functors and (Q, Π) -natural transformations. We want to compare this to $(Q, \Pi)\text{-}\mathfrak{GS}\mathfrak{Cat}$, the 2-category of graded (Q, Π) -supercategories, graded superfunctors and homogeneous even supernatural transformations of degree zero. Like in (5.3), there is a strict 2-functor

$$\mathbb{E} : (Q, \Pi)\text{-}\mathfrak{GS}\mathfrak{Cat} \rightarrow (Q, \Pi)\text{-}\mathfrak{Cat} \quad (6.3)$$

sending a graded (Q, Π) -supercategory \mathcal{A} to the underlying category $\underline{\mathcal{A}}$, which is a (Q, Π) -category thanks to Corollary 6.7(i). It sends a graded superfunctor $F : \mathcal{A} \rightarrow \mathcal{B}$ to the restriction $\underline{F} : \underline{\mathcal{A}} \rightarrow \underline{\mathcal{B}}$, made into a (Q, Π) -functor as in Corollary 6.7(ii). It sends a homogeneous graded supernatural transformation $x : F \Rightarrow G$ of degree zero to $\underline{x} : \underline{F} \Rightarrow \underline{G}$ defined from $\underline{x}_\lambda := x_\lambda$, which is a (Q, Π) -natural transformation thanks to Corollary 6.7(iii).

Theorem 6.13. *The 2-functor \mathbb{E} from (6.3) is a 2-equivalence of 2-categories.*

Theorem 6.13 is proved in a similar way to Theorem 5.3. The key point of course is to define the appropriate strict 2-functor \mathbb{D} in the opposite direction. We just go briefly through the definition of this, since there are a few subtleties. So let \mathcal{A} be a (Q, Π) -category. Let $Q^n := Q \cdots Q$ (n times) if $n \geq 0$ or $Q^{-1} \cdots Q^{-1}$ ($-n$ times) if $n \leq 0$. Given any composition C of r of the functors Q and s of the functors Q^{-1} (in any order), there is an isomorphism $c : C \xrightarrow{\cong} Q^{r-s}$ defined by repeatedly applying $\mathbb{1}$ and \mathbb{j} to cancel $Q^{-1}Q$ - or QQ^{-1} -pairs. The isomorphism c is independent of the particular order chosen for these cancellations. In particular, we obtain in this way a canonical isomorphism $c_{m,n} : Q^m Q^n \xrightarrow{\cong} Q^{m+n}$ for any $m, n \in \mathbb{Z}$, and deduce that

$$c_{l,m+n} \circ Q^l c_{m,n} = c_{l+m,n} \circ c_{l,m} Q^n \quad (6.4)$$

in $\text{Hom}(Q^l Q^m Q^n, Q^{l+m+n})$. Next, let $F : \mathcal{A} \rightarrow \mathcal{A}'$ be a (Q, Π) -functor between two (Q, Π) -categories. For each $n \in \mathbb{Z}$, we define an isomorphism $\gamma_F^n : (Q')^n F \xrightarrow{\cong} F Q^n$ as follows: set $\gamma_F^0 := 1_F$; then for $n \geq 1$ recursively define

$$\begin{aligned} \gamma_F^n &:= \gamma_F^{(n-1)} Q \circ (Q')^{n-1} \gamma_F, \\ \gamma_F^{-n} &:= \gamma_F^{1-n} Q^{-1} \circ (Q')^{1-n} \mathbb{1}' Q^{-1} \circ (Q')^{-n} (\gamma_F)^{-1} Q^{-1} \circ (Q')^{-n} F \mathbb{j}^{-1}. \end{aligned}$$

