REPRESENTATIONS OF THE ORIENTED SKEIN CATEGORY

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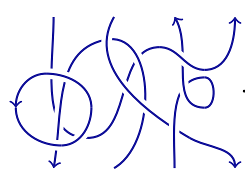
ABSTRACT. The *oriented skein category* $\mathcal{OS}(z,t)$ is a ribbon category which underpins the definition of the HOMFLY-PT invariant of an oriented link, in the same way that the Temperley-Lieb category underpins the Jones polynomial. In this article, we develop its representation theory using a highest weight theory approach. This allows us to determine the Grothendieck ring of its additive Karoubi envelope for all possible choices of parameters, including the (already well-known) semisimple case, and all non-semisimple situations. Then we construct a graded lift of $\mathcal{OS}(z,t)$ by realizing it as a 2-representation of a Kac-Moody 2-category. We also discuss the degenerate analog of $\mathcal{OS}(z,t)$, which is the *oriented Brauer category* $\mathcal{OB}(\delta)$.

1. Introduction

1.1. We begin by recalling briefly the definition of the category $\mathcal{FOT}$ of *framed oriented tangles*; this is the framed analog of the oriented tangle category $\mathcal{OT}$ introduced by Turaev in [T2] and also appears in [EGNO, Remark 8.10.3] where it is denoted $\mathcal{FT}$. By definition, it is the strict monoidal category with objects given by the set $\langle\uparrow,\downarrow\rangle$ of all words in the letters $\uparrow$ and $\downarrow$. Tensor product of objects is given by concatenation, e.g., $\uparrow\otimes\uparrow\otimes\downarrow\;=\;\uparrow\uparrow\downarrow$, and the unit object $\mathbb{1}$ is the empty word $\varnothing$. For two words $\mathsf{a}=\mathsf{a}_m\cdots\mathsf{a}_1, \mathsf{b}=\mathsf{b}_n\cdots\mathsf{b}_1\in\langle\uparrow,\downarrow\rangle$, morphisms $f:\mathsf{a}\to\mathsf{b}$ are isotopy classes of framed oriented tangles in $[0,1]\times[0,1]\times\mathbb{R}$ with boundary

$$\left\{\left(\tfrac{m+1-i}{m+1},0,0\right)\;\middle|\;i=1,\dots,m\right\}\cup\left\{\left(\tfrac{n+1-j}{n+1},1,0\right)\;\middle|\;j=1,\dots,n\right\},$$

such that the orientation in the y-direction near the boundary points $\left(\frac{m+1-i}{m+1},0,0\right)$ and $\left(\frac{n+1-j}{n+1},1,0\right)$ are a_i and b_j, respectively. We will draw such tangles by projecting onto the xy-plane in such a way that the implicit framing is "blackboard," and there are no triple intersections or tangencies; we also keep track of "over" or "under" data at each crossing. We call the resulting diagrams (a,b)-*ribbons* for short. For example, here is a $(\downarrow\uparrow\uparrow\downarrow,\downarrow\downarrow\uparrow\uparrow)$-ribbon:

 . (1.1)

Isotopy translates into the equivalence relation on diagrams generated by planar isotopy fixing the boundary, together with the oriented Reidemeister moves (FRI) (*not* the full (RI) due to framing!), (RII) and (RIII) displayed in Figure 1. Composition of morphisms in $\mathcal{FOT}$ is given by vertically stacking diagrams, i.e., $f\circ g:=\begin{smallmatrix}f\\g\end{smallmatrix}$, while tensor product is given by horizontal concatenation, i.e., $f\otimes g:=fg$.

Now let $\Bbbk$ be some fixed commutative ground ring and fix parameters $z,t\in\Bbbk^\times$. The *extended oriented skein category* $\widehat{\mathcal{OS}}(z,t)$ is the quotient of the $\Bbbk$-linearization of $\mathcal{FOT}$

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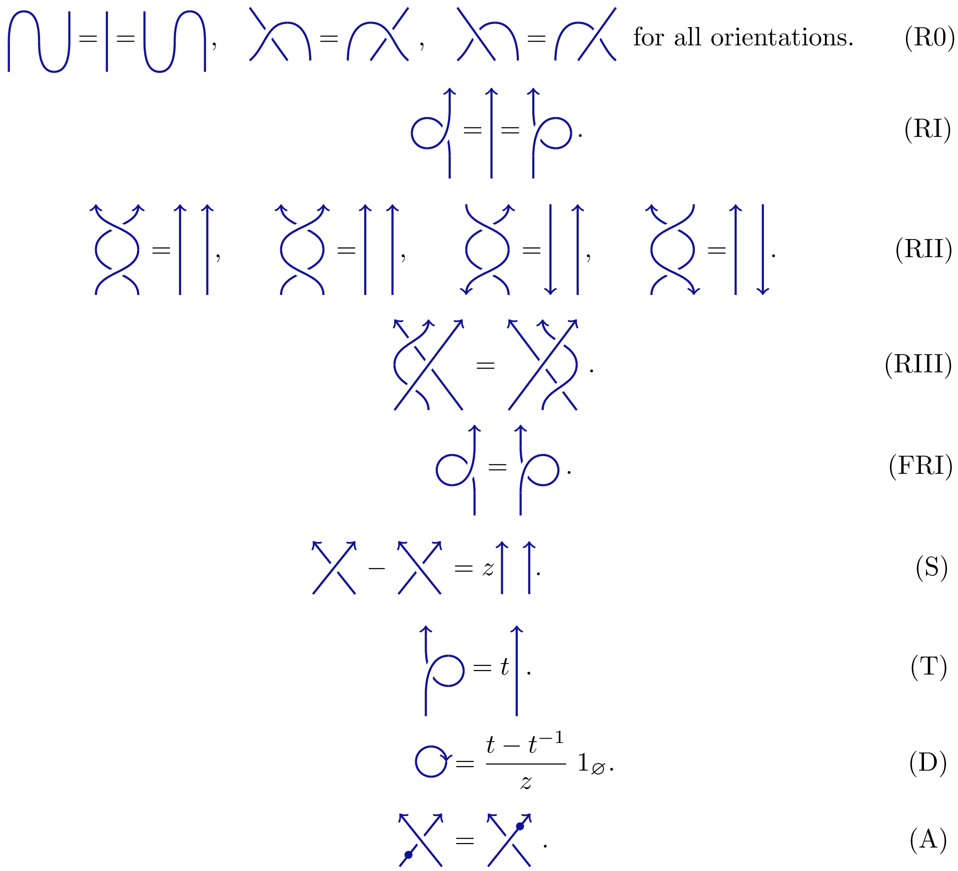
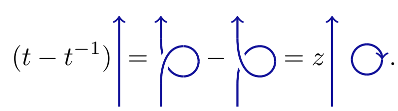


FIGURE 1. Reidemeister-type relations

by the $\Bbbk$-linear tensor ideal generated by the Conway skein relation (S) and the twist relation (T), both of which are displayed in Figure 1. These relations imply that



(1.2)

Usually, we will impose the additional dimension relation (D) from Figure 1. We call the resulting category the (reduced) *oriented skein category*, and denote it simply by $\mathcal{OS}(z,t)$. This is the main object of study in this article.

With a different normalization of crossings, the category $\mathcal{OS}(z,t)$ was introduced in [T2, §5.2], where it is called the *Hecke category*. In [QS, Definition 2.1] it is called the *quantized oriented Brauer category*. Others call $\mathcal{OS}(z,t)$ the *framed HOMFLY-PT skein category*.

1.2. Let us state some foundational results about $\mathcal{OS}(z,t)$. The first one gives an efficient monoidal presentation. It is a corollary of a more general result of Turaev [T3, Lemma I.3.3] which gives a presentation for the category $\mathcal{FOT}$.

Theorem 1.1. *The oriented skein category $\mathcal{OS}(z,t)$ is isomorphic to the strict $\Bbbk$-linear monoidal category generated by objects E and F and morphisms*

$$S : E \otimes E \to E \otimes E, \quad T : F \otimes E \to E \otimes F, \quad C : \mathbb{1} \to F \otimes E, \quad D : E \otimes F \to \mathbb{1},$$

subject to the following relations:

(1) $S^2 = zS + 1_E \otimes 1_E$;
(2) $(S \otimes 1_E) \circ (1_E \otimes S) \circ (S \otimes 1_E) = (1_E \otimes S) \circ (S \otimes 1_E) \circ (1_E \otimes S)$;
(3) $(D \otimes 1_E) \circ (1_E \otimes C) = 1_E$, $(1_F \otimes D) \circ (C \otimes 1_F) = 1_F$;
(4) $T^{-1} = (1_F \otimes 1_E \otimes D) \circ (1_F \otimes S \otimes 1_F) \circ (C \otimes 1_E \otimes 1_F)$ *(two-sided inverse);*
(5) $tD \circ T \circ C = \frac{t-t^{-1}}{z} 1_{\mathbb{1}}$.

An explicit functor giving an isomorphism from the monoidal category with this presentation to $\mathcal{OS}(z,t)$ sends $E \mapsto \uparrow$, $F \mapsto \downarrow$, and the generating morphisms S, T, C and D to , , and , respectively. We strongly encourage the reader to verify that the relations (1)–(5) from the theorem all hold in $\mathcal{OS}(z,t)$ by drawing the appropriate pictures!

1.3. The next theorem gives bases for morphism spaces. Again, this is due to Turaev [T2, Theorems 5.1 and 5.2.3]; Turaev notes that it was also proved independently by Morton and Traczyk. To formulate it, given $\mathsf{a} = \mathsf{a}_m \cdots \mathsf{a}_1, \mathsf{b} = \mathsf{b}_n \cdots \mathsf{b}_1 \in \langle \uparrow, \downarrow \rangle$, an (a, b)-*matching* means a bijection

$$\left\{ \left(\tfrac{m+1-i}{m+1}, 0, 0\right) \;\middle|\; \mathsf{a}_i = \uparrow \right\} \cup \left\{ \left(\tfrac{n+1-j}{n+1}, 1, 0\right) \;\middle|\; \mathsf{b}_j = \downarrow \right\}$$
$$\xrightarrow{\sim} \left\{ \left(\tfrac{m+1-i}{m+1}, 0, 0\right) \;\middle|\; \mathsf{a}_i = \downarrow \right\} \cup \left\{ \left(\tfrac{n+1-j}{n+1}, 1, 0\right) \;\middle|\; \mathsf{b}_j = \uparrow \right\}.$$

There are no such bijections unless the domain and codomain have the same size d, in which case there are $d!$ possibilities. An (a, b)-ribbon is a *lift* of a given (a, b)-matching if the boundary of each strand in the ribbon consists of a pair of points which correspond under the matching; in particular, it contains no "floating bubbles." An (a, b)-ribbon is *reduced* if no strand crosses itself and no two strands cross more than once.

Theorem 1.2. *The morphism space* $\mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathsf{a}, \mathsf{b})$ *is free as a* $\Bbbk$*-module with basis given by any set consisting of a reduced lift for each of the* (a, b)*-matchings. The same is true in* $\widehat{\mathcal{OS}}(z,t)$ *with one exception: if* $\mathsf{a} = \mathsf{b} = \varnothing$ *then the morphism space* $\mathrm{Hom}_{\widehat{\mathcal{OS}}(q,t)}(\varnothing, \varnothing)$ *is free of rank two with basis* $\left\{ 1_\varnothing, \right\}$.

The algebra $\mathrm{Hom}_{\widehat{\mathcal{OS}}(z,t)}(\varnothing, \varnothing)$ appearing in Theorem 1.2 is known in the literature as the *Conway skein module* [T1] or the *framed HOMFLY-PT skein module* [MS, Definition 2.1] of the manifold $\mathbb{R}^3$. The basis $\left\{ 1_\varnothing, \right\}$ for it described in the theorem is implicit already in [HOMFLY, PT], indeed, the existence of the HOMFLY-PT polynomial for oriented links constructed in those papers follows easily from this result. To explain this briefly, let L be an oriented link diagram, and define writhe(L) as usual to be the number of positive crossings minus the number of negative crossings. Viewing L as a $(\varnothing, \varnothing)$-ribbon, there is a unique scalar $H(L) \in \Bbbk$ such that

$$t^{-\operatorname{writhe}(L)} L = H(L)$$

in $\mathrm{End}_{\widehat{\mathcal{OS}}(q,t)}(\varnothing)$. The scalar $H(L)$ is invariant under the Reidemeister moves (RI), (RII) and (RIII); for all but (RI), this is automatic from the defining relations in $\mathcal{FOT}$, while (RI) follows from (T) and (FRI). The relation (S) implies that

$$tH(L_+) - t^{-1} H(L_-) = zH(L_0)$$

for oriented link diagrams L_+, L_- and L_0 which agree except in one place, which is a positive crossing in L_+, a negative crossing in L_-, and the crossing is resolved in L_0. This is exactly the skein relation defining the HOMFLY-PT polynomial. Finally, observe that $H(L) = 1$ in case L is the unknot. Hence, taking $\Bbbk := \mathbb{Z}[z, z^{-1}, t, t^{-1}]$, the scalar $H(L)$ is exactly the HOMFLY-PT polynomial of L.

1.4. Let H_r be the *Iwahori-Hecke algebra* of the symmetric group $\mathfrak{S}_r$ with quadratic relation $S^2 = zS+1$; if $z = q-q^{-1}$ this can be written equivalently as $(S-q)(S+q^{-1}) = 0$. There is a homomorphism

$$\imath_r : H_r \to \mathrm{End}_{\mathcal{OS}(z,t)}(\uparrow^r) \tag{1.3}$$

sending the generator for H_r that corresponds to the ith basic transposition to the positive crossing  of the ith and $(i+1)$th strand, numbering strands by $1, \dots, r$ from right to left. The main step in Turaev's proof of Theorem 1.2 is to show that $\imath_r$ is an isomorphism. This is deduced ultimately from Jimbo's *quantized Schur-Weyl reciprocity* from [J], which connects H_r to the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$.

In fact, quantized Schur-Weyl reciprocity can be upgraded to the following well-known result. For a $\Bbbk$-linear category $\mathcal{C}$, we write $\dot{\mathcal{C}}$ for its *additive Karoubi envelope*, that is, the idempotent completion of its additive envelope; in case $\mathcal{C}$ is monoidal, $\dot{\mathcal{C}}$ is monoidal too.

Theorem 1.3. *Assume that $\Bbbk$ is a field of characteristic zero, $q \in \Bbbk^\times$ is not a root of unity, $z = q - q^{-1}$, and $t = q^{\varepsilon n}$ for $n \in \mathbb{N}$ and $\varepsilon \in \{\pm\}$. There is a full $\Bbbk$-linear monoidal functor $\Psi : \mathcal{OS}(z,t) \to \mathrm{Rep}\, U_q(\mathfrak{gl}_n)$ sending $\uparrow$ to the natural $U_q(\mathfrak{gl}_n)$-module and $\downarrow$ to its dual. It induces a monoidal equivalence*

$$\bar{\Psi} : \dot{\mathcal{OS}}(z,t)/\mathcal{N} \xrightarrow{\approx} \mathrm{Rep}\, U_q(\mathfrak{gl}_n), \tag{1.4}$$

where $\mathcal{N}$ is the tensor ideal of $\dot{\mathcal{OS}}(z,t)$ consisting of negligible morphisms (see [De, §6.1]*). As an additive $\Bbbk$-linear tensor ideal, $\mathcal{N}$ is generated by $\imath_{n+1}(e)$ where $e \in H_{n+1}$ is the Young symmetrizer associated to the sign representation if $\varepsilon = +$ or the trivial representation if $\varepsilon = -$.*

The evident ribbon structure on $\mathcal{OS}(z,t)$ induces a ribbon structure on $\mathrm{Rep}\, U_q(\mathfrak{gl}_n)$ so that $\bar{\Psi}$ is an equivalence of ribbon categories. This induced ribbon structure depends on the sign ε; we denote the resulting ribbon category by $\mathrm{Rep}\, U_q(\mathfrak{gl}_{\varepsilon n})$. When $\Bbbk = \mathbb{C}$, q is not a root of unity, and δ is any complex number, the category

$$\underline{\mathrm{Rep}}\, U_q(\mathfrak{gl}_\delta) := \dot{\mathcal{OS}}(q-q^{-1}, q^\delta) \tag{1.5}$$

is the q-analog of the Deligne category $\underline{\mathrm{Rep}}\, GL_\delta$ introduced in [DM] (see also [De, §10]) and studied recently in [CW, EHS]. In $\underline{\mathrm{Rep}}\, U_q(\mathfrak{gl}_\delta)$, relation (D) implies that the objects $\uparrow$ and $\downarrow$ have categorical dimension

$$[\delta]_q := \frac{q^\delta - q^{-\delta}}{q - q^{-1}}.$$

For $n \in \mathbb{Z}$, $[n]_q$ is the usual quantum integer, and Theorem 1.3 shows that the ribbon category $\mathrm{Rep}\, U_q(\mathfrak{gl}_n)$ is a quotient of $\underline{\mathrm{Rep}}\, U_q(\mathfrak{gl}_n)$. Thus, the categories $\underline{\mathrm{Rep}}\, U_q(\mathfrak{gl}_\delta)$ for $\delta \in \mathbb{C}$ interpolate between the categories $\mathrm{Rep}\, U_q(\mathfrak{gl}_n)$. It is also known that the category $\underline{\mathrm{Rep}}\, U_q(\mathfrak{gl}_\delta)$ is semisimple when $\delta \notin \mathbb{Z}$; we will say more about this shortly.

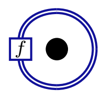
Most of the recent literature on diagrammatic approaches to $\mathrm{Rep}\, U_q(\mathfrak{gl}_n)$ focuses instead on variants of the "SL_n-spider" of Cautis, Kamnitzer and Morrison from [CKM]. In [QS, Definition 6.4], this $\mathbb{C}$-linear monoidal category is upgraded to a ribbon category $\mathsf{Sp}(\delta)$ depending on $q \in \mathbb{C}^\times$ (not a root of unity) and a parameter $\delta \in \mathbb{C}$; we prefer to denote $\mathsf{Sp}(\delta)$ by $\mathcal{W}eb(\delta)$. According to [QS, Proposition 6.7], $\mathcal{W}eb(\delta)$ is a thickening (i.e., a partial idempotent completion) of $\mathcal{OS}(q-q^{-1}, q^\delta)$, so that the Deligne category $\underline{\mathrm{Rep}}\, U_q(\mathfrak{gl}_\delta)$ may also be realized as the additive Karoubi envelope of $\mathcal{W}eb(\delta)$. Subsequent developments in the literature have revolved around 2-categorifications related to Khovanov-Rozansky homology; e.g., see [MW].

1.5. We are interested here instead in the decategorification of $\mathcal{OS}(z,t)$. There are two basic ways to understand this: either by taking the trace, or by passing to the Grothendieck ring. Let us briefly recall these definitions.

By the *trace* of a $\Bbbk$-linear category $\mathcal{C}$, we mean the $\Bbbk$-module

$$\operatorname{Tr}(\mathcal{C}) := \bigoplus_{X \in \operatorname{ob}\mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(X,X) \Big/ \left\langle f \circ g - g \circ f \;\middle|\; \begin{array}{l} \text{for all } X, Y \in \operatorname{ob}\mathcal{C} \text{ and} \\ f : X \to Y, g : Y \to X \end{array} \right\rangle.$$

One can represent the image $[f] \in \operatorname{Tr}(\mathcal{C})$ of $f \in \operatorname{Hom}_{\mathcal{C}}(X,X)$ diagrammatically by drawing f in an annulus:



. (1.6)

If $\mathcal{C}$ is a monoidal category, then $\operatorname{Tr}(\mathcal{C})$ is a $\Bbbk$-algebra with $[f][g] := [f \otimes g]$. Note also that $\operatorname{Tr}(\mathcal{C})$ and $\operatorname{Tr}(\dot{\mathcal{C}})$ may be identified; see [BGHL, Proposition 3.2].

The *Grothendieck group* $K_0(\dot{\mathcal{C}})$ is the $\mathbb{Z}$-module generated by isomorphism classes $[X]$ of objects X in $\dot{\mathcal{C}}$ modulo the relations $[X \oplus Y] = [X] + [Y]$. Also, a *$\mathcal{C}$-module* means a $\Bbbk$-linear functor from $\mathcal{C}^{\mathrm{op}}$ to the category of $\Bbbk$-modules; we write Mod-$\mathcal{C}$ for the category of all such modules. The Yoneda embedding induces an equivalence between $\dot{\mathcal{C}}$ and the full subcategory pMod-$\mathcal{C}$ of Mod-$\mathcal{C}$ consisting of finitely generated projective $\mathcal{C}$-modules. So any finitely generated projective $\mathcal{C}$-module M also defines a class $[M] \in K_0(\dot{\mathcal{C}})$.

