Transformation Groups, Vol. ?, No. ?, ??, pp. 1–?? ©Birkhäuser Boston (??)

COHOMOLOGY OF SPALTENSTEIN VARIETIES

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To Professor Tonny Springer on the occasion of his eighty-fifth birthday

Abstract. We give a presentation for the cohomology algebra of the Spaltenstein variety of all partial flags annihilated by a fixed nilpotent matrix, generalizing the description of the cohomology algebra of the Springer fiber found by De Concini, Procesi and Tanisaki.

1. Introduction

Throughout the article, we fix integers $n, d \geq 0$ and write $\Lambda(n, d)$ for the set of all n-part *compositions* $\mu = (\mu_1, \ldots, \mu_n)$ of d, so the μ_i's are non-negative integers summing to d. Let $\Lambda^+(n, d) \subseteq \Lambda(n, d)$ denote the n-part *partitions* of d, i.e. the $\lambda \in \Lambda(n, d)$ satisfying $\lambda_1 \geq \cdots \geq \lambda_n$. For $\lambda \in \Lambda^+(n, d)$, we write λ^T for the transpose partition (which may have more than n non-zero parts). Let X be the complex projective variety of flags $(U_0, \ldots, U_d)$ in $\mathbb{C}^d$, so

$$\{0\} = U_0 < U_1 < \cdots < U_d = \mathbb{C}^d, \qquad \dim U_i/U_{i-1} = 1.$$

By a classical result of Borel, the cohomology algebra $H^*(X, \mathbb{C})$ is isomorphic to the *coinvariant algebra*, which is the quotient P/I of the polynomial algebra $P := \mathbb{C}[x_1, \ldots, x_d]$, graded so that each x_i is in degree 2, by the ideal I generated by the homogeneous symmetric polynomials of positive degree. To fix a specific isomorphism, let $\widetilde{U}_i$ be the sub-bundle of the trivial vector bundle $\mathbb{C}^d \times X \to X$ having fiber U_i over the point $(U_0, \ldots, U_d) \in X$. Then there is a unique graded algebra isomorphism

$$\varphi : C \xrightarrow{\sim} H^*(X, \mathbb{C})$$

with $\varphi(x_i) = -c_1(\widetilde{U}_i/\widetilde{U}_{i-1}) \in H^2(X, \mathbb{C})$ for each $i = 1, \ldots, d$; see e.g. [F, §10.2, Proposition 3].

Associated to $\lambda \in \Lambda^+(n, d)$, we have the *Springer fiber* X^λ, which is the closed subvariety of X consisting of all flags annihilated by the nilpotent matrix x^λ of Jordan type λ^T, so

$$X^\lambda = \{(U_0, \ldots, U_d) \in X \mid x^\lambda U_i \subseteq U_{i-1} \text{ for each } i = 1, \ldots, d\}.$$

*Supported by NSF grant DMS-0654147
**Supported by NSF grant DMS-0602263
Received December 16, 2010. Accepted April 25, 2011.

The pull-back homomorphism $i^* : H^*(X, \mathbb{C}) \to H^*(X^\lambda, \mathbb{C})$ arising from the inclusion $i : X^\lambda \hookrightarrow X$ is surjective. In [DCP], De Concini and Procesi computed generators for the ideal of C that maps to $\ker i^*$ under the isomorphism φ, thus obtaining an explicit presentation for $H^*(X^\lambda, \mathbb{C})$. Soon after that, a slightly simplified description was given by Tanisaki in [T], as follows. Let $C^\lambda := P/I^\lambda$ where I^λ is the ideal generated by the elementary symmetric functions

$$\left\{ e_r(x_{i_1}, \dots, x_{i_m}) \;\middle|\; \begin{array}{l} m \geq 1, 1 \leq i_1 < \cdots < i_m \leq d, \\ r > m - \lambda_{d-m+1} - \cdots - \lambda_n \end{array} \right\}.$$

Then there is a unique isomorphism $\bar\varphi : C^\lambda \xrightarrow{\sim} H^*(X^\lambda, \mathbb{C})$ making the diagram

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & H^*(X, \mathbb{C}) \\ \downarrow & & \downarrow{\scriptstyle i^*} \\ C^\lambda & \xrightarrow{\bar\varphi} & H^*(X^\lambda, \mathbb{C}) \end{array} \tag{1.1}$$

commute, where the left hand map is the canonical quotient map which makes sense because $I \subseteq I^\lambda$.

The symmetric group S_d acts on P, hence also on the quotients C and C^λ, by algebra automorphisms so that $w \cdot x_i = x_{w(i)}$ for $w \in S_d$. Using the isomorphism φ, we get induced an S_d-action on $H^*(X, \mathbb{C})$, which has (at least) two geometric definitions: the classical one as in [J, §13.1], and another more sophisticated construction originating in the remarkable work of Springer [S1, S2] (see also [J, §13.6]). Springer's construction yields also an action of S_d on $H^*(X^\lambda, \mathbb{C})$, uniquely determined by the property that the surjection i^* is S_d-equivariant (see [HS, Theorem 1.1] or [J, §13.13]). This plays an essential role in the derivation of the De Concini-Procesi-Tanisaki presentation. It is also one of the reasons the cohomology algebras $H^*(X^\lambda, \mathbb{C})$ are so interesting: the top degree cohomology is isomorphic as an S_d-module to the irreducible representation $S(\lambda^T)$ indexed by the transpose partition λ^T (see [HS, Proposition 2.7] or [J, §13.16]).

The main goal of this article is to prove a parabolic analogue of (1.1). Fix $\lambda \in \Lambda^+(n, d)$ as before and also $\mu \in \Lambda(n, d)$. We are going to replace the flag variety X by the *partial flag variety* X_μ consisting of flags $(V_0, \dots, V_n)$ of type μ in $\mathbb{C}^d$, so

$$\{0\} = V_0 \leq V_1 \leq \cdots \leq V_n = \mathbb{C}^d, \qquad \dim V_i/V_{i-1} = \mu_i,$$

and to replace the Springer fiber X^λ by the *Spaltenstein variety*

$$X^\lambda_\mu = \{(V_0, \dots, V_n) \in X_\mu \mid x^\lambda V_i \subseteq V_{i-1} \text{ for each } i = 1, \dots, n\}.$$

Our main result gives an explicit presentation for the cohomology algebra of the Spaltenstein variety.

To formulate this, we must first recall the classical description of the cohomology algebra $H^*(X_\mu, \mathbb{C})$. Let $P_\mu := P^{S_\mu\text{-inv}}$ be the subalgebra of P consisting of all S_μ-invariants, where S_μ denotes the parabolic subgroup $S_{\mu_1} \times \cdots \times S_{\mu_n}$ of S_d. For $1 \leq i_1 < \cdots < i_m \leq n$ and $r \geq 1$, we let $e_r(\mu; i_1, \dots, i_m)$ and $h_r(\mu; i_1, \dots, i_m)$

denote the rth elementary and complete symmetric functions in the variables $X_{i_1} \cup \cdots \cup X_{i_m}$, where

$$X_j = \{x_k \mid \mu_1 + \cdots + \mu_{j-1} + 1 \leq k \leq \mu_1 + \cdots + \mu_j\}.$$

We interpret $e_r(\mu; i_1, \dots, i_m)$ and $h_r(\mu; i_1, \dots, i_m)$ as 1 if $r = 0$ and as 0 if $r < 0$. Note also that

$$e_r(\mu; i_1, \dots, i_m) = \sum_{r_1 + \cdots + r_m = r} e_{r_1}(\mu; i_1) \cdots e_{r_m}(\mu; i_m), \tag{1.2}$$

$$h_r(\mu; i_1, \dots, i_m) = \sum_{r_1 + \cdots + r_m = r} h_{r_1}(\mu; i_1) \cdots h_{r_m}(\mu; i_m). \tag{1.3}$$

The algebra P_μ is freely generated either by $\{e_r(\mu; i) \mid 1 \leq i \leq n, 1 \leq r \leq \mu_i\}$ or by $\{h_r(\mu; i) \mid 1 \leq i \leq n, 1 \leq r \leq \mu_i\}$. Let I_μ be the ideal of P_μ generated by homogeneous symmetric polynomials of positive degree. Set $C_\mu := P_\mu / I_\mu$. Then there is a unique isomorphism

$$\psi : C_\mu \xrightarrow{\sim} H^*(X_\mu, \mathbb{C})$$

with $\psi(e_r(\mu; i)) = (-1)^r c_r(\widetilde{V}_i / \widetilde{V}_{i-1})$ for each $i = 1, \dots, n$ and $r > 0$, where $\widetilde{V}_i$ denotes the sub-bundle of the trivial vector bundle $\mathbb{C}^d \times X_\mu \to X_\mu$ with fiber V_i over $(V_0, \dots, V_n) \in X_\mu$.

Let $C_\mu^\lambda := P_\mu / I_\mu^\lambda$ where I_μ^λ is the ideal of P_μ generated by the elements

$$\left\{ h_r(\mu; i_1, \dots, i_m) \;\middle|\; \begin{array}{l} m \geq 1,\ 1 \leq i_1 < \cdots < i_m \leq n, \\ r > \lambda_1 + \cdots + \lambda_m - \mu_{i_1} - \cdots - \mu_{i_m} \end{array} \right\}. \tag{1.4}$$

Equivalently (see [B, Lemma 2.2]) I_μ^λ is generated by

$$\left\{ e_r(\mu; i_1, \dots, i_m) \;\middle|\; \begin{array}{l} m \geq 1,\ 1 \leq i_1 < \cdots < i_m \leq n, \\ r > \mu_{i_1} + \cdots + \mu_{i_m} - \lambda_{l+1} - \cdots - \lambda_n \\ \text{where } l := \#\{i \mid \mu_i > 0, i \neq i_1, \dots, i_m\} \end{array} \right\}, \tag{1.5}$$

from which it is easy to see that $I_\mu^\lambda = I^\lambda$, hence $C_\mu^\lambda = C^\lambda$, if μ is regular (all parts ≤ 1). Finally let $j : X_\mu^\lambda \hookrightarrow X_\mu$ be the inclusion, and note that the pull-back $j^* : H^*(X_\mu, \mathbb{C}) \to H^*(X_\mu^\lambda, \mathbb{C})$ is surjective; see §2.

Theorem 1.1. *There is a unique isomorphism* $\bar{\psi} : C_\mu^\lambda \xrightarrow{\sim} H^*(X_\mu^\lambda, \mathbb{C})$ *making the diagram*

$$\begin{array}{ccc} C_\mu & \xrightarrow{\psi} & H^*(X_\mu, \mathbb{C}) \\ \downarrow & & \downarrow j^* \\ C_\mu^\lambda & \xrightarrow{\bar{\psi}} & H^*(X_\mu^\lambda, \mathbb{C}) \end{array} \tag{1.6}$$

commute, where the left hand map is the canonical quotient map which makes sense because $I_\mu \subseteq I_\mu^\lambda$.

The precise form of the relations (1.4)–(1.5) was originally worked out in [B], which is concerned with the centers of integral blocks of parabolic category $\mathcal{O}$ for the general linear Lie algebra. These centers provide another natural occurrence of the algebras $C_\mu^\lambda \cong H^*(X_\mu^\lambda, \mathbb{C})$; see also [St, Theorem 4.1.1] (which treats regular μ) and [BLPPW, Remark 9.10] (for all μ). The latter work shows moreover that the *equivariant* cohomology of X_μ^λ is isomorphic to the center of the universal deformation of the corresponding block of parabolic category $\mathcal{O}$, and extends the duality between equivariant cohomology rings of partial flag varieties and Springer fibers discovered by Goresky and MacPherson [GM] to Spaltenstein varieties.

We next formulate a result which plays an essential role in the proof of Theorem 1.1. Let $p : X \to X_\mu$ be the projection sending $(U_0, \dots, U_d) \in X$ to $(V_0, \dots, V_n) \in X_\mu$ where $V_i = U_{\mu_1 + \cdots + \mu_i}$ for each i. It is classical that the pull-back p^* defines a graded algebra isomorphism between $H^*(X_\mu, \mathbb{C})$ and $H^*(X, \mathbb{C})^{S_\mu\text{-inv}}$. Moreover the diagram

$$\begin{array}{ccc} C_\mu & \xrightarrow{\sim} & C^{S_\mu\text{-inv}} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ H^*(X_\mu, \mathbb{C}) & \xrightarrow{p^*} & H^*(X, \mathbb{C})^{S_\mu\text{-inv}} \end{array} \tag{1.7}$$

commutes, where the top map is induced by the inclusion $P_\mu \hookrightarrow P$. Using Poincaré duality, it also makes sense to consider the push-forward p_* as a map in cohomology. Let

$$d_\mu := \dim X - \dim X_\mu = \tfrac{1}{2} \sum_{i=1}^{n} \mu_i(\mu_i - 1).$$

Then p_* restricts to give an $H^*(X_\mu, \mathbb{C})$-module isomorphism between the space $H^*(X, \mathbb{C})^{S_\mu\text{-anti}}$ of S_μ-anti-invariants in $H^*(X, \mathbb{C})$ and the module $H^*(X_\mu, \mathbb{C})$. Thus, there is a commuting diagram

$$\begin{array}{ccc} C^{S_\mu\text{-anti}} & \xrightarrow{\sim} & C_\mu[-2d_\mu] \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ H^*(X, \mathbb{C})^{S_\mu\text{-anti}} & \xrightarrow{p_*} & H^*(X_\mu, \mathbb{C})[-2d_\mu], \end{array} \tag{1.8}$$

where for a graded vector space M we write $M[i]$ for the graded vector space obtained by shifting the grading down by i, i.e. $M[i]_j = M_{i+j}$. Viewed as a C_μ-module via the isomorphism $C_\mu \cong C^{S_\mu\text{-inv}}$ from (1.7), $C^{S_\mu\text{-anti}}$ is free of rank one generated by the element

$$\varepsilon_\mu := \frac{1}{|S_\mu|} \sum_{\substack{1 \le i < j \le d, \\ \text{same } S_\mu\text{-orbit}}} (x_i - x_j).$$

The isomorphism at the top of (1.8) sends $x\varepsilon_\mu \mapsto x$ for each $x \in C_\mu$; see e.g. [B, Lemma 3.2].

Theorem 1.2. *There is a unique homogeneous linear map $\bar{p}_*$ of degree zero making the following diagram commute:*

$$\begin{array}{ccc} H^*(X,\mathbb{C}) & \xrightarrow{p_*} & H^*(X_\mu,\mathbb{C})[-2d_\mu] \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ H^*(X^\lambda,\mathbb{C}) & \xrightarrow{\bar{p}_*} & H^*(X^\lambda_\mu,\mathbb{C})[-2d_\mu]. \end{array} \tag{1.9}$$

Moreover the restriction of $\bar{p}_*$ is an isomorphism of graded vector spaces $\bar{p}_* : H^*(X^\lambda,\mathbb{C})^{S_\mu\text{-anti}} \xrightarrow{\sim} H^*(X^\lambda_\mu,\mathbb{C})[-2d_\mu]$.

The existence of an isomorphism $H^*(X^\lambda,\mathbb{C})^{S_\mu\text{-anti}} \cong H^*(X^\lambda_\mu,\mathbb{C})[-2d_\mu]$ was established already by Borho and Macpherson [BM2, Corollary 3.6(b)]. The point of Theorem 1.2 is to make this isomorphism canonical; see §4. Granted Theorem 1.2, one possible approach to the proof of Theorem 1.1 is sketched in [B, Remark 4.6]. The proof of Theorem 1.1 given in §5 below is quite different and gives a more natural explanation of the relations; it is a generalization of Tanisaki's original argument in [T]. In the course of the proof, we also obtain a homogeneous algebraic basis for $H^*(X^\lambda_\mu,\mathbb{C})$ indexed by certain λ-tableaux of type μ, which is of independent interest; see §§2–3.