One can show that

$$\gamma_F^{m+n} \circ c'_{m,n} F = F c_{m,n} \circ \gamma_F^m (Q^n) \circ (Q')^m \gamma_F^n \quad (6.5)$$

in $\text{Hom}((Q')^m (Q')^n F, F Q^{m+n})$. In particular, taking $F := \Pi : \mathcal{A} \rightarrow \mathcal{A}$, this gives us an isomorphism $\gamma_\Pi^n : Q^n \Pi \xrightarrow{\cong} \Pi Q^n$; let $\beta_{Q^n} : \Pi Q^n \xrightarrow{\cong} Q^n \Pi$ be its inverse. This together with $\gamma_{Q^n} := c_{n,1}^{-1} \circ c_{1,n} : Q Q^n \xrightarrow{\cong} Q^n Q$ makes Q^n into a (Q, Π) -functor, i.e. we have that

$$\xi Q^n = Q^n \xi \circ \beta_{Q^n} \Pi \circ \Pi \beta_{Q^n}. \quad (6.6)$$

We note also that

$$c_{m,n} \Pi \circ \beta_{Q^{m+n}} = Q^m \beta_{Q^n} \circ \beta_{Q^m} Q^n \circ \Pi c_{m,n}. \quad (6.7)$$

Now, for a (Q, Π) -category \mathcal{A} , we are ready to define the *associated graded (Q, Π) -supercategory* $\widehat{\mathcal{A}}$. It has the same objects as \mathcal{A} , and morphisms $\text{Hom}_{\widehat{\mathcal{A}}}(\lambda, \mu)_{m,a} := \text{Hom}_{\mathcal{A}}(\lambda, Q^m \Pi^a \mu)$. The composition $\widehat{g} \circ \widehat{f}$ of \widehat{f}, \widehat{g} coming from $f : \lambda \rightarrow Q^m \Pi^a \mu, g : \mu \rightarrow Q^n \Pi^b \nu$, respectively, is obtained from $(Q^m \Pi^a g) \circ f : \lambda \rightarrow Q^m \Pi^a Q^n \Pi^b \nu$ by first using β_{Q^n} to commute Q^n past Π^a if necessary, then using ξ and $c_{m,n}$ to simplify $Q^m Q^n \Pi^a \Pi^b \nu$ to $Q^{m+n} \Pi^{a+b} \nu$. The check that this is associative uses (6.4), (6.6)–(6.7) and the identity $\xi \Pi = \Pi \xi$. For a (Q, Π) -functor $F : \mathcal{A} \rightarrow \mathcal{A}'$, we get $\widehat{F} : \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{A}'}$ by composing $F f : F \lambda \rightarrow F Q^m \Pi^a \mu$ with the map $F Q^m \Pi^a \mu \rightarrow Q^m \Pi^a F \mu$ obtained using β_F and γ_F^n . The check that $\widehat{F}(\widehat{g} \circ \widehat{f}) = (\widehat{F} \widehat{g}) \circ (\widehat{F} \widehat{f})$ uses (6.5). In particular, since Π, Q and Q^{-1} are all (Q, Π) -functors, this gives us the functors $\widehat{\Pi}, \widehat{Q}$ and \widehat{Q}^{-1} needed to make $\widehat{\mathcal{C}}$ into a graded (Q, Π) -supercategory.

Definition 6.14. A (Q, Π) -2-category is a \mathbb{k} -linear 2-category \mathfrak{A} plus families $\pi = (\pi_\lambda : \lambda \rightarrow \lambda)$, $q = (q_\lambda : \lambda \rightarrow \lambda)$ and $q^{-1} = (q_\lambda^{-1} : \lambda \rightarrow \lambda)$ of 1-morphisms, families $\beta = (\beta_{\mu, \lambda} : \pi_\mu - \xrightarrow{\sim} -\pi_\lambda)$ and $\gamma = (\gamma_{\mu, \lambda} : q_\mu - \xrightarrow{\sim} -q_\lambda)$ of natural isomorphisms, and families $\xi = (\xi_\lambda : \pi_\lambda^2 \xrightarrow{\sim} \mathbb{1}_\lambda)$, $\mathfrak{1} = (\mathfrak{1}_\lambda : q_\lambda^{-1} q_\lambda \xrightarrow{\sim} \mathbb{1}_\lambda)$ and $\mathfrak{J} = (\mathfrak{J}_\lambda : q_\lambda q_\lambda^{-1} \xrightarrow{\sim} \mathbb{1}_\lambda)$ of 2-isomorphisms, such that the following hold (assuming strictness):