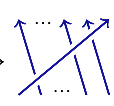
The notions of trace and Grothendieck group are related by the *character map*

$$h : K_0(\dot{\mathcal{C}}) \otimes_{\mathbb{Z}} \Bbbk \to \operatorname{Tr}(\mathcal{C}), \qquad [X] \mapsto [1_X]. \tag{1.7}$$

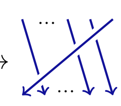
Typically, this map is injective, e.g., it is so if $\Bbbk$ is an algebraically closed field and all of the morphism spaces of $\mathcal{C}$ are finite-dimensional; see [BHLW, Proposition 2.4]. If in addition $\dot{\mathcal{C}}$ is semisimple then h is an isomorphism; see [BHLW, Proposition 2.5] for a more general statement here. In case $\mathcal{C}$ is monoidal, $K_0(\dot{\mathcal{C}})$ is a ring with $[X][Y] := [X \otimes Y]$, and h is a ring homomorphism.

The trace of $\mathcal{OS}(z,t)$ was computed originally by Turaev [T1, Theorem 2], albeit from a rather different point of view: it is exactly the Conway skein module of the solid torus, as follows by contemplating the picture (1.6). Turaev's result can be formulated as follows.

Theorem 1.4. *The algebra* $\operatorname{Tr}(\mathcal{OS}(z,t))$ *is the free polynomial algebra* $\Bbbk[u_n, v_n \,|\, n \geq 1]$ *generated by the trace classes* u_n *and* v_n *of the following "cycles" for all* $n \geq 1$*:*

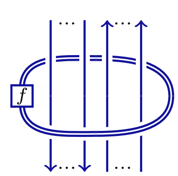
$u_n \leftrightarrow$  ,

n strands

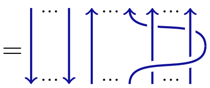
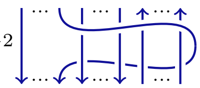
$v_n \leftrightarrow$  .

n strands

Theorem 1.2 implies that the algebra $B_{r,s} := \operatorname{End}_{\mathcal{OS}(z,t)}(\downarrow^s \uparrow^r)$ is free as a $\Bbbk$-module of rank $(r+s)!$. This is the *quantized walled Brauer algebra* introduced originally by Kosuda and Murakami in [KM1, KM2]. As pointed out by Morton [M], any $[f] \in \operatorname{Tr}(\mathcal{OS}(z,t))$ defines a central element



in $B_{r,s}$. Morton conjectured that these elements generate the entire center $Z(B_{r,s})$. Morton's conjecture has recently been proved in [JK] assuming $\Bbbk$ is a field of characteristic zero and z, t are generic. In fact, Jung and Kim show that $Z(B_{r,s})$ is generated already by the supersymmetric power sums $p_n(X_1, \dots, X_r | Y_1, \dots, Y_s) = X_1^n + \cdots + X_r^n - Y_1^n - \cdots - Y_s^n$ in the *Jucys-Murphy elements*

$X_i :=$  $,\qquad Y_j := t^{-2}$  $,$ (1.8)

where the interesting strand is the ith or $(r+j)$th from the right, respectively. These elements were also introduced by Morton (extending an observation from [Ra] in the case of the Iwahori-Hecke algebra): up to an obvious symmetry and rescaling they are the elements T and U from the proof of [M, Theorem 1]; see [JK, Remark 6.7]. (Jung and Kim also prove a version of [SS, Conjecture 7.4] in the degenerate case.) Later in the article, we will give a more conceptual interpretation of Jucys-Murphy elements based on another monoidal category, the *affine oriented skein category* $\mathcal{AOS}(z,t)$, which is of independent interest.

1.7. *For the remainder of the introduction*, we assume that $\Bbbk$ is a field and $z = q - q^{-1}$ for $q \in \Bbbk^\times \setminus \{\pm 1\}$. The next theorem describes $K_0(\dot{\mathcal{OS}}(z,t))$ in all semisimple cases. It is also possible to compute the irreducible characters $h(\chi_{\boldsymbol\lambda}) \in \Bbbk[u_n, v_n \,|\, n \geq 1]$ by an algorithm involving Starkey's rule [Ge].

To state the theorem, let $\mathrm{Bip} = \coprod_{r,s \geq 0} \mathrm{Bip}_{r,s}$ where $\mathrm{Bip}_{r,s}$ consists of *bipartitions* $\boldsymbol\lambda = (\lambda^\uparrow, \lambda^\downarrow)$ for $\lambda^\uparrow \vdash r$ and $\lambda^\downarrow \vdash s$. Let Sym be the ring of symmetric functions and denote the Schur function associated to a partition λ by χ_λ. The structure constants for this basis of Sym are the Littlewood-Richardson coefficients: $\chi_\mu \chi_\nu = \sum_\lambda LR^\lambda_{\mu,\nu} \chi_\lambda$. Let $\lambda^{\mathtt{t}}$ denote the conjugate partition to λ.

Theorem 1.5. *The category $\dot{\mathcal{OS}}(z,t)$ is semisimple if and only if q is not a root of unity and $t \notin \{\pm q^n | n \in \mathbb{Z}\}$. Assuming this is the case, the isomorphism classes of indecomposable objects in $\dot{\mathcal{OS}}(z,t)$ are parametrized in a canonical way by* Bip. *Moreover, the rings $K_0(\dot{\mathcal{OS}}(z,t))$ and $\mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$ may be identified so that the isomorphism class of the indecomposable indexed by $\boldsymbol\lambda \in \mathrm{Bip}_{r,s}$ identifies with*

$$\chi_{\boldsymbol\lambda} := \sum_{\substack{0 \leq d \leq \min(r,s) \\ \boldsymbol\mu \in \mathrm{Bip}_{r-d,s-d}}} N^{\boldsymbol\lambda}_{\boldsymbol\mu} \chi_{\mu^\uparrow} \otimes \chi_{\mu^\downarrow} \quad \textit{where} \quad N^{\boldsymbol\lambda}_{\boldsymbol\mu} := (-1)^d \sum_{\nu \vdash d} LR^{\lambda^\uparrow}_{\mu^\uparrow,\nu} LR^{\lambda^\downarrow}_{\mu^\downarrow,\nu^{\mathtt{t}}}. \tag{1.9}$$

The standard technique to prove Theorem 1.5 is to deduce it from Theorem 1.3 by similar arguments to [De, Proposition 10.6]; see also [CW, Theorems 4.8.1 and 7.1.1]. In other words, one uses Schur-Weyl duality and well-known properties of $\operatorname{Rep} U_q(\mathfrak{gl}_n)$ for sufficiently large n. We will take a completely different approach to the proof of Theorem 1.5 and the representation theory of $\mathcal{OS}(z,t)$ in general (even in positive characteristic or at roots of unity) based on the simple observation that it has a *triangular decomposition*. This allows us to adapt the usual arguments of highest weight theory in a way that is reminiscent of the general framework of [HN, BT].

In this triangular decomposition, the "Cartan subalgebra" $\mathcal{OS}^\circ(z,t)$ is the monoidal subcategory consisting of all objects, and morphisms spanned by diagrams containing neither caps nor cups in which all upward propagating strands pass underneath downward propagating strands. The "positive Borel subalgebra" $\mathcal{OS}^\sharp(z,t)$ is defined similarly, allowing also cups but no caps. Inflation from $\mathcal{OS}^\circ(z,t)$ to $\mathcal{OS}^\sharp(z,t)$ followed by induction from there to $\mathcal{OS}(z,t)$ defines an exact *standardization functor*

$$\Delta : \text{Mod-}\mathcal{OS}^\circ(z,t) \to \text{Mod-}\mathcal{OS}(z,t). \tag{1.10}$$

Moreover, there is an obvious equivalence of categories

$$\text{Mod-}\mathcal{OS}^\circ(z,t) \approx \prod_{r,s\geq 0} \text{Mod-}H_r \otimes H_s. \tag{1.11}$$

Since the Hecke algebra is semisimple when q is not a root of unity, the semisimplicity part of Theorem 1.5 is a consequence of the following more general result.

Theorem 1.6. *If $t \notin \{\pm q^n \mid n \in \mathbb{Z}\}$ then Δ is an equivalence of categories.*

The other basic observation used to compute $K_0(\dot{\mathcal{OS}}(z,t))$ as a ring is:

Theorem 1.7. *The inclusion $\mathcal{OS}^\circ(z,t) \to \mathcal{OS}(z,t)$ induces a ring isomorphism*

$$K_0(\dot{\mathcal{OS}}^\circ(z,t)) \xrightarrow{\sim} K_0(\dot{\mathcal{OS}}(z,t)).$$

Now we describe $K_0(\dot{\mathcal{OS}}(z,t))$ for all choices of q and t. There are four cases.

- Suppose first that q is not a root of unity. Up to isomorphism, the irreducible representations of the (semisimple) Hecke algebra H_r are the *Specht modules* parametrized by partitions of r. Using the Morita equivalence (1.11), we deduce that the irreducible $\mathcal{OS}^\circ(z,t)$-modules are parametrized by bipartitions; we denote them $\{\mathrm{S}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathrm{Bip}\}$. Their standardizations give us a family of $\mathcal{OS}(z,t)$-modules $\{\Delta(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathrm{Bip}\}$.
  - When $t \notin \{\pm q^n \mid n \in \mathbb{Z}\}$ (so that $\dot{\mathcal{OS}}(z,t)$ is semisimple), the modules $\Delta(\boldsymbol{\lambda})$ give a full set of pairwise inequivalent indecomposable $\mathcal{OS}(z,t)$-modules. This is the labelling from Theorem 1.5: in the identification of $K_0(\dot{\mathcal{OS}}(z,t))$ with $\mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$ we have that
  $$[\Delta(\boldsymbol{\lambda})] \leftrightarrow \chi_{\boldsymbol{\lambda}}. \tag{1.12}$$
  - When $t = \pm q^n$ for $n \in \mathbb{Z}$, we will show that Mod-$\mathcal{OS}(z,t)$ has the structure of an *upper-finite highest weight category* with standard modules $\{\Delta(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathrm{Bip}\}$. This is a slight generalization of the usual notion of highest weight category; e.g., see [EL, §6.1.2]. Each standard module $\Delta(\boldsymbol{\lambda})$ has a unique irreducible quotient $\mathrm{L}(\boldsymbol{\lambda})$, and these give a full set of pairwise inequivalent irreducible $\mathcal{OS}(z,t)$-modules. Moreover, the projective cover $\mathrm{P}(\boldsymbol{\lambda})$ of $\Delta(\boldsymbol{\lambda})$ has a finite Δ-flag with multiplicities satisfying BGG reciprocity; however, unlike for usual highest weight categories, standard modules have infinite length. In this situation, the Grothendieck ring $K_0(\dot{\mathcal{OS}}(z,t))$ is identified with the *same* ring $\mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$ as for generic t so that
  $$[\mathrm{P}(\boldsymbol{\mu})] \leftrightarrow \sum_{\substack{0\leq d\leq \min(r,s)\\ \boldsymbol{\lambda}\in \mathrm{Bip}_{r-d,s-d}}} [\Delta(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})]\chi_{\boldsymbol{\lambda}} \tag{1.13}$$
  for $\boldsymbol{\mu} \in \mathrm{Bip}_{r,s}$. It remains to compute the numbers $[\Delta(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})]$. This turns out to be quite straightforward: they are all either 0 or 1 and can be computed using the cup diagrams of [BS]. The combinatorics is discussed in detail elsewhere; e.g., see [CW, EHS] (with Theorem 1.12 in mind).
- Now suppose that q^2 is a primitive eth root of unity for $e > 1$. Then the situation is more complicated as the Hecke algebras are no longer semisimple. Let $e\text{-Bip} = \coprod_{r,s\geq 0} e\text{-Bip}_{r,s}$ be the set of *e-restricted bipartitions*. By [DJ1] and (1.11), the "Specht module" $\mathrm{S}(\boldsymbol{\lambda})$ has irreducible head $\mathrm{D}(\boldsymbol{\lambda})$ if $\boldsymbol{\lambda}$ is e-restricted, and the modules $\{\mathrm{D}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in e\text{-Bip}\}$ give a full set of pairwise inequivalent irreducible $\mathcal{OS}^\circ(z,t)$-modules. Also let $\mathrm{Y}(\boldsymbol{\lambda})$ be a projective cover of $\mathrm{D}(\boldsymbol{\lambda})$. Applying the standardization functor to $\mathrm{D}(\boldsymbol{\lambda})$ and $\mathrm{Y}(\boldsymbol{\lambda})$ gives us $\mathcal{OS}(z,t)$-modules denoted $\bar{\Delta}(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\lambda})$, respectively.

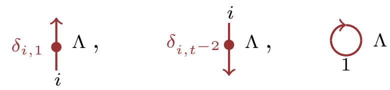
- When $t \notin \{\pm q^n \mid n \in \mathbb{Z}\}$, the modules $\{\Delta(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in e\text{-Bip}\}$ give a full set of pairwise inequivalent indecomposable projective $\mathcal{OS}(z,t)$-modules, and $K_0(\dot{\mathcal{OS}}(z,t))$ is identified with a proper subring of $\mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$ so that
$$[\Delta(\boldsymbol{\lambda})] \leftrightarrow \sum_{\boldsymbol{\kappa} \in \mathrm{Bip}_{r,s}} [\mathrm{S}(\boldsymbol{\kappa}) : \mathrm{D}(\boldsymbol{\lambda})] \chi_{\boldsymbol{\kappa}} \tag{1.14}$$
for $\boldsymbol{\lambda} \in e\text{-Bip}_{r,s}$ and $r, s \geq 0$. The decomposition numbers $[\mathrm{S}(\boldsymbol{\kappa}) : \mathrm{D}(\boldsymbol{\lambda})]$ are known providing $\Bbbk$ is of characteristic zero, since they are products of the decomposition numbers of Hecke algebras determined by Ariki [A].
- When $t = \pm q^n$, the category of $\mathcal{OS}(z,t)$-modules is an *upper-finite standardly stratified category* with standard modules $\{\Delta(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in e\text{-Bip}\}$ and proper standard modules $\{\bar{\Delta}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in e\text{-Bip}\}$; see [LW, §2] and [EL, §6.2.1]. The proper standard module $\bar{\Delta}(\boldsymbol{\lambda})$ has irreducible head $\mathrm{L}(\boldsymbol{\lambda})$, and these modules give a full set of pairwise inequivalent irreducible $\mathcal{OS}(z,t)$-modules. The projective cover $\mathrm{P}(\boldsymbol{\mu})$ of $\mathrm{L}(\boldsymbol{\mu})$ has a finite Δ-flag with multiplicities satisfying $(\mathrm{P}(\boldsymbol{\mu}) : \Delta(\boldsymbol{\lambda})) = [\bar{\Delta}(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})]$. Then $K_0(\dot{\mathcal{OS}}(z,t))$ is identified with the *same* subring of $\mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$ as for generic t so that
$$[\mathrm{P}(\boldsymbol{\mu})] \leftrightarrow \sum_{\substack{0 \leq d \leq \min(r,s) \\ \boldsymbol{\lambda} \in e\text{-Bip}_{r-d,s-d} \\ \boldsymbol{\kappa} \in \mathrm{Bip}_{r-d,s-d}}} [\bar{\Delta}(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})][\mathrm{S}(\boldsymbol{\kappa}) : \mathrm{D}(\boldsymbol{\lambda})] \chi_{\boldsymbol{\kappa}} \tag{1.15}$$
for $\boldsymbol{\mu} \in e\text{-Bip}_{r,s}$ and $r, s \geq 0$. It means that as well as the decomposition numbers for Hecke algebras, one also wants to determine the composition multiplicities $[\bar{\Delta}(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})]$. This is still an open problem even when $\Bbbk$ is of characteristic zero; we will make some further comments at the end of the next subsection.

1.8. Suppose either that q is not a root of unity and $e = 0$, or q^2 is a primitive eth root of unity for $e > 1$. Let $I := \{q^{2n}, t^{-2}q^{-2n} \mid n \in \mathbb{Z}\} \subset \Bbbk$ and $\mathfrak{g}$ be the (complex) Kac-Moody algebra with Cartan matrix $(c_{i,j})_{i,j \in I}$ defined from
$$c_{i,j} := \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = q^2 j \text{ or } i = q^{-2} j \text{ but not both}, \\ -2 & \text{if } i = q^2 j = q^{-2} j \text{ (which is possible only if } e = 2), \\ 0 & \text{otherwise.} \end{cases} \tag{1.16}$$
There are four cases paralleling the discussion of K_0 in the previous subsection: when $e = 0$ then $\mathfrak{g} \cong \mathfrak{sl}_\infty \oplus \mathfrak{sl}_\infty$ if $t \notin \{\pm q^n \mid n \in \mathbb{Z}\}$ and $\mathfrak{g} \cong \mathfrak{sl}_\infty$ otherwise; when $e > 0$ then $\mathfrak{g} \cong \widehat{\mathfrak{sl}}_e \oplus \widehat{\mathfrak{sl}}_e$ if $t \notin \{\pm q^n \mid n \in \mathbb{Z}\}$ and $\mathfrak{g} \cong \widehat{\mathfrak{sl}}_e$ otherwise. We denote the weight lattice of $\mathfrak{g}$ by P and its fundamental dominant weights by Λ_i $(i \in I)$. Let $V(-\Lambda_1 | \Lambda_{t^{-2}})$ be the tensor product of the integrable lowest weight module of lowest weight $-\Lambda_1$ and the integrable highest weight module of highest weight $\Lambda_{t^{-2}}$. This is an irreducible $\mathfrak{g}$-module if and only if $t \notin \{\pm q^n \mid n \in \mathbb{Z}\}$.

Theorem 1.8. *The category of $\mathcal{OS}(z,t)$-modules admits the structure of a tensor product categorification of the $\mathfrak{g}$-module $V(-\Lambda_1 | \Lambda_{t^{-2}})$ in the general sense of Losev and Webster* [LW].

This means in particular that $\dot{\mathcal{OS}}(z,t)$ is a 2-representation of the *Kac-Moody 2-category* $\mathfrak{U}(\mathfrak{g})$ of Khovanov, Lauda and Rouquier [KL, Ro]: there is a strict $\Bbbk$-linear 2-functor from $\mathfrak{U}(\mathfrak{g})$ to the 2-category of $\Bbbk$-linear categories taking objects to blocks of $\dot{\mathcal{OS}}(z,t)$, 1-morphisms to functors between these blocks, and 2-morphisms to natural transformations between these functors. The functors E_i $(i \in I)$ arise from the summands of the endofunctor $\uparrow\otimes ?$ defined by the generalized i-eigenspaces of Jucys-Murphy elements. For more background on 2-representations, we refer to [BD], whose notation and conventions we follow closely. Diagrams representing 2-morphisms in $\mathfrak{U}(\mathfrak{g})$ will be drawn in red to distinguish them from diagrams in $\mathcal{OS}(z,t)$.

For any weight $\Lambda \in P$, there is a universal 2-representation $\mathcal{R}(\Lambda)$ of $\mathfrak{U}(\mathfrak{g})$ with weight subcategories $\mathcal{R}(\Lambda)_\omega := \mathcal{H}om_{\mathfrak{U}(\mathfrak{g})}(\Lambda,\omega)$ for each $\omega \in P$; see [Ro, §5.1.2] and also [BD, §4.2]. Now we set $\Lambda := \Lambda_{t^{-2}} - \Lambda_1$ and let $\mathcal{I}$ be the invariant ideal ("full sub-2-representation") of $\mathcal{R}(\Lambda)$ generated by the 2-morphisms



(1.17)

for all $i \in I$ (the last generator is needed only in case $t = \pm 1$). The quotient 2-representation

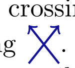
$$\mathcal{V}(-\Lambda_1|\Lambda_{t^{-2}}) := \mathcal{R}(\Lambda)/\mathcal{I} \tag{1.18}$$

is a special one of Webster's *generalized cyclotomic quotients* of $\mathfrak{U}(\mathfrak{g})$ introduced in [W2, Proposition 5.6]; see also [BD, Construction 4.13] where it is denoted $\mathcal{L}_{\min}(-\Lambda_1|\Lambda_{t^{-2}})$. It is a $\Bbbk$-linear category which is not monoidal in any obvious way.