We point out finally that the algebras $H^*(X^\lambda_\mu,\mathbb{C})$ arise in Ginzburg's construction of analogues of the Springer representations for the general linear Lie algebra; see [G1, BG]. In particular, there is a natural way to define an action of $\mathfrak{gl}_n(\mathbb{C})$ on the direct sum of the $H^*(X^\lambda_\mu,\mathbb{C})$'s for all $\mu \in \Lambda(n,d)$, so that the top cohomology

$$\bigoplus_{\mu\in\Lambda(n,d)} H^{2d_\lambda-2d_\mu}(X^\lambda_\mu,\mathbb{C})$$

is irreducible of highest weight λ. Following the reformulation by Braverman and Gaitsgory [BG] (see also [G2, §7]), this action can be constructed by applying the signed Schur functor $V_{\text{sgn}}^{\otimes d}\otimes_{\mathbb{C}S_d}?$ to the S_d-module $H^*(X^\lambda,\mathbb{C})$. In more detail, let $V_{\text{sgn}}^{\otimes d}$ denote the dth tensor power of the natural $\mathfrak{gl}_n(\mathbb{C})$-module viewed as a right $\mathbb{C}S_d$-module so that $w \in S_d$ acts by

$$(v_1 \otimes \cdots \otimes v_d)w = \text{sgn}(w) v_{w(1)} \otimes \cdots \otimes v_{w(d)}.$$

For $\mu \in \Lambda(n,d)$, the μ-weight space of $V_{\text{sgn}}^{\otimes d} \otimes_{\mathbb{C}S_d} H^*(X^\lambda,\mathbb{C})$ is canonically isomorphic to $H^*(X^\lambda,\mathbb{C})^{S_\mu\text{-anti}}$; see [B, Lemma 3.1]. Using also Theorem 1.2, we get a canonical vector space isomorphism

$$V_{\text{sgn}}^{\otimes d} \otimes_{\mathbb{C}S_d} H^*(X^\lambda,\mathbb{C}) \cong \bigoplus_{\mu\in\Lambda(n,d)} H^*(X^\lambda_\mu,\mathbb{C}), \tag{1.10}$$

hence can transport the $\mathfrak{gl}_n(\mathbb{C})$-module structure from the left to the right hand space. This yields the desired action. Using the presentation for the algebras $H^*(X^\lambda_\mu,\mathbb{C})$ from Theorem 1.1, it is possible to give a purely algebraic construction (no cohomology) of these representations, in the same spirit as the approach of Garsia and Procesi to the type A Springer representations in [GP]. This is pursued further in [B, §4], where one can find explicit algebraic formulae for the actions of the Chevalley generators of $\mathfrak{gl}_n(\mathbb{C})$ on the space on the right hand side of (1.10).

2. Affine paving

By an *affine paving* of a variety M, we mean a partition of M into disjoint subsets M_i indexed by some finite poset $(I, \leq)$, such that the following hold for all $i \in I$:

- $\bigcup_{j \leq i} M_j$ is closed in M;
- M_i (which is automatically locally closed) is isomorphic as a variety to $\mathbb{A}^{d_i}$ for some $d_i \geq 0$.

Suppose we are given $\lambda \in \Lambda^+(n,d)$ and $\mu \in \Lambda(n,d)$. The Spaltenstein variety X^λ_μ from the introduction was defined and studied originally in [Spa]. In that paper, Spaltenstein constructed an explicit affine paving of the Springer fiber X^λ, and deduced from that a parametrization of the irreducible components of X^λ_μ. The goal in this section is to extend Spaltenstein's argument slightly to produce an affine paving of X^λ_μ itself; actually we explain the dual construction which produces a more convenient parametrization from a combinatorial point of view. This is a widely known piece of folk-lore, but still we could not find it explicitly written in the literature.

We begin with a few more conventions regarding partitions. We draw the Young diagram of a partition λ in the usual English way as in [M]. The sum of the parts of λ is $|\lambda|$, and $h(\lambda)$ denotes the *height* of λ, that is, the number of non-zero parts. We write $\mathscr{P}_k$ for the set of all partitions of height at most k, and $\mathscr{P}_{k,l}$ for the set of all partitions fitting into a $k \times l$-rectangle, i.e. $h(\lambda) \leq k$ and $\lambda_1 \leq l$. For partitions λ and μ, we write $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i. Also $\leq$ denotes the usual dominance ordering. For $\mu \in \Lambda(n,d)$, we let $\mu^+ \in \Lambda^+(n,d)$ be the unique partition obtained from μ by rearranging the parts in weakly decreasing order.

Given $0 \leq k \leq d$, let $\mathrm{Gr}_{k,d}$ be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^d$. To a partition $\gamma \in \mathscr{P}_k$, we associate its *column sequence* $(c_1, \dots, c_k)$ defined from

$$\gamma_{k+1-i} = c_i - i. \tag{2.1}$$

This gives a bijection between $\mathscr{P}_{k,d-k}$ and the set of all sequences $(c_1, \dots, c_k)$ with $1 \leq c_1 < \cdots < c_k \leq d$. Given a fixed basis $f_1, \dots, f_d$ for $\mathbb{C}^d$ and $\gamma \in \mathscr{P}_{k,d-k}$, we have the *Schubert variety*

$$Y_\gamma := \{U \in \mathrm{Gr}_{k,d} \mid \dim U \cap \langle f_1, \dots, f_{c_i} \rangle \geq i \text{ for } i = 1, \dots, k\}, \tag{2.2}$$

where $(c_1, \dots, c_k)$ is the column sequence associated to γ; see e.g. [F, §9.4]. This is an irreducible closed subvariety of dimension $|\gamma|$. Also $Y_\gamma \subseteq Y_{\gamma'}$ if and only if $\gamma \subseteq \gamma'$. The *Schubert cells* $Y^\circ_\gamma := Y_\gamma \setminus \bigcup_{\gamma' \subsetneq \gamma} Y_{\gamma'}$ give an affine paving of $\mathrm{Gr}_{k,d}$ indexed by the poset $(\mathscr{P}_{k,d-k}, \subseteq)$. In terms of coordinates, every $U \in Y^\circ_\gamma$ can be represented as the span of the vectors

$$f_{c_i} + \sum_{\substack{1 \leq j \leq c_i \\ j \neq c_1, \dots, c_i}} a_{i,j} f_j \qquad (i = 1, \dots, k) \tag{2.3}$$

for unique $a_{i,j} \in \mathbb{C}$. In particular, $Y^\circ_\gamma \cong \mathbb{A}^{|\gamma|}$.

In this section, we fix the basis $f_1, \ldots, f_d$ as follows. Let $e_1, \ldots, e_d$ be the standard basis for $\mathbb{C}^d$. If we identify the basis vectors $e_1, \ldots, e_d$ with the boxes of the Young diagram of λ working up columns starting from the first (leftmost) column, then the nilpotent matrix x^λ of Jordan type λ^T is the endomorphism of $\mathbb{C}^d$ sending a basis vector to the one immediately below it in the diagram, or to zero if it is at the bottom of its column. Then let $f_1, \ldots, f_d$ be the basis for $\mathbb{C}^d$ obtained by reading the boxes of the Young diagram in order along rows starting from the first (top) row. For example:

$$\lambda = (4,3,2) \qquad \begin{array}{|c|c|c|c|}\hline e_3 & e_6 & e_8 & e_9 \\ \hline e_2 & e_5 & e_7 \\ \cline{1-3} e_1 & e_4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline f_1 & f_2 & f_3 & f_4 \\ \hline f_5 & f_6 & f_7 \\ \cline{1-3} f_8 & f_9 \\ \cline{1-2}\end{array} \qquad x^\lambda = \downarrow \tag{2.4}$$

Also let $(.,.)$ be the symmetric bilinear form on $\mathbb{C}^d$ such that $(f_i, f_j) = \delta_{i,j}$. The following proposition provides the key induction step.

Proposition 2.1. *Suppose $n \geq 1$ and that $k := \mu_n, s := \lambda_1$ and $t := \lambda_n$ satisfy $s \geq k \geq t$. Let $\pi : X_\mu \to \mathrm{Gr}_{k,d}$ be the morphism $(V_0, \ldots, V_n) \mapsto V_{n-1}^\perp$. Let $\overline{\mu} \in \Lambda(n-1, d-k)$ be obtained from μ by forgetting the last part μ_n, let $\beta := ((s-k)^{k-t}) \in \mathscr{P}_{k,d-k}$, and take any $\gamma \in \mathscr{P}_{k,d-k}$ with column sequence $(c_1, \ldots, c_k)$.*

(i) *The restriction of π defines a morphism $\bar{\pi} : X_\mu^\lambda \to Y_\beta$.*

(ii) *There is an isomorphism of varieties $f_\gamma : \pi^{-1}(Y_\gamma^\circ) \xrightarrow{\sim} Y_\gamma^\circ \times X_{\overline{\mu}}$.*

Assume in addition that $\gamma \subseteq \beta$. Let $\overline{\lambda} \in \Lambda^+(n-1, d-k)$ be the partition whose Young diagram is obtained by removing one box from the bottom of each of the columns numbered $c_1 = 1, \ldots, c_t = t, c_{t+1}, \ldots, c_k$ in the Young diagram of λ, then re-ordering the columns to get a proper partition shape.

(iii) *For $(V_0, \ldots, V_n) \in \pi^{-1}(Y_\gamma^\circ)$, we have $x^\lambda(V_{n-1}) \subseteq x^\lambda(V_n) \subseteq V_{n-1}$, and the restriction of x^λ to V_{n-1} is of Jordan type $(\overline{\lambda})^T$.*

(iv) *The isomorphism f_γ can be chosen so that it restricts to an isomorphism $\bar{f}_\gamma : \bar{\pi}^{-1}(Y_\gamma^\circ) \xrightarrow{\sim} Y_\gamma^\circ \times X_{\overline{\mu}}^{\overline{\lambda}}$.*

Proof. Note that $Y_\beta = \{U \in \mathrm{Gr}_{k,d} \mid \langle f_1, \ldots, f_t \rangle \subseteq U \subseteq \langle f_1, \ldots, f_s \rangle\}$. Take $(V_0, \ldots, V_n) \in X_\mu^\lambda$. Then $x^\lambda(V_n) \subseteq V_{n-1}$ and $(x^\lambda)^{n-1}(V_{n-1}) = \{0\}$, or equivalently, $\langle f_1, \ldots, f_t \rangle = (\ker (x^\lambda)^{n-1})^\perp \subseteq V_{n-1}^\perp \subseteq (\mathrm{im}\, x^\lambda)^\perp = \langle f_1, \ldots, f_s \rangle$. This shows $\pi(X_\mu^\lambda) \subseteq Y_\beta$, establishing (i).

Now we'll prove (ii), (iii) and (iv) all under the assumption that $\gamma \subseteq \beta$, noting that (ii) for more general γ is actually a particular case of (iv) on replacing λ by the partition of d with just one non-zero part.

The subspace $\langle f_{c_1}, \ldots, f_{c_k} \rangle$ belongs to Y_γ°. Take $U \in Y_\gamma^\circ$ represented by the coordinates $(a_{i,j})$ according to (2.3). The non-zero vectors of the form $(x^\lambda)^i(f_j)$ for $i \geq 0$ and $1 \leq j \leq s$ form a basis for $\mathbb{C}^d$. Hence there exists a unique matrix $g(U) \in GL_d(\mathbb{C})$ such that

- $g(U)(f_{c_i}) = f_{c_i} + \sum_{j \in \{1,\ldots,c_i\} \setminus \{c_1,\ldots,c_i\}} a_{i,j} f_j$ for each $i = 1, \ldots, k$;
- $g(U)(f_j) = f_j$ for each $j \in \{1, \ldots, s\} \setminus \{c_1, \ldots, c_k\}$;
- $g(U)(x^\lambda(f_j)) = x^\lambda(g(U)(f_j))$ for any $1 \leq j \leq d$ such that $x^\lambda(f_j) \neq 0$.

The transpose matrix $(x^\lambda)^T$ annihilates $f_1, \dots, f_s$, and $(x^\lambda)^T(x^\lambda(f_j)) = f_j$ whenever $x^\lambda(f_j) \neq 0$. Using this we see that $g(U)$ commutes with $(x^\lambda)^T$, or equivalently, $g(U)^T$ commutes with x^λ. Moreover $g(U)(\langle f_{c_1}, \dots, f_{c_k}\rangle) = U$, hence $g(U)^T(U^\perp) = \langle f_{c_1}, \dots, f_{c_k}\rangle^\perp$.

The restriction of x^λ to the space $\langle f_{c_1}, \dots, f_{c_k}\rangle^\perp$ is of Jordan type $(\overline{\lambda})^T$. Hence we can pick an isomorphism $\theta : \langle f_{c_1}, \dots, f_{c_k}\rangle^\perp \to \mathbb{C}^{d-k}$ such that $\theta \circ x^\lambda = x^{\overline{\lambda}} \circ \theta$. Define $f_\gamma : \pi^{-1}(Y_\gamma^\circ) \to Y_\gamma^\circ \times X_{\overline{\mu}}$ to be the map sending $(V_0, \dots, V_n) \in \pi^{-1}(Y_\gamma^\circ)$ to $(U, (\overline{V}_0, \dots, \overline{V}_{n-1})) \in Y_\gamma^\circ \times X_{\overline{\mu}}$, where $U = V_{n-1}^\perp$ and $\overline{V}_i = \theta(g(U)^T(V_i))$. This makes sense because $g(U)^T(V_{n-1}) = \langle f_{c_1}, \dots, f_{c_k}\rangle^\perp$. It is clear that the map f_γ is invertible, hence it is an isomorphism of varieties as in (ii).

To deduce (iii), take $(V_0, \dots, V_n) \in \pi^{-1}(Y_\gamma^\circ)$. We have that $V_{n-1}^\perp \in Y_\gamma^\circ \subseteq Y_\beta$, hence $V_{n-1}^\perp \subseteq \langle f_1, \dots, f_s\rangle$. This shows that $x^\lambda(V_n) = \langle f_1, \dots, f_s\rangle^\perp \subseteq V_{n-1}$, so in particular x^λ leaves V_{n-1} invariant. Moreover multiplication by $g(V_{n-1}^\perp)^T$ is an x^λ-equivariant isomorphism between V_{n-1} and $\langle f_{c_1}, \dots, f_{c_k}\rangle^\perp$. Hence the Jordan type of x^λ on V_{n-1} is the same as its Jordan type on $\langle f_{c_1}, \dots, f_{c_k}\rangle^\perp$, namely $(\overline{\lambda})^T$.

Finally, for (iv), suppose we are given $(V_0, \dots, V_n) \in \pi^{-1}(Y_\gamma^\circ)$ mapping to $(U, (\overline{V}_0, \dots, \overline{V}_{n-1}))$ under the isomorphism f_γ. We need to show that $(V_0, \dots, V_n)$ belongs to X_μ^λ if and only if $(\overline{V}_0, \dots, \overline{V}_{n-1})$ belongs to $X_{\overline{\mu}}^{\overline{\lambda}}$. By (iii), we know automatically that $x^\lambda(V_n) \subseteq V_{n-1}$. Hence we just need to check for each $i = 1, \dots, n-1$ that $x^\lambda(V_i) \subseteq V_{i-1}$ if and only if $x^{\overline{\lambda}}(\overline{V}_i) \subseteq \overline{V}_{i-1}$. This is clear as $x^{\overline{\lambda}} \circ \theta \circ g(U)^T = \theta \circ g(U)^T \circ x^\lambda$ on V_{n-1}. □

By a *column-strict* (resp. *semi-standard*) λ-tableau we mean a filling of the boxes of the Young diagram of λ by integers so that entries are strictly increasing down columns (resp. strictly increasing down columns and weakly increasing along rows). Let Col_μ^λ (resp. Std_μ^λ) denote the set of all column-strict (resp. semi-standard) λ-tableaux that have exactly μ_i entries equal to i for each $i = 1, \dots, n$. The following definition will also be needed in the next section.