- (i) (π, β) and (q, γ) are objects in the Drinfeld center of \mathfrak{A} ;
- (ii) $(\beta_{\lambda, \lambda})_{\pi_\lambda} = -1_{\pi_\lambda^2}$, $(\gamma_{\lambda, \lambda})_{q_\lambda} = 1_{q_\lambda^2}$ and $(\gamma_{\lambda, \lambda})_{\pi_\lambda} = ((\beta_{\lambda, \lambda})_{q_\lambda})^{-1}$;
- (iii) $\xi_\mu F \xi_\lambda^{-1} = (\beta_{\mu, \lambda})_F \pi_\lambda \circ \pi_\mu (\beta_{\mu, \lambda})_F$ for all 1-morphisms $F : \lambda \rightarrow \mu$;
- (iv) $q_\lambda \mathfrak{1}_\lambda = \mathfrak{J}_\lambda q_\lambda$ and $\mathfrak{1}_\lambda q_\lambda^{-1} = q_\lambda^{-1} \mathfrak{J}_\lambda$.

The story here continues just as it did for Π -2-supercategories and Π -2-categories. For a graded (Q, Π) -2-supercategory \mathfrak{A} , its *underlying 2-category* $\underline{\mathfrak{A}}$, consisting of the same objects and 1-morphisms but just the even 2-morphisms of degree zero, is a (Q, Π) -2-category. Conversely, for a (Q, Π) -2-category \mathfrak{A} , there is a construction of its *associated graded (Q, Π) -2-supercategory* $\widehat{\mathfrak{A}}$, which we leave to the reader. The constructions $\mathfrak{A} \mapsto \underline{\mathfrak{A}}$ and $\mathfrak{A} \mapsto \widehat{\mathfrak{A}}$ are mutual inverses (up to isomorphism), so that (Q, Π) -2-categories and graded (Q, Π) -2-supercategories are equivalent notions. Again, we leave it to the reader to formalize this statement by writing down the appropriate analog of Theorem 5.5.

Finally, we discuss Grothendieck groups/rings in the graded setting:

- For a graded supercategory \mathcal{A} , let $\text{GSKar}(\mathcal{A})$ denote $\text{Kar}(\underline{\mathcal{A}}_{q, \pi})$, that is, the additive Karoubi envelope of the underlying category to the (Q, Π) -envelope of \mathcal{A} . This is a (Q, Π) -category that is additive and idempotent complete. Its Grothendieck group $K_0(\text{GSKar}(\mathcal{A}))$ is a $\mathbb{Z}^\pi[q, q^{-1}]$ -module with π acting as $[\Pi]$ and q acting as $[Q]$.
- For a graded 2-supercategory \mathfrak{A} , let $\text{GSKar}(\mathfrak{A}) := \text{Kar}(\underline{\mathfrak{A}}_{q, \pi})$ denote the additive Karoubi envelope of the (Q, Π) -2-category underlying the (Q, Π) -envelope of \mathfrak{A} . It is an additive, idempotent complete (Q, Π) -2-category. Its Grothendieck ring $K_0(\text{GSKar}(\mathfrak{A}))$ is naturally a locally unital ring with a distinguished system of mutually orthogonal idempotents $\{1_\lambda \mid \lambda \in \text{ob } \mathfrak{A}\}$. Moreover this ring is actually a $\mathbb{Z}^\pi[q, q^{-1}]$ -algebra with π and q acting on $1_\mu K_0(\text{GSKar}(\mathfrak{A})) 1_\lambda$ by left multiplication by $[\pi_\mu]$ and $[q_\mu]$ (equivalently, right multiplication by $[\pi_\lambda]$ and $[q_\lambda]$), respectively.

This construction will be used in particular in [BE] in order to pass from the Kac-Moody 2-supercategory $\mathfrak{U}(\mathfrak{g})$ introduced there to the modified integral form of the corresponding covering quantum group $U_{q, \pi}(\mathfrak{g})$ as in [C].

APPENDIX A. ODD TEMPERLEY-LIEB

In this appendix, we prove Theorem 1.18. Throughout we let $\varepsilon := -1$. If instead one takes $\varepsilon := +1$ and works in the purely even setting, replacing the quantum superalgebra $\dot{U}_q(\mathfrak{osp}_{1|2})$ with the quantum algebra $\dot{U}_q(\mathfrak{sl}_2)$, the arguments below may be used to recover the classical result for the Temperley-Lieb category $\mathcal{TL}(\delta)$. We assume some familiarity with the combinatorics from that story; e.g. see [W].