Theorem 1.9. *Evaluation on the unit object defines a full strongly equivariant functor ("morphism of 2-representations")* $\Theta : \mathcal{R}(\Lambda_{t^{-2}} - \Lambda_1) \to \dot{\mathcal{OS}}(z,t)$. *This factors through* $\mathcal{V}(-\Lambda_1|\Lambda_{t^{-2}})$ *to induce a strongly equivariant equivalence*

$$\bar{\Theta} : \dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t^{-2}}) \xrightarrow{\approx} \dot{\mathcal{OS}}(z,t).$$

This is significant because the finite-dimensional category $\mathcal{V}(-\Lambda_1|\Lambda_{t^{-2}})$ possesses a natural $\mathbb{Z}$-grading. When the ground field is of characteristic zero, this grading is known to be *mixed* in the sense of [W2, Definition 1.11] in three of the four cases discussed above: it is trivial in the semisimple case; it may be deduced in an elementary way from the Koszulity of the Khovanov arc algebra K^∞_∞ studied in [BS] when $e = 0$ and $t \in \{\pm q^n \mid n \in \mathbb{Z}\}$; and it follows from [VV] when $e > 0$ and $t \notin \{\pm q^n \mid n \in \mathbb{Z}\}$. We conjecture that it is also mixed in the fourth case. As discussed in [W2, §8], the truth of this conjecture implies that the classes $[\mathrm{P}(\boldsymbol{\lambda})]$ coincide with Lusztig's canonical basis for $V(-\Lambda_1|\Lambda_{t^{-2}})$.

1.9. There is a parallel story in the *degenerate case* $z = 0$. In this case, relation (S) says simply that the positive and negative crossings are *equal*; it is natural to denote them both by the same "singular" crossing . The relation (T) forces $t^2 = 1$; we assume actually that $t = 1$ since the other possibility $t = -1$ produces an isomorphic object. In place of the relation (D) (which no longer makes any sense) we impose that

 $= \delta\, 1_\varnothing$

for some $\delta \in \Bbbk$. The resulting category is the *oriented Brauer category* $\mathcal{OB}(\delta)$ from [BCNR], which is the free $\Bbbk$-linear symmetric monoidal category generated by the dual pair of objects $\uparrow$ and $\downarrow$ of dimension δ. Like in (1.3), there is a homomorphism

$$\imath_r : \Bbbk\mathfrak{S}_r \to \mathrm{End}_{\mathcal{OS}(\delta)}(\uparrow^r) \tag{1.19}$$

sending the transposition $(i\ i{+}1)$ to the crossing of the ith and $(i+1)$th strands. The degenerate analogs of Theorems 1.1 and 1.2 are discussed in [BCNR]; the latter shows in particular that $\imath_r$ is an isomorphism.

In this paragraph suppose that $\Bbbk = \mathbb{C}$. The category

$$\underline{\mathrm{Rep}}\, GL_\delta := \dot{\mathcal{OB}}(\delta) \tag{1.20}$$

is the Deligne category mentioned before. As in Theorem 1.3, for $n \in \mathbb{N}$ and any sign $\varepsilon \in \{\pm\}$, the category $\operatorname{Rep} GL_n$ of (finite-dimensional) rational representations of GL_n over $\Bbbk$ is monoidally equivalent to the quotient of the Deligne category $\underline{\operatorname{Rep}}\, GL_{\varepsilon n}$ by the tensor ideal of negligible morphisms. This is proved in [De, Théorème 10.4]; the induced symmetric monoidal structure on $\operatorname{Rep} GL_n$ is the usual one when $\varepsilon = +$, and comes from super vector spaces when $\varepsilon = -$. The following extends this result to include fields of positive characteristic.

Theorem 1.10. *For $n \in \mathbb{N}$ and $\varepsilon \in \{\pm\}$, there is a full $\Bbbk$-linear monoidal functor $\Psi : \mathcal{OB}(\varepsilon n) \to \operatorname{Rep} GL_n$ sending $\uparrow$ and $\downarrow$ to the natural GL_n-module V and its dual V^*, respectively, the crossing*  *to the homomorphism $V \otimes V \to V \otimes V, v \otimes w \mapsto \varepsilon w \otimes v$, and the cap*  *to $V \otimes V^* \to \Bbbk, v \otimes f \mapsto \varepsilon f(v)$. It induces a monoidal equivalence*

$$\bar{\Psi} : \dot{\mathcal{OB}}(\varepsilon n)/\mathcal{N} \xrightarrow{\approx} \operatorname{Tilt}' GL_n, \tag{1.21}$$

where $\mathcal{N}$ is the additive $\Bbbk$-linear tensor ideal of $\dot{\mathcal{OB}}(\varepsilon n)$ generated by

$$x := \begin{cases} \sum_{g \in \mathfrak{S}_{n+1}} \operatorname{sgn}(g) \imath_{n+1}(g) & \text{if } \varepsilon = +, \\ \sum_{g \in \mathfrak{S}_{n+1}} \imath_{n+1}(g) & \text{if } \varepsilon = -, \end{cases}$$

and $\operatorname{Tilt}' GL_n$ is the full subcategory of $\operatorname{Rep} GL_n$ consisting of modules that are isomorphic to direct sums of summands of tensor products of V and V^*.

The other results discussed above can also be adapted quite easily to $\mathcal{OB}(\delta)$. For example, the degenerate analog of Theorem 1.5 gives that $\dot{\mathcal{OB}}(\delta)$ is semisimple if and only if $\Bbbk$ is of characteristic zero and $\delta \notin \mathbb{Z}$. This is proved in [De]; non-semisimplicity in positive characteristic is clear from (1.3). The analog of Theorem 1.6 needs $\delta \notin \mathbb{Z} \cdot 1_\Bbbk$. Then there is a description of $K_0(\dot{\mathcal{OB}}(\delta))$ with four cases similar to the above: replace the representation theory of Hecke algebras with that of symmetric groups.

Let us discuss the degenerate analogs of Theorem 1.8–1.9 in a little more detail. Assume now that $\Bbbk$ is a field of characteristic $p \geq 0$. Let $I := \{n, -n-\delta \mid n \in \mathbb{Z}\} \subseteq \Bbbk$, and $\mathfrak{g}$ be the (complex) Kac-Moody algebra with Cartan matrix $(c_{i,j})_{i,j \in I}$ defined from

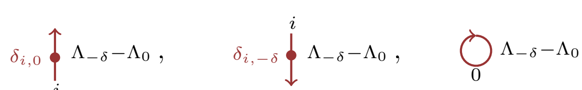
$$c_{i,j} := \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j+1 \text{ or } i = j-1 \text{ but not both}, \\ -2 & \text{if } i = j+1 = j-1 \text{ (which is possible only if } p = 2), \\ 0 & \text{otherwise.} \end{cases} \tag{1.22}$$

As before, there are four possibilities depending on p and δ: $\mathfrak{g} \cong \mathfrak{sl}_\infty \oplus \mathfrak{sl}_\infty, \mathfrak{sl}_\infty, \widehat{\mathfrak{sl}}_p \oplus \widehat{\mathfrak{sl}}_p$ or $\widehat{\mathfrak{sl}}_p$. The degenerate analog of Theorem 1.8 shows that $\dot{\mathcal{OB}}(\delta)$ admits the structure of a tensor product categorification of the $\mathfrak{g}$-module $V(-\Lambda_0|\Lambda_{-\delta})$; see also [E, Theorem 10.2.1] for a closely related (actually, Ringel dual) statement in the case $p = 0$. The following is the degenerate analog of Theorem 1.9; it was conjectured originally in discussions with Stroppel and Webster.

Theorem 1.11. *Evaluation on the unit induces a strongly equivariant equivalence*

$$\bar{\Theta} : \dot{\mathcal{V}}(-\Lambda_0|\Lambda_{-\delta}) \xrightarrow{\approx} \dot{\mathcal{OB}}(\delta) \tag{1.23}$$

where $\mathcal{V}(-\Lambda_0|\Lambda_{-\delta})$ is the quotient of the universal 2-representation $\mathcal{R}(\Lambda_{-\delta} - \Lambda_0)$ of $\mathfrak{U}(\mathfrak{g})$ by the invariant ideal generated by the 2-morphisms



(1.24)

for all $i \in I$ (the last generator is needed only in case $\delta = 0$).

Corollary 1.12. *Assume that $\Bbbk = \mathbb{C}$, q is not a root of unity, and $\delta \in \mathbb{C}$ is arbitrary. Then there is a $\mathbb{C}$-linear equivalence of categories $\underline{\mathrm{Rep}}\, U_q(\mathfrak{gl}_\delta) \xrightarrow{\approx} \underline{\mathrm{Rep}}\, GL_\delta$.*

Proof. When $e = p = 0$, we have that $\mathfrak{g} \cong \mathfrak{sl}_\infty \oplus \mathfrak{sl}_\infty$ if $\delta \notin \mathbb{Z}$ or $\mathfrak{sl}_\infty$ if $\delta \in \mathbb{Z}$. Now observe that the category $\mathcal{V}(-\Lambda_0|\Lambda_{-\delta})$ in Theorem 1.11 is isomorphic to the category $\mathcal{V}(-\Lambda_1|\Lambda_{t^{-2}})$ in Theorem 1.9 by a relabelling of the Dynkin diagram. □

The equivalence constructed in Corollary 1.12 is *not* monoidal, but it is a strongly equivariant equivalence of 2-representations. Hence, it is compatible with the endofunctors $\uparrow \otimes -$ and $\downarrow \otimes -$, and it induces a ring isomorphism between the Grothendieck rings preserving the labellings of isomorphism classes of indecomposable objects. Etingof has suggested that such an equivalence could also be constructed using KZ equations in the spirit of the Drinfeld-Kohno theorem.

For our final corollary, we assume $\Bbbk$ is a field of positive characteristic p and take δ to be the image in $\Bbbk$ of some $n \in \mathbb{N}$, so that $\mathfrak{g} \cong \widehat{\mathfrak{sl}}_p$. There is a well-known categorical action making the category $\mathrm{Rep}\, GL_n$ of rational representations of GL_n over $\Bbbk$ into a 2-representation of $\mathfrak{U}(\mathfrak{g})$; see [CR, §7.5.1] and [RW, §6]. The full subcategory $\mathrm{Tilt}\, GL_n$ of $\mathrm{Rep}\, GL_n$ consisting of all tilting modules (e.g., see [Do]) is a Karoubian sub-2-representation. As explained in detail in [RW, Proposition 6.5], there is a $\mathfrak{g}$-module isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{Tilt}\, GL_n) \cong \bigwedge^n \mathrm{Nat}_p, \tag{1.25}$$

where Nat_p is the level zero representation of $\mathfrak{g}$ with basis $\{m_r \mid r \in \mathbb{Z}\}$ on which the Chevalley generators of $\mathfrak{g}$ act via

$$e_i m_r = \begin{cases} m_{r+1} & \text{if } i \equiv r \pmod p, \\ 0 & \text{otherwise;} \end{cases} \qquad f_i m_{r+1} = \begin{cases} m_r & \text{if } i \equiv r \pmod p, \\ 0 & \text{otherwise.} \end{cases}$$

Using the defining relations for the cyclic module $V(-\Lambda_0|\Lambda_{-n})$ (e.g., see [BD, (3.6)]), it is easy to check that there is a $\mathfrak{g}$-module homomorphism

$$V(-\Lambda_0|\Lambda_{-n}) \to \bigwedge^n \mathrm{Nat}_p$$

sending the generator of $V(-\Lambda_0|\Lambda_{-n})$ (= the class of the unit object under (1.23)) to $m_0 \wedge m_{-1} \wedge \cdots \wedge m_{1-n}$ (= the class of the trivial module under (1.25)). This map is surjective, i.e., $m_0 \wedge m_{-1} \wedge \cdots \wedge m_{1-n}$ generates $\bigwedge^n \mathrm{Nat}_p$ as a $\mathfrak{g}$-module, if and only if $p > n$. This is also exactly the requirement on p needed to ensure that the subcategory $\mathrm{Tilt}'\, GL_n$ from Theorem 1.10 is all of $\mathrm{Tilt}\, GL_n$. Indeed, when $p > n$ all of the exterior powers $V, \bigwedge^2 V, \dots, \det = \bigwedge^n V$ plus $\det^{-1} = \bigwedge^n V^*$ are summands of corresponding tensor powers of V or V^* so they lie in $\mathrm{Tilt}'\, GL_n$. Every indecomposable tilting module arises as a summand of some tensor product of these fundamental tilting modules by highest weight considerations.

Corollary 1.13. *Suppose that $n \in \mathbb{N}$ and $\Bbbk$ is a field of characteristic $p > n$. There is a strongly equivariant equivalence*

$$\Phi : \dot{\mathcal{V}}(-\Lambda_0|\Lambda_{-n})/\mathcal{J} \xrightarrow{\approx} \mathrm{Tilt}\, GL_n$$

where $\mathcal{J}$ is the invariant ideal of $\dot{\mathcal{V}}(-\Lambda_0|\Lambda_{-n})$ generated by

$$y := \delta_{p,n+1} \underset{-n}{\uparrow\!\!\bullet}\ \underset{1-n}{\uparrow}\ \cdots\ \underset{-1}{\uparrow}\ \underset{0}{\uparrow}\ \Lambda_{-n}-\Lambda_0.$$

Proof. Let Ψ be the functor from Theorem 1.10 taking $\varepsilon = +$. By the definitions of the categorical actions, it is strongly equivariant. The tensor ideal $\mathcal{N}$ in Theorem 1.10 is generated by the quasi-idempotent $x = \sum_{g \in \mathfrak{S}_{n+1}} \mathrm{sgn}(g) \imath_{n+1}(g)$. Since $\dot{\mathcal{OB}}(n)$ is symmetric monoidal, it is actually generated by x just as a *left* tensor ideal.

Let $\bar{\Theta}^{-1}$ be quasi-inverse to the strongly equivariant equivalence $\bar{\Theta}$ from Theorem 1.11. We claim that it maps the generator x of $\mathcal{N}$ (as a left tensor ideal) to a non-zero multiple of the generator y of $\mathcal{J}$ (as an invariant ideal). To prove this, the definition of $\bar{\Theta}$ from the proof of Theorem 1.11 means that on endomorphisms of $\uparrow^{n+1}$ the map induced by $\bar{\Theta}^{-1}$ arises from the isomorphism between the group algebra $\Bbbk\mathfrak{S}_{n+1}$ and the corresponding cyclotomic quiver Hecke algebra constructed in [BK]. So the claim follows from [HM, Proposition 6.7] applied to the standard tableau $\mathfrak{s}$ that is a single row of length $(n+1)$. (This argument works when $p \leq n$ too producing a slightly more complicated formula for y.)

From the claim and Theorem 1.10, it follows that $\bar{\Theta}$ induces a strongly equivariant equivalence $\dot{\mathcal{V}}(-\Lambda_0|\Lambda_{-n})/\mathcal{J} \xrightarrow{\approx} \dot{\mathcal{OB}}(n)/\mathcal{N}$. To get Φ, it just remains to compose this with the strongly equivariant equivalence $\bar{\Psi}$ from Theorem 1.10, noting that $\mathrm{Tilt}'\, GL_n = \mathrm{Tilt}\, GL_n$ due to the assumption $p > n$. □

The category $\dot{\mathcal{V}}(-\Lambda_0|\Lambda_{-n})/\mathcal{J}$ appearing in Corollary 1.13 has a natural $\mathbb{Z}$-grading. So we have constructed a graded lift of $\mathrm{Tilt}\, GL_n$. In [RW], Riche and Williamson have constructed graded lifts of all regular blocks of $\mathrm{Tilt}\, GL_n$ via the diagrammatic Hecke category of [EW]; see also [EL]. We expect that the graded lifts of such blocks arising from Corollary 1.13 are equivalent to the ones of *loc. cit.*.

We leave it to the reader to formulate the q-analog of Corollary 1.13; see Remark 3.4.

1.10. The remainder of the article is organized as follows. Sections 2 and 3 are expository in nature and contain proofs of Theorems 1.1,1.2 and 1.3, thereby making the connection to $U_q(\mathfrak{gl}_n)$. For Theorem 1.4, which is not needed in the remainder of the article, we refer the reader to Turaev's original article [T1]. Then Section 4 discusses the affine oriented skein category $\mathcal{AOS}(z,t)$ and the resulting Jucys-Murphy elements. The triangular decomposition of $\mathcal{OS}(z,t)$ is introduced in section 5, and the highest weight approach to representations is developed there. Another noteworthy result in this section is Theorem 5.9, which is used both to prove Theorem 1.7 and to obtain the description of $K_0(\dot{\mathcal{OS}}(z,t))$ (Theorem 5.18). Then in section 6 we study certain induction and restriction functors E_i and F_i which give rise to the categorical action on $\mathcal{OS}(z,t)$-modules. A novel result here is Theorem 6.11, which uses these induction and restriction functors to compute the formal characters of the standardizations of Specht modules. This is used to prove Theorem 1.6, also completing the proof of Theorem 1.5. In section 7, Theorem 6.11 is used again to prove a linkage principle for the decomposition numbers $[\bar{\Delta}(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})]$ (Theorem 7.4). In Theorem 7.8, we introduce the highest weight/standardly stratified structure on $\mathcal{OS}(z,t)$-modules. Then we prove Theorems 1.8–1.9. Finally, in section 8, we discuss the degenerate case. Theorem 1.10 is proved by the same argument as Theorem 1.3. After that, we just highlighting the main differences in the degenerate case compared to the quantum case, which arise because the Jucys-Murphy elements need different treatment. Theorem 1.11 then follows.

Acknowledgements. This article is based in part on the PhD thesis of Andrew Reynolds [Re], who developed the representation theory of the oriented Brauer category using the highest weight approach. In particular, Reynolds proved the degenerate analog of Theorem 1.8. Many of the arguments were inspired by Ben Webster's work [W3] and the ideas of [LW]. I also thank Stephen Donkin, Stephen Doty, Michael Ehrig, Inna Entova-Aizenbud, Pavel Etingof, Catharina Stroppel and Geordie Williamson for helpful questions and answers.

2. Generators and relations

The monoidal category $\mathcal{FOT}$ from the introduction is the category $\mathcal{R}ib_{\mathcal{V}}$ from [T3, §I.2.3], taking $\mathcal{V}$ to be the trivial monoidal category with just one object $*$ and one morphism $1_* : * \to *$. Our generating objects $\uparrow$ and $\downarrow$ are Turaev's $(*, -)$ and $(*, +)$. As we explained in the introduction, objects in $\mathcal{FOT}$ are words $\mathsf{a}, \mathsf{b}, \dots$ in the letters $\uparrow, \downarrow$, i.e., elements of the free monoid $\langle \uparrow, \downarrow \rangle$ generated by these symbols. Morphisms $\mathsf{a} \to \mathsf{b}$ are isotopy classes of (a, b)-ribbons.

We say that an (a, b)-ribbon is *generic* if all of its critical points (= points of slope zero) are local maxima and minima, and all crossings occur away from the critical points. Thus, a generic (a, b)-ribbon can only involve "identity lines" of non-zero slope, two sorts of cup (left/right), two sorts of cap (left/right), and eight sorts of crossing (up/right/down/left and positive/negative). Any (a, b)-ribbon is isotopic to a generic one. Moreover, isotopy of generic (a, b)-ribbons is generated by *rectilinear isotopy*, i.e., planar isotopy that fixes the boundary and preserves genericity, plus the oriented Reidemeister moves (R0), (FRI), (RII) and (RIII) from Figure 1. This is justified carefully in [T3, §I.4.6].

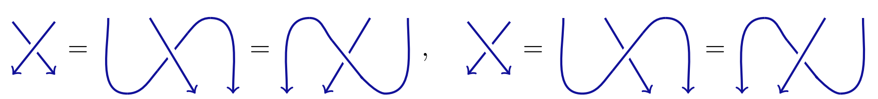
The following theorem giving an explicit monoidal presentation for $\mathcal{FOT}$ follows from this discussion; see also [T2, Theorem 3.2] for the analogous result without framing, and [T3, §I.4.2] for more background about generators and relations for strict monoidal categories.