Definition 2.2. Suppose that $n \geq 1$ and we are given $\mathtt{T} \in \mathrm{Col}_\mu^\lambda$. The existence of $\mathtt{T}$ implies that the integers k, s, t defined as in Proposition 2.1 satisfy $s \geq k \geq t$. We define $(c_1, \dots, c_k)$, γ, $\overline{\mu}$, $\overline{\mathtt{T}}$, and $\overline{\lambda}$ as follows:

- $1 \leq c_1 < \cdots < c_k \leq s$ index the columns of $\mathtt{T}$ containing entry n;
- $\gamma \in \mathscr{P}_{k,d-k}$ is the partition with column sequence $(c_1, \dots, c_k)$;
- $\overline{\mu} \in \Lambda(n-1, d-k)$ is defined by forgetting the last part of μ;
- $\overline{\mathtt{T}}$ is obtained by removing all boxes labelled by n from $\mathtt{T}$, then repeatedly interchanging columns i and $(i+1)$ whenever the ith column contains fewer boxes than the $(i+1)$th column for some i;
- $\overline{\lambda} \in \Lambda^+(n-1, d-k)$ is the shape of $\overline{\mathtt{T}}$, so $\overline{\mathtt{T}} \in \mathrm{Col}_{\overline{\mu}}^{\bar{\lambda}}$.

Example 2.3. (i) If $n = 6$ and $\mathtt{T} = \begin{array}{|c|c|c|c|}\hline 2&1&2&2\\ \hline 3&2&4\\ \cline{1-3} 4&4&6\\ \cline{1-3} 6&5\\ \cline{1-2}\end{array}$ then $(c_1, c_2) = (1, 3)$, $\gamma = (1)$ and

$$\overline{\mathtt{T}} = \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\ \hline 2&3&4\\ \cline{1-3} 4&4\\ \cline{1-2} 5\\ \cline{1-1}\end{array}.$$

(ii) If $n = 4$ and $\mathtt{T} = \begin{array}{|c|c|c|c|c|c|}\hline 3&1&1&2&1&4\\\hline 4&2&3&4&2\\\cline{1-5}\end{array}$ then $(c_1, c_2, c_3) = (1, 4, 6)$, $\gamma = (3, 2)$ and $\overline{\mathtt{T}} = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&3&2\\\hline 2&3&2\\\cline{1-3}\end{array}$.

Now we make several definitions by induction on n. Suppose first that $n = d = 0$. We define the *degree* $\deg(\mathtt{T})$ of the unique tableau $\mathtt{T} \in \mathrm{Col}^\lambda_\mu$ (the empty tableau) to be zero, let $Y^\circ_{\mathtt{T}} := X^\lambda_\mu$ (which is a single point), and let $\preceq$ be the trivial partial order on Col^λ_μ. Now suppose that $n \geq 1$, take $\mathtt{T} \in \mathrm{Col}^\lambda_\mu$ and define γ and $\overline{\mathtt{T}}$ as in Definition 2.2. Then set

$$\deg(\mathtt{T}) := |\gamma| + \deg(\overline{\mathtt{T}}). \tag{2.5}$$

For example, the tableau $\mathtt{T}$ from Example 2.3(i) is of degree 2 and the one from Example 2.3(ii) is of degree 8. Also define a subset $Y^\circ_{\mathtt{T}} \subseteq X^\lambda_\mu$ by setting

$$Y^\circ_{\mathtt{T}} := \bar{f}_\gamma^{-1}(Y^\circ_\gamma \times Y^\circ_{\overline{\mathtt{T}}}), \tag{2.6}$$

where $\bar{f}_\gamma$ is the map chosen in Proposition 2.1(iv). Finally define a partial order $\preceq$ on Col^λ_μ by declaring that $\mathtt{T} \preceq \mathtt{T}'$ if $\gamma \subsetneq \gamma'$, or $\gamma = \gamma'$ and $\overline{\mathtt{T}} \preceq \overline{\mathtt{T}}'$, where γ' and $\overline{\mathtt{T}}'$ are defined using Definition 2.2 again but starting from $\mathtt{T}'$.

Theorem 2.4. *The sets $Y^\circ_{\mathtt{T}}$ give an affine paving of X^λ_μ indexed by the poset $(\mathrm{Col}^\lambda_\mu, \preceq)$, such that $Y^\circ_{\mathtt{T}} \cong \mathbb{A}^{\deg(\mathtt{T})}$. Moreover the complement $X_\mu \setminus X^\lambda_\mu$ also has an affine paving.*

Proof. Consider the map $\bar{\pi} : X^\lambda_\mu \to Y_\beta$ from Proposition 2.1(i). Since Y_β is paved by the affine spaces Y°_γ for $\gamma \subseteq \beta$, it suffices to show that the inverse images $\bar{\pi}^{-1}(Y^\circ_\gamma)$ all have affine pavings. This is follows by induction on n using Proposition 2.1(iv) and the definition (2.6). The same induction gives also that $Y^\circ_{\mathtt{T}} \cong \mathbb{A}^{\deg(\mathtt{T})}$. Finally, to see that the complement $X_\mu \setminus X^\lambda_\mu$ has an affine paving, we use also Proposition 2.1(ii) to see that the complement is the disjoint union of the spaces $\pi^{-1}(Y^\circ_\gamma) \setminus \bar{\pi}^{-1}(Y^\circ_\gamma) \cong Y^\circ_\gamma \times (X_{\overline{\mu}} \setminus X^{\overline{\lambda}}_{\overline{\mu}})$ for each $\gamma \subseteq \beta$, each of which has an affine paving by induction, together with $\pi^{-1}(Y^\circ_\gamma) \cong Y^\circ_\gamma \times X_{\overline{\mu}}$ for $\gamma \not\subseteq \beta$, which have affine pavings too. □

Corollary 2.5. *The cohomology $H^*(X^\lambda_\mu, \mathbb{C})$ vanishes in all odd degrees, and in even degrees $\dim H^{2r}(X^\lambda_\mu, \mathbb{C})$ is equal to the number of $\mathtt{T} \in \mathrm{Col}^\lambda_\mu$ with $\deg(\mathtt{T}) = r$. Moreover the pull-back $j^* : H^*(X_\mu, \mathbb{C}) \to H^*(X^\lambda_\mu, \mathbb{C})$ is surjective.*

Proof. This is a standard consequence of the existence of an affine paving as in Theorem 2.4; see e.g. [HS, Corollary 2.3] or the discussion at the bottom of [J, p.163]. □

Corollary 2.6. *The variety X^λ_μ is non-empty if and only if $\lambda \geq \mu^+$. Assuming that is the case, X^λ_μ is connected.*

Proof. The first statement follows because Col^λ_μ is non-empty if and only if $\lambda \geq \mu^+$. For the second statement, we observe that there is a unique $\mathtt{T} \in \mathrm{Col}^\lambda_\mu$ of degree zero, hence $\dim H^0(X^\lambda_\mu, \mathbb{C}) = 1$ by Corollary 2.5. □

Although not really needed in the rest of the article, we end the section by explaining for completeness how to recover Spaltenstein's classification of the irreducible components of X^λ_μ from Theorem 2.4. Suppose we are given $(V_0,\dots,V_n)\in X^\lambda_\mu$. Assuming $n\geq 1$, Proposition 2.1(iii) shows that the restriction of x^λ to V_{n-1} is of Jordan type $(\overline{\lambda})^T$ for some $\overline{\lambda}\in\Lambda^+(n-1,d-\mu_n)$ such that $\lambda_{i+1}\leq\overline{\lambda}_i\leq\lambda_i$ for all $1\leq i\leq n-1$. By induction we deduce for $j=0,1,\dots,n$ that the restriction of x^λ to V_j is of Jordan type $(\lambda^{(j)})^T$ for $\lambda^{(j)}\in\Lambda^+(j,\mu_1+\cdots+\mu_j)$ satisfying $\lambda^{(j+1)}_{i+1}\leq\lambda^{(j)}_i\leq\lambda^{(j+1)}_i$ for all $1\leq i\leq j$. It follows that there is a well-defined map

$$J: X^\lambda_\mu\to \mathrm{Std}^\lambda_\mu,\qquad (V_0,\dots,V_n)\mapsto \mathtt{S}, \tag{2.7}$$

where $\mathtt{S}$ is the semi-standard tableau having entry j in all boxes belonging to the Young diagram of $\lambda^{(j)}$ but not of $\lambda^{(j-1)}$ for $j=1,\dots,n$.

Theorem 2.7. *For $\mathtt{S}\in\mathrm{Std}^\lambda_\mu$, we have that $J^{-1}(\mathtt{S})$ is a locally closed, smooth, irreducible subvariety of X^λ_μ of dimension $d_\lambda-d_\mu$. Moreover $Y^\circ_{\mathtt{S}}$ is a dense open subset of $J^{-1}(\mathtt{S})$.*

Proof. For $\mathtt{T}\in\mathrm{Col}^\lambda_\mu$, define $\mathtt{T}^+\in\mathrm{Std}^\lambda_\mu$ as follows. Let $\overline{\mathtt{T}}\in\mathrm{Col}^{\overline{\lambda}}_{\overline{\mu}}$ be as in Definition 2.2. Then let $\mathtt{T}^+$ be obtained from the recursively defined $(\overline{\mathtt{T}})^+\in\mathrm{Std}^{\overline{\lambda}}_{\overline{\mu}}$ by adding the entry n to all boxes belonging to the Young diagram of λ but not of $\overline{\lambda}$. For example if $\mathtt{T}$ is as in Example 2.3(ii) then $\mathtt{T}^+=\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&2&3&4\\\hline 2&2&3&4&4\\\cline{1-5}\end{array}$. All points of $Y^\circ_{\mathtt{T}}$ map to $\mathtt{T}^+$ under the map J. Hence for $\mathtt{S}\in\mathrm{Std}^\lambda_\mu$ we have that

$$J^{-1}(\mathtt{S})=\bigcup_{\mathtt{T}\in\Omega(\mathtt{S})}Y^\circ_{\mathtt{T}}\quad\text{where}\quad\Omega(\mathtt{S}):=\{\mathtt{T}\in\mathrm{Col}^\lambda_\mu\mid \mathtt{T}^+=\mathtt{S}\}. \tag{2.8}$$

In particular, $J^{-1}(\mathtt{S})$ is locally closed as each $Y^\circ_{\mathtt{T}}$ is so by Theorem 2.4. Note further that $\mathtt{S}$ belongs to $\Omega(\mathtt{S})$, and all other $\mathtt{T}\in\Omega(\mathtt{S})$ are strictly smaller than $\mathtt{S}$ in the partial order $\preceq$, so Theorem 2.4 implies $Y^\circ_{\mathtt{S}}$ is open in $J^{-1}(\mathtt{S})$.

It remains to prove that $J^{-1}(\mathtt{S})$ is smooth and irreducible of the given dimension. Define γ and $\overline{\mathtt{S}}\in\mathrm{Std}^{\overline{\lambda}}_{\overline{\mu}}$ as in Definition 2.2, using $\mathtt{S}$ in place of $\mathtt{T}$. Let $(c_1,\dots,c_k)$ be the column sequence of γ. In other words, the c_i's index the columns of $\mathtt{S}$ containing the entry n. Also define $\bar{\pi}$ as in Proposition 2.1(i). Let $\Omega_0(\mathtt{S}):=\{\mathtt{T}\in\Omega(\mathtt{S})\mid\mathtt{T}\text{ has entry }n\text{ in columns }c_1,\dots,c_k\}$, so that

$$M:=J^{-1}(\mathtt{S})\cap\bar{\pi}^{-1}(Y^\circ_\gamma)=\bigcup_{\mathtt{T}\in\Omega_0(\mathtt{S})}Y^\circ_{\mathtt{T}}. \tag{2.9}$$

Observe that every element of $\Omega(\mathtt{S})\setminus\Omega_0(\mathtt{S})$ is smaller than every element of $\Omega_0(\mathtt{S})$ in the partial order $\preceq$. Hence using Theorem 2.4 again, we deduce that M is an open subset of $J^{-1}(\mathtt{S})$. The map $\mathtt{T}\mapsto\overline{\mathtt{T}}$ is a bijection between $\Omega_0(\mathtt{S})$ and $\Omega(\overline{\mathtt{S}})$. So, comparing (2.9) with (2.8) for $\overline{\mathtt{S}}$, we see that the isomorphism $\bar{f}_\gamma$ from Proposition 2.1(iv) restricts to an isomorphism $M\overset{\sim}{\to}Y^\circ_\gamma\times J^{-1}(\overline{\mathtt{S}})$. By induction, we deduce that M is a smooth irreducible variety of dimension $|\gamma|+d_{\overline{\lambda}}-d_{\overline{\mu}}=d_\lambda-d_\mu$.

Finally let Z be the centralizer of x^λ in $GL_d(\mathtt{C})$. This is a connected algebraic group which acts naturally on X^λ_μ. Moreover Z leaves the subset $J^{-1}(\mathtt{S})$ invariant, so the action restricts to a morphism

$$m : Z \times M \to J^{-1}(\mathtt{S}).$$

To complete the proof of the theorem, we just need to show that this map is surjective. Take $(V_0, \dots, V_n) \in J^{-1}(\mathtt{S})$. By (2.8), it lies in $Y^\circ_{\mathtt{T}}$ for some $\mathtt{T} \in \Omega(\mathtt{S})$. Suppose that the entries of $\mathtt{T}$ equal to n are in columns $1 \leq b_1 < \cdots < b_k \leq s$. There is a permutation of the columns of equal height in the Young diagram of λ sending columns $b_1, \dots, b_k$ to columns $c_1, \dots, c_k$. Let $x \in GL_d(\mathbb{C})$ be the matrix inducing the associated permutation of the basis $f_1, \dots, f_d$, labelling boxes as in (2.4). So we have that $x(\langle f_{b_1}, \dots, f_{b_k}\rangle^\perp) = \langle f_{c_1}, \dots, f_{c_k}\rangle^\perp$, and obviously $x \in Z$. Also, as in the proof of Proposition 2.1, there is an element $y := g(V^\perp_{n-1})^T \in Z$ such that $y(V_{n-1}) = \langle f_{b_1}, \dots, f_{b_k}\rangle^\perp$. Hence $xy(V_{n-1}) = \langle f_{c_1}, \dots, f_{c_k}\rangle^\perp$, so $xy \in Z$ maps $(V_0, \dots, V_n)$ to a point of M. This implies that m is surjective. □

Corollary 2.8 (Spaltenstein). *Assuming $\lambda \geq \mu^+$, X^λ_μ is equidimensional of dimension $d_\lambda - d_\mu$, with irreducible components $Y_{\mathtt{S}} := \overline{Y^\circ_{\mathtt{S}}} = \overline{J^{-1}(\mathtt{S})}$ for $\mathtt{S} \in \mathrm{Std}^\lambda_\mu$ (closure in the Zariski topology).*

Proof. By Theorem 2.7, the subvarieties $J^{-1}(\mathtt{S})$ for $\mathtt{S} \in \mathrm{Std}^\lambda_\mu$ are irreducible of dimension $d_\lambda - d_\mu$, and they partition X^λ_μ into disjoint subsets. Hence their closures give all the irreducible components. Finally, $Y^\circ_{\mathtt{S}}$ has the same closure as $J^{-1}(\mathtt{S})$ as it is a dense subset. □

Corollary 2.9. *For $\mathtt{T} \in \mathrm{Col}^\lambda_\mu$, we have that $\deg(\mathtt{T}) \leq d_\lambda - d_\mu$, with equality if and only if $\mathtt{T}$ is semi-standard.*

Proof. Let $\mathtt{S} := \mathtt{T}^+$, notation as in the proof of Theorem 2.7. By Theorem 2.7 and the decomposition (2.8), $Y^\circ_{\mathtt{T}}$ is an irreducible subset of the irreducible variety $J^{-1}(\mathtt{S})$, and it is dense in $J^{-1}(\mathtt{S})$ if and only if $\mathtt{T}$ is semi-standard (equivalently, $\mathtt{T} = \mathtt{S}$). The corollary follows from this since $\dim J^{-1}(\mathtt{S}) = d_\lambda - d_\mu$ and $\dim Y^\circ_{\mathtt{T}} = \deg(\mathtt{T})$. (It is not hard to supply a purely combinatorial proof of this corollary.) □

3. Algebraic basis

Continue with fixed $\lambda \in \Lambda^+(n, d)$ and $\mu \in \Lambda(n, d)$. Recall the definition of the elements $h_r(\mu; i_1, \dots, i_m) \in P_\mu$ and the algebra $C^\lambda_\mu := P_\mu / I^\lambda_\mu$ from the introduction.