Let \mathbb{k} be a field of characteristic different from 2, and $q \in \mathbb{k}^\times$ be a scalar that is not a root of unity. For any $n \in \mathbb{Z}$, let $[n]$ denote $\frac{q^n - (\varepsilon q)^{-n}}{q - \varepsilon q^{-1}}$. For $n \in \mathbb{N}$, the element $[n]$ is the same as $[n]_{q, \varepsilon}$ from (1.10), and $[-n] = -\varepsilon^n [n]$. Also set $\delta := -[2] = -(q + \varepsilon q^{-1})$. Recall that $\mathcal{STL}(\delta)$ is the strict monoidal supercategory with one generating object \cdot and two odd generating morphisms \cup and \cap , subject

to the following relations:

$$\begin{array}{c} \cup \\ \cap \end{array} = |, \quad \begin{array}{c} \cup \\ \cap \end{array} = \varepsilon |, \quad \bigcirc = \delta.$$

We denote the n -fold tensor product of the generating object \cdot by n and its identity endomorphism by e_n .

Using the string calculus, any crossingless matching connecting m points on the bottom boundary and n points on the top boundary can be interpreted as a morphism $m \rightarrow n$ in $\mathcal{STL}(\delta)$. In view of the super interchange law, isotopic crossingless matchings produce the same morphism up to a sign. Moreover, to get a spanning set for $\text{Hom}_{\mathcal{STL}(\delta)}(m, n)$, one just has to pick a system of representatives for the isotopy classes of crossingless matchings. Our first claim is that any such spanning set is actually a basis for $\text{Hom}_{\mathcal{STL}(\delta)}(m, n)$. For example, this assertion implies that $\text{Hom}_{\mathcal{STL}(\delta)}(3, 3)$ is of dimension 5 (the third Catalan number) with basis

$$\begin{array}{c} \cup \\ \cup \end{array} |, \quad | \begin{array}{c} \cup \\ \cup \end{array}, \quad \begin{array}{c} \cup \\ \cup \\ \cup \end{array}, \quad \begin{array}{c} \cup \\ \cup \\ \cup \end{array}, \quad |||.$$

To prove it, we construct an explicit representation of $\mathcal{STL}(\delta)$.

Lemma A.1. *Let V be the vector superspace on basis v_1, v_{-1} , where v_1 is even and v_{-1} is odd. There is a monoidal superfunctor $G : \mathcal{STL}(\delta) \rightarrow \mathcal{SVec}$ with $G(n) = V^{\otimes n}$ and*

$$\begin{array}{ll} G(\cup) : \mathbb{k} \rightarrow V \otimes V, & 1 \mapsto v_{-1} \otimes v_1 - qv_1 \otimes v_{-1}; \\ G(\cap) : V \otimes V \rightarrow \mathbb{k}, & v_1 \otimes v_1 \mapsto 0, \quad v_1 \otimes v_{-1} \mapsto 1, \\ & v_{-1} \otimes v_1 \mapsto -\varepsilon q^{-1}, \quad v_{-1} \otimes v_{-1} \mapsto 0. \end{array}$$

Proof. Check the three relations. \square

Theorem A.2. *Any set of representatives for the isotopy classes of crossingless matchings from m points to n points defines a basis for $\text{Hom}_{\mathcal{STL}(\delta)}(m, n)$.*

Proof. We just need to prove linear independence. There is a linear map

$$\text{Hom}_{\mathcal{STL}(\delta)}(m, n) \rightarrow \text{Hom}_{\mathcal{STL}(\delta)}(m+n, 0), \quad f \mapsto c_n \circ (f \otimes e_n),$$

where $c_n \in \text{Hom}_{\mathcal{STL}(\delta)}(2n, 0)$ is the morphism defined by n nested caps. Using this, one reduces to proving the result in the special case that m is even and $n = 0$, i.e. our crossingless matchings consist of $m/2$ caps. Let S be a set of representatives for such matchings. For $s \in S$, let $\theta_s : V^{\otimes m} \rightarrow \mathbb{k}$ be the linear map obtained by applying the monoidal superfunctor G from Lemma A.1 to the morphism in $\mathcal{STL}(\delta)$ that is defined by s . It suffices to show that the linear maps $\{\theta_s \mid s \in S\}$ are linearly independent.