Lemma 2.1. *The category $\mathcal{FOT}$ is the free strict monoidal category generated by the objects $\uparrow$ and $\downarrow$ and the morphisms* , , , , , , , , , , *and* , *subject only to the relations* (R0), (FRI), (RII) *and* (RIII).

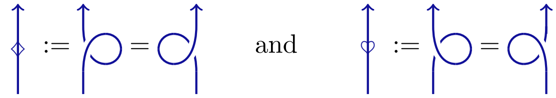
There are many redundancies in the presentation for $\mathcal{FOT}$ just given. In fact, the morphisms , , , , and suffice to generate all other morphisms, since the other crossings can be expressed in terms of them:


(2.1)

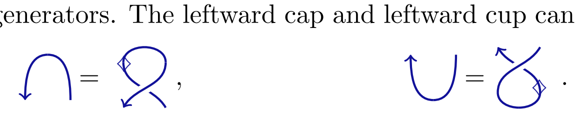

(2.2)


(2.3)

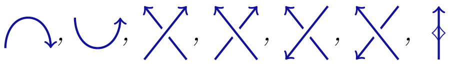
Alternatively, as observed in [T3, Lemma I.3.1.1], the morphisms , , , , , together with

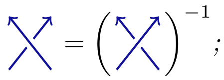

(2.4)

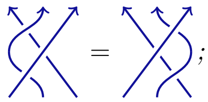
give a system of generators. The leftward cap and leftward cup can then be recovered:

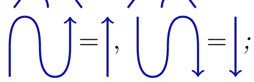

(2.5)

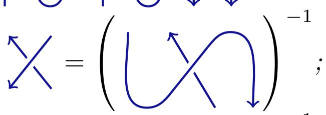
The following theorem due to Turaev gives a more efficient set of relations between this set of generators.

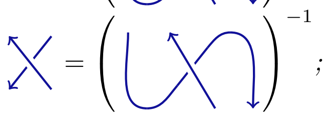
Theorem 2.2. *The category $\mathcal{FOT}$ is generated by the objects $\uparrow, \downarrow$ and the morphisms*  *and*  *subject only to the following relations:*

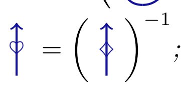
(i) 

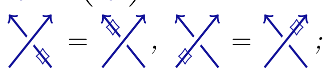
(ii) 

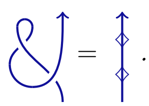
(iii) 

(iv) 

(v) 

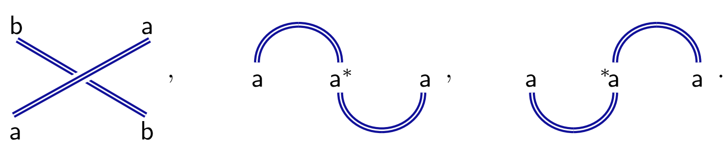
(vi) 

(vii) 

(viii) 

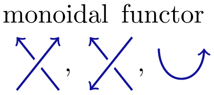
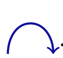
Proof. This is [T3, Lemma I.3.3] except that we have rotated Turaev's generators and relations by 180° around a horizontal axis. □

The category $\mathcal{FOT}$ is a ribbon category in the sense of [TV, §3.3.2]: it is braided and pivotal, and (FRI) ensures that the right and left twists are equal. Like in [T3, §XII.2.2], the braiding $\tau_{\mathsf{a},\mathsf{b}} : \mathsf{a} \otimes \mathsf{b} \rightarrow \mathsf{b} \otimes \mathsf{a}$ is defined by the first of the following diagrams; the right dual of $\mathsf{a} = \mathsf{a}_n \cdots \mathsf{a}_1 \in \langle \uparrow, \downarrow \rangle$ is $\mathsf{a}^* := \mathsf{a}_1^* \cdots \mathsf{a}_n^*$ where $\uparrow^* = \downarrow$ and $\downarrow^* = \uparrow$ with structure maps $\mathsf{a} \otimes \mathsf{a}^* \rightarrow \mathbb{1} \rightarrow \mathsf{a}^* \otimes \mathsf{a}$ defined from the second of the following diagrams; the left dual ${}^*\mathsf{a}$ is the same object as the right dual with structure maps ${}^*\mathsf{a} \otimes \mathsf{a} \rightarrow \mathbb{1} \rightarrow \mathsf{a} \otimes {}^*\mathsf{a}$ defined by the third diagram.

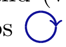


(In these diagrams, the double lines labelled a denote parallel thin lines oriented in order from left to right according to the letters of the word a.) The right and left duality functors are both defined on diagrams by rotating the xy-plane through 180°, hence, they are equal, and we have equipped $\mathcal{FOT}$ with a strictly pivotal structure.

The ribbon structure on $\mathcal{FOT}$ induces a ribbon structure on the oriented skein category $\mathcal{OS}(z,t)$ too. In particular, it also possesses a strictly pivotal structure.

Proof of Theorem 1.1. Let $\mathcal{C}$ be the strict monoidal category defined by the generators and relations (1)–(5) from Theorem 1.1. We first define a strict monoidal functor $\Phi : \mathcal{C} \rightarrow \mathcal{OS}(z,t)$ sending $E \rightarrow \uparrow, F \rightarrow \downarrow$, and S, T, C and D to  and . To check this is well defined, one needs to verify that the relations from

Theorem 1.1(1)–(5) all hold in $\mathcal{OS}(z,t)$. We already set this as an exercise for the reader in the introduction, and are not about to spoil the fun here!

Next we construct a strict monoidal functor $\Psi : \mathcal{OS}(z,t) \to \mathcal{C}$ in the other direction. For this we use the presentation for the strict $\Bbbk$-linear monoidal category $\mathcal{OS}(z,t)$ arising from Theorem 2.2, with eight generating morphisms and relations (i)–(viii) from the theorem, plus the relations (S), (T) and (D). Then (2.5) becomes the *definition* of the leftward cap and leftward cup. We set $C' := tT \circ C$ and $D' := tD \circ T$, then define Ψ by sending $\uparrow \mapsto E, \downarrow \mapsto F$ and the eight generating morphisms in the order listed to $D, C, S, S - z1_E \otimes 1_E, T, T + zC' \circ D'$, $t1_E$ and $t^{-1}1_E$, respectively. To verify that this is well defined, we must check the images of the eleven relations hold in $\mathcal{C}$. The relations (i) and (S) follow from (1). Relations (ii), (iii), (iv), (vi) and (vii) are equally easy using (2), (3) and the definitions. For relation (D), Ψ maps  to $D \circ C' = D' \circ C = tD \circ T \circ C = \frac{t-t^{-1}}{z} 1_{\mathbb{1}}$ by (5). Consider relation (v). From (4), we see that the image of the negative rightward crossing is $(T^{-1} - zC \circ D)$, and we must check that this has two-sided inverse $(T + zC' \circ D')$. This follows easily using the identities established so far plus $T^{-1} \circ C' = tC, D' \circ T^{-1} = tD$. For (viii), we note that $(1_F \otimes S) \circ (C \otimes 1_E) = (T^{-1} \otimes 1_E) \circ (1_E \otimes C)$. So Ψ maps the & symbol to

$$\begin{aligned}
&((D \circ T + zD \circ C' \circ D') \otimes 1_E) \circ (T^{-1} \otimes 1_E) \circ (1_E \otimes C) \\
&\qquad = ((t^{-1}D' + (t - t^{-1})D') \otimes 1_E) \circ (T^{-1} \otimes 1_E) \circ (1_E \otimes C) \\
&\qquad = t((D' \circ T^{-1}) \otimes 1_E) \circ (1_E \otimes C) = t^2(D \otimes 1_E) \circ (1_E \otimes C) = t^2 1_E,
\end{aligned}$$

as required. Finally, for (T), the image of the positive right curl is

$$(D \otimes 1_E) \circ (1_E \otimes T^{-1}) \circ (1_E \otimes C') = t(D \otimes 1_E) \circ (1_E \otimes C) = t1_E.$$

To complete the proof of Theorem 1.1, it remains to observe that the functors Φ and Ψ are two-sided inverses. This depends on (2.4)–(2.5). $\square$

To conclude the section, we briefly list some further symmetries of the monoidal categories $\mathcal{OS}(z,t)$. There is an isomorphism

$$\tau : \mathcal{OS}(z,t) \xrightarrow{\sim} \mathcal{OS}(z,t)^{\mathrm{op}} \tag{2.6}$$

which fixes objects, and rotates diagrams for morphisms though 180° around a horizontal axis then reverses all orientations. Thus, the vertical crossings are fixed and leftward crossings are switched with rightward crossings (preserving whether they are positive or negative), while rightward and leftward caps are switched with leftward and rightward cups, respectively. Composing τ with duality, we obtain an isomorphism

$$\phi : \mathcal{OS}(z,t) \xrightarrow{\sim} \mathcal{OS}(z,t)^{\mathrm{rev}}. \tag{2.7}$$

This fixes sideways crossings, cups and caps, but switches upward crossings with downward crossings (preserving whether they are positive or negative). Finally, there are isomorphisms

$$\rho : \mathcal{OS}(z,t) \xrightarrow{\sim} \mathcal{OS}(z,t), \tag{2.8}$$

$$\sigma : \mathcal{OS}(z,t) \xrightarrow{\sim} \mathcal{OS}(-z,-t), \tag{2.9}$$

$$\omega : \mathcal{OS}(z,t) \xrightarrow{\sim} \mathcal{OS}(-z,t^{-1}), \tag{2.10}$$

$$\pi : \mathcal{OS}(z,t) \xrightarrow{\sim} \mathcal{OS}(z,-t). \tag{2.11}$$

These reverse all orientations, scale by $(-1)^{\#\text{crossings}}$, switch all positive crossings with negative crossings, and scale by $(-1)^{\#\text{leftward cups}+\#\text{leftward caps}}$, respectively. Let

$$\# : \mathcal{OS}(z,t) \xrightarrow{\sim} \mathcal{OS}(z,t^{-1}) \tag{2.12}$$

denote $\sigma \circ \omega \circ \pi$.

3. Connection to $\operatorname{Rep} U_q(\mathfrak{gl}_n)$

In this section, we assume until the final proof that $\Bbbk$ is a field of characteristic 0, $q \in \Bbbk^\times$ is not a root of unity, and $z = q - q^{-1}$. Fix $n \in \mathbb{N}$ and let $U_q(\mathfrak{gl}_n)$ be the usual quantized enveloping algebra over $\Bbbk$; we include the possibility that $n = 0$ by interpreting $U_q(\mathfrak{gl}_0)$ as $\Bbbk$. We denote the standard generators of $U_q(\mathfrak{gl}_n)$ by $\left\{e_i, f_i, d_j^{\pm} \mid 1 \leq i < n, 1 \leq j \leq n\right\}$. This is a well-known object, so the reader should have no trouble surmising the relations on being told that the usual diagonal generator k_i of $U_q(\mathfrak{sl}_n)$ is $d_i d_{i+1}^{-1}$. We have the natural $U_q(\mathfrak{gl}_n)$-module V^+ on basis $\left\{v_i^+ \mid 1 \leq i \leq n\right\}$ and the dual natural module V^- on basis $\left\{v_i^- \mid 1 \leq i \leq n\right\}$. The actions of the generators on these bases are given by the following formulae:

$$
\begin{aligned}
f_i v_j^+ &= \delta_{i,j} v_{i+1}^+, & e_i v_j^+ &= \delta_{i+1,j} v_i^+, & d_i v_j^+ &= q^{\delta_{i,j}} v_j^+,\\
f_i v_j^- &= \delta_{i+1,j} v_i^-, & e_i v_j^- &= \delta_{i,j} v_{i+1}^-, & d_i v_j^- &= q^{-\delta_{i,j}} v_j^-.
\end{aligned}
$$

We use the comultiplication $\Delta : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$ defined from

$$
\Delta(f_i) = 1 \otimes f_i + f_i \otimes d_i d_{i+1}^{-1}, \quad \Delta(e_i) = d_i^{-1} d_{i+1} \otimes e_i + e_i \otimes 1, \quad \Delta(d_i) = d_i \otimes d_i.
$$

The corresponding antipode is given by $\mathrm{S}(e_i) = -d_i d_{i+1}^{-1} e_i, \mathrm{S}(f_i) = -f_i d_i^{-1} d_{i+1}$ and $\mathrm{S}(d_i) = d_i^{-1}$; for the user of [L] we note that Lusztig's v and K_i are our q^{-1} and $k_i^{-1} = d_i^{-1} d_{i+1}$.

Let $\operatorname{Rep} U_q(\mathfrak{gl}_n)$ be the category of finite-dimensional $U_q(\mathfrak{gl}_n)$-modules that are isomorphic to finite direct sums of summands of the modules obtained by taking tensor products of V^+ and V^-; in the trivial case $n = 0$, we mean the category of finite-dimensional vector spaces. In general, $\operatorname{Rep} U_q(\mathfrak{gl}_n)$ is the usual category of finite-dimensional representations of $U_q(\mathfrak{gl}_n)$ that are semisimple of type $\mathbf{1}$ over its diagonal subalgebra. It is well known that $\operatorname{Rep} U_q(\mathfrak{gl}_n)$ is a ribbon category, but we do not want to fix a ribbon structure yet. Instead, we are going to use Theorem 1.1 classify monoidal functors $\Phi : \mathcal{OS}(z,t) \to \operatorname{Rep} U_q(\mathfrak{gl}_n)$ that take $\uparrow$ to V^+ and $\downarrow$ to V^-.

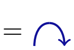
There is a unique (up to scalars) non-degenerate bilinear pairing

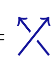
$$
\langle \cdot, \cdot \rangle : V^+ \times V^- \to \Bbbk
$$

satisfying $\langle uv^+, v^- \rangle = \langle v^+, \mathrm{S}(u) v^- \rangle$. Since there is freedom to rescale the basis vectors v_j^- by a global scalar, we may assume this is given explicitly by the formula $\langle v_i^+, v_j^- \rangle := (-1)^i q^{-i} \delta_{i,j}$. The associated evaluation and coevaluation maps will be denoted

$$
\mathrm{ev} : V^+ \otimes V^- \to \Bbbk, \qquad v_i^+ \otimes v_j^- \mapsto (-1)^i q^{-i} \delta_{i,j}, \tag{3.1}
$$

$$
\mathrm{coev} : \Bbbk \to V^- \otimes V^+, \qquad 1 \mapsto \sum_{j=1}^n (-1)^j q^j v_j^- \otimes v_j^+. \tag{3.2}
$$

Then if we define $\Phi(C) :=$ coev and $\Phi(D) :=$ ev, where $C =$  and $D =$ , the relation (3) from Theorem 1.1 is satisfied.

Next we choose a candidate for the image of $S =$ . This should be an isomorphism $V^+ \otimes V^+ \xrightarrow{\sim} V^+ \otimes V^+$ satisfying the relations (1) and (2) from Theorem 1.1. One possible choice is to take $\Psi(S) := R$ where

$$
R(v_i^+ \otimes v_j^+) := \begin{cases} v_j^+ \otimes v_i^+ & \text{if } i < j,\\ q v_j^+ \otimes v_i^+ & \text{if } i = j,\\ v_j^+ \otimes v_i^- + (q - q^{-1}) v_i^+ \otimes v_j^+ & \text{if } i > j. \end{cases} \tag{3.3}
$$

This formula is the R-matrix from [L, §32.1.4]. The only other possibility for us would be to take $\Psi(S) := -R^{-1}$, but there is no loss in generality in choosing the former, since one can twist with the isomorphism $\# : \mathcal{OS}(z,t) \xrightarrow{\sim} \mathcal{OS}(z,t^{-1})$ from

(2.12) which switches S and $-S^{-1}$. To see that $-R^{-1}$ is indeed the only other option, recall that endomorphism algebra of $V^+ \otimes V^+$ is two-dimensional, so any isomorphism $V^+ \otimes V^+ \to V^+ \otimes V^+$ takes the form $aR + b$ for scalars a and b. Then a simple computation shows there are only two choices for these scalars which satisfy the relation (1): $a = 1, b = 0$ or $a = -1, b = q - q^{-1}$.

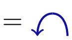
Using the relation (4), we can determine the image of $T =$ , as follows. We want to have $\Psi(T^{-1}) = (1_{V^-} \otimes 1_{V^+} \otimes \mathrm{ev}) \circ (1_{V^-} \otimes R \otimes 1_{V^-}) \circ (\mathrm{coev} \otimes 1_{V^+} \otimes 1_{V^-})$. Computing the right hand side explicitly gives that

$$\Psi(T^{-1})(v_i^+ \otimes v_j^-) = \begin{cases} v_j^- \otimes v_i^+ & \text{if } i \neq j, \\ q v_i^- \otimes v_i^+ + (q - q^{-1}) \displaystyle\sum_{k=j+1}^{n} (-q)^{k-i} v_k^- \otimes v_k^+ & \text{if } i = j; \end{cases}$$

Inverting this map then gives us $\Psi(T)$:

$$\Psi(T)(v_i^- \otimes v_j^+) = \begin{cases} v_j^+ \otimes v_i^- & \text{if } i \neq j, \\ q^{-1} v_i^+ \otimes v_i^- - (q - q^{-1}) \displaystyle\sum_{k=i+1}^{n} (-q)^{i-k} v_k^+ \otimes v_k^- & \text{if } i = j. \end{cases}$$

Now the relation (4) holds.

The images under Ψ of $C' =$  and $D' =$  must come from another non-degenerate pairing

$$\langle \cdot, \cdot \rangle' : V^- \times V^+ \to \Bbbk$$

such that $\langle uv^-, v^+ \rangle' = \langle v^-, \mathrm{S}(u) v^+ \rangle'$. There is a unique (up to scalars) such pairing, namely, $\langle v_i^-, v_j^+ \rangle' := \varepsilon (-1)^i q^{i-n-1} \delta_{i,j}$ for $\varepsilon \in \Bbbk^\times$. We denote the corresponding evaluation and coevaluation maps by

$$\mathrm{ev}' : V^- \otimes V^+ \to \Bbbk, \qquad v_i^- \otimes v_j^+ \mapsto \varepsilon (-1)^i q^{i-n-1} \delta_{i,j}, \tag{3.4}$$

$$\mathrm{coev}' : \Bbbk \to V^+ \otimes V^-, \qquad 1 \mapsto \varepsilon^{-1} \sum_{j=1}^{n} (-1)^j q^{n+1-j} v_j^+ \otimes v_j^-. \tag{3.5}$$

Then, for some choice of ε, we have that $\Phi(C') = \mathrm{coev}'$ and $\Phi(D') = \mathrm{ev}'$. To determine the possibilities for ε, from the first equation in (2.5), we know that $\mathrm{ev}' = t\, \mathrm{ev} \circ \Psi(T)$. Applying this equation to the vector $v_n^- \otimes v_n^+$ quickly produces the equation $\varepsilon(-1)^n q^{-1} = t q^{-1} (-1)^n q^{-n}$, hence, $t = \varepsilon q^n$. From the second equation in (2.5), we know that $\mathrm{coev}' = t \Psi(T) \circ \mathrm{coev}$. Looking at the $v_1^+ \otimes v_1^-$-coefficient of the image of 1 under the two sides of this equation gives $-\varepsilon^{-1} q^n = -t$, hence, $t = \varepsilon^{-1} q^n$. We deduce that $\varepsilon = \varepsilon^{-1}$, i.e., $\varepsilon = \pm 1$. Since we can twist with the isomorphism $\pi : \mathcal{OS}(z,t) \xrightarrow{\sim} \mathcal{OS}(z,-t)$ from (2.11) which takes C' to $-C'$ and D' to $-D'$, we are reduced without loss of generality to the case that $\varepsilon = +1$ and $t = q^n$. It can then be checked that (2.5) holds fully.

Finally, we have that $\mathrm{coev} \circ \mathrm{ev}' = [n]_q$, so that relation (5) from Theorem 1.1 holds too, and the theorem implies that the functor Ψ is well defined. We have proved the following lemma, a version of which was used already in [T2].