Lemma 3.1. *The algebra C^λ_μ is non-zero if and only if $\lambda \geq \mu^+$.*

Proof. Since everything is graded, C^λ_μ is non-zero if and only if all the generators from (1.4) are of positive degree, i.e. $\lambda_1 + \cdots + \lambda_m \geq \mu_{i_1} + \cdots + \mu_{i_m}$ for all $m \geq 1$ and $1 \leq i_1 < \cdots < i_m \leq n$. By the definition of the dominance ordering on partitions, this is the statement that $\lambda \geq \mu^+$. □

For $\mathtt{T} \in \mathrm{Col}^\lambda_\mu$, we inductively define an element $h(\mathtt{T}) \in P_\mu$ as follows. If $n = d = 0$ then $\mathtt{T}$ is the empty tableau and we simply set $h(\mathtt{T}) := 1$. If $n \geq 1$, we let $\gamma \in \mathscr{P}_k$ and $\overline{\mathtt{T}} \in \mathrm{Col}^{\overline{\lambda}}_{\overline{\mu}}$ be as in Definition 2.2. The natural embedding $\mathbb{C}[x_1, \dots, x_{d-k}] \hookrightarrow \mathbb{C}[x_1, \dots, x_d]$ induces an embedding $P_{\overline{\mu}} \hookrightarrow P_\mu$. This allows us to view the recursively defined element $h(\overline{\mathtt{T}}) \in P_{\overline{\mu}}$ as an element of P_μ. Then we set

$$h(\mathtt{T}) := h(\overline{\mathtt{T}})h_\gamma(\mu; n) \qquad \text{where} \qquad h_\gamma(\mu; n) := \prod_{i=1}^k h_{\gamma_i}(\mu; n), \tag{3.1}$$

where each $h_{\gamma_i}(\mu; n)$ is as defined in the introduction. Recalling (2.5), $h(\mathtt{T})$ is homogeneous of degree $2\deg(\mathtt{T})$. For example, if $\mathtt{T}$ is as in Example 2.3(i) then $h(\mathtt{T}) = h_1(\mu; 3)h_1(\mu; 6) = x_6(x_{11} + x_{12})$, while for Example 2.3(ii) we get that $h(\mathtt{T}) = h_1(\mu; 2)h_2(\mu; 3)h_1(\mu; 3)h_3(\mu; 4)h_2(\mu; 4)$. We use the same notation for the canonical image of $h(\mathtt{T})$ in the quotient C^λ_μ.

Theorem 3.2. *The elements $\{h(\mathtt{T}) \mid \mathtt{T} \in \mathrm{Col}^\lambda_\mu\}$ give a basis for C^λ_μ.*

In the remainder of the section, we will prove the spanning part of Theorem 3.2, postponing the proof of linear independence to §5. The approach is similar to [T, Lemmas 2–3] where the case of regular μ was treated, but the general case turns out to be considerably more delicate.

The argument goes by induction on n. The theorem is trivial in the case $n = 0$, so assume for the rest of the section that $n \geq 1$ and that we have proved the spanning part of Theorem 3.2 for all smaller n. Let $k := \mu_n$, $s := \lambda_1$ and $t := \lambda_n$. In view of Lemma 3.1, we may as well assume that $\lambda \geq \mu^+$, hence we have that $s \geq k \geq t$. Set $\beta := ((s-k)^{k-t}) \in \mathscr{P}_k$, and let $\overline{\mu} \in \Lambda(n-1, d-k)$ be obtained from μ by forgetting the last part (cf. Proposition 2.1). Let $\trianglerighteq$ be the partial order on the set $\mathscr{P}_k$ of partitions of height at most k such such that $\gamma \trianglerighteq \kappa$ if either $|\gamma| > |\kappa|$, or $|\gamma| = |\kappa|$ and $\gamma \geq \kappa$ in the dominance ordering. For $\gamma \in \mathscr{P}_k$, let $J_{\trianglerighteq\gamma}$ (resp. $J_{\triangleright\gamma}$) denote the ideal of P_μ generated by I^λ_μ and all $h_\kappa(\mu; n)$ for $\kappa \trianglerighteq \gamma$ (resp. $\kappa \triangleright \gamma$).

Lemma 3.3. *Fix $m \geq 0$, $1 \leq i_1 < \cdots < i_m \leq n-1$, $r \geq 0$ and $\gamma \in \mathscr{P}_k$. Suppose we are given $1 \leq c \leq k$ such that*

$$r + c > \lambda_1 + \cdots + \lambda_m - \mu_{i_1} - \cdots - \mu_{i_m}, \tag{3.2}$$

$$r + \gamma_c > \lambda_1 + \cdots + \lambda_m + \lambda_{m+1} - \mu_{i_1} - \cdots - \mu_{i_m} - \mu_n. \tag{3.3}$$

Then $h_r(\mu; i_1, \dots, i_m)h_\gamma(\mu; n) \in J_{\triangleright\gamma}$.

Proof. We first formulate and prove two technical claims. By a *marked partition* we mean a pair $(\pi; p)$ consisting of a partition π and a non-zero part p of π. We write $\pi \cup \{q\}$ for the partition obtained by adding one extra part equal to q to the partition π, so that $(\pi \cup \{q\}; q)$ is a marked partition.

Claim 1. *Suppose we are given vectors $v(\pi; p)$ for each marked partition $(\pi; p)$ with $1 \leq |\pi| \leq c$, where c is fixed as in the statement of the lemma. For any partition π with $|\pi| \leq c$, let $v(\pi)$ denote $\sum_p v(\pi; p)$ summing over the set of all non-zero parts p of π; in particular $v(\varnothing) := 0$. Assume for $1 \leq b \leq c$ that*

$$v(\pi) + \sum_{q=1}^b v(\pi \cup \{q\}; q) = 0 \tag{3.4}$$

for each partition π *of* $(c-b)$*. Then we have that* $\sum_{|\pi|=c}(-1)^{h(\pi)}v(\pi)=0$.
To see this, we note for $i=1,\dots,c$ that

$$\begin{aligned}\sum_{\substack{|\pi|<c\\h(\pi)=i-1}}(-1)^{h(\pi)}v(\pi)&=\sum_{\substack{|\pi|<c\\h(\pi)=i-1}}\sum_{q=1}^{c-|\pi|}(-1)^{h(\pi)+1}v(\pi\cup\{q\};q)\\&=\sum_{\substack{|\pi|<c\\h(\pi)=i}}(-1)^{h(\pi)}v(\pi)+\sum_{\substack{|\pi|=c\\h(\pi)=i}}(-1)^{h(\pi)}v(\pi),\end{aligned}$$

using (3.4) for the first equality and the definition of $v(\pi)$ for the second. Using this identity repeatedly, we get that

$$\begin{aligned}v(\varnothing)&=\sum_{\substack{|\pi|<c\\h(\pi)=0}}(-1)^{h(\pi)}v(\pi)+\sum_{\substack{|\pi|=c\\h(\pi)\le 0}}(-1)^{h(\pi)}v(\pi)\\&=\sum_{\substack{|\pi|<c\\h(\pi)=1}}(-1)^{h(\pi)}v(\pi)+\sum_{\substack{|\pi|=c\\h(\pi)\le 1}}(-1)^{h(\pi)}v(\pi)\\&\;\;\vdots\\&=\sum_{\substack{|\pi|<c\\h(\pi)=c}}(-1)^{h(\pi)}v(\pi)+\sum_{\substack{|\pi|=c\\h(\pi)\le c}}(-1)^{h(\pi)}v(\pi)=\sum_{|\pi|=c}(-1)^{h(\pi)}v(\pi).\end{aligned}$$

Since $v(\varnothing)=0$ this establishes Claim 1.

Call a function $f:\{1,\dots,c\}\to\{1,\dots,c\}$ a *quasi-permutation* of *descent* $b\ge 0$ if there exist distinct integers $1\le j_1,\dots,j_{b+1}\le c$ and a bijection $\bar{f}:\{1,\dots,c\}\setminus\{j_1,\dots,j_b\}\to\{1,\dots,c\}\setminus\{j_1,\dots,j_b\}$ such that

- $j_1=c$;
- $f(j_1)=j_2,f(j_2)=j_3,\dots,f(j_b)=j_{b+1}$;
- $f(j)=\bar{f}(j)$ for each $j\in\{1,\dots,c\}\setminus\{j_1,\dots,j_b\}$.

Note b, $j_1,\dots,j_{b+1}$ and $\bar{f}$ are uniquely determined by the quasi-permutation f, and quasi-permutations of descent 0 are ordinary permutations belonging to the symmetric group S_c. The *marked cycle type* of the quasi-permutation f is the marked partition $(\pi;p)$ defined by letting π be the partition of $(c-b)$ recording the usual cycle type of the permutation $\bar{f}$ of $\{1,\dots,c\}\setminus\{j_1,\dots,j_b\}$ and p be the length of the cycle that involves j_{b+1} when $\bar{f}$ is written as a product of disjoint cycles. For example, the quasi-permutation

$$\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\2&1&5&6&7&10&8&9&7&4&12&13&14&11&3\end{pmatrix}$$

of $c=15$ has marked cycle type $((4,3^2,2);3)$. Let $S_c(\pi;p)$ denote the set of all quasi-permutations of marked cycle type $(\pi;p)$. In particular, $S_c((1^c);1)=\{\mathrm{id}\}$.

Given $f \in S_c(\pi; p)$, let

$$\omega(f) := (1 - f(1), 2 - f(2), \dots, c - f(c)) \in \mathbb{Z}^c,$$
$$|\omega(f)| := (1 - f(1)) + \cdots + (c - f(c)) \in \mathbb{Z}_{\geq 0},$$
$$h_{\gamma - \omega(f)}(\mu; n) := \prod_{a=1}^{c} h_{\gamma_a - a + f(a)}(\mu; n) \prod_{a=c+1}^{k} h_{\gamma_a}(\mu; n).$$

Claim 2. *For each marked partition $(\pi; p)$ with $1 \leq |\pi| \leq c$, we set*

$$v(\pi; p) := \sum_{f \in S_c(\pi; p)} h_{r + |\omega(f)|}(\mu; i_1, \dots, i_m) h_{\gamma - \omega(f)}(\mu; n).$$

Also define $v(\pi)$ as in Claim 1. Then, for $1 \leq b \leq c$ and each partition π of $(c - b)$, we have that $v(\pi) + \sum_{q=1}^{b} v(\pi \cup \{q\}; q) \in J_{\rhd \gamma}$.

To establish this, fix $1 \leq b \leq c$ and a partition π of $(c - b)$, and let

$$\Omega := \left\{ (\underline{j}; g) \;\middle|\; \begin{array}{l} \underline{j} = (j_1, \dots, j_b) \text{ a tuple of distinct integers with} \\ j_1 = c \text{ and } 1 \leq j_2, \dots, j_b < c; \quad g \text{ a permutation} \\ \text{of } \{1, \dots, c\} \setminus \{j_1, \cdots, j_b\} \text{ of cycle type } \pi \end{array} \right\}.$$

Given $(\underline{j}; g) \in \Omega$ and $1 \leq i \leq c$, we let $f_i(\underline{j}; g) : \{1, \dots, c\} \to \{1, \dots, c\}$ be the function mapping $j_1 \mapsto j_2, \dots, j_{b-1} \mapsto j_b, j_b \mapsto i$ and $j \mapsto g(j)$ for all $j \in \{1, \dots, c\} \setminus \{j_1, \dots, j_b\}$. If $i \in \{1, \dots, c\} \setminus \{j_1, \dots, j_b\}$ then $f_i(\underline{j}; g)$ is a quasi-permutation of descent b and marked cycle type $(\pi; p)$ where p is the length of the cycle of g containing i. Otherwise, we have that $i = j_{b-q+1}$ for some $1 \leq q \leq b$, and $f_i(\underline{j}; g)$ is a quasi-permutation of descent $(b - q)$ and marked cycle type $(\pi \cup \{q\}; q)$. It follows that $|\omega(f_i(\underline{j}; g))| = c - i$ and

$$v(\pi) + \sum_{q=1}^{b} v(\pi \cup \{q\}; q) = \sum_{(\underline{j}; g) \in \Omega} \sum_{i=1}^{c} h_{r+c-i}(\mu; i_1, \dots, i_m) h_{\gamma - \omega(f_i(\underline{j}; g))}(\mu; n).$$

Therefore it suffices to show for each $(\underline{j}; g) \in \Omega$ that

$$x := \sum_{i=1}^{c} h_{r+c-i}(\mu; i_1, \dots, i_m) h_{\gamma - \omega(f_i(\underline{j}; g))}(\mu; n)$$

belongs to $J_{\rhd \gamma}$. Let

$$y := \prod_{\substack{a=1 \\ a \neq j_b}}^{c} h_{\gamma_a - a + f_i(\underline{j}; g)(a)}(\mu; n) \prod_{a=c+1}^{k} h_{\gamma_a}(\mu; n)$$

so that

$$x = \sum_{i=1}^{c} h_{r+c-i}(\mu; i_1, \dots, i_m) h_{\gamma_{j_b} - j_b + i}(\mu; n) y.$$

Using (1.3), we can expand

$$h_{r+c+\gamma_{j_b}-j_b}(\mu;i_1,\dots,i_m,n)=\sum_{i=j_b-\gamma_{j_b}}^{r+c}h_{r+c-i}(\mu;i_1,\dots,i_m)h_{\gamma_{j_b}-j_b+i}(\mu;n).$$

As $j_b\le c$, we have that $c+\gamma_{j_b}-j_b\ge\gamma_c$, hence using (3.3) and (1.4) we get that $h_{r+c+\gamma_{j_b}-j_b}(\mu;i_1,\dots,i_m,n)\in I^\lambda_\mu$. Using (3.2) we have that $h_{r+c-i}(\mu;i_1,\dots,i_m)\in I^\lambda_\mu$ for $i\le 0$. We deduce that

$$\sum_{i=1}^{r+c}h_{r+c-i}(\mu;i_1,\dots,i_m)h_{\gamma_{j_b}-j_b+i}(\mu;n)\in I^\lambda_\mu.$$

As $I^\lambda_\mu\subseteq J_{\rhd\gamma}$, this implies that

$$x\equiv-\sum_{i=c+1}^{c+r}h_{r+c-i}(\mu;i_1,\dots,i_m)h_{\gamma_{j_b}-j_b+i}(\mu;n)y\pmod{J_{\rhd\gamma}}.$$

But for $i>c$ the term $h_{\gamma_{j_b}-j_b+i}(\mu;n)y$ is of the form $h_\kappa(\mu;n)$ for some $\kappa\in\mathscr{P}_k$ with $|\kappa|>|\gamma|$, so it belongs to $J_{\rhd\gamma}$. This proves Claim 2.