By writing $+1$ underneath the left hand vertex and -1 underneath the right hand vertex of each cap of $s \in S$ then reading off the resulting sequence, we obtain a function from S to the set of *Dyck sequences* (s_1, \dots, s_m) with $s_1, \dots, s_m \in \{\pm 1\}$ and $s_1 + \dots + s_k \geq 0$ for each $k = 1, \dots, m$. As s can be recovered uniquely (up to isotopy) from its Dyck sequence, the vectors $\{v_s := v_{s_1} \otimes \dots \otimes v_{s_m} \in V^{\otimes m} \mid s \in S\}$ are linearly independent. Finally, we observe that $\theta_s(v_s) = \pm 1$ and $\theta_s(v_t) = 0$ unless $t \leq s$, where \leq is the partial order defined by $s \leq t$ if and only if the corresponding Dyck sequences satisfy $s_1 + \dots + s_k \leq t_1 + \dots + t_k$ for each $k = 1, \dots, m$. The required linear independence follows. \square

Now we can prove Theorem 1.18:

Theorem A.3. *For δ as above, $\text{SKar}(\mathcal{STL}(\delta))$ is a semisimple Abelian category. Moreover, as a based ring, $K_0(\text{SKar}(\mathcal{STL}(\delta)))$ is isomorphic to the subring of $\mathbb{Z}^\pi[x, x^{-1}]$ with basis $\{[n+1]_{x,\pi}, \pi[n+1]_{x,\pi} \mid n \in \mathbb{N}\}$.*

Proof. We begin by defining super analogs of the Jones-Wenzl projectors

$$f_n = \begin{array}{|c|} \hline \cdots \\ \hline n \\ \hline \cdots \\ \hline \end{array} \in \text{End}_{\mathcal{STL}(\delta)}(n).$$

These are defined recursively by setting $f_0 := \mathbb{1}$ and

$$\begin{array}{|c|} \hline \cdots \\ \hline n+1 \\ \hline \cdots \\ \hline \end{array} := \begin{array}{|c|} \hline \cdots \\ \hline n \\ \hline \cdots \\ \hline \end{array} + \frac{[n]}{[n+1]} \begin{array}{|c|} \hline \cdots \\ \hline n \\ \hline \cdots \\ \hline n \\ \hline \cdots \\ \hline \end{array}.$$

Clearly, f_n is equal to e_n plus a linear combination of diagrams with at least one cup and cap. Hence, using Theorem A.2, f_n is non-zero. By (1.11), we have that $[n][2] = [n+1] + \varepsilon[n-1]$. Using this, an easy but crucial inductive calculation shows that

$$\begin{array}{|c|} \hline \cdots \\ \hline n \\ \hline \cdots \\ \hline \end{array} \begin{array}{|c|} \hline \cdots \\ \hline n \\ \hline \cdots \\ \hline \end{array} = -\frac{[n+1]}{[n]} \begin{array}{|c|} \hline \cdots \\ \hline n-1 \\ \hline \cdots \\ \hline \end{array},$$

and each f_n is an idempotent. Moreover, one gets zero if one vertically composes f_n on top (resp. bottom) with any diagram involving a cap (resp. a cup).

To prove the semisimplicity, we find it convenient to replace the supercategory $\mathcal{STL}(\delta)$ with the superalgebra

$$A := \bigoplus_{m,n \in \mathbb{N}} \text{Hom}_{\mathcal{STL}(\delta)}(m, n),$$