Lemma 3.1. *Assume $\Bbbk$ is of characteristic zero, $z = q - q^{-1}$ for generic $q \in \Bbbk^\times$, and $t = q^n$ for $n \in \mathbb{N}$. There is a $\Bbbk$-linear monoidal functor $\Psi : \mathcal{OS}(z,t) \to \operatorname{Rep} U_q(\mathfrak{gl}_n)$ sending $\uparrow \mapsto V^+, \downarrow \mapsto V^-$, the positive upward crossing to the R-matrix from (3.3), the rightward cap and rightward cup to the maps* ev *and* coev *from (3.1)–(3.2), and the leftward cap and leftward cup to the maps* ev′ *and* coev′ *from (3.4)–(3.5) taking $\varepsilon = +1$.*

Remark 3.2. Pre-composing the functor Ψ with one or both of the isomorphisms $\#$ and π from (2.11)–(2.12) gives three more such functors with $t = q^n$ replaced by $q^{-n}, -q^n$ or $-q^{-n}$. The arguments above actually show that these four functors constitute essentially all possible $\Bbbk$-linear monoidal functors $\mathcal{OS}(z,t) \to \operatorname{Rep} U_q(\mathfrak{gl}_n)$ taking $\uparrow \mapsto V^+$ and $\downarrow \mapsto V^-$. Each of the four choices for this functor corresponds to a ribbon structure on the monoidal category $\operatorname{Rep} U_q(\mathfrak{gl}_n)$. The standard ribbon structure on $\operatorname{Rep} U_q(\mathfrak{gl}_n)$ is the one coming from Ψ itself, i.e., $t = q^n$, and we will only use this from now it. Taking $\Psi \circ \#$, i.e., $t = q^{-n}$, gives a non-standard ribbon structure on $\operatorname{Rep} U_q(\mathfrak{gl}_n)$ with positive upward crossing $-R^{-1}$, rightward cap and cup being ev and coev, and leftward cap and cup being ev$'$ and coev$'$ with $\varepsilon = -1$; this is the ribbon category denoted $\operatorname{Rep} U_q(\mathfrak{gl}_{-n})$ in the introduction.

In order to prove Theorems 1.2 and 1.3 from the introduction, we also need the *Iwahori-Hecke algebra* H_r associated to the symmetric group $\mathfrak{S}_r$. This is the associative $\Bbbk$-algebra with generators $S_1, \dots, S_{r-1}$ subject to the relations

$$(S_i - q)(S_i + q^{-1}) = 0, \quad S_i S_j = S_j S_i \text{ if } |i-j| > 1, \quad S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}. \tag{3.6}$$

As is well known, H_r has dimension $r!$, with basis $\{S_w \mid w \in \mathfrak{S}_r\}$ defined as usual by letting S_w be the word in the generators S_i arising from any reduced expression for w. It is obvious from the defining relations that there is a homomorphism

$$\imath_r : H_r \to \operatorname{End}_{\mathcal{OS}(z,t)}(\uparrow^r) \tag{3.7}$$

sending S_i to the positive crossing  of the ith and $(i+1)$th strands, numbering strands in increasing order from right to left. Also for $\lambda \vdash r$ we let

$$x_\lambda := \sum_{w \in \mathfrak{S}_\lambda} q^{\ell(w)} S_w, \qquad y_\lambda := \sum_{w \in \mathfrak{S}_\lambda} (-q)^{-\ell(w)} S_w, \tag{3.8}$$

where $\mathfrak{S}_\lambda$ denotes the usual parabolic subgroup $\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots$ of $\mathfrak{S}_r$. Assuming q is not a root of unity, there is a unique (up to sign) idempotent

$$e_\lambda \in y_{\lambda^t} H_r x_\lambda. \tag{3.9}$$

This is the *Young symmetrizer*. For example,

$$e_{(n)} = \frac{q^{-\frac{1}{2}n(n-1)}}{[n]_q^!} \sum_{w \in \mathfrak{S}_n} q^{\ell(w)} h_w, \qquad e_{(1^n)} = \frac{q^{\frac{1}{2}n(n-1)}}{[n]_q^!} \sum_{w \in \mathfrak{S}_n} (-q)^{-\ell(w)} h_w, \tag{3.10}$$

which correspond to the trivial and the sign representations of H_r, respectively. The algebra involution

$$\# : H_r \to H_r, \qquad S_i \mapsto -S_i^{-1} \tag{3.11}$$

interchanges $e_{(n)}$ and $e_{(1^n)}$. Given an H_r-module M, the H_r-module $M^\#$ obtained from M by twisting the action by $\#$ gives the q-analog of "tensoring with sign."

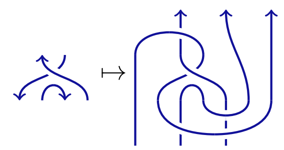
Proof of Theorem 1.3. Since we can compose with $\#$, which switches $t = q^n$ with $t = q^{-n}$ and $e_{(n)}$ with $e_{(1^n)}$, we are reduced just to proving the theorem in the case that $\varepsilon = +$. Then the appropriate monoidal functor Ψ is the one constructed in Lemma 3.1.

In this paragraph, we show that Ψ is full. Take $\mathsf{a}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle$ such that x (resp. x') letters of a and y (resp. y') letters of b are equal to $\downarrow$ (resp. $\uparrow$). The space $\operatorname{Hom}_{\mathcal{OS}(z,t)}(\mathsf{a}, \mathsf{b})$ is zero unless $r := x' + y = x + y'$, so we may assume that is the case. Let $a : \uparrow^{x'} \to \mathsf{a}\uparrow^x$ be the unique morphism that consists of x nested rightward cups on top of x' vertical upward strands. Let $b : \uparrow^y \mathsf{b} \to \uparrow^{y'}$ be the unique morphism that consists of y nested rightward caps on top of y' vertical strands. The linear map

$$\theta : \operatorname{Hom}_{\mathcal{OS}(z,t)}(\mathsf{a}, \mathsf{b}) \to \operatorname{Hom}_{\mathcal{OS}(z,t)}(\uparrow^r, \uparrow^r), \tag{3.12}$$

$$f \mapsto (b \otimes 1_{\uparrow^x}) \circ (1_{\uparrow^y} \otimes f \otimes 1_{\uparrow^x}) \circ (1_{\uparrow^y} \otimes a)$$

has an obvious two-sided inverse, hence, it is a vector space isomorphism. For example, taking $\mathsf{a} = \downarrow\uparrow\downarrow\uparrow$ and $\mathsf{b} = \uparrow\downarrow$, the map θ sends



Since Ψ is a monoidal functor, there is an isomorphism

$$\phi : \mathrm{Hom}_{U_q(\mathfrak{gl}_n)}(\Psi(\mathsf{a}), \Psi(\mathsf{b})) \xrightarrow{\sim} \mathrm{Hom}_{U_q(\mathfrak{gl}_n)}((V^+)^{\otimes r}, (V^+)^{\otimes r}),$$
$$g \mapsto (\Psi(b) \otimes 1_{V^+}^{\otimes x}) \circ (1_{V^+}^{\otimes y} \otimes g \otimes 1_{V^+}^{\otimes x}) \circ (1_{V^+}^{\otimes y} \otimes \Psi(a))$$

making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathsf{a},\mathsf{b}) \arrow[r, "\sim", "\theta"'] \arrow[d, "\Psi"'] & \mathrm{Hom}_{\mathcal{OS}(z,t)}(\uparrow^r, \uparrow^r) \arrow[d, "\Psi"] & H_r \arrow[l, "\imath_r"'] \\
\mathrm{Hom}_{U_q(\mathfrak{gl}_n)}(\Psi(\mathsf{a}), \Psi(\mathsf{b})) \arrow[r, "\sim", "\phi"'] & \mathrm{Hom}_{U_q(\mathfrak{gl}_n)}((V^+)^{\otimes r}, (V^+)^{\otimes r}). &
\end{tikzcd} \tag{3.13}$$

The composition $\jmath_r : H_r \to \mathrm{End}_{U_q(\mathfrak{gl}_n)}((V^+)^{\otimes r})$ of $\imath_r$ and the right hand Ψ is a homomorphism studied in [J] in the context of "quantized Schur-Weyl reciprocity." It is shown there that $\jmath_r$ is surjective. Hence, the right hand Ψ is surjective. The commutativity of the diagram then implies the analogous statement for the left hand Ψ.

As $\mathrm{Rep}\, U_q(\mathfrak{gl}_n)$ is additive Karoubian, the functor Ψ extends to a full functor $\dot{\Psi} : \dot{\mathcal{OS}}(z,t) \to \mathrm{Rep}\, U_q(\mathfrak{gl}_n)$. Let $\mathcal{N}$ be the tensor ideal of $\dot{\mathcal{OS}}(z,t)$ of negligible morphisms. Since $\mathrm{Rep}\, U_q(\mathfrak{gl}_n)$ is absolutely semisimple, $\dot{\Psi}$ induces a fully faithful functor $\bar{\Psi} : \dot{\mathcal{OS}}(z,t)/\mathcal{N} \to \mathrm{Rep}\, U_q(\mathfrak{gl}_n)$ by the argument from the proof of [De, Théorème 6.2]. This functor is also dense since every object of $\mathrm{Rep}\, U_q(\mathfrak{gl}_n)$ is a summand of some tensor product of the modules V^+ and V^-. So it is a monoidal equivalence.

To complete the proof, we need to show that $\mathcal{N}$ is generated as an additive $\Bbbk$-linear tensor ideal of $\dot{\mathcal{OS}}(z,t)$ by the morphism $\imath_{n+1}(e_{(1^{n+1})})$. It suffices to show that the kernel[1] of the original functor Ψ is the $\Bbbk$-linear tensor ideal $\mathcal{I}$ of $\mathcal{OS}(z,t)$ generated by $\imath_{n+1}(e_{(1^{n+1})})$. Jimbo's results show that the homomorphism $\jmath_r$ introduced above is injective when $r \leq n$, and that $\ker \jmath_r$ is the ideal of H_r generated by $e_{(1^{n+1})}$ (viewed as an element of H_r via the natural embedding $H_{n+1} \hookrightarrow H_r$) when $r > n$. In particular, taking $r = n+1$, this shows that $\Psi(\imath_{n+1}(e_{1^{n+1}})) = 0$, so Ψ induces a monoidal functor $\tilde{\Psi} : \mathcal{OS}(q, q^n)/\mathcal{I} \to \mathrm{Rep}\, U_q(\mathfrak{gl}_n)$. The commuting diagram (3.13) becomes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{OS}(z,t)/\mathcal{I}}(\mathsf{a},\mathsf{b}) \arrow[r, "\sim", "\tilde{\theta}"'] \arrow[d, "\tilde{\Psi}"'] & \mathrm{Hom}_{\mathcal{OS}(z,t)/\mathcal{I}}(\uparrow^r, \uparrow^r) \arrow[d, "\tilde{\Psi}"] & H_r/I_r \arrow[l, "\tilde{\imath}_r"'] \\
\mathrm{Hom}_{U_q(\mathfrak{gl}_n)}(\Psi(\mathsf{a}), \Psi(\mathsf{b})) \arrow[r, "\sim", "\phi"'] & \mathrm{Hom}_{U_q(\mathfrak{gl}_n)}((V^+)^{\otimes r}, (V^+)^{\otimes r}), &
\end{tikzcd}$$

where $I_r := \{0\}$ if $r \leq n$ and $I_r := \langle e_{(1^{n+1})} \rangle$ if $r > n$. The isomorphism $\tilde{\theta}$ in this diagram is defined in the same way as θ, indeed, it is induced by θ in an obvious way. Also when $r > n$ the map $\imath_r$ takes $e_{(1^{n+1})} \in H_r$ to $\uparrow^{r-n-1} \imath_{n+1}(e_{(1^{n+1})})$. This morphism lies in $\mathcal{I}$, showing that $\imath_r$ induces the homomorphism $\tilde{\imath}_r$ indicated in the diagram. Now the composition of $\tilde{\imath}_r$ and the right hand $\tilde{\Psi}$ is an isomorphism. Also it is obvious from

[1] We mean the tensor ideal of $\mathcal{OS}(z,t)$ defined by the kernels of the maps $\Psi : \mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathsf{a},\mathsf{b}) \twoheadrightarrow \mathrm{Hom}_{U_q(\mathfrak{gl}_n)}(\Psi(\mathsf{a}), \Psi(\mathsf{b}))$ for all $\mathsf{a}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle$.

the definition of $\mathcal{OS}(z,t)$ that $\imath_r$, hence, $\tilde{\imath}_r$ is surjective. We deduce that the right hand $\tilde{\Psi}$ is an isomorphism, hence, the left hand one is too. This shows that $\mathcal{I}$ is indeed the kernel of Ψ. $\square$

Remark 3.3. Let notation be as in Theorem 1.3, taking $\varepsilon = +$. If $\mathsf{a}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle$ are objects such that x (resp. x') letters of a and y (resp. y') letters of b are equal to $\downarrow$ (resp. $\uparrow$), and $r := x' + y = x + y'$ satisfies $r \leq n$, then Ψ is injective on $\mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathsf{a}, \mathsf{b})$ and

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathsf{a}, \mathsf{b}) = \dim H_r = r!. \tag{3.14}$$

These assertions follow from the proof just explained: when $r \leq n$ the map $\jmath_r$ is an isomorphism so all of the vertical maps in (3.13) are isomorphisms too. (Theorem 1.2 implies that the formula (3.14) holds without the restriction $r \leq n$, but we will use this special case in its proof.)

Proof of Theorem 1.2. In this proof, we are going to allow $\Bbbk$ to vary, so may add an additional subscript, denoting $\mathcal{OS}(z,t)$ and $\widehat{\mathcal{OS}}(z,t)$ instead by $\mathcal{OS}(z,t)_{\Bbbk}$ and $\widehat{\mathcal{OS}}(z,t)_{\Bbbk}$, respectively. We first establish the result for the morphism spaces of $\mathcal{OS}$. Take $\mathsf{a}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle$ such that x (resp. x') letters of a and y (resp. y') letters of b are equal to $\downarrow$ (resp. $\uparrow$), and $r := x' + y = x + y'$. Let $B(\mathsf{a}, \mathsf{b})$ be some set of reduced lifts of the (a, b)-matchings, so that $|B(\mathsf{a}, \mathsf{b})| = r!$. It is straightforward to see for any $\Bbbk, z$ and t that $B(\mathsf{a}, \mathsf{b})$ spans $\mathrm{Hom}_{\mathcal{OS}(z,t)_{\Bbbk}}(\mathsf{a}, \mathsf{b})$. We need to show that it is also linearly independent.

Consider first the case that $\Bbbk = \mathbb{Z}[z, z^{-1}, t, t^{-1}]$. Take a linear relation

$$\sum_{b \in B(\mathsf{a}, \mathsf{b})} c_b(z,t) b = 0$$

for $c_b(z,t) \in \mathbb{Z}[z, z^{-1}, t, t^{-1}]$. For any $n \geq r$, we can consider the obvious strict $\mathbb{Z}$-linear monoidal functor $\omega : \mathcal{OS}(z,t)_{\mathbb{Z}[z,z^{-1},t,t^{-1}]} \to \mathcal{OS}(q - q^{-1}, q^n)_{\mathbb{Q}(q)}$ sending $z \mapsto q - q^{-1}, t \mapsto q^n$, and generating morphisms to the generating morphisms with the same names. This functor maps $B(\mathsf{a}, \mathsf{b})$ to a spanning set for $\mathrm{Hom}_{\mathcal{OS}(q-q^{-1},q^n)_{\mathbb{Q}(q)}}(\mathsf{a}, \mathsf{b})$. Since $|B(\mathsf{a}, \mathsf{b})| = r!$, we deduce from (3.14) that $\omega(B(\mathsf{a}, \mathsf{b}))$ is linearly independent too. Hence, $c_b(q - q^{-1}, q^n) = 0$ for each $b \in B(\mathsf{a}, \mathsf{b})$. Since this is true for infinitely many values of n, it follows that each $c_b(z,t) = 0$.

Now take an arbitrary commutative ground ring $\Bbbk$ and parameters $\bar{z}, \bar{t} \in \Bbbk^{\times}$. Viewing $\Bbbk$ as a $\mathbb{Z}[z, z^{-1}, t, t^{-1}]$-module so z and t act via $\bar{z}$ and $\bar{t}$, there is an obvious strict $\Bbbk$-linear monoidal functor $\mathcal{OS}(\bar{z}, \bar{t})_{\Bbbk} \to \mathcal{OS}(z,t)_{\mathbb{Z}[z,z^{-1},t,t^{-1}]} \otimes_{\mathbb{Z}[z,z^{-1},t,t^{-1}]} \Bbbk$ sending generating morphisms to the generating morphisms with the same name tensored with $1_{\Bbbk}$. This functor sends $B(\mathsf{a}, \mathsf{b})$ to a set of morphisms which we already know is linearly independent thanks to the previous paragraph. Hence $B(\mathsf{a}, \mathsf{b})$ itself is linearly independent. This completes the proof for $\mathcal{OS}$.

It remains to treat the extended category $\widehat{\mathcal{OS}}$. Again, it is clear that the morphisms from the statement of Theorem 1.2 span, so we just need to establish linear independence. For all but the case $\mathsf{a} = \mathsf{b} = \varnothing$, this follows immediately since we have already established linear independence in the quotient category $\mathcal{OS}(z,t)_{\Bbbk}$. Thus, we are left with showing that $1_{\varnothing}$ and  are linearly independent in $\mathrm{Hom}_{\widehat{\mathcal{OS}}(z,t)_{\Bbbk}}(\varnothing, \varnothing)$. By the same arguments as in the previous two paragraphs, this follows if we can check it in $\mathrm{Hom}_{\widehat{\mathcal{OS}}(q-q^{-1},q^n)_{\mathbb{Q}(q)}}(\varnothing, \varnothing)$ for infinitely many values of n. This is done in the final paragraph of the proof.

So assume that $\Bbbk = \mathbb{Q}(q)$, $z = q - q^{-1}$ and $t = q^n$ for $n \in \mathbb{N}$. We define a new strict $\Bbbk$-linear monoidal category $\mathcal{C}$. Its objects are as in $\mathcal{OS}(z,t)$ with the same tensor product, and its morphisms are defined from

$$\mathrm{Hom}_{\mathcal{C}}(\mathsf{a},\mathsf{b}) := \begin{cases} \mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathsf{a},\mathsf{b}) & \text{if } \mathsf{a}\neq\varnothing \text{ or } \mathsf{b}\neq\varnothing,\\ \mathrm{Hom}_{\mathcal{OS}(z,t)}(\varnothing,\varnothing)\oplus\Bbbk & \text{if } \mathsf{a}=\mathsf{b}=\varnothing. \end{cases}$$

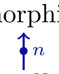
So $\mathrm{Hom}_{\mathcal{C}}(\varnothing,\varnothing)$ is two-dimensional with basis $(1_\varnothing, 0)$ and $(0, 1_\Bbbk)$. Horizontal and vertical composition of most of the morphisms in $\mathcal{C}$ is induced by the compositions in $\mathcal{OS}(z,t)$ in the obvious way; the horizontal and vertical composition of $(0,1_\Bbbk)$ with any morphism in $\mathrm{Hom}_{\mathcal{C}}(\mathsf{a},\mathsf{b})$ is zero if $\mathsf{a}\neq\varnothing$ or $\mathsf{b}\neq\varnothing$; the horizontal and vertical composition of $(0,1_\Bbbk)$ with $(a1_\varnothing, b1_\Bbbk)$ is $(0, b1_\Bbbk)$. Now the point is that there is a strict $\Bbbk$-linear monoidal functor $\widehat{\mathcal{OS}}(z,t)\to\mathcal{C}$ sending objects and generating morphisms to their images under the quotient functor to $\mathcal{OS}(z,t)$ embedded (non-unitally) into $\mathcal{C}$. Due to the relation (D) in $\mathcal{OS}(z,t)$, the morphism  $\in \mathrm{Hom}_{\widehat{\mathcal{OS}}(z,t)}(\varnothing,\varnothing)$ maps to $([n]_q 1_\varnothing, 0)$, while the identity element $1_\varnothing \in \mathrm{Hom}_{\widehat{\mathcal{OS}}(z,t)}(\varnothing,\varnothing)$ must map to the identity element $(1_\varnothing, 1_\Bbbk)\in\mathrm{Hom}_{\mathcal{C}}(\varnothing,\varnothing)$. Since $([n]_q1_\varnothing,0)$ and $(1_\varnothing,1_\Bbbk)$ are linearly independent, it follows that  and $1_\varnothing$ are linearly independent in $\mathrm{Hom}_{\widehat{\mathcal{OS}}(z,t)}(\varnothing,\varnothing)$. $\square$

Remark 3.4. A modified version of Theorem 1.3 holds over any field $\Bbbk$ for any $q\in\Bbbk^\times\setminus\{\pm1\}$. Let $q\text{-}GL_n$ be the quantum general linear group over $\Bbbk$ at parameter q; its coordinate algebra $\Bbbk[q\text{-}GL_n]$ is the localization of Manin's quantized coordinate algebra of $n\times n$ matrices at the quantum determinant as in [PW]. Let $\mathrm{Rep}\,q\text{-}GL_n$ be the category of rational $q\text{-}GL_n$-modules (=finite-dimensional $\Bbbk[q\text{-}GL_n]$-comodules). Then there is a full $\Bbbk$-linear monoidal functor $\Psi:\mathcal{OS}(q-q^{-1},t^n)\to\mathrm{Rep}\,q\text{-}GL_n$ sending $\uparrow\mapsto V^+$ and $\downarrow\mapsto V^-$ defined just like in Lemma 3.1. It induces a monoidal equivalence

$$\bar\Psi:\dot{\mathcal{OS}}(q-q^{-1},q^n)/\mathcal{N}\xrightarrow{\approx}\mathrm{Tilt}'\,q\text{-}GL_n \tag{3.15}$$

where $\mathcal{N}$ is the additive $\Bbbk$-linear tensor ideal generated by $\imath_{n+1}(e_{(1^{n+1})})$, and $\mathrm{Tilt}'\,q\text{-}GL_n$ is the full subcategory of $\mathrm{Rep}\,q\text{-}GL_n$ consisting of all modules isomorphic to direct sums of summands of tensor powers of V^+ and V^-. The proof of this is similar to the proof of Theorem 1.3, using the generalization of Schur-Weyl duality from [DPS, Theorem 6.2] and [H, Theorem 4].