Now we can complete the proof of the lemma. Let $v(\pi;p)$ be as in Claim 2 and then define $v(\pi)$ as in Claim 1. From Claims 1 and 2, we get that

$$\sum_{|\pi|=c}(-1)^{h(\pi)-c}v(\pi)\in J_{\rhd\gamma}.$$

For $|\pi|=c$, the set $S_c(\pi;p)$ appearing in the definition of $v(\pi;p)$ consists of quasi-permutations of descent zero, i.e. ordinary permutations. So the above sum is equal to $v((1^c))=h_r(\mu;i_1,\dots,i_m)h_\gamma(\mu;n)$ plus a linear combination of terms $h_{r+|\omega(f)|}(\mu;i_1,\dots,i_m)h_{\gamma-\omega(f)}(\mu;n)$ for ordinary permutations $f\neq\mathrm{id}$. It remains to observe for all such f that $\omega(f)<0$ in the dominance ordering on $\mathbb{Z}^c$, hence $h_{\gamma-\omega(f)}(\mu;n)\in J_{\rhd\gamma}$. □

Lemma 3.4. *For $\gamma\in\mathscr{P}_k$ with $\gamma\not\subseteq\beta$, we have that $J_{\unrhd\gamma}=J_{\rhd\gamma}$. Moreover $J_{\unrhd\gamma}=I^\lambda_\mu$ if $|\gamma|>k(s-k)$.*

Proof. If $\gamma_1>s-k=\lambda_1-\mu_n$ then $h_{\gamma_1}(\mu;n)\in I^\lambda_\mu$ by (1.4), hence $h_\gamma(\mu;n)\in I^\lambda_\mu\subseteq J_{\rhd\gamma}$. In particular, this shows $J_{\unrhd\gamma}=I^\lambda_\mu$ when $|\gamma|>k(s-k)$, as in that case all $\gamma\unlhd\kappa\in\mathscr{P}_k$ satisfy $\kappa_1>s-k$. It remains to show that $h_\gamma(\mu;n)\in J_{\rhd\gamma}$ for $\gamma\not\subseteq\beta$ with $\gamma_1\le s-k$. In that case, we have that $t\ge1$ and $\gamma_{k-t+1}>0$. Now apply Lemma 3.3, taking $r=0,m=n-1$ and $c=k-t+1$ and noting the right hand sides of (3.2) and (3.3) equal $k-t$ and 0, respectively. □

Lemma 3.5. *Take $\gamma\in\mathscr{P}_k$ such that $\gamma\subseteq\beta$. Define $(c_1,\dots,c_k)$ and $\overline{\lambda}\in\Lambda^+(n-1,d-k)$ as in the statement of Proposition 2.1. The quotient $J_{\unrhd\gamma}/J_{\rhd\gamma}$ is spanned by the images of the elements $h(\mathtt{T})$ for $\mathtt{T}\in\mathrm{Col}^\lambda_\mu$ such that $\mathtt{T}$ has entry n in each of columns $c_1,\dots,c_k$.*

Proof. Consider the homomorphism $\mathbb{C}[x_1, \dots, x_d] \to \mathbb{C}[x_1, \dots, x_{d-k}]$ which maps $x_i \mapsto x_i$ for $1 \leq i \leq d-k$ and $x_i \mapsto 0$ for $d-k+1 \leq i \leq d$. It restricts to a homomorphism $P_\mu \to P_{\overline{\mu}}$ such that $h_r(\mu; i) \mapsto h_r(\overline{\mu}; i)$ for $1 \leq i \leq n-1$, and $h_r(\mu; n) \mapsto 0$ for $r \geq 1$. Let $\Phi : P_\mu \to C_{\overline{\mu}}^{\overline{\lambda}}$ be the composite of this homomorphism with the natural quotient map $P_{\overline{\mu}} \to C_{\overline{\mu}}^{\overline{\lambda}}$.

The P_μ-module $J_{\trianglerighteq\gamma}/J_{\triangleright\gamma}$ is cyclic, generated by the image of $h_\gamma(\mu; n)$. We claim that the action of P_μ on $J_{\trianglerighteq\gamma}/J_{\triangleright\gamma}$ factors through Φ to make $J_{\trianglerighteq\gamma}/J_{\triangleright\gamma}$ into a well-defined cyclic $C_{\overline{\mu}}^{\overline{\lambda}}$-module. To prove this, we need to show that $(\ker \Phi) J_{\trianglerighteq\gamma} \subseteq J_{\triangleright\gamma}$. By definition, $\ker \Phi$ is generated by the elements $\{h_r(\mu; n) \mid r > 0\}$ together with the elements

$$\left\{ h_r(\mu; i_1, \dots, i_m) \;\middle|\; \begin{array}{l} m \geq 1, 1 \leq i_1 < \cdots < i_m \leq n-1, \\ r > \overline{\lambda}_1 + \cdots + \overline{\lambda}_m - \mu_{i_1} - \cdots - \mu_{i_m} \end{array} \right\}.$$

So we need to show that multiplication by either of these families of elements sends $h_\gamma(\mu; n)$ into $J_{\triangleright\gamma}$.

The first family is easy to deal with: for $r > 0$ the element $h_r(\mu; n) h_\gamma(\mu; n)$ is a symmetric polynomial in the variables $x_{d-k+1}, \dots, x_d$ of degree strictly greater than $|\gamma|$. Hence it can be expressed as a linear combination of $h_\kappa(\mu; n)$'s with $|\kappa| > |\gamma|$, as the $h_\kappa(\mu; n)$'s for $\kappa \in \mathscr{P}_k$ give a basis for the space of all symmetric polynomials in $x_{d-k+1}, \dots, x_d$.

To deal with the second family, we need to show that

$$h_r(\mu; i_1, \dots, i_m) h_\gamma(\mu; n) \in J_{\triangleright\gamma}$$

for $m \geq 1$, $1 \leq i_1 < \cdots < i_m \leq n-1$ and $r > \overline{\lambda}_1 + \cdots + \overline{\lambda}_m - \mu_{i_1} - \cdots - \mu_{i_m}$. Let $c := \lambda_1 + \cdots + \lambda_m - \overline{\lambda}_1 - \cdots - \overline{\lambda}_m$. If $c = 0$ then $h_r(\mu; i_1, \dots, i_m) \in I_\mu^\lambda$ already, so there is nothing to do. So we may assume that $c \geq 1$, and are in the situation of Lemma 3.3. The hypothesis (3.2) is immediate, so we are left with checking (3.3). To see that, note that $\overline{\lambda}$ has c fewer boxes than λ on the first m rows. So by the definition of $\overline{\lambda}$, we must have that $k + 1 - c + \gamma_c > \lambda_{m+1}$. Hence

$$\begin{aligned} r + \gamma_c &> \overline{\lambda}_1 + \cdots + \overline{\lambda}_m - \mu_{i_1} - \cdots - \mu_{i_m} + \lambda_{m+1} - k \\ &\geq \lambda_1 + \cdots + \lambda_m + \lambda_{m+1} - \mu_{i_1} - \cdots - \mu_{i_m} - \mu_n, \end{aligned}$$

and the claim is proved.

Now to prove the lemma, we have shown that $J_{\trianglerighteq\gamma}/J_{\triangleright\gamma}$ is a cyclic $C_{\overline{\mu}}^{\overline{\lambda}}$-module generated by the image of $h_\gamma(\mu; n)$. By the induction hypothesis which we have been assuming since the paragraph before Lemma 3.3, we know that $C_{\overline{\mu}}^{\overline{\lambda}}$ is spanned by the elements $h(\overline{\mathtt{T}})$ for $\overline{\mathtt{T}} \in \mathrm{Col}_{\overline{\mu}}^{\overline{\lambda}}$. Hence J_γ is spanned by the images of the elements $h(\mathtt{T}) = h(\overline{\mathtt{T}}) h_\gamma(\mu; n)$ as described in the statement of the lemma. □

Now we can complete the proof of the spanning part of Theorem 3.2. Enumerate the partitions $\gamma \subseteq \beta$ as $\gamma^{(1)} = \varnothing, \gamma^{(2)}, \dots, \gamma^{(N-1)}, \gamma^{(N)} = \beta$ so that $\gamma^{(i)} \trianglelefteq \gamma^{(j)}$ implies $i \leq j$. Let J_i be the ideal of C_μ^λ generated by $\{h_{\gamma^{(j)}}(\mu; n) \mid i \leq j \leq N\}$. Then

$$C_\mu^\lambda = J_1 \geq \cdots \geq J_N \geq J_{N+1} := \{0\}$$

is a filtration of C^λ_μ. Lemma 3.4 implies for $i = 1, \dots, N$ that the canonical map $J_{\trianglerighteq\gamma^{(i)}} \to J_i$ induces a surjection $J_{\trianglerighteq\gamma^{(i)}}/J_{\triangleright\gamma^{(i)}} \twoheadrightarrow J_i/J_{i+1}$. Hence by Lemma 3.5 we get that J_i/J_{i+1} is spanned by the images of the $h(\mathtt{T})$ for $\mathtt{T} \in \mathrm{Col}^\lambda_\mu$ with entry n in each of columns $c_1, \dots, c_k$, where $(c_1, \dots, c_k)$ is the column sequence of $\gamma^{(i)}$. Hence the $h(\mathtt{T})$ for all $\mathtt{T} \in \mathrm{Col}^\lambda_\mu$ span C^λ_μ itself.

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we need to exploit the construction of the Springer representations via perverse sheaves. As is recalled in detail below, there are two basic approaches, one (by restriction) due to Lusztig, Borho and Macpherson [BM1], and the other (by Fourier transform) due to Kashiwara and Brylinski [Bry]. We refer to [J, ch.13] and [G2, §6] for more recent accounts of these two approaches, and also [KW, §VI.15] which explains the relationship between them. (Some of these references work in terms of étale cohomology so require some translation before they can be used in our complex setting.)

For any complex variety Y, we let $D^b(Y)$ be the bounded derived category of constructible sheaves of $\mathbb{C}$-vector spaces on Y; see e.g. [G2, §3]. Let

$$\mathbf{D} : D^b(Y) \to D^b(Y)$$

be the Verdier duality functor. Let $\mathrm{Perv}(Y)$ be the (abelian) full subcategory of $D^b(Y)$ consisting of perverse sheaves; see e.g. [G2, §4]. The constant sheaf on Y is denoted $\mathbb{C}_Y$, which we often view as an object in $D^b(Y)$ concentrated in degree zero. For $\mathcal{M} \in D^b(Y)$, $\mathcal{M}[i]$ denotes the object obtained from $\mathcal{M}$ by translating down by i. If $V = \bigoplus_{j\in\mathbb{Z}} V_j$ is a graded vector space, we'll write $\mathcal{M} \otimes V$ for $\bigoplus_{j\in\mathbb{Z}} \mathcal{M}[-j] \otimes V_j$, so $\mathcal{M} \otimes (V[i]) = (\mathcal{M}[i]) \otimes V = (\mathcal{M} \otimes V)[i]$.

Assume from now on that Y is irreducible and smooth. In that case, $\mathbb{C}_Y[\dim Y]$ is a perverse sheaf. For a holomorphic vector bundle $E \to Y$, let $D^b_{mon}(E)$ be the derived category of the category of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on E whose cohomology sheaves are monodromic, i.e. locally constant over orbits of the natural $\mathbb{C}^\times$-action on E. Let $\mathrm{Perv}_{mon}(E)$ be the full subcategory of $\mathrm{Perv}(E)$ consisting of the monodromic perverse sheaves. We will need the Fourier transform

$$\mathbf{F} : D^b_{mon}(E) \to D^b_{mon}(E^*),$$

where $E^* \to Y$ is the dual bundle; see [G2, §8] or [Bry, §6] for its definition (our $\mathbf{F}$ is the normalized Fourier transform denoted $\widetilde{\mathcal{F}}$ on [Bry, p.69]). The Fourier transform induces an equivalence of categories

$$\mathbf{F} : \mathrm{Perv}_{mon}(E) \to \mathrm{Perv}_{mon}(E^*)$$

(see [Bry, Corollaire 7.23]), which corresponds under the Riemann-Hilbert correspondence to the formal Fourier transform on holonomic D-modules with regular singularities (see [Bry, Théorème 7.24]).

Lemma 4.1. *Let $E \to Y$ be a vector bundle on the smooth irreducible variety Y as above. Let $\iota : V \hookrightarrow E$ be a sub-bundle with annihilator $\bar\iota : V^\circ \hookrightarrow E^*$. Also* *let* $\hat{\iota} : \mathbf{0} \hookrightarrow E$ *be the zero sub-bundle. The unit of adjunction* $\mathrm{Id} \to \bar{\iota}_* \circ \bar{\iota}^{-1}$ *defines a canonical map* $\mathrm{res} : \mathbb{C}_{E^*} \to \bar{\iota}_* \mathbb{C}_{V^\circ}$*. Similarly there is a canonical map* $\mathrm{res} : \iota_* \mathbb{C}_V \to \hat{\iota}_* \mathbb{C}_{\mathbf{0}}$ *defined by applying* ι_* *to the unit of adjunction for the inclusion* $\mathbf{0} \hookrightarrow V$*. There are unique (up to scalars) horizontal isomorphisms in the following diagram:*

$$\begin{array}{ccc} \mathbf{DF}(\hat{\iota}_* \mathbb{C}_{\mathbf{0}}[\dim V]) & \xrightarrow{\sim} & \mathbb{C}_{E^*}[\dim V^\circ] \\ {\scriptstyle \mathbf{DF}(\mathrm{res})}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathbf{DF}(\iota_* \mathbb{C}_V[\dim V]) & \xrightarrow{\sim} & \bar{\iota}_* \mathbb{C}_{V^\circ}[\dim V^\circ] \end{array}$$

Moreover the scalars can be chosen so the diagram commutes.