whose multiplication is induced by composition in $\mathcal{STL}(\delta)$. Note that A is a *locally unital superalgebra* with distinguished idempotents $\{e_n \mid n \in \mathbb{N}\}$. Moreover, it is *locally finite dimensional* in the sense that each $e_n A e_m$ is a finite-dimensional super-space. Consider the Π -supercategory $\mathcal{SMod}\text{-}A$ consisting of right A -supermodules V which are themselves locally unital in the sense that $V = \bigoplus_{n \in \mathbb{N}} V e_n$. Like in Example 1.17(i), there is an equivalence between $\text{SKar}(\mathcal{STL}(\delta))$ and the full subcategory of $\mathcal{SMod}\text{-}A$ consisting of all finitely generated projective supermodules. Thus, we are reduced to working in $\mathcal{SMod}\text{-}A$. Let $P(n) := f_n A$, which is a projective supermodule. Let L be any irreducible A -supermodule. Let $n \in \mathbb{N}$ be minimal such that $L e_n \neq 0$. The minimality of n ensures that any basis element of A with a cup in its diagram acts as zero on $L e_n$. We deduce that $\text{Hom}_A(P(n), L) \cong L f_n = L e_n \neq 0$, demonstrating that L is a quotient of $P(n)$ or $\Pi P(n)$. Moreover, $\text{End}_A(P(n)) = f_n A f_n \cong \mathbb{k}$, so $P(n)$ is indecomposable; equivalently, f_n is a primitive idempotent. Also for $m \neq n$, we have that $\text{Hom}_A(P(m), P(n)) = f_n A f_m = 0$. These observations together imply that every A -supermodule is completely reducible, and each irreducible A -supermodule is evenly isomorphic to a unique one of the supermodules $\{P(n), \Pi P(n) \mid n \in \mathbb{N}\}$, which are themselves irreducible.

The previous paragraph implies that $\text{SKar}(\mathcal{STL}(\delta))$ is a semisimple Abelian category. Moreover, we get a basis for $K_0(\text{SKar}(\mathcal{STL}(\delta)))$ by taking the isomorphism classes in $\text{SKar}(\mathcal{STL}(\delta))$ corresponding to the primitive idempotents $\{(f_n)_0^0, (f_n)_1^1 \mid n \in \mathbb{N}\}$. Thus, we can identify $K_0(\text{SKar}(\mathcal{STL}(\delta)))$ with the ring in the statement of the theorem using the correspondence $(f_n)_0^0 \leftrightarrow [n+1]_{x,\pi}$ and $(f_n)_1^1 \leftrightarrow \pi[n+1]_{x,\pi}$. To complete the proof of the theorem, it remains to check that the ring structures agree. Since $[n]_{x,\pi}[2]_{x,\pi} = [n+1]_{x,\pi} + \pi[n-1]_{x,\pi}$ by (1.11), we

must show that the idempotents $(f_{n-1})_0^{\bar{0}} \otimes (f_1)_0^{\bar{0}}$ and $(f_n)_0^{\bar{0}} + (f_{n-2})_1^{\bar{1}}$ are equivalent for each $n \geq 2$. We have that $(f_{n-1})_0^{\bar{0}} \otimes (f_1)_0^{\bar{0}} = (f_{n-1} \otimes f_1)_0^{\bar{0}} = (f_n)_0^{\bar{0}} + (g_n)_0^{\bar{0}}$ where

$$g_n := -\frac{[n-1]}{[n]} \begin{array}{c} \boxed{\dots} \\ \boxed{n-1} \\ \boxed{\dots} \\ \boxed{\dots} \\ \boxed{n-1} \\ \boxed{\dots} \end{array}.$$

Using the properties from the first paragraph of the proof, we have that $g_n \circ g_n = g_n$ and $g_n \circ f_n = f_n \circ g_n = 0$, so $(f_n)_0^{\bar{0}}$ and $(g_n)_0^{\bar{0}}$ are orthogonal idempotents. It just remains to observe that

$$u_n := -\frac{[n-1]}{[n]} \begin{array}{c} \boxed{\dots} \\ \boxed{n-1} \\ \boxed{\dots} \end{array}, \quad v_n := \begin{array}{c} \boxed{\dots} \\ \boxed{n-1} \\ \boxed{\dots} \end{array}$$

are odd morphisms in $\mathcal{STL}(\delta)$ such that $u_n \circ v_n = g_n$ and $v_n \circ u_n = f_{n-2}$. Hence, we get that $(v_n)_0^{\bar{1}} \circ (g_n)_0^{\bar{0}} \circ (u_n)_0^{\bar{0}} = (f_{n-2})_1^{\bar{1}}$, i.e. $(g_n)_0^{\bar{0}}$ is equivalent to $(f_{n-2})_1^{\bar{1}}$ in $\text{SKar}(\mathcal{STL}(\delta))$, as required. \square