4. The affine oriented skein category

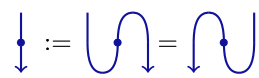
In this section, $\Bbbk$ is a commutative ground ring and $z,t\in\Bbbk^\times$ are arbitrary. The *affine oriented skein category* $\mathcal{AOS}(z,t)$ is the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{OS}(z,t)$ by adjoining an additional generating morphism  and a two-sided inverse of this morphism, subject to the additional relation (A) from Figure 1. For any $n\in\mathbb{Z}$, we write  for the nth power of this additional generator. The relation (A) comes from the *affine Hecke algebra* AH_r, which is generated by the Iwahori-Hecke algebra H_r from (3.6) plus additional elements $X_1^{\pm1},\dots,X_r^{\pm1}$ subject to the relations

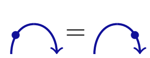
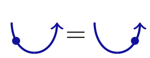
$$X_iX_j = X_jX_i, \qquad\qquad S_iX_iS_i = X_{i+1} \tag{4.1}$$

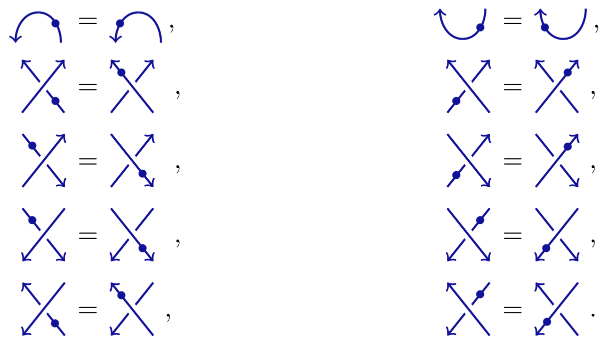
for all i,j. There is an algebra homomorphism

$$\jmath_r : AH_r\to\mathrm{End}_{\mathcal{AOS}(z,t)}(\uparrow^r) \tag{4.2}$$

defined on $S_1,\dots,S_{r-1}$ in the same way as for the homomorphism $\imath_r$ from (3.7), and sending X_i to the dot on the ith strand from the right.

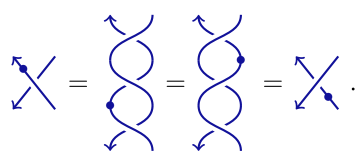
Lemma 4.1. *In $\mathcal{AOS}(z,t)$, we have that*  *. Moreover, all of the following relations hold:*

 ,  , (4.3)

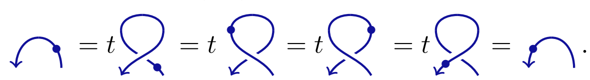


(4.4)
(4.5)
(4.6)
(4.7)
(4.8)

Proof. Define  to be the left hand expression from the main identity that we are trying to prove; we still need to show that it is equal to the right hand expression. The relations (4.5) follow from (A) and (RII). Using the relations from (R0) involving rightward cups and rightward caps, the relations (4.3), (4.6) and (4.7) are then easy to check too. Here is the proof of the first equality from (4.8); the second equality can be proved similarly:



Then we use these identities plus (2.5) to check the first equality from (4.4):



The remaining equality from (4.4), and the equality of  with the second expression from the main identity, now follow easily using (R0) once more. $\square$

There is an obvious monoidal functor

$$\alpha : \mathcal{OS}(z,t) \to \mathcal{AOS}(z,t)$$

taking objects and morphisms to the same things in $\mathcal{AOS}(z,t)$. The following lemma shows that α is faithful (and also that the functor β from the lemma is full).

Lemma 4.2. *There is a unique* $\mathbb{k}$*-linear (but not monoidal!) functor*

$$\beta : \mathcal{AOS}(z,t) \to \mathcal{OS}(z,t)$$

such that $\beta \circ \alpha = \mathrm{Id}_{\mathcal{OS}(z,t)}$ *and* $\beta\Big(1_{\mathsf{a}} \otimes$  $\Big) = 1_{\mathsf{a}} \otimes \uparrow$ *for all* $\mathsf{a} \in \langle \uparrow, \downarrow \rangle$*. The effect of* β *on dots on other strands is given by*

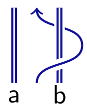
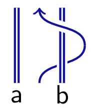
(4.9)

for any $\mathsf{a}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle$.

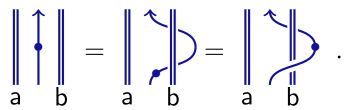
Proof. We already have a presentation for $\mathcal{AOS}(z,t)$ as a $\mathbb{k}$-linear monoidal category, with generators and relations coming from Theorem 1.1 plus the additional generator $O :=$  and its two-sided inverse O^{-1} subject to (A). Since we are trying to construct a $\mathbb{k}$-linear functor that is not monoidal, we need to convert this into a presentation for $\mathcal{AOS}(z,t)$ as a $\mathbb{k}$-linear category, as explained in [BCNR, §2.6]. The generators are all morphisms of the form $1_{\mathsf{a}} \otimes S \otimes 1_{\mathsf{b}}$, $1_{\mathsf{a}} \otimes T \otimes 1_{\mathsf{b}}$, $1_{\mathsf{a}} \otimes C \otimes 1_{\mathsf{b}}$, $1_{\mathsf{a}} \otimes D \otimes 1_{\mathsf{b}}$ and $1_{\mathsf{a}} \otimes O \otimes 1_{\mathsf{b}}$

for all $\mathsf{a}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle$. The relations are derived from the monoidal relations by tensoring with 1_a and 1_b in a similar way, plus we also need *commuting relations* replacing the interchange law.

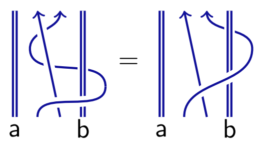
Using this new presentation, we can establish the existence of β: it is the identity on objects, and must send the generating morphisms $1_\mathsf{a} \otimes S \otimes 1_\mathsf{b}, 1_\mathsf{a} \otimes T \otimes 1_\mathsf{b}, 1_\mathsf{a} \otimes C \otimes 1_\mathsf{b}$ and $1_\mathsf{a} \otimes D \otimes 1_\mathsf{b}$ to the same morphisms in $\mathcal{OS}(z,t)$ since we want $\beta \circ \alpha = \mathrm{Id}_{\mathcal{OS}(z,t)}$. It sends $1_\mathsf{a} \otimes O \otimes 1_\mathsf{b}$ and its two-sided inverse $1_\mathsf{a} \otimes O^{-1} \otimes 1_\mathsf{b}$ to


and

,

respectively. Again, there is no choice here, since in $\mathcal{AOS}(z,t)$ we have that

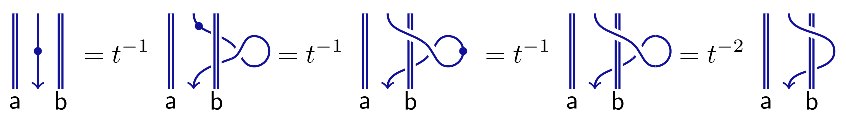


Now we check the relations. All the ones that do not involve O hold automatically. For the rest, the following checks (A):

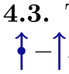


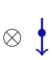
The commuting relations involving O and any other generating morphism are equally straightforward.

So now we have constructed β and proved its uniqueness. The composition $\beta \circ \alpha$ is $\mathrm{Id}_{\mathcal{OS}(z,t)}$ since it is the identity on objects and generating morphisms. It remains to see that the second equality in (4.9) holds:



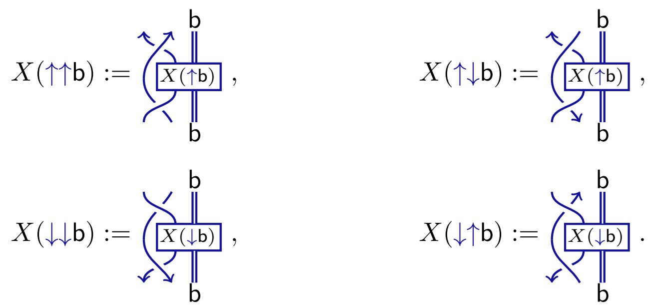
$\square$

Remark 4.3. The kernel of β is the left tensor ideal of $\mathcal{AOS}(z,t)$ generated by the morphism . This follows because the quotient of $\mathcal{AOS}(z,t)$ by this left tensor ideal is a $\mathbb{k}$-linear category which maps surjectively onto $\mathcal{OS}(z,t)$ via the functor induced by β, and also $\mathcal{OS}(z,t)$ maps surjectively onto it via the functor induced by α. This identifies $\mathcal{OS}(z,t)$ with the simplest "level one" example of a *cyclotomic quotient* of $\mathcal{AOS}(z,t)$.

The images of the morphisms $1_\mathsf{a} \otimes$  $\otimes 1_\mathsf{b}$ and $1_\mathsf{a} \otimes$  $\otimes 1_\mathsf{b}$ under β are the *Jucys-Murphy elements* of $\mathcal{OS}(z,t)$. The elements (1.8) in the introduction are a special case. In the sequel, we will often need these elements in the special case that $\mathsf{a} = \varnothing$, adopting the following notation:

$X(\uparrow\mathsf{b}) := \beta\Big($  $\otimes 1_\mathsf{b}\Big)$, $\qquad X(\downarrow\mathsf{b}) := \beta\Big($  $\otimes 1_\mathsf{b}\Big)$. (4.10)

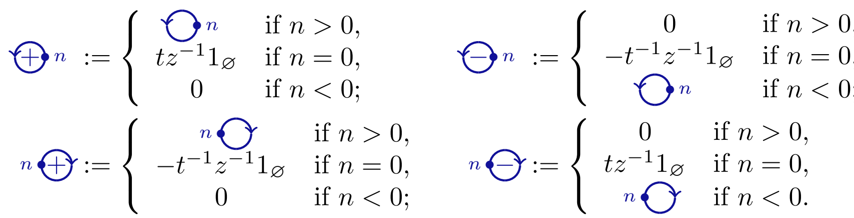
We can use (4.9) to obtain a recursive formula: first $X(\uparrow) := 1_\uparrow$, $X(\downarrow) := t^{-2}1_\downarrow$; then for any $\mathsf{b} \in \langle \uparrow, \downarrow \rangle$ we have that



(4.11)

(4.12)

The following lemma will not be needed in this article. It suggests $\mathrm{End}_{\mathcal{AOS}(z,t)}(\varnothing)$ consists of two copies of the ring of symmetric functions. Define the positive and negative bubbles with n dots by

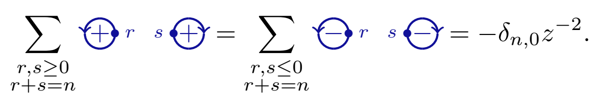


(4.13)

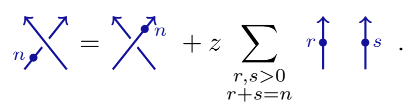
(4.14)

Note that  due to the relation (D).

Lemma 4.4. *For any $n \in \mathbb{Z}$, we have that*

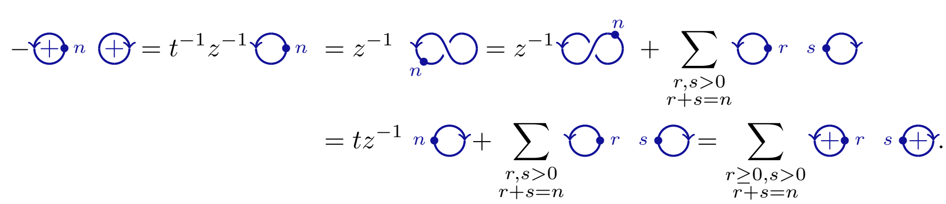


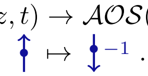
Proof. Consider the relation involving positive bubbles. It is immediate in case $n \le 0$. If $n > 0$, we need the following relation which may be proved by induction using the relations (A) and (S):



(4.15)

Then we calculate:



The relation involving for negative bubbles may now be deduced by applying the automorphism $\rho : \mathcal{AOS}(z,t) \to \mathcal{AOS}(z,t)$ which is defined on generators in the same way as (2.8) plus it maps  . $\square$

The category $\mathcal{AOS}(z,t)$ is studied further in [B]: it is the $k = 0$ case of the q-Heisenberg category $\mathcal{H}_k(z,t)$ introduced there.

5. Shortest word theory

For the next few sections, we assume that $\Bbbk$ is a field of characteristic $p \geq 0$ and $z = q - q^{-1}$ for $q \in \Bbbk^\times \setminus \{\pm 1\}$. Since we are going to be discussing linear representations rather than tensor ones, it is natural to replace the category $\mathcal{OS}(z,t)$ with the associative algebra

$$OS = \bigoplus_{\mathsf{a},\mathsf{b} \in \langle \uparrow, \downarrow \rangle} 1_{\mathsf{a}} OS 1_{\mathsf{b}} \qquad \text{where} \qquad 1_{\mathsf{a}} OS 1_{\mathsf{b}} := \mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathsf{b}, \mathsf{a}),$$

multiplication being induced by composition in $\mathcal{OS}(z,t)$. This algebra is *locally unital* rather than unital, with a distinguished system of local units given by the mutually orthogonal idempotents $\{1_{\mathsf{a}} \mid \mathsf{a} \in \langle \uparrow, \downarrow \rangle\}$. The functor category Mod-$\mathcal{OS}(z,t)$ of $\mathcal{OS}(z,t)$-modules as defined in the introduction may be identified with the usual algebraic category Mod-OS of all *right OS-modules M* which are locally unital in the sense that $M = \bigoplus_{\mathsf{a} \in \langle \uparrow, \downarrow \rangle} M1_{\mathsf{a}}$. The additive Karoubi envelope $\dot{\mathcal{OS}}(z,t)$ is equivalent to the full subcategory pMod-OS of Mod-OS consisting of *finitely generated projective modules*, i.e., modules isomorphic to finite direct sums of right ideals eOS for idempotents $e \in OS$. This means that we may identify $K_0(\dot{\mathcal{OS}}(z,t))$ with the split Grothendieck group $K_0(\text{pMod-}OS)$; the resulting ring structure on $K_0(\text{pMod-}OS)$ is determined by

$$[eOS][fOS] = [(e \otimes f)OS]$$

for idempotents $e, f \in OS$.

Each $1_{\mathsf{a}} OS 1_{\mathsf{b}}$ is finite-dimensional by Theorem 1.2, hence, OS is *locally finite-dimensional*, and Mod-OS is a *locally Schurian category* in the sense of [BD, §2]. We will freely use the general language and results about such categories explained there, most of which boil down to the observation that Mod-OS is a Grothendieck category. In particular, we let lfdMod-OS be the subcategory of Mod-OS consisting of all *locally finite-dimensional modules*, i.e., the OS-modules M such that $\dim_{\Bbbk} M1_{\mathsf{a}} < \infty$ for all $\mathsf{a} \in \langle \uparrow, \downarrow \rangle$. These are exactly the modules that have finite composition multiplicities. Any finitely generated OS-module M is locally finite-dimensional, so that pMod-OS is a subcategory of lfdMod-OS.

In this section, we are going to classify the irreducible OS-modules. The key to our approach is that the algebra OS has a *triangular decomposition*. Any ribbon has three sorts of component:

- cups whose boundary is on $y = 1$;
- caps whose boundary is on $y = 0$;
- propagating strands whose boundary intersects both $y = 0$ and $y = 1$.

The cups and caps carry an overall orientation that is either leftward or rightward, while the propagating strands are either upward strands or downward strands. Introduce the following locally unital subalgebras of OS:

$OS^\circ_{r,s}$: The $\Bbbk$-span of all ribbons that have r propagating upward strands and s propagating downward strands, no components that are cups or caps, and in which propagating upward strands only cross underneath propagating downward strands.

OS°: $\bigoplus_{r,s \geq 0} OS^\circ_{r,s}$.

OS^+: The $\Bbbk$-span of all ribbons that have no components that are caps and in which no two propagating strands cross.

$OS^\sharp$: The $\Bbbk$-span of all ribbons with no components that are caps and in which propagating upward strands only cross underneath propagating downward strands.

OS^-: The $\Bbbk$-span of all ribbons that have no components that are cups and in which no two propagating strands cross.

$OS^\flat$: The $\Bbbk$-span of all ribbons with no components that are cups and in which propagating upward strands only cross underneath propagating downward strands.

It is obvious that these are all locally unital subalgebras of OS. Also, OS is graded as $OS = \bigoplus_{d\in\mathbb{Z}} OS[d]$ with $OS[d]$ spanned by all ribbons such that the total number of cups minus the total number of caps equals d. This induces a grading on each of the subalgebras above. Moreover, $OS^\sharp[0] = OS^\flat[0] = OS^\circ$. We have already introduced the right OS-module categories $\text{Mod-}OS, \text{lfdMod-}OS$ and $\text{pMod-}OS$. We adopt analogous notation for all of these other locally unital algebras. Also let

$$\text{fdMod-}OS^\circ = \coprod_{r,s\geq 0} \text{fdMod-}OS^\circ_{r,s}$$

be the category of (globally) finite-dimensional right OS°-modules.

In the following lemma, we take tensor products of locally unital modules over the locally unital algebra $\mathbb{K} := \bigoplus_{\mathsf{a}\in\langle\uparrow,\downarrow\rangle} \Bbbk 1_{\mathsf{a}} < OS$. In terms of the usual tensor product $\otimes$ over the ground field $\Bbbk$, we have that $V\otimes_{\mathbb{K}} W = \bigoplus_{\mathsf{a}\in\langle\uparrow,\downarrow\rangle} V1_{\mathsf{a}}\otimes 1_{\mathsf{a}}W$.

Lemma 5.1. *Multiplication define a vector space isomorphism*

$$OS^+\otimes_{\mathbb{K}} OS^\circ \otimes_{\mathbb{K}} OS^- \xrightarrow{\sim} OS.$$

Similarly, there are isomorphisms $OS^+\otimes_{\mathbb{K}} OS^\circ \xrightarrow{\sim} OS^\sharp$ *and* $OS^\circ\otimes_{\mathbb{K}} OS^- \xrightarrow{\sim} OS^\flat$.