Proof. As $\bar{\iota}_* \mathbb{C}_{V^\circ}[\dim V^\circ]$ is self-dual, the existence of the bottom isomorphism amounts to the assertion that $\mathbf{F}(\iota_* \mathbb{C}_V[\dim V]) \cong \bar{\iota}_* \mathbb{C}_{V^\circ}[\dim V^\circ]$, which is a basic property of Fourier transform; see [G2, Proposition 8.3(4)] or [KW, Corollary III.13.4] in the étale setting. The uniqueness of the bottom isomorphism follows as $\mathrm{End}(\bar{\iota}_* \mathbb{C}_{V^\circ}) \cong \mathbb{C}$. Existence and uniqueness of the top isomorphism is proved in a similar way. Finally the commutativity of the diagram is justified in [KW, Remark III.13.6] (we have applied $\mathbf{D}$ to the statement there). □

Now let $\mathfrak{g} := \mathfrak{gl}_d(\mathbb{C})$ and $\mathfrak{b}$ be the Borel subalgebra of upper triangular matrices. As always X and X_μ are the varieties of full flags and partial flags of type μ in $\mathbb{C}^d$. We abbreviate

$$r := 2 \dim X = d(d-1), \qquad r_\mu := 2 \dim X_\mu = d(d-1) - 2d_\mu.$$

We will always identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the trace form, noting that $\mathfrak{b}^\perp$ is the nilpotent radical $\mathfrak{n}$ of $\mathfrak{b}$. We can view $\mathfrak{g}$ as a self-dual vector bundle over a point, so that the Fourier transform for $\mathfrak{g}$ gives a self-equivalence

$$\mathbf{F} : \mathrm{Perv}_{mon}(\mathfrak{g}) \to \mathrm{Perv}_{mon}(\mathfrak{g}).$$

We also work with the trivial vector bundles $\mathfrak{g} \times X \to X$ and $\mathfrak{g} \times X_\mu \to X_\mu$. These are again identified with their duals, so Fourier transform gives two more self-equivalences

$$\mathbf{F} : \mathrm{Perv}_{mon}(\mathfrak{g} \times X) \to \mathrm{Perv}_{mon}(\mathfrak{g} \times X),$$
$$\mathbf{F} : \mathrm{Perv}_{mon}(\mathfrak{g} \times X_\mu) \to \mathrm{Perv}_{mon}(\mathfrak{g} \times X_\mu).$$

Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$, and set

$$\widetilde{\mathcal{N}} := \{(x, (U_0, \dots, U_d)) \in \mathcal{N} \times X \mid xU_i \subseteq U_{i-1} \text{ for each } i = 1, \dots, d\},$$
$$\widetilde{\mathfrak{g}} := \{(x, (U_0, \dots, U_d)) \in \mathfrak{g} \times X \mid xU_i \subseteq U_i \text{ for each } i = 1, \dots, d\}.$$

These are sub-bundles of $\mathfrak{g} \times X \to X$, with fibers isomorphic to $\mathfrak{n}$ and $\mathfrak{b}$, respectively. In fact, $\widetilde{\mathfrak{g}}$ is the annihilator of $\widetilde{\mathcal{N}}$ in the self-dual bundle $\mathfrak{g} \times X$, while $\widetilde{\mathcal{N}}$ is canonically isomorphic to the cotangent bundle T^*X of the flag variety; see e.g. [J,

§6.5]. Hence both varieties are smooth and irreducible, with $\dim \widetilde{\mathcal{N}} = 2\dim X = r$ and $\dim \widetilde{\mathfrak{g}} = \dim \mathfrak{g} = d^2$. Analogously in the parabolic case, we consider

$$\begin{aligned} \widetilde{\mathcal{N}}_\mu &:= \{(x,(V_0,\dots,V_n)) \in \mathcal{N} \times X_\mu \mid xV_i \subseteq V_{i-1} \text{ for each } i = 1,\dots,n\}, \\ \widetilde{\mathfrak{g}}_\mu &:= \{(x,(V_0,\dots,V_n)) \in \mathfrak{g} \times X_\mu \mid xV_i \subseteq V_i \text{ for each } i = 1,\dots,n\}, \end{aligned}$$

which are sub-bundles of $\mathfrak{g} \times X_\mu \to X_\mu$ such that $\widetilde{\mathfrak{g}}_\mu$ is the annihilator of $\widetilde{\mathcal{N}}_\mu$, and $\widetilde{\mathcal{N}}_\mu$ is isomorphic to the cotangent bundle T^*X_μ. So $\widetilde{\mathcal{N}}_\mu$ and $\widetilde{\mathfrak{g}}_\mu$ are smooth, irreducible varieties with $\dim \widetilde{\mathcal{N}}_\mu = 2\dim X_\mu = r_\mu$ and $\dim \widetilde{\mathfrak{g}}_\mu = \dim \mathfrak{g} = d^2$, respectively.

We will often exploit the following inclusions of sub-bundles:

$$\begin{tikzcd} \mathfrak{g}\times X & \widetilde{\mathfrak{g}} \arrow[l, hook', "\iota"'] & \widetilde{\mathcal{N}} \arrow[l, hook'] \arrow[ll, bend right, "\bar\iota"'] & \{0\}\times X \arrow[l, hook'] \arrow[lll, bend left, "\hat\iota"] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{g}\times X_\mu & \widetilde{\mathfrak{g}}_\mu \arrow[l, hook', "\iota'"'] & \widetilde{\mathcal{N}}_\mu \arrow[l, hook'] \arrow[ll, bend right, "\bar\iota'"'] & \{0\}\times X_\mu \arrow[l, hook'] \arrow[lll, bend left, "\hat\iota'"] \end{tikzcd}$$

Composing the named maps with the first projections $\mathrm{pr} : \mathfrak{g}\times X \to \mathfrak{g}$ and $\mathrm{pr}' : \mathfrak{g}\times X_\mu \to \mathfrak{g}$ gives the named maps in the following diagrams:

$$\begin{tikzcd} \mathfrak{g} & \widetilde{\mathfrak{g}} \arrow[l, "\pi"'] & \widetilde{\mathcal{N}} \arrow[l, hook'] \arrow[ll, bend right, "\bar\pi"'] & \{0\}\times X \arrow[l, hook'] \arrow[lll, bend left, "\hat\pi"] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{g} & \widetilde{\mathfrak{g}}_\mu \arrow[l, "\pi'"'] & \widetilde{\mathcal{N}}_\mu \arrow[l, hook'] \arrow[ll, bend right, "\bar\pi'"'] & \{0\}\times X_\mu \arrow[l, hook'] \arrow[lll, bend left, "\hat\pi'"] \end{tikzcd}$$

The morphism $\pi : \widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ here is part of Grothendieck's simultaneous resolution of the quotient of $\mathfrak{g}$ by the adjoint action of $G := GL_d(\mathbb{C})$. It is a projective, hence proper, morphism which is small in the sense of Goresky-Macpherson (see e.g. [J, Lemma 13.2]). Similarly the map $\pi' : \widetilde{\mathfrak{g}}_\mu \twoheadrightarrow \mathfrak{g}$ in the parabolic case is small. Hence $R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]$ and $R\pi'_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}[d^2]$ are (monodromic) perverse sheaves on $\mathfrak{g}$, which we call the *Grothendieck sheaf* and the *partial Grothendieck sheaf*, respectively.

There is a natural action of the Weyl group S_d on the Grothendieck sheaf. To recall some of the details of the construction of this action, let $\mathfrak{g}^{rs}$ denote the set of regular semisimple elements in $\mathfrak{g}$, and set $\widetilde{\mathfrak{g}}^{rs} := \pi^{-1}(\mathfrak{g}^{rs})$. Let π^{rs} be the restriction of π to $\widetilde{\mathfrak{g}}^{rs}$. Then $\pi^{rs} : \widetilde{\mathfrak{g}}^{rs} \twoheadrightarrow \mathfrak{g}^{rs}$ is a regular covering whose automorphism group is canonically identified with the Weyl group S_d; see e.g. [G2, Proposition 9.3]. So S_d acts freely on $\widetilde{\mathfrak{g}}^{rs}$ by deck transformations, and there is a unique isomorphism between $\mathfrak{g}^{rs}$ and the quotient $S_d\backslash\widetilde{\mathfrak{g}}^{rs}$ making the following diagram commute:

$$\begin{array}{ccc} & \widetilde{\mathfrak{g}}^{rs} & \\ {}^{\pi^{rs}}\swarrow & & \searrow^{\mathrm{can}} \\ \mathfrak{g}^{rs} & \xrightarrow{\sim} & S_d\backslash\widetilde{\mathfrak{g}}^{rs}. \end{array} \tag{4.1}$$

The action of S_d on $\widetilde{\mathfrak{g}}^{rs}$ induces an action on the local system $\pi_*^{rs}\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}}$. Let IC be the intersection cohomology functor from local systems on $\mathfrak{g}^{rs}$ to perverse sheaves on $\mathfrak{g}$. Applying it to $\pi_*^{rs}\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}}$, we get an induced action of S_d on the perverse sheaf $IC(\pi_*^{rs}\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}})$. Finally, as in [G2, Corollary 9.2] or [J, Theorem 13.5], the smallness of π implies that there is a unique isomorphism

$$\kappa : IC(\pi_*^{rs}\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}}) \xrightarrow{\sim} R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2] \tag{4.2}$$

such that the composition

$$\pi_*^{rs}\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}} \xrightarrow{\sim} \mathcal{H}^{-d^2}(IC(\pi_*^{rs}\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}}))|_{\mathfrak{g}^{rs}} \xrightarrow{\sim} \mathcal{H}^0(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}})|_{\mathfrak{g}^{rs}} \xrightarrow{\sim} \pi_*^{rs}\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}}$$

is the identity, where the first and last isomorphisms are canonical and the middle one is induced by κ. Using the isomorphism κ, we transport the action of S_d to get the desired action on $R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]$. It is important to note that this action induces an algebra isomorphism

$$\mathbb{C}S_d \xrightarrow{\sim} \mathrm{End}(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]). \tag{4.3}$$

This follows by the perverse continuation principle (see e.g. [G2, Proposition 4.5]) as the action on $\pi_*^{rs}\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}}$ obviously defines an isomorphism $\mathbb{C}S_d \xrightarrow{\sim} \mathrm{End}(\pi_*^{rs}\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}})$.

The partial Grothendieck sheaf was studied in detail by Borho and Macpherson, who explained how to identify it with S_μ-invariants in the Grothendieck sheaf; see especially [BM2, Proposition 2.7(a)]. We need to reformulate this result slightly. Consider the map $\tilde{p} : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}_\mu$ arising from the restriction of the map $\mathrm{id} \times p : \mathfrak{g} \times X \to \mathfrak{g} \times X_\mu$. Applying $R\pi'_*$ to the unit of adjunction $\mathrm{Id} \to \tilde{p}_* \circ \tilde{p}^{-1}$ evaluated at $\mathbb{C}_{\widetilde{g}_\mu}$, we get a map

$$BM : R\pi'_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}[d^2] \to R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]. \tag{4.4}$$

Note also as $\pi^{-1}(\{0\}) = \{0\} \times X \cong X$ and ${\pi'}^{-1}(\{0\}) = \{0\} \times X_\mu \cong X_\mu$ that the proper base change theorem gives us canonical isomorphisms $H^*(X, \mathbb{C}) \cong (R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}})_0$ and $H^*(X_\mu, \mathbb{C}) \cong (R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}})_0$ between $H^*(X, \mathbb{C})$ and $H^*(X_\mu, \mathbb{C})$ (viewed here as a complexes of vector spaces rather than graded vector spaces) and the stalks of the Grothendieck and partial Grothendieck sheaves at the origin. Moreover, under the isomorphism $H^*(X, \mathbb{C}) \cong (R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}})_0$, the classical action of S_d on $H^*(X, \mathbb{C})$ agrees with the action of S_d on $(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}})_0$ induced by its action on the Grothendieck sheaf; see e.g. [J, Lemma 13.6].

Theorem 4.2. *The morphism BM just defined gives an isomorphism of perverse sheaves*

$$BM : R\pi'_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}[d^2] \xrightarrow{\sim} R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]^{S_\mu\text{-inv}}.$$

Moreover, letting BM_0 denote the induced map between stalks at the origin, the following diagram commutes:

$$\begin{array}{ccc} H^*(X_\mu, \mathbb{C}) & \xrightarrow{p^*} & H^*(X, \mathbb{C}) \\ {\scriptstyle\mathrm{can}}\downarrow & & \downarrow{\scriptstyle\mathrm{can}} \\ (R\pi'_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu})_0 & \xrightarrow[BM_0]{} & (R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}})_0 \end{array} \tag{4.5}$$

Proof. The second statement is clear from the definition (4.4) and naturality of proper base change.

For the first statement, the idea is to compare our morphism BM with the isomorphism $R\pi'_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}[d^2] \xrightarrow{\sim} R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]^{S_\mu\text{-inv}}$ constructed in [BM2, Proposition 2.7(a)]. Let $S_\mu\backslash\widetilde{\mathfrak{g}}^{rs}$ denote the quotient of $\widetilde{\mathfrak{g}}^{rs}$ by the action of the parabolic subgroup S_μ. Let $\widetilde{\mathfrak{g}}_\mu^{rs} := \pi'^{-1}(\mathfrak{g}^{rs})$ and $\pi'^{\,rs}$ be the restriction of π' to $\mathfrak{g}^{rs}$. The key point is that there is a unique isomorphism $\widetilde{g}_\mu^{rs} \xrightarrow{\sim} S_\mu\backslash\widetilde{\mathfrak{g}}^{rs}$ making the following diagram

$$\begin{array}{ccc} & \widetilde{\mathfrak{g}}^{rs} & \\ {}^{\tilde{p}^{rs}}\swarrow & & \searrow^{\text{can}} \\ \widetilde{\mathfrak{g}}_\mu^{rs} & \xrightarrow{\sim} & S_\mu\backslash\widetilde{\mathfrak{g}}^{rs} \\ {\scriptstyle \pi'^{\,rs}}\downarrow & & \downarrow{\scriptstyle \text{can}} \\ \mathfrak{g}^{rs} & \xrightarrow{\sim} & S_d\backslash\widetilde{\mathfrak{g}}^{rs} \end{array}$$

commute, where the bottom isomorphism comes from (4.1) and $\tilde{p}^{rs}$ is the restriction of $\tilde{p}$. This isomorphism induces an isomorphism of local systems $\pi'^{\,rs}_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu^{rs}} \xrightarrow{\sim} \left(\pi^{rs}_*\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}}\right)^{S_\mu\text{-inv}}$. Now apply the functor IC to get an isomorphism of perverse sheaves

$$IC(\pi'^{\,rs}_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu^{rs}}) \xrightarrow{\sim} IC(\pi^{rs}_*\mathbb{C}_{\widetilde{\mathfrak{g}}^{rs}})^{S_\mu\text{-inv}}.$$

The Borho-Macpherson isomorphism $R\pi'_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}[d^2] \xrightarrow{\sim} R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]^{S_\mu\text{-inv}}$ is obtained from this by composing on the right with the inverse of the isomorphism κ from (4.2) and on the left with the analogously defined isomorphism $\kappa' : IC(\pi'^{\,rs}_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu^{rs}}) \xrightarrow{\sim} R\pi'_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}[d^2]$ from [BM2, Proposition 2.6].

To complete the proof it remains to observe that our map BM is equal to the map from the previous paragraph. This follows by the perverse continuation principle, since the two maps agree on restriction to $\mathfrak{g}^{rs}$. □

As in the statement of Lemma 4.1, the units of adjunction associated to the maps $\bar{\iota} : \widetilde{\mathcal{N}} \hookrightarrow \mathfrak{g} \times X$ and $\hat{\iota} : \{0\} \times X \hookrightarrow \mathfrak{g} \times X$ induce canonical maps $\text{res} : \mathbb{C}_{\mathfrak{g}\times X} \to \bar{\iota}_*\mathbb{C}_{\widetilde{\mathcal{N}}}$ and $\text{res} : \hat{\iota}_*\mathbb{C}_{\{0\}\times X} \to \iota_*\mathbb{C}_{\widetilde{\mathfrak{g}}}$. Applying the derived functor $R\,\text{pr}_*$, we get morphisms

$$\alpha : \mathbb{C}_{\mathfrak{g}} \otimes H^*(X, \mathbb{C}) \to R\bar{\pi}_*\mathbb{C}_{\widetilde{\mathcal{N}}}, \tag{4.6}$$

$$\beta : R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}} \to \delta_0 \otimes H^*(X, \mathbb{C}), \tag{4.7}$$

where we are viewing $H^*(X, \mathbb{C})$ as a graded vector space with $H^i(X, \mathbb{C})$ in degree i, δ_0 denotes the constant skyscraper sheaf on $\mathfrak{g}$ supported at the origin, and we have implicitly composed with the canonical isomorphisms $R\,\text{pr}_*\mathbb{C}_{\mathfrak{g}\times X} \cong \mathbb{C}_{\mathfrak{g}} \otimes H^*(X, \mathbb{C})$, $R\,\text{pr}_*(\bar{\iota}_*\mathbb{C}_{\widetilde{\mathcal{N}}}) \cong R\bar{\pi}_*\mathbb{C}_{\widetilde{\mathcal{N}}}$, $R\,\text{pr}_*(\hat{\iota}_*\mathbb{C}_{\{0\}\times X}) \cong R\hat{\pi}_*\mathbb{C}_{\{0\}\times X} \cong \delta_0 \otimes H^*(X, \mathbb{C})$ and $R\,\text{pr}_*(\iota_*\mathbb{C}_{\widetilde{\mathfrak{g}}}) \cong R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}$. Also let

$$\gamma : \mathbf{DF}(\delta_0) \otimes H_*(X, \mathbb{C}) \to \mathbf{DF}(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}) \tag{4.8}$$

be the morphism obtained from β by applying the composite functor $\mathbf{DF}$, noting that $\mathbf{DF}(\delta_0 \otimes H^*(X, \mathbb{C})) \cong \mathbf{DF}(\delta_0) \otimes H_*(X, \mathbb{C})$ if $H_*(X, \mathbb{C})$ is identified with the graded dual of the graded vector space $H^*(X,\mathbb{C})$ (so $H_i(X,\mathbb{C})$ is in degree $-i$). Repeating all this in exactly the same way in the parabolic case, we get analogous morphisms

$$\alpha' : \mathbb{C}_{\mathfrak{g}} \otimes H^*(X_\mu,\mathbb{C}) \to R\bar{\pi}'_* \mathbb{C}_{\widetilde{\mathcal{N}}_\mu}, \tag{4.9}$$

$$\beta' : R\pi'_* \mathbb{C}_{\widetilde{\mathfrak{g}}_\mu} \to \delta_0 \otimes H^*(X_\mu,\mathbb{C}), \tag{4.10}$$

$$\gamma' : \mathbf{DF}(\delta_0) \otimes H_*(X_\mu,\mathbb{C}) \to \mathbf{DF}(R\pi'_* \mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}). \tag{4.11}$$

Finally note that there exists a unique (up to scalars) isomorphism

$$\sigma : \mathbf{DF}(\delta_0[d^2]) \xrightarrow{\sim} \mathbb{C}_{\mathfrak{g}}, \tag{4.12}$$

as follows for example from the top isomorphism in Lemma 4.1 taking $V = E = E^* = \mathfrak{g}$. We fix a choice of such a map. In the regular case the following proposition is essentially [KW, Theorem VI.15.1]. We are repeating some of the details of the proof to make it clear that it extends to the parabolic case.