To explain what is really going on here, assume finally that the ground field \mathbb{k} is of characteristic zero. Let $U = \dot{U}_q(\mathfrak{osp}_{1|2})$ be the locally unital superalgebra with homogeneous distinguished idempotents $\{1_n \mid n \in \mathbb{Z}\}$ and odd generators $E_n \in 1_{n+2}U1_n$ and $F_n \in 1_{n-2}U1_n$, subject to the relations

$$E_{n-2}F_n - \varepsilon F_{n+2}E_n = [n]1_n.$$

This is the idempotent form of the quantum supergroup $U_q(\mathfrak{osp}_{1|2})$ introduced in [CW]⁷. Let \mathcal{C} be the Π -supercategory of all finite-dimensional left U -supermodules V which are locally unital in the sense that $V = \bigoplus_{n \in \mathbb{Z}} 1_n V$. By [CW], the underlying Π -category $\underline{\mathcal{C}}$ is a semisimple Abelian category, and a complete set of pairwise inequivalent irreducible objects is given by $\{V(n), \Pi V(n) \mid n \in \mathbb{N}\}$, where $V(n)$ is defined as follows. It has a homogeneous basis $v_n, v_{n-2}, \dots, v_{-n}$ with $|v_i| = (n-i)/2 \pmod{2}$. We have that $1_i v_i = v_i$. The appropriate E 's and F 's act on the basis by the following scalars:

$$\begin{aligned} E &: v_n \xleftarrow{[n]} v_{n-2} \xleftarrow{[n-1]} \dots \xleftarrow{[2]} v_{2-n} \xleftarrow{[1]} v_{-n}, \\ F &: v_n \xrightarrow{[1]} v_{n-2} \xrightarrow{\varepsilon[2]} v_{n-4} \xrightarrow{[3]} \dots \xrightarrow{\varepsilon^{n-1}[n]} v_{-n}. \end{aligned}$$

For example: $(E_{n-2}F_n - \varepsilon F_{n+2}E_n)v_n = E_{n-2}v_{n-2} = [n]v_n$.

We wish next to make U into a Hopf superalgebra by introducing a comultiplication Δ and counit ε defined on generators by the following:

$$\Delta(1_n) = \sum_{a+b=n} 1_a \otimes 1_b, \quad \varepsilon(1_n) = \delta_{n,0}1,$$

$$\Delta(E_n) = \sum_{a+b=n} (E_a \otimes 1_b + q^{-a}1_a \otimes E_b), \quad \varepsilon(E_n) = 0,$$

$$\Delta(F_n) = \sum_{a+b=n} (\varepsilon^a 1_a \otimes F_b + q^b F_a \otimes 1_b), \quad \varepsilon(F_n) = 0.$$

⁷More precisely, our U is the idempotent form of the algebra from [CW] as defined in [C]. Also, we are using a different convention for quantum integers compared to [CW, C]: our q is the same as the parameter q^{-1} of [CW] or the parameter v^{-1} of [C].

However, some of these formulae involve infinite sums, so don't make sense yet: we need some completions! If $A = \bigoplus_{x,y \in X} 1_x A 1_y$ is any locally unital (super)algebra with distinguished idempotents indexed by some set X , we can form the completion \widehat{A} consisting of all elements $(a_{xy})_{x,y \in X} \in \prod_{x,y \in X} 1_x A 1_y$ such that for each x there are only finitely many y with $a_{xy} \neq 0$, and for each y there are only finitely many x with $a_{xy} = 0$. Clearly the multiplication on A extends to \widehat{A} to make it into a (super)algebra with $1 = \sum_{x \in X} 1_x$. Applying this construction to U , we get the completion \widehat{U} ; applying it to the superalgebra $U \otimes U$, which is locally unital with distinguished idempotents $\{1_m \otimes 1_n \mid m, n \in \mathbb{Z}\}$, we get $\widehat{U \otimes U}$; the triple tensor product $U \otimes U \otimes U$ may be completed similarly. Now the formulae above extend canonically to define superalgebra homomorphisms $\Delta : \widehat{U} \rightarrow \widehat{U \otimes U}$ and $\varepsilon : \widehat{U} \rightarrow \mathbb{k}$, satisfying completed versions of the usual coassociativity and counit axioms. This makes \widehat{U} into a Hopf superalgebra in a completed sense. (We remark there are several other possible choices of coproduct here; see [C, §2.4].)