Proof. We apply Theorem 1.2 to pick a basis for OS^+ consisting of cap-free generic reduced lifts of matchings. Similarly, we pick a basis for OS^-. Finally, we pick a basis for OS° consisting of cup- and cap-free generic ribbons, all of whose rightward crossings are positive and leftward crossings are negative. To prove the lemma, it remains to observe that the non-zero images of the pure tensors in these basis elements under the multiplication $OS^+\otimes_{\mathbb{K}} OS^\circ\otimes_{\mathbb{K}} OS^- \to OS$ give a basis for OS itself: it is just another of the bases from Theorem 1.2 consisting of generic reduced lifts with all cups at the top, all crossings of propagating strands in the middle, and all caps at the bottom of the picture. $\square$

Recall the Iwahori-Hecke algebras H_r from (3.6) and the homomorphism $\imath_r$ from (3.7). Theorem 1.2 shows that this is an isomorphism. More generally, let

$$H_{r,s} := H_r \otimes H_s \tag{5.1}$$

for any $r,s\geq 0$. Then there is an injective (but no longer surjective) homomorphism

$$\imath_{r,s} : H_{r,s} \to 1_{\downarrow^s\uparrow^r} OS 1_{\downarrow^s\uparrow^r} \tag{5.2}$$

sending $S_i\otimes 1$ to the positive crossing  of the ith and $(i+1)$th strands and $1\otimes S_j$ to the positive crossing  of the $(r+j)$th and $(r+j+1)$th strands, again numbering strands from right to left.

Lemma 5.2. *The map* $\imath_{r,s}$ *is an algebra isomorphism* $H_{r,s} \xrightarrow{\sim} 1_{\downarrow^s\uparrow^r} OS^\circ_{r,s} 1_{\downarrow^s\uparrow^r}$. *Moreover,* $OS^\circ_{r,s}$ *is isomorphic to the matrix algebra* $\mathrm{Mat}_{\binom{r+s}{r}}(H_{r,s})$. *Hence, the functor*

$$\Upsilon_{r,s} := ?\otimes_{H_{r,s}} 1_{\downarrow^s\uparrow^r} OS^\circ_{r,s} : \text{Mod-}H_{r,s} \to \text{Mod-}OS^\circ_{r,s}.$$

is an equivalence of categories, with quasi-inverse defined by right multiplication by the idempotent $1_{\downarrow^s\uparrow^r}$.

Proof. The first statement follows from Theorem 1.2. To deduce the second statement, let $\langle\uparrow,\downarrow\rangle_{r,s}$ denote the $\binom{r+s}{r}$ different words which have r letters $\uparrow$ and s letters $\downarrow$. Note that $OS^\circ_{r,s} = \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle_{r,s}} 1_{\mathsf{a}} OS^\circ_{r,s} 1_{\mathsf{b}}$. For each $\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle_{r,s}$, let $e_{\mathsf{a},\mathsf{b}} \in 1_{\mathsf{a}} OS^\circ_{r,s} 1_{\mathsf{b}}$

be the unique (up to planar isotopy) reduced (b,a)-ribbon which only involves positive rightward crossings and negative leftward crossings, and has no cups, caps or upward/downward crossings. For example, $e_{\mathsf{a},\mathsf{a}} = 1_{\mathsf{a}}$ for each a. We have that $e_{\mathsf{a},\mathsf{b}} e_{\mathsf{b}',\mathsf{c}} = \delta_{\mathsf{b},\mathsf{b}'} e_{\mathsf{a},\mathsf{c}}$. Hence, the map

$$H_{r,s} \to 1_{\mathsf{a}} OS^{\circ}_{r,s} 1_{\mathsf{b}}, \qquad f \mapsto e_{\mathsf{a},\downarrow^s\uparrow^r} \imath_{r,s}(f) e_{\downarrow^s\uparrow^r,\mathsf{b}}$$

is a bijection, and $OS^{\circ}_{r,s} = \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle_{r,s}} H_{r,s} e_{\mathsf{a},\mathsf{b}}$ is the matrix algebra as claimed. □

Next we are going to mimic the usual arguments of highest weight theory for semisimple Lie algebras, with the role of "Borel subalgebra" being played by $OS^{\sharp}$, and the role of "Cartan subalgebra" being played by OS°. To parametrize the isomorphism classes of irreducible representations of OS°, we need some facts about the representation theory of the Iwahori-Hecke algebra H_r; e.g. see [DJ1].

The algebra H_r is semisimple if q is not a root of unity, with irreducible representations being the Specht modules S_λ parametrized by partitions $\lambda \vdash r$. To construct S_λ explicitly, recall the elements x_λ and y_λ from (3.8). The right ideals $x_\lambda H_r$ and $y_\lambda H_r$ are the *permutation module* M_λ and the *signed permutation module* N_λ, respectively. By [DJ1, Theorem 3.3], the space $\mathrm{Hom}_{H_r}(M_\lambda, N_{\lambda^{\mathrm{t}}})$ is one-dimensional. Then the *Specht module* S_λ is the image of any non-zero homomorphism in this space.

The definition just given also makes sense when q is a root of unity (remembering $q^2 \neq 1$); the resulting Specht module S_λ is related to the module S^λ constructed in [DJ1] by $\mathrm{S}_\lambda \cong (\mathrm{S}^{\lambda^{\mathrm{t}}})^{\#}$. In general, we define e to be the smallest positive integer such that $q^{2e} = 1$, setting $e := 0$ if q is not a root of unity. A partition λ is *e-restricted* if either $e = 0$, or $e > 0$ and $\lambda_i - \lambda_{i+1} < e$ for each $i = 1, 2, \dots$. For e-restricted $\lambda \vdash r$, the Specht module S_λ has irreducible head D_λ, and these modules for all e-restricted $\lambda \vdash r$ give a complete set of pairwise inequivalent irreducible right H_r-modules.

Let Y_λ be a projective cover of D_λ; since H_r is a symmetric algebra, this is also an injective hull. If $e = 0$ we have simply that $\mathrm{D}_\lambda = \mathrm{S}_\lambda = \mathrm{Y}_\lambda$, and they are all equal to the right ideal $e_\lambda H_r$ where e_λ is the Young symmetrizer from (3.9). In general, it is known that Y_λ has a finite filtration[2] with sections S_μ, each appearing $[\mathrm{S}_\mu : \mathrm{D}_\lambda]$ times. Consequently,

$$[\mathrm{Y}_\lambda] = \sum_{\mu \vdash r} [\mathrm{S}_\mu : \mathrm{D}_\lambda][\mathrm{S}_\mu]$$

in the Grothendieck group $K_0(\text{fdMod-}H_r)$ of the Abelian category fdMod-H_r. We refer to this decomposition as *Brauer reciprocity*; it may also be proved by lifting idempotents.

Now let $\mathrm{Bip}_{r,s} := \{\boldsymbol{\lambda} = (\lambda^\uparrow, \lambda^\downarrow) \mid \lambda \vdash r, \mu \vdash s\}$ be the set of *bipartitions* of (r,s), and let $e\text{-}\mathrm{Bip}_{r,s} \subseteq \mathrm{Bip}_{r,s}$ be the e-restricted ones. In particular, we denote the empty bipartition $(\varnothing, \varnothing)$ by $\boldsymbol{\varnothing}$. From the previous paragraph, it follows that $e\text{-}\mathrm{Bip}_{r,s}$ parametrizes the irreducible $H_{r,s}$-modules. Applying Lemma 5.2, we get from them irreducible OS°-modules

$$\mathrm{D}(\boldsymbol{\lambda}) := (\mathrm{D}_{\lambda^\uparrow} \boxtimes \mathrm{D}_{\lambda^\downarrow}) \otimes_{H_{r,s}} 1_{\downarrow^s\uparrow^r} OS^{\circ}_{r,s}. \tag{5.3}$$

Define $\mathrm{S}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathrm{Bip}_{r,s}$ and $\mathrm{Y}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{r,s}$ in similar ways, starting from $\mathrm{S}_{\lambda^\uparrow} \boxtimes \mathrm{S}_{\lambda^\downarrow}$ or $\mathrm{Y}_{\lambda^\uparrow} \boxtimes \mathrm{Y}_{\lambda^\downarrow}$, respectively. For $\boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{r,s}$, $\mathrm{Y}(\boldsymbol{\lambda})$ is a projective cover and an injective hull of $\mathrm{D}(\boldsymbol{\lambda})$, and it has a filtration with sections $\mathrm{S}(\boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \mathrm{Bip}_{r,s}$, each appearing $[\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\lambda})] = [\mathrm{S}_{\mu^\uparrow} : \mathrm{D}_{\lambda^\uparrow}][\mathrm{S}_{\mu^\downarrow} : \mathrm{D}_{\lambda^\downarrow}]$ in the filtration. Finally, we put these modules all together: setting $\mathrm{Bip} := \coprod_{r,s\geq 0} \mathrm{Bip}_{r,s}$ and $e\text{-}\mathrm{Bip} := \coprod_{r,s\geq 0} e\text{-}\mathrm{Bip}_{r,s}$, the modules $\{\mathrm{D}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}\}$ are a complete set of pairwise inequivalent irreducible OS°-modules.

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[2] This is proved by applying the "Schur functor" to the analogous result for the q-Schur algebra.

The projection of $OS^\sharp$ onto its degree zero component $OS^\sharp[0]$ is a surjective homomorphism $OS^\sharp \twoheadrightarrow OS^\circ$. Using this, we can view any OS°-module instead as an $OS^\sharp$-module. We denote the resulting inflation functor by $\mathrm{infl}^\sharp$; similarly, there is an inflation functor $\mathrm{infl}^\flat$ taking OS°-modules to $OS^\flat$-modules. Define

$$\bar{\Delta}(\boldsymbol{\lambda}) := \mathrm{infl}^\sharp\,\mathrm{D}(\boldsymbol{\lambda}) \otimes_{OS^\sharp} OS, \tag{5.4}$$

$$\tilde{\Delta}(\boldsymbol{\lambda}) := \mathrm{infl}^\sharp\,\mathrm{S}(\boldsymbol{\lambda}) \otimes_{OS^\sharp} OS, \tag{5.5}$$

$$\Delta(\boldsymbol{\lambda}) := \mathrm{infl}^\sharp\,\mathrm{Y}(\boldsymbol{\lambda}) \otimes_{OS^\sharp} OS, \tag{5.6}$$

for $\lambda \in e\text{-Bip}, \mathrm{Bip}$ and $e\text{-Bip}$, respectively. We call $\bar{\Delta}(\boldsymbol{\lambda})$ the *proper standard module* and $\Delta(\boldsymbol{\lambda})$ the *standard module* associated to $\boldsymbol{\lambda} \in e\text{-Bip}$. These are locally finite-dimensional but not "globally" finite-dimensional OS-modules. Note also if $e = 0$ that $\bar{\Delta}(\boldsymbol{\lambda}) = \tilde{\Delta}(\boldsymbol{\lambda}) = \Delta(\boldsymbol{\lambda})$ for each $\boldsymbol{\lambda} \in \mathrm{Bip}$.

Any OS-module M decomposes as $\bigoplus_{\mathsf{a} \in \langle \uparrow, \downarrow \rangle} M1_{\mathsf{a}}$. We refer to the direct sum of the subspaces $M1_{\mathsf{a}}$ taken over all $\mathsf{a} \in \langle \uparrow, \downarrow \rangle$ of minimal length such that $M1_{\mathsf{a}} \neq 0$ as the *shortest word space* of M. It is automatically an $OS^\sharp$-submodule of M on which $\bigoplus_{d>0} OS^\sharp[d]$ acts as zero. We say that M is a *shortest word module* of type $\boldsymbol{\lambda} \in e\text{-Bip}$ if M is generated as an OS-module by its shortest word space, and this space is isomorphic to $\mathrm{D}(\boldsymbol{\lambda})$ as an OS°-module. Then, since $\mathrm{D}(\boldsymbol{\lambda})$ is irreducible, M is actually generated by any non-zero vector in its shortest word space.

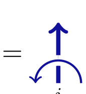
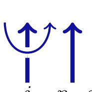
Theorem 5.3. *For* $\boldsymbol{\lambda} \in e\text{-Bip}$, *the proper standard module* $\bar{\Delta}(\boldsymbol{\lambda})$ *has a unique maximal submodule* $\mathrm{rad}\,\bar{\Delta}(\boldsymbol{\lambda})$. *Setting* $\mathrm{L}(\boldsymbol{\lambda}) := \bar{\Delta}(\boldsymbol{\lambda})/\mathrm{rad}\,\bar{\Delta}(\boldsymbol{\lambda})$, *we obtain a complete set of pairwise inequivalent irreducible OS-modules* $\{\mathrm{L}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in e\text{-Bip}\}$. *Moreover, each* $\mathrm{L}(\boldsymbol{\lambda})$ *is absolutely irreducible.*

Proof. By Lemma 5.1, we have that $\bar{\Delta}(\boldsymbol{\lambda}) = \mathrm{D}(\boldsymbol{\lambda}) \otimes_{OS^\sharp} OS = \mathrm{D}(\boldsymbol{\lambda}) \otimes_{\mathbb{K}} OS^-$ as a right $\mathbb{K}$-module. Hence, it is a non-zero shortest word module of type $\boldsymbol{\lambda}$. Since any non-zero vector in $\mathrm{D}(\boldsymbol{\lambda}) \otimes OS^-[0]$ generates all of $\bar{\Delta}(\boldsymbol{\lambda})$, any proper submodule of $\bar{\Delta}(\boldsymbol{\lambda})$ must be a subspace of $\bigoplus_{d<0} \mathrm{D}(\boldsymbol{\lambda}) \otimes_{\mathbb{K}} OS^-[d]$. This implies that the sum of all proper submodules of $\bar{\Delta}(\boldsymbol{\lambda})$ is itself proper, hence, it is the unique maximal submodule $\mathrm{rad}\,\bar{\Delta}(\boldsymbol{\lambda})$ of $\bar{\Delta}(\boldsymbol{\lambda})$. Thus, the quotient modules $\mathrm{L}(\boldsymbol{\lambda}) := \bar{\Delta}(\boldsymbol{\lambda})/\mathrm{rad}\,\Delta(\boldsymbol{\lambda})$ are irreducible.

Now let L be any irreducible OS-module. Pick an irreducible OS°-submodule $L^\circ \cong \mathrm{D}(\boldsymbol{\lambda})$ of its shortest word space. Then by Frobenius reciprocity we get an OS-module homomorphism $\bar{\Delta}(\boldsymbol{\lambda}) \to L$ lifting an isomorphism $\mathrm{D}(\boldsymbol{\lambda}) \xrightarrow{\sim} L^\circ$. This map is surjective since L is irreducible, hence, we get that $L \cong \mathrm{L}(\boldsymbol{\lambda})$.

Finally, if $\mathrm{L}(\boldsymbol{\lambda}) \cong \mathrm{L}(\boldsymbol{\mu})$, then their shortest word spaces must be isomorphic as OS°-modules, so $\boldsymbol{\lambda} = \boldsymbol{\mu}$. Also, since $\mathrm{End}_{OS}(\mathrm{L}(\boldsymbol{\lambda})) \cong \mathrm{End}_{OS^\circ}(\mathrm{D}(\boldsymbol{\lambda}))$, the absolute irreducibility follows from the analogous assertion for Hecke algebras, which is well known. $\square$

Example 5.4. To illustrate "shortest word theory," we use it to prove the existence of a non-zero homomorphism $\bar{\Delta}(\boldsymbol{\mu}) \to \bar{\Delta}(\boldsymbol{\lambda})$ when $t = q^n$ for $n \in \mathbb{N}$, $\boldsymbol{\lambda} := ((1^n), \varnothing)$ and $\boldsymbol{\mu} = ((1^{n+1}), (1))$. The irreducible $OS^\circ_{n,0}$-module $\mathrm{D}(\boldsymbol{\lambda})$ comes from the "sign representation" of H_n. So it is one-dimensional and is spanned by a vector on which $1_{\uparrow^n}$ acts as the identity and any positive upward crossing acts as $-q^{-1}$. Let v be the generator of $\bar{\Delta}(\boldsymbol{\lambda}) = \mathrm{D}(\boldsymbol{\lambda}) \otimes_{OS^\sharp} OS$ defined by this vector tensored $1_{\uparrow^n} \in OS$. Also for $i, j = 0, \dots, n$ define

$a_i :=$  , $b_j :=$  .

A calculation with relations shows that

$$va_ib_j = \begin{cases} [n]_q v & \text{if } i=j, \\ q^n(-q)^{i-j+1}v & \text{if } i<j, \\ q^{-n}(-q)^{i-j-1}v & \text{if } i>j. \end{cases}$$

Now consider the vector

$$w := \sum_{i=0}^{n} (-q)^i v a_i \in \bar{\Delta}(\boldsymbol{\lambda}).$$

This is non-zero by Lemma 5.1. We claim:

(1) $wb_j = 0$ for each $j = 0, 1, \dots, n$;
(2) $w\imath_{n+1,1}(S_i) = -q^{-1}w$ for $i = 1, \dots, n$.

To see (1), we have that

$$wb_j = \left[(-q)^j [n]_q + \sum_{i=0}^{j-1} q^n (-q)^{i-j+1}(-q)^i + \sum_{i=j+1}^{n} q^{-n}(-q)^{i-j-1}(-q)^i \right] v,$$

which is zero since $[n]_q = \sum_{i=0}^{j-1} q^{-n+2i-2j+1} + \sum_{i=j+1}^{n} q^{-n+2i-2j-1}$. We leave the check of (2) to the reader. It means that w spans a one-dimensional $H_{n+1,1}$-submodule of $\bar{\Delta}(\boldsymbol{\lambda})$ isomorphic to its "sign representation," so the OS°-submodule generated by w is a copy of $\mathrm{D}(\boldsymbol{\mu})$. In view of (1), it is actually an $OS^\sharp$-submodule isomorphic to $\mathrm{infl}^\sharp \mathrm{D}(\boldsymbol{\mu})$. Finally, by Frobenius reciprocity, we get a non-zero OS-module homomorphism $\bar{\Delta}(\boldsymbol{\mu}) \to \bar{\Delta}(\boldsymbol{\lambda})$.

The various flavors of standard module introduced in (5.4)–(5.6) are obtained by applying the *standardization functor*

$$\Delta := (\mathrm{infl}^\sharp -) \otimes_{OS^\sharp} OS : \text{Mod-}OS^\circ \to \text{Mod-}OS \tag{5.7}$$

to the OS°-modules $\mathrm{D}(\boldsymbol{\lambda}), \mathrm{S}(\boldsymbol{\lambda})$ and $\mathrm{Y}(\boldsymbol{\lambda})$. By Lemma 5.1, the composition of this functor followed by the forgetful functor to vector spaces is isomorphic to $- \otimes_{\mathbb{K}} OS^-$. Hence, Δ is exact. There is also the *costandardization functor*

$$\nabla := \bigoplus_{\mathsf{a} \in \langle \uparrow, \downarrow \rangle} \mathrm{Hom}_{OS^\flat}(1_{\mathsf{a}} OS, \mathrm{infl}^\flat -) : \text{Mod-}OS^\circ \to \text{Mod-}OS, \tag{5.8}$$

where the action of $a \in 1_{\mathsf{a}} OS 1_{\mathsf{b}}$ on $f \in \mathrm{Hom}_{OS^\flat}(1_{\mathsf{a}'} OS, \mathrm{infl}^\flat M)$ is zero unless $\mathsf{a} = \mathsf{a}'$, in which case, it is the element of $\mathrm{Hom}_{OS^\flat}(1_{\mathsf{b}} OS, \mathrm{infl}^\flat M)$ defined from $(fa)(b) := f(ab)$. This functor is exact since

$$1_{\mathsf{a}} OS \cong 1_{\mathsf{a}} OS^+ \otimes_{\mathbb{K}} OS^\flat \cong \bigoplus_{\mathsf{b} \in \langle \uparrow, \downarrow \rangle} \dim_{\Bbbk}(1_{\mathsf{a}} OS^+ 1_{\mathsf{b}}) 1_{\mathsf{b}} OS^\flat$$

as a right $OS^\flat$-module, which is finitely generated and projective. We refer to the modules

$$\bar{\nabla}(\boldsymbol{\lambda}) := \nabla \mathrm{D}(\boldsymbol{\lambda}), \qquad \nabla(\boldsymbol{\lambda}) := \nabla \mathrm{Y}(\boldsymbol{\lambda}) \tag{5.9}$$

as the *proper costandard* and *costandard modules*, respectively.