Proposition 4.3. *There exist unique isomorphisms τ and τ' making the following diagrams commute:*

$$\begin{array}{ccc} \mathbf{DF}(\delta_0[d^2]) \otimes H_*(X,\mathbb{C}) & \xrightarrow[\sigma\otimes PD]{\sim} & \mathbb{C}_{\mathfrak{g}} \otimes H^*(X,\mathbb{C})[r] \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \mathbf{DF}(R\pi_* \mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]) & \xrightarrow[\tau]{\sim} & R\bar{\pi}_* \mathbb{C}_{\widetilde{\mathcal{N}}}[r] \end{array} \tag{4.13}$$

$$\begin{array}{ccc} \mathbf{DF}(\delta_0[d^2]) \otimes H_*(X_\mu,\mathbb{C}) & \xrightarrow[\sigma\otimes PD']{\sim} & \mathbb{C}_{\mathfrak{g}} \otimes H^*(X_\mu,\mathbb{C})[r_\mu] \\ {\scriptstyle\gamma'}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ \mathbf{DF}(R\pi'_* \mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}[d^2]) & \xrightarrow[\tau']{\sim} & R\bar{\pi}'_* \mathbb{C}_{\widetilde{\mathcal{N}}_\mu}[r_\mu] \end{array} \tag{4.14}$$

Here $PD : H_i(X,\mathbb{C}) \to H^{r-i}(X,\mathbb{C})$ and $PD' : H_i(X_\mu,\mathbb{C}) \to H^{r_\mu - i}(X_\mu,\mathbb{C})$ are the isomorphisms defined by Poincaré duality.

Proof. We just explain the proof for τ, since the argument for τ' is similar.

For existence, we apply Lemma 4.1 with $E = \mathfrak{g} \times X = E^*$, $V = \widetilde{\mathfrak{g}}$ and $V^\circ = \widetilde{\mathcal{N}}$ to get a commuting diagram

$$\begin{array}{ccc} \mathbf{DF}(\hat{\iota}_* \mathbb{C}_{\{0\}\times X}[d^2]) & \xrightarrow{\sim} & \mathbb{C}_{\mathfrak{g}\times X}[r] \\ {\scriptstyle\mathbf{DF}(\mathrm{res})}\downarrow & & \downarrow{\scriptstyle\mathrm{res}} \\ \mathbf{DF}(\iota_* \mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]) & \xrightarrow{\sim} & \bar{\iota}_* \mathbb{C}_{\widetilde{\mathcal{N}}}[r] \end{array}$$

Now apply $R\,\mathrm{pr}_*$, using Verdier duality and the fact that Fourier transform commutes with direct images (see [Bry, Proposition 6.8] or [KW, Theorem III.13.3] in the étale setting) which together give us an isomorphism of functors $\mathbf{DF} \circ R\,\mathrm{pr}_* \xrightarrow{\sim} R\,\mathrm{pr}_* \circ \mathbf{DF}$. We get the following commutative diagram:

$$\begin{array}{ccccc} \mathbf{DF}(R\,\mathrm{pr}_*(\hat\iota_*\mathbb{C}_{\{0\}\times X}[d^2])) & \xrightarrow{\sim} & R\,\mathrm{pr}_*(\mathbf{DF}(\hat\iota_*\mathbb{C}_{\{0\}\times X}[d^2])) & \xrightarrow{\sim} & R\,\mathrm{pr}_*(\mathbb{C}_{\mathfrak{g}\times X}[r]) \\ \downarrow{\scriptstyle \mathbf{DF}(R\,\mathrm{pr}_*(\mathrm{res}))} & & \downarrow{\scriptstyle R\,\mathrm{pr}_*(\mathbf{DF}(\mathrm{res}))} & & \downarrow{\scriptstyle R\,\mathrm{pr}_*(\mathrm{res})} \\ \mathbf{DF}(R\,\mathrm{pr}_*(\iota_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2])) & \xrightarrow{\sim} & R\,\mathrm{pr}_*(\mathbf{DF}(\iota_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2])) & \xrightarrow{\sim} & R\,\mathrm{pr}_*(\bar\iota_*\mathbb{C}_{\widetilde{\mathcal{N}}}[r]) \end{array}$$

Recalling the identifications made in the definitions of α, β and γ above, this gives the commuting square (4.13) on checking that the top map in the diagram constructed is equal up to a scalar to the map $\sigma \otimes PD$ at the top of (4.13). The latter statement amounts to showing that the graded vector space maps $\zeta : H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C})$ and $\xi : H_*(X,\mathbb{C}) \to H^*(X,\mathbb{C})[r]$ defined by the following commutative diagrams are equal (up to scalars) to the maps id and PD, respectively (the vertical identifications in these diagrams are various maps induced by unique up to scalars isomorphisms):

$$\begin{array}{ccc} \mathbf{F}(R\,\mathrm{pr}_*(\hat\iota_*\mathbb{C}_{\{0\}\times X}[d^2])) & \xrightarrow{\sim} & R\,\mathrm{pr}_*(\mathbf{F}(\hat\iota_*\mathbb{C}_{\{0\}\times X}[d^2])) \\ \| & & \| \\ \mathbb{C}_{\mathfrak{g}} \otimes H^*(X,\mathbb{C})[2d^2] & \xrightarrow{\mathrm{id}\otimes\zeta} & \mathbb{C}_{\mathfrak{g}} \otimes H^*(X,\mathbb{C})[2d^2] \end{array}$$

$$\begin{array}{ccc} \mathbf{D}(R\,\mathrm{pr}_*(\mathbb{C}_{\mathfrak{g}\times X}[2d^2])) & \xrightarrow{\sim} & R\,\mathrm{pr}_*(\mathbf{D}(\mathbb{C}_{\mathfrak{g}\times X}[2d^2])) \\ \| & & \| \\ \mathbb{C}_{\mathfrak{g}} \otimes H_*(X,\mathbb{C}) & \xrightarrow{\mathrm{id}\otimes\xi} & \mathbb{C}_{\mathfrak{g}} \otimes H^*(X,\mathbb{C})[r] \end{array}$$

To see that ξ is proportional to PD, push-forward to a point and use the usual connection between Verdier duality and Poincaré duality. It is more difficult to see that ζ is proportional to id because the isomorphism $R\,\mathrm{pr}_* \circ \mathbf{F} \cong \mathbf{F} \circ R\,\mathrm{pr}_*$ is only defined implicitly. One way to avoid this issue is to apply the functor $\rho^!$ where $\rho : \{0\} \hookrightarrow \mathfrak{g}$ is the inclusion, noting that the induced isomorphism $\rho^! \circ R\,\mathrm{pr}_* \circ \mathbf{F} \cong \rho^! \circ \mathbf{F} \circ R\,\mathrm{pr}_*$ of triangulated functors is actually unique (up to scalars) as both functors are isomorphic to the global hypercohomology functor $\mathbf{H}^*(\mathfrak{g} \times X, ?)$.

For the uniqueness of τ, it suffices to show that the map

$$\mathrm{End}\left(\mathbf{DF}(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2])\right) \to \mathrm{Hom}\left(\mathbf{DF}(\delta_0[d^2]) \otimes H_*(X,\mathbb{C}), \mathbf{DF}(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2])\right)$$

mapping f to $f \circ \gamma$ is injective. Equivalently, the map

$$\mathrm{End}\left(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]\right) \to \mathrm{Hom}\left(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2], \delta_0[d^2] \otimes H^*(X,\mathbb{C})\right)$$

mapping f to $\beta \circ f$ is injective. To see the latter statement, suppose that $\beta \circ f = 0$. Then the induced map $(\beta \circ f)_0 = \beta_0 \circ f_0$ between stalks at 0 is also zero. But the stalks at 0 of all the spaces under consideration are naturally identified with

$H^*(X,\mathbb{C})$ in such a way that $\beta_0 = \mathrm{id}$, hence we get that $f_0 = 0$. It remains to observe that the map

$$\mathrm{End}(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}) \to \mathrm{End}(H^*(X,\mathbb{C})), \qquad f \mapsto f_0$$

is injective. This follows from (4.3), recalling that $H^*(X,\mathbb{C})$ is isomorphic to the group algebra $\mathbb{C}S_d$ as a module over the symmetric group thanks to a classical result of Chevalley [Ch] about the coinvariant algebra. (The analogous statement needed in the parabolic case, namely, that the map

$$\mathrm{End}(R\pi'_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}) \to \mathrm{End}(H^*(X_\mu,\mathbb{C})), \qquad f \mapsto f_0$$

is injective, is a consequence of the injectivity just established in the regular case, since $R\pi'_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}$ is isomorphic to a summand of $R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}$ by Theorem 4.2 and the semisimplicity of $\mathbb{C}S_d$.) □

In view of Proposition 4.3, $R\bar{\pi}_*\mathbb{C}_{\widetilde{\mathcal{N}}}[r]$ and $R\bar{\pi}'_*\mathbb{C}_{\widetilde{\mathcal{N}}_\mu}[r_\mu]$ are (monodromic) perverse sheaves on $\mathfrak{g}$, which we call the *Springer sheaf* and the *Spaltenstein sheaf*, respectively. Letting $h : \mathcal{N} \hookrightarrow \mathfrak{g}$ be the inclusion and $\tilde{\pi} : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the usual Springer resolution, we have canonical isomorphisms

$$R\bar{\pi}_*\mathbb{C}_{\widetilde{\mathcal{N}}}[r] \cong h_*(R\tilde{\pi}_*\mathbb{C}_{\widetilde{\mathcal{N}}}[r]) \cong h_*(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[r]|_{\mathcal{N}}), \tag{4.15}$$

the first of which follows as $\bar{\pi} = h \circ \tilde{\pi}$, and the second comes from the proper base change theorem, noting that $\widetilde{\mathcal{N}} = \pi^{-1}(\mathcal{N})$. If we apply the functor $h_* \circ h^{-1}$ to the action of S_d on the Grothendieck sheaf, we get an action of S_d on the perverse sheaf on the right hand side of (4.15). Using the above isomorphisms we lift this to an action of S_d on the Springer sheaf itself.

Proposition 4.4. *The maps α and β from (4.6)–(4.7) are S_d-equivariant, where the actions of S_d on $\mathbb{C}_{\mathfrak{g}} \otimes H^*(X,\mathbb{C})$ and $\delta_0 \otimes H^*(X,\mathbb{C})$ are induced by its natural action on $H^*(X,\mathbb{C})$.*

Proof. This follows as in [KW, Proposition VI.14.1] (our maps α and β are compositions of pairs of the maps considered there). □

The following important corollary is essentially [KW, Corollary VI.15.2].

Corollary 4.5. *The isomorphism $\tau : \mathbf{DF}(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2]) \xrightarrow{\sim} R\bar{\pi}_*\mathbb{C}_{\widetilde{\mathcal{N}}}[r]$ in (4.13) is S_d-equivariant up to a twist by sign, i.e. we have that $\tau \circ w = \mathrm{sgn}(w) w \circ \tau$ for each $w \in S_d$, where the action of S_d on $\mathbf{DF}(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2])$ is defined by applying the functor $\mathbf{DF}$ to its action on the Grothendieck sheaf.*

Proof. Proposition 4.4 implies that the vertical maps α and γ in the diagram (4.13) are S_d-equivariant, where the action on $\mathbf{DF}(\delta_0[d^2]) \otimes H_*(X,\mathbb{C})$ arises from the dual of the natural action on $H^*(X,\mathbb{C})$. Also the map $\sigma \otimes PD$ at the top of the diagram (4.13) is S_d-equivariant up to a twist by sign because the top cohomology $H^r(X,\mathbb{C})$ is the sign representation of the symmetric group. The corollary now follows from the commutativity of the diagram and the uniqueness of τ in Proposition 4.3. □

Now we are ready for the main theorem of the section, which roughly speaking is the Fourier transform of Borho-Macpherson's Theorem 4.2. Recall the isomorphisms τ and τ' from Proposition 4.3, and the morphism BM from (4.4). Let

$$\overline{BM} := \tau' \circ \mathbf{DF}(BM) \circ \tau^{-1} : R\bar{\pi}_* \mathbb{C}_{\widetilde{\mathcal{N}}}[r] \to R\bar{\pi}'_* \mathbb{C}_{\widetilde{\mathcal{N}}_\mu}[r_\mu]. \tag{4.16}$$

Although τ and τ' depend up to a scalar on the choice of the map σ in (4.12), it is clear that the map $\overline{BM}$ is independent of this choice.

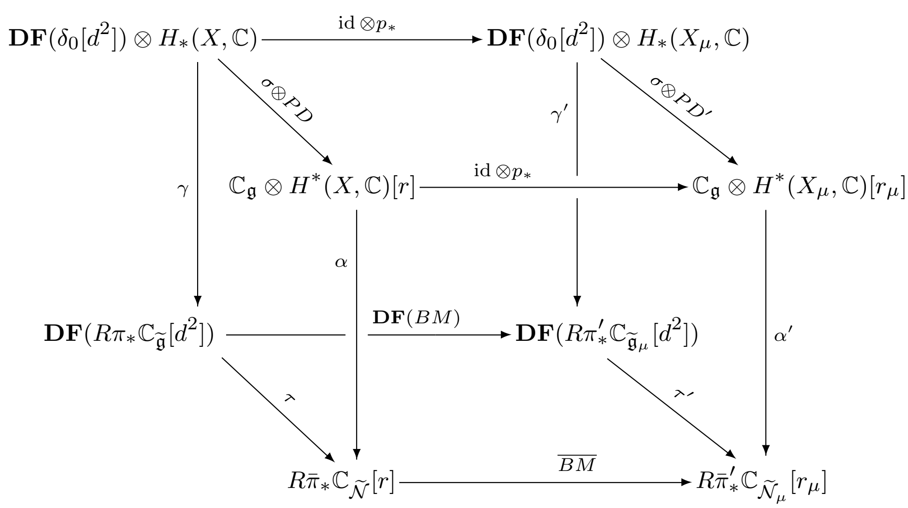
Theorem 4.6. *The morphism $\overline{BM}$ just defined restricts to an isomorphism of perverse sheaves*

$$\overline{BM} : R\bar{\pi}_* \mathbb{C}_{\widetilde{\mathcal{N}}}[r]^{S_\mu\text{-anti}} \xrightarrow{\sim} R\bar{\pi}'_* \mathbb{C}_{\widetilde{\mathcal{N}}_\mu}[r_\mu].$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}_{\mathfrak{g}} \otimes H^*(X, \mathbb{C})[r] & \xrightarrow[\mathrm{id} \otimes p_*]{} & \mathbb{C}_{\mathfrak{g}} \otimes H^*(X_\mu, \mathbb{C})[r_\mu] \\ \alpha \downarrow & & \downarrow \alpha' \\ R\bar{\pi}_* \mathbb{C}_{\widetilde{\mathcal{N}}}[r] & \xrightarrow[\overline{BM}]{} & R\bar{\pi}'_* \mathbb{C}_{\widetilde{\mathcal{N}}_\mu}[r_\mu] \end{array} \tag{4.17}$$

Proof. Consider the following cube:



In order to prove the second statement of the theorem, we need to show that the front face commutes, which follows if we can show that all the other faces commute. The commutativity of the top face is clear, the left and right faces commute by Proposition 4.3, and the bottom face commutes by the definition of $\overline{BM}$. Thus it remains to show that the back face commutes. Equivalently, by the definitions of γ and γ', we need to show that the following diagram commutes:

$$\begin{array}{ccc} \delta_0 \otimes H^*(X, \mathbb{C}) & \xleftarrow[\mathrm{id} \otimes p^*]{} & \delta_0 \otimes H^*(X_\mu, \mathbb{C}) \\ \beta \uparrow & & \uparrow \beta' \\ R\pi_* \mathbb{C}_{\widetilde{\mathfrak{g}}} & \xleftarrow[BM]{} & R\pi'_* \mathbb{C}_{\widetilde{\mathfrak{g}}_\mu} \end{array}$$

To see this, we just need to check commutativity of all the induced diagrams at the level of stalks. This is trivial apart from the stalk at 0, and at 0 it follows from (4.5).