Given $V, W \in \text{ob } \mathcal{C}$, the tensor product $V \otimes W$ is naturally a $U \otimes U$ -supermodule. Since it is finite dimensional, it is a $\widehat{U \otimes U}$ -supermodule too, hence using Δ we can view it as a U -supermodule. This makes \mathcal{C} into a monoidal Π -supercategory equipped with a *fiber functor* $\nu : \mathcal{C} \rightarrow \mathcal{SVec}$, namely, the obvious forgetful superfunctor. Setting $V := V(1)$, we also have the monoidal superfunctor $G : \mathcal{STL}(\delta) \rightarrow \mathcal{SVec}$ from Lemma A.1.

Theorem A.4. *There is a unique monoidal superfunctor $F : \mathcal{STL}(\delta) \rightarrow \mathcal{C}$ such that $G = \nu \circ F$:*

$$\begin{array}{ccc} \mathcal{STL}(\delta) & \xrightarrow{G} & \mathcal{SVec} \\ & \searrow F & \nearrow \nu \\ & \mathcal{C} & \end{array} .$$

Moreover, F induces a monoidal equivalence $\tilde{F} : \text{SKar}(\mathcal{STL}(\delta)) \rightarrow \underline{\mathcal{C}}$.

Proof. All of the superspaces $V^{\otimes n}$ are naturally objects of \mathcal{C} . Moreover, the linear maps defined in Lemma A.1 are U -supermodule homomorphisms. This proves the existence and uniqueness of F .

The proof of Theorem A.2 shows that F is faithful. Hence, so is the induced functor $\tilde{F} : \text{SKar}(\mathcal{STL}(\delta)) \rightarrow \underline{\mathcal{C}}$. Both $\text{SKar}(\mathcal{STL}(\delta))$ and $\underline{\mathcal{C}}$ are semisimple Abelian. So, to prove that \tilde{F} is an equivalence, we just need to show that the induced \mathbb{Z}^π -algebra homomorphism $K_0(\text{SKar}(\mathcal{STL}(\delta))) \rightarrow K_0(\underline{\mathcal{C}})$ sends the canonical basis coming from the classes of irreducibles in $\text{SKar}(\mathcal{STL}(\delta))$ to that of $\underline{\mathcal{C}}$.

In view of Theorem A.3, we may identify $K_0(\text{SKar}(\mathcal{STL}(\delta)))$ with the subring of $\mathbb{Z}^\pi[x, x^{-1}]$ having canonical basis $\{[n+1]_{x,\pi}, \pi[n+1]_{x,\pi} \mid n \in \mathbb{N}\}$. Note this is generated as a \mathbb{Z}^π -algebra just by $[2]_{x,\pi}$, which corresponds to the object 1 in $\mathcal{STL}(\delta)$. To understand $K_0(\underline{\mathcal{C}})$, consider the map sending a finite-dimensional U -supermodule M to its *supercharacter*

$$\text{SCh } M := \sum_{n \in \mathbb{Z}} (\dim(1_n M)_0 x^n + \dim(1_n M)_1 \pi x^n) \in \mathbb{Z}^\pi[x, x^{-1}].$$

We have that $\text{SCh } V(n) = [n+1]_{x,\pi}$. Hence, SCh induces a \mathbb{Z}^π -algebra isomorphism between $K_0(\underline{\mathcal{C}})$ and the same based subring of $\mathbb{Z}^\pi[x, x^{-1}]$ as $K_0(\text{SKar}(\mathcal{STL}(\delta)))$. Moreover, the generator $[2]_{x,\pi}$ is the supercharacter of V . It remains to observe that $F(1) = V$. \square

Corollary A.5. *The irreducible U -supermodule $V(n)$ is isomorphic to the image of the idempotent $F(f_n) \in \text{End}_U(V^{\otimes n})$, where f_n is the Jones-Wenzl projector from the proof of Theorem A.3.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403, USA
E-mail address: brundan@uoregon.edu, apellis@gmail.com