There is a well-known duality functor $\circledast$ on finite-dimensional modules over the Hecke algebra with $\mathrm{D}_\lambda^\circledast \cong \mathrm{D}_\lambda$, hence, $\mathrm{Y}_\lambda^\circledast \cong \mathrm{Y}_\lambda$. The corresponding duality $\circledast$ on fdMod-OS° takes a right module to its linear dual with the natural left action twisted into a right action using the antiautomorphism arising from the restriction of the isomorphism τ from (2.6). In an entirely analogous way, τ gives rise to a duality, also denoted $\circledast$, on the category lfdMod-OS; this sends a module to the direct sum of the linear duals of its word spaces. Since $\mathrm{D}(\boldsymbol{\lambda})^\circledast \cong \mathrm{D}(\boldsymbol{\lambda})$ and $\mathrm{Y}(\boldsymbol{\lambda})^\circledast \cong \mathrm{Y}(\boldsymbol{\lambda})$, the following lemma implies that

$$\mathrm{L}(\boldsymbol{\lambda})^\circledast \cong \mathrm{L}(\boldsymbol{\lambda}), \qquad \Delta(\boldsymbol{\lambda})^\circledast \cong \nabla(\boldsymbol{\lambda}), \qquad \bar{\Delta}(\boldsymbol{\lambda})^\circledast \cong \bar{\nabla}(\boldsymbol{\lambda}), \tag{5.10}$$

for any $\boldsymbol{\lambda} \in e\text{-Bip}$. In particular, this means that $\mathrm{L}(\boldsymbol{\lambda})$ can also be realized as the irreducible socle of $\bar{\nabla}(\boldsymbol{\lambda})$.

Lemma 5.5. *The functors Δ and ∇ send finite-dimensional OS°-modules to locally finite-dimensional OS-modules. Moreover, the functors $\circledast \circ \Delta$ and $\nabla \circ \circledast$ are isomorphic on finite-dimensional OS°-modules.*

Proof. The first statement is easy to see from the definitions; for ∇, one needs to know that $1_{\mathsf{a}} OS$ is a finitely generated right $OS^\flat$-module by Lemma 5.1. Then, for a finite-dimensional OS°-module M, we define an OS-module homomorphism

$$(\mathrm{infl}^\sharp M \otimes_{\mathcal{OS}^\sharp} OS)^\circledast \to \bigoplus_{\mathsf{a} \in \langle \uparrow, \downarrow \rangle} \mathrm{Hom}_{OS^\flat}(1_{\mathsf{a}} OS, \mathrm{infl}^\flat(M^\circledast)), \qquad \theta \mapsto \widehat{\theta}$$

where $\widehat{\theta}(f)(v) = \theta(v \otimes \tau(f))$ for $v \in M, f \in 1_{\mathsf{a}} OS$. There is a two-sided inverse

$$\bigoplus_{\mathsf{a} \in \langle \uparrow, \downarrow \rangle} \mathrm{Hom}_{OS^\flat}(1_{\mathsf{a}} OS, \mathrm{infl}^\flat(M^\circledast)) \to (\mathrm{infl}^\sharp M \otimes_{\mathcal{OS}^\sharp} OS)^\circledast, \qquad \psi \mapsto \widetilde{\psi}$$

where $\widetilde{\psi}(v \otimes f) = \psi(\tau(f))(v)$. $\square$

We say that an OS-module M has a *finite Δ-flag* if it has a finite filtration whose sections are isomorphic to standard modules $\Delta(\boldsymbol{\lambda})$ for various $\boldsymbol{\lambda} \in e\text{-Bip}$. Let $\Delta\mathrm{Mod}\text{-}OS$ be the full subcategory of $\mathrm{Mod}\text{-}OS$ consisting of all modules with a finite Δ-flag. We view it as an exact category with admissible sequences being the ones that are exact in $\mathrm{Mod}\text{-}OS$. The next three lemmas are all well known in this sort of situation.

Lemma 5.6. *The restriction of $\Delta(\boldsymbol{\lambda})$ to $OS^\flat$ is isomorphic to $\mathrm{Y}(\boldsymbol{\lambda}) \otimes_{OS^\circ} OS^\flat$. These modules give all of the indecomposable projective $OS^\flat$-modules (up to isomorphism).*

Proof. The first statement is clear from Lemma 5.1. For the second, observe that the $OS^\flat$-modules $\mathrm{Y}(\boldsymbol{\lambda}) \otimes_{OS^\circ} OS^\flat$ are induced from the projective OS°-modules, hence, they are projective and every indecomposable projective $OS^\flat$-module is isomorphic to a summand of one of them. It remains to show that $\mathrm{Y}(\boldsymbol{\lambda}) \otimes_{OS^\circ} OS^\flat$ is indecomposable. This follows because $\mathrm{End}_{OS^\flat}(\mathrm{Y}(\boldsymbol{\lambda}) \otimes_{OS^\circ} OS^\flat) \cong \mathrm{End}_{OS^\circ}(\mathrm{Y}(\boldsymbol{\lambda}))$, which is local as $\mathrm{Y}(\boldsymbol{\lambda})$ is indecomposable. $\square$

In view of the following lemma, the Grothendieck group $K_0(\Delta\mathrm{Mod}\text{-}OS)$ of the exact category $\Delta\mathrm{Mod}\text{-}OS$ is the free $\mathbb{Z}$-module on basis $\{[\Delta(\boldsymbol{\lambda})] \mid \boldsymbol{\lambda} \in e\text{-Bip}\}$.

Lemma 5.7. *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in e\text{-Bip}$ and $d \geq 0$, we have that*

$$\dim \mathrm{Ext}^d_{OS}(\Delta(\boldsymbol{\lambda}), \bar{\nabla}(\boldsymbol{\mu})) = \begin{cases} 1 & \text{if } d = 0 \text{ and } \boldsymbol{\lambda} = \boldsymbol{\mu}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for any $M \in \mathrm{Mod}\text{-}OS$ with a finite Δ-flag, the multiplicity $(M : \Delta(\boldsymbol{\lambda}))$ of $\Delta(\boldsymbol{\lambda})$ as a section of such a flag is well defined independent of the particular choice of flag, and it satisfies $(M : \Delta(\boldsymbol{\lambda})) = \dim \mathrm{Hom}_{OS}(M, \bar{\nabla}(\boldsymbol{\lambda}))$.

Proof. For the first statement, we have natural isomorphisms

$$\begin{aligned} \mathrm{Ext}^d_{OS}(\Delta(\boldsymbol{\lambda}), \bar{\nabla}(\boldsymbol{\mu})) &\cong \mathrm{Ext}^d_{OS}\left(\Delta(\boldsymbol{\lambda}), \bigoplus_{\mathsf{a} \in \langle \uparrow, \downarrow \rangle} \mathrm{Hom}_{OS^\flat}\left(1_{\mathsf{a}} OS, \mathrm{infl}^\flat \mathrm{D}(\boldsymbol{\mu})\right)\right) \\ &\cong \mathrm{Ext}^d_{OS^\flat}(\mathrm{Y}(\boldsymbol{\lambda}) \otimes_{OS^\circ} OS^\flat, \mathrm{infl}^\flat \mathrm{D}(\boldsymbol{\mu})) \\ &\cong \mathrm{Ext}^d_{OS^\circ}(\mathrm{Y}(\boldsymbol{\lambda}), \mathrm{D}(\boldsymbol{\mu})). \end{aligned}$$

This is zero unless $\boldsymbol{\lambda} = \boldsymbol{\mu}$ and $d = 0$ as $\mathrm{Y}(\boldsymbol{\lambda})$ is the projective cover of $\mathrm{D}(\boldsymbol{\lambda})$. It follows that $\dim \mathrm{Hom}_{OS}(M, \bar{\nabla}(\boldsymbol{\lambda}))$ counts the multiplicity of $\Delta(\boldsymbol{\lambda})$ in a Δ-flag of M, giving the second statement. $\square$

The next lemma shows that ΔMod-OS is Karoubian.

Lemma 5.8. *An OS-module M has a finite Δ-flag if and only if it is finitely generated and projective as an $OS^\flat$-module. Hence, any direct summand of a module with a finite Δ-flag also has a finite Δ-flag.*

Proof. If M has a finite Δ-flag, then it is finitely generated and projective over $OS^\flat$ thanks to Lemma 5.6. Conversely, suppose that M is finitely generated and projective over $OS^\flat$, so that the restriction of M to $OS^\flat$ is isomorphic to a direct sum of some number n of $OS^\flat$-modules of the form $\mathrm{Y}(\boldsymbol{\lambda})\otimes_{OS^\circ} OS^\flat$. We show that M has a finite Δ-flag by induction on n. The case $n=0$ is trivial. If $n>0$, we choose $r,s\geq 0$ with $r+s$ minimal such that the restriction of M to $OS^\flat$ has a summand $M'\cong \mathrm{Y}(\boldsymbol{\lambda})\otimes_{OS^\circ} OS^\flat$ for some $\boldsymbol{\lambda}\in e\text{-}\mathrm{Bip}_{r,s}$. The OS°-module homomorphism $\mathrm{Y}(\boldsymbol{\lambda})\cong \mathrm{Y}(\boldsymbol{\lambda})\otimes_{OS^\circ} OS^\flat[0]\hookrightarrow M'$ is actually an $OS^\sharp$-module homomorphism $\mathrm{infl}^\sharp\,\mathrm{Y}(\boldsymbol{\lambda})\hookrightarrow M$ since its image is in the shortest word space of M. Hence, we get induced an OS-module homomorphism $\Delta(\boldsymbol{\lambda})\to M$ with image M'. This shows that M' is actually an OS-submodule of M and $M'\cong\Delta(\boldsymbol{\lambda})$. The quotient M/M' is finitely generated and projective over $OS^\flat$ with one fewer indecomposable summand. It remains to apply the induction hypothesis to deduce that M/M' has a finite Δ-flag, hence, so does M. $\square$

Now we look at projectives. Let $\mathrm{P}(\boldsymbol{\lambda})$ be a projective cover of $\mathrm{L}(\boldsymbol{\lambda})$. The classes $\{[\mathrm{P}(\boldsymbol{\lambda})]\mid \boldsymbol{\lambda}\in e\text{-}\mathrm{Bip}\}$ give a basis for $K_0(\mathrm{pMod}\text{-}OS)$. The OS-module

$$\mathrm{Q}(\boldsymbol{\lambda}) := \mathrm{Y}(\boldsymbol{\lambda})\otimes_{OS^\circ} OS \tag{5.11}$$

described by the following theorem should be viewed as a first approximation to $\mathrm{P}(\boldsymbol{\lambda})$.

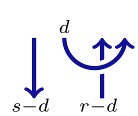
Theorem 5.9. *For $\boldsymbol{\lambda}\in e\text{-}\mathrm{Bip}_{r,s}$, the module $\mathrm{Q}(\boldsymbol{\lambda})$ has a canonical filtration with sections indexed by $d=0,1,\dots,\min(r,s)$ appearing in order from top to bottom, such that the dth section is isomorphic to*

$$\bigoplus_{\boldsymbol{\mu}\in e\text{-}\mathrm{Bip}_{r-d,s-d}} \Delta(\boldsymbol{\mu})^{\oplus M^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(e,p)} \tag{5.12}$$

where $M^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(e,p) := \sum_{\boldsymbol{\nu}\in e\text{-}\mathrm{Bip}_{d,d}} \left[\mathrm{D}_{\mu^\uparrow}\circ \mathrm{D}_{\nu^\uparrow} : \mathrm{D}_{\lambda^\uparrow}\right]\left[\mathrm{D}_{\mu^\downarrow}\circ \mathrm{D}_{\nu^\downarrow} : \mathrm{D}_{\lambda^\downarrow}\right]\left[\mathrm{Y}_{\nu^\uparrow} : \mathrm{D}_{\nu^\downarrow}\right]$ (which depends on e and the characteristic p of the field $\Bbbk$).

Proof. Take $0\leq d\leq \min(r,s)$. We always view H_d as a subalgebra of H_r or H_s via the natural embeddings. We also need the "unnatural" embeddings $H_{r-d}\hookrightarrow H_r$ and $H_{s-d}\hookrightarrow H_s$ which send $S_i\mapsto S_{d+i}$; the images of these embeddings centralize H_d. Let $\mathrm{Hom}_{H_d}(H_s,H_r)$ be the space of all right H_d-module homomorphisms. Using the unnatural embeddings for H_{r-d} and H_{s-d}, this is an $(H_r\otimes H_{s-d}, H_{r-d}\otimes H_s)$-bimodule. Let $\tau\text{-}\mathrm{Hom}_{H_d}(H_s,H_r)$ be the the same space but with the left action of H_{s-d} and right action of H_s twisted into right and left actions, respectively, using the anti-automorphism τ which sends $S_w\mapsto S_{w^{-1}}$. This means that $\tau\text{-}\mathrm{Hom}_{H_d}(H_s,H_r)$ is an $(H_{r,s},H_{r-d,s-d})$-bimodule. The space $1_{\downarrow^s\uparrow^r} OS^\sharp 1_{\downarrow^{s-d}\uparrow^{r-d}}$ is also naturally an $(H_{r,s},H_{r-d,s-d})$-bimodule. We claim that these two bimodules are isomorphic.

To prove the claim, H_s is a free right H_d-module with basis $\{S_y\mid y\in\mathfrak{D}\}$ where $\mathfrak{D}$ is the set of minimal length $\mathfrak{S}_s/\mathfrak{S}_d$-coset representatives. So the H_d-module homomorphisms $\{f_{x,y}: H_s\to H_r\mid x\in\mathfrak{S}_r, y\in\mathfrak{D}\}$ defined from $f_{x,y}(S_z) := \delta_{y,z}S_x$ for $x\in\mathfrak{S}_r, y,z\in\mathfrak{D}$ give a linear basis for $\tau\text{-}\mathrm{Hom}_{H_d}(H_s,H_r)$. Let

$c :=$  $\in 1_{\downarrow^s\uparrow^r} OS^\sharp 1_{\downarrow^{s-d}\uparrow^{r-d}},$

where the thick arrows labelled by a number represent that number of parallel thin ones. We will show that the linear map

$$\theta : \tau\text{-Hom}_{H_d}(H_s, H_r) \to 1_{\downarrow^s\uparrow^r} OS^\sharp 1_{\downarrow^{s-d}\uparrow^{r-d}}, \qquad f_{x,y} \mapsto \imath_{r,s}(S_x \otimes S_y)c$$

is an $(H_{r,s}, H_{r-d,s-d})$-bimodule isomorphism. To see this, it is clear from Theorem 1.2 that θ is a vector space isomorphism. We must check that it is a bimodule homomorphism. This is straightforward for the left action of H_r and the right action of H_{r-d}. In the next two paragraphs, we check it for the left action of H_s and the right action of H_{s-d}, respectively.

To show θ is a left H_s-module homomorphism, take $x \in \mathfrak{S}_r, y \in \mathfrak{D}$ and $1 \leq i < s$. By [DJ1, Lemma 1.1], exactly one of the following holds: (a) $s_i y \in \mathfrak{D}$; (b) $s_{y^{-1}(i)} \in \mathfrak{S}_d$. In case (a), $S_i S_y = S_{s_i y}$ if $\ell(s_i y) > \ell(y)$ or $S_{s_i y} + (q - q^{-1})S_y$ if $\ell(s_i y) < \ell(y)$. In case (b), $S_i S_y = S_y S_{y^{-1}(i)}$ and

$$\imath_{r,s}(S_x \otimes S_i S_y)c = \imath_{r,s}(S_x \otimes S_y S_{y^{-1}(i)})c = \imath_{r,s}(S_x S_{y^{-1}(i)} \otimes S_y)c.$$

We deduce for $z \in \mathfrak{D}$ that

$$\begin{aligned}
(\theta^{-1}(S_i\theta(f_{x,y})))(S_z) &= (\theta^{-1}(\imath_{r,s}(S_x \otimes S_i S_y)c))(S_z) \\
&= \begin{cases} f_{x,s_i y}(S_z) & \text{if } s_i y \in \mathfrak{D}, \ell(s_i y) > \ell(y), \\ (f_{x,s_i y} + (q-q^{-1})f_{x,y})(S_z) & \text{if } s_i y \in \mathfrak{D}, \ell(s_i y) < \ell(y), \\ f_{xs_{y^{-1}(i)},y}(S_z) & \text{if } s_i y \notin \mathfrak{D}, \ell(xs_{y^{-1}(i)}) > \ell(x), \\ (f_{xs_{y^{-1}(i)},y} + (q-q^{-1})f_{x,y})(S_z) & \text{if } s_i y \notin \mathfrak{D}, \ell(xs_{y^{-1}(i)}) < \ell(x) \end{cases} \\
&= \begin{cases} \delta_{s_i y, z} S_x & \text{if } s_i y \in \mathfrak{D}, \ell(s_i y) > \ell(y), \\ \delta_{s_i y, z} S_x + (q-q^{-1})\delta_{y,z} S_x & \text{if } s_i y \in \mathfrak{D}, \ell(s_i y) < \ell(y), \\ \delta_{y,z} S_x S_{y^{-1}(i)} & \text{if } s_i y \notin \mathfrak{D}. \end{cases}
\end{aligned}$$

We need to show this is equal to

$$(S_i f_{x,y})(S_z) = f_{x,y}(S_i S_z) = \begin{cases} \delta_{y,s_i z} S_x & \text{if } s_i z \in \mathfrak{D}, \ell(s_i z) > \ell(z), \\ \delta_{y,s_i z} S_x + (q-q^{-1})\delta_{y,z} S_x & \text{if } s_i z \in \mathfrak{D}, \ell(s_i z) < \ell(z), \\ \delta_{y,z} S_x S_{y^{-1}(i)} & \text{if } s_i z \notin \mathfrak{D}. \end{cases}$$

This follows easily by considering several cases: (a) $y = z$; (b) $y = s_i z$; (c) $s_i y = z$; (d) none of the above.

To show that θ is a right H_{s-d}-module homomorphism, take $x \in \mathfrak{S}_r, y, z \in \mathfrak{D}$ and $1 \leq i < s - d$. We must show that $(\theta^{-1}(\theta(f_{x,y})S_i))(S_z) = (f_{x,y}S_{d+i})(S_z)$, i.e.,

$$(\theta^{-1}(\imath_{r,s}(S_x \otimes S_y S_{d+i})c)(S_z) = f_{x,y}(S_z S_{d+i}).$$

This time, $ys_{d+i} \in \mathfrak{D}$ always. So the left hand side of the identity to be proved equals

$$\begin{cases} \delta_{ys_{d+i},z} S_x & \text{if } \ell(ys_{d+i}) > \ell(y), \\ \delta_{ys_{d+i},z} S_x + \delta_{y,z}(q-q^{-1})S_x & \text{if } \ell(ys_{d+i}) < \ell(y). \end{cases}$$

Similarly, the right hand side is

$$\begin{cases} \delta_{y,zs_{d+i}} S_x & \text{if } \ell(zs_{d+i}) > \ell(z), \\ \delta_{y,zs_{d+i}} S_x + \delta_{y,z}(q-q^{-1})S_x & \text{if } \ell(zs_{d+i}) < \ell(z). \end{cases}$$

The two sides are equal by considering four cases like before.

We have now proved the claim made in the opening paragraph. To prove the theorem, we must construct the filtration of $\mathrm{Q}(\boldsymbol{\lambda})$. By transitivity of induction,

$$\mathrm{Q}(\boldsymbol{\lambda}) = (\mathrm{Y}(\boldsymbol{\lambda}) \otimes_{OS^\circ} OS^\sharp) \otimes_{OS^\sharp} OS.$$

Since $OS^\sharp$ is positively graded, the grading gives us a filtration of $\mathrm{Y}(\boldsymbol{\lambda}) \otimes_{OS^\circ} OS^\sharp$ as an $OS^\sharp$-module with sections $\mathrm{Y}(\boldsymbol{\lambda}) \otimes_{OS^\circ} OS^\sharp[d]$ for $d = 0, 1, \ldots$ appearing in order from top to bottom. The dth section is clearly zero unless $d \leq \min(r,s)$. Since the functor $? \otimes_{OS^\sharp} OS$ is exact, we are thus reduced to checking for $0 \leq d \leq \min(r,s)$ that

$$\mathrm{Y}(\boldsymbol{\lambda}) \otimes_{OS^\circ} OS^\sharp[d] \cong \bigoplus_{\boldsymbol{\mu} \in e\text{-}\mathrm{Bip}_{r-d,s-d}} \mathrm{Y}(\boldsymbol{\mu})^{\oplus M^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(e,p)}$$

as a right OS°-module. Using Lemma 5.2, we show equivalently that

$$(\mathrm{Y}_{\lambda^\uparrow} \boxtimes \