To deduce the first statement, consider the bottom square in our commuting cube. Applying the functor $\mathbf{DF}$ to the first statement of Theorem 4.2, we know already that $\mathbf{DF}(BM)$ restricts to an isomorphism between $\mathbf{DF}(R\pi_*\mathbb{C}_{\widetilde{\mathfrak{g}}}[d^2])^{S_\mu\text{-inv}}$ and $\mathbf{DF}(R\pi'_*\mathbb{C}_{\widetilde{\mathfrak{g}}_\mu}[d^2])$. Now use Corollary 4.5. □

Remark 4.7. Although we are only considering the case $\mathfrak{g} = \mathfrak{gl}_d(\mathbb{C})$ here, Theorem 4.6 remains true (with appropriate notational changes) on replacing $\mathfrak{g}$ by an arbitrary semisimple Lie algebra. The proof in general is essentially the same as above.

Finally we explain how to deduce Theorem 1.2 from Theorem 4.6. The uniqueness in the statement of Theorem 1.2 is clear as the map i^* is surjective. To prove the existence of $\bar{p}_*$, apply the functor $\mathcal{H}^{i-r}(?)_{x^\lambda}$ (the stalk of the $(i-r)$th cohomology sheaf at the point x^λ) to the commutative square (4.17), and note by proper base change that there are canonical isomorphisms

$$\mathcal{H}^{i-r}(R\bar{\pi}_*\mathbb{C}_{\widetilde{\mathcal{N}}}[r])_{x^\lambda} \cong H^i(\bar{\pi}^{-1}(x^\lambda), \mathbb{C}_{\widetilde{\mathcal{N}}}) \cong H^i(X^\lambda, \mathbb{C}), \tag{4.18}$$

$$\mathcal{H}^{i-r}(R\bar{\pi}'_*\mathbb{C}_{\widetilde{\mathcal{N}}_\mu}[r_\mu])_{x^\lambda} \cong H^{i-2d_\mu}(\bar{\pi}'^{-1}(x^\lambda), \mathbb{C}_{\widetilde{\mathcal{N}}_\mu}) \cong H^{i-2d_\mu}(X^\lambda_\mu, \mathbb{C}). \tag{4.19}$$

Under these isomorphisms, the maps induced by α and α' correspond to the maps i^* and j^* in (1.9), respectively, as a consequence of naturality in the proper base change theorem. The last statement of Theorem 1.2 follows from the first statement of Theorem 4.6, since by Proposition 4.4 the action of S_d on $H^i(X^\lambda, \mathbb{C})$ induced by its action on $R\bar{\pi}_*\mathbb{C}_{\widetilde{\mathcal{N}}}[r]$ via (4.18) is exactly the Springer action being considered in Theorem 1.2.

5. Proof of Theorems 1.1 and 3.2

Now everything is ready to prove Theorem 1.1 (and complete the proof of Theorem 3.2). Consider the diagram (1.6). In order to show that the composition $j^* \circ \psi$ factors through the quotient to induce a homorphism $\bar{\psi} : C^\lambda \to H^*(X^\lambda, \mathbb{C})$, we need to show that

$$j^*(\psi(e_r(\mu; i_1, \dots, i_m))) = 0 \tag{5.1}$$

for $m, i_1, \dots, i_m$ and r as in (1.5). Let $\nu \in \Lambda(n, d)$ be a composition obtained by rearranging the parts of μ so that $\nu_j = \mu_{i_j}$ for each $j = 1, \dots, m$. Let $w \in S_d$ be the unique permutation of minimal length such that $wS_\nu w^{-1} = S_\mu$. The natural action of w on P induces an algebra isomorphism $w : C_\nu \xrightarrow{\sim} C_\mu$.

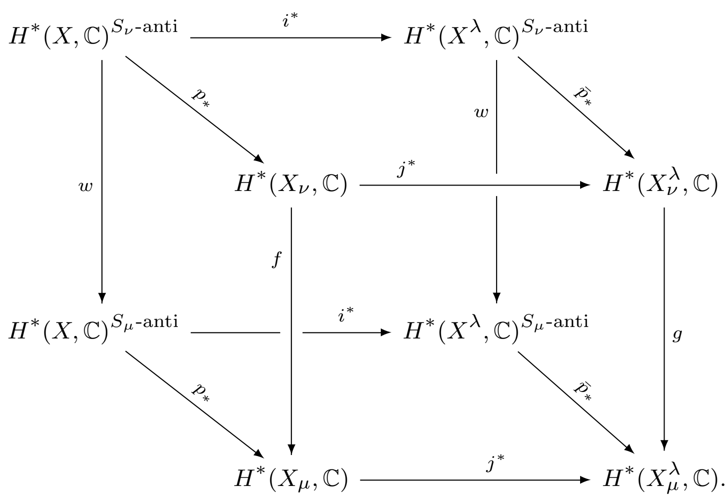
Lemma 5.1. *There exist unique algebra isomorphisms f and g making the following diagram commute:*

$$\begin{array}{ccccc}
C_\nu & \xrightarrow{\psi} & H^*(X_\nu, \mathbb{C}) & \xrightarrow{j^*} & H^*(X^\lambda_\nu, \mathbb{C}) \\
{\scriptstyle w}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
C_\mu & \xrightarrow{\psi} & H^*(X_\mu, \mathbb{C}) & \xrightarrow{j^*} & H^*(X^\lambda_\mu, \mathbb{C}).
\end{array}$$

Proof. The existence and uniqueness of f making the left hand square commute is clear, as the other three maps in this square are algebra isomorphisms. For the right hand square, we first observe that the diagram

$$\begin{array}{ccc} C^{S_\nu\text{-anti}} & \xrightarrow{\sim} & C_\nu \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle w} \\ C^{S_\mu\text{-anti}} & \xrightarrow{\sim} & C_\mu \end{array}$$

commutes, where the horizontal maps are as in (1.8) and the map w is induced by the natural action of w on C. This follows using the algebraic description of the horizontal maps given just after (1.8), together with the observation that $w(\varepsilon_\nu) = \varepsilon_\mu$ by the choice of w. Combining this with (1.8), we deduce that the left face of the following cube commutes:



The vertical maps on the back face are induced by the natural action of w on $H^*(X,\mathbb{C})$ and on $H^*(X^\lambda,\mathbb{C})$, respectively, and it is clear that this face commutes because $i^* : H^*(X,\mathbb{C}) \twoheadrightarrow H^*(X^\lambda,\mathbb{C})$ is S_d-equivariant. Also the top and bottom faces commute by Theorem 1.2. Finally, we let g be the unique isomorphism making the right face commute. Then the front face commutes too, and this establishes the existence of the map g as in the statement of the lemma. The uniqueness of g and the fact that it is an algebra homomorphism follow because f is a homomorphism and j^* is surjective according to Corollary 2.5. □

Continuing with the notation fixed just before Lemma 5.1, we observe that $w(e_r(\nu;1,\dots,m)) = e_r(\mu;i_1,\dots,i_m)$. So Lemma 5.1 implies that

$$j^*(\psi(e_r(\mu;i_1,\dots,i_m))) = g(j^*(\psi(e_r(\nu;1,\dots,m)))).$$

Replacing μ by ν if necessary, this reduces the proof of (5.1) to the special case that $i_j = j$ for each $j = 1,\dots,m$, i.e. we just need to show that

$$j^*(\psi(e_r(x_1,\dots,x_k))) = 0 \tag{5.2}$$

for all $r \geq k - h$, where $k := \mu_1 + \cdots + \mu_m$, $h := \lambda_{l+1} + \cdots + \lambda_n$ and $l := \#\{i = m+1, \dots, n \mid \mu_i > 0\}$. In view of Corollary 2.6, we may as well assume further that $\lambda \geq \mu^+$, hence that $h \leq k$.

In order to establish (5.2), we follow Tanisaki's original argument from [T] closely. We work again with the Schubert varieties $Y_\gamma \subseteq \mathrm{Gr}_{k,d-k}$ from (2.2). However we switch to using the basis $f_1, \dots, f_d$ for $\mathbb{C}^d$ obtained by reading the boxes of the Young diagram in the opposite order to §2, for example in the situation of (2.4) we now use the labelling

$$\begin{array}{|c|c|c|c|}\hline f_9 & f_8 & f_7 & f_6 \\ \hline f_5 & f_4 & f_3 \\ \cline{1-3} f_2 & f_1 \\ \cline{1-2}\end{array} \qquad\qquad x^\lambda = \downarrow. \tag{5.3}$$

The fundamental classes $[Y_\gamma] \in H_{2|\gamma|}(\mathrm{Gr}_{k,d}, \mathbb{C})$ give a basis for $H_*(\mathrm{Gr}_{k,d}, \mathbb{C})$. Let $\{\sigma_\gamma \mid \gamma \in \mathscr{P}_{k,d-k}\}$ be the dual basis for $H^*(\mathrm{Gr}_{k,d}, \mathbb{C})$.

Lemma 5.2. *Let $s : X_\mu \to \mathrm{Gr}_{k,d}$, $(V_0, \dots, V_n) \mapsto V_m$ be the natural projection. We have that $s^*(\sigma_\gamma) = \psi(s_\gamma(x_1, \dots, x_k))$ for each $\gamma \in \mathscr{P}_{k,d-k}$, where $s_\gamma(x_1, \dots, x_k)$ is the image in C_μ of the Schur function associated to the partition γ.*

Proof. This follows from the analogous statement for the full flag variety X, which is classical. To give a little more detail, consider the projection $r : X \to \mathrm{Gr}_{k,d}$, $(U_0, \dots, U_d) \mapsto U_k$. By [F, §10.6], we have that $r^*(\sigma_\gamma) = \varphi(s_\gamma(x_1, \dots, x_k))$. Now use (1.7) and the fact that $r = s \circ p$. □

Set $\alpha := ((d-k)^{h-k}) \in \mathscr{P}_{k,d-k}$ and consider the Schubert variety

$$Y_\alpha = \{U \in \mathrm{Gr}_{k,d} \mid \langle f_1, \dots, f_h \rangle \subseteq U\}.$$

It is obvious that $Y_\alpha \cong \mathrm{Gr}_{k-h,d-h}$.

Lemma 5.3. *The projection $s : X_\mu \to \mathrm{Gr}_{k,d}$ from Lemma 5.2 restricts to a morphism $\bar{s} : X_\mu^\lambda \to Y_\alpha$.*

Proof. Take $(V_0, \dots, V_n) \in X_\mu^\lambda$. We need to show that $\langle f_1, \dots, f_h \rangle \subseteq V_m$. We have that $(x^\lambda)^l(V_n) \subseteq V_m$ by the definition of l. The image of $(x^\lambda)^l$ is the span of the basis vectors f_i labelling boxes in rows $> l$ in the Young diagram of λ. Because of our choice of labelling (recall (5.3)) this is exactly $\langle f_1, \dots, f_h \rangle$. Hence $\langle f_1, \dots, f_h \rangle \subseteq V_m$. □

Now we can establish (5.2) for any $r > k - h$. Clearly we can assume that $r \leq k < d$. Set $\delta := (1^r) \in \mathscr{P}_{k,d-k}$ and observe that $s_\delta(x_1, \dots, x_k) = e_r(x_1, \dots, x_k)$. So in view of Lemma 5.2 we just need to show that $j^*(s^*(\sigma_\delta)) = 0$. Letting $v : Y_\alpha \hookrightarrow \mathrm{Gr}_{k,d-k}$ be the inclusion, we have from Lemma 5.3 that $s \circ j = v \circ \bar{s}$. So $j^*(s^*(\sigma_\delta)) = \bar{s}^*(v^*(\sigma_\delta))$ and we are done if we can show that $v^*(\sigma_\delta) = 0$. But this is clear as the classes $\{v_*([Y_\gamma]) \mid \gamma \subseteq \alpha\}$ form a basis for $H_*(Y_\alpha, \mathbb{C})$ and $\delta \not\subseteq \alpha$.

Finally we can complete the proof of Theorem 1.1. We have shown so far that there is a unique homomorphism $\bar{\psi} : C_\mu^\lambda \to H^*(X_\mu^\lambda, \mathbb{C})$ making the diagram (1.6) commute. Moreover $\bar{\psi}$ is surjective since j^* is surjective by Corollary 2.5. It remains to observe that $\dim C_\mu^\lambda \leq |\operatorname{Col}_\mu^\lambda|$ by the spanning part of Theorem 3.2 established in §3, while $\dim H^*(X_\mu^\lambda, \mathbb{C}) = |\operatorname{Col}_\mu^\lambda|$ according to Corollary 2.5. Hence $\bar{\psi}$ is actually an isomorphism, and Theorem 1.1 is proved. At the same time, we have shown that $\dim C_\mu^\lambda = |\operatorname{Col}_\mu^\lambda|$, so the vectors $\{h(\mathtt{T}) \mid \mathtt{T} \in \operatorname{Col}_\mu^\lambda\}$ which are already known to span C_μ^λ must actually be linearly independent. Thus we have also proved Theorem 3.2.

Remark 5.4. Theorem 1.1 also holds over the integers. More precisely, if one replaces every occurence of $\mathbb{C}$ with $\mathbb{Z}$ in the definitions of the algebras C^λ and C_μ^λ in the introduction, then the statement of Theorem 1.1 with $\mathbb{C}$ replaced by $\mathbb{Z}$ is still true. So $H^*(X_\mu^\lambda, \mathbb{Z})$ is isomorphic to the analogue of C_μ^λ over the integers. Moreover, the latter algebra is a free $\mathbb{Z}$-module with basis given by the elements listed in Theorem 3.2. To see all this, we just note that the proof of Corollary 2.5 works over $\mathbb{Z}$, giving in particular that $H^*(X_\mu^\lambda, \mathbb{Z})$ is a free $\mathbb{Z}$-module. Also the proof of the spanning part of Theorem 3.2 given in §3 is valid over $\mathbb{Z}$. Finally, the analogue of the key relation (5.1) over $\mathbb{Z}$, which is all that is needed to adapt the argument in the previous paragraph, follows from the relation established over $\mathbb{C}$ thanks to the freeness of $H^*(X_\mu^\lambda, \mathbb{Z})$.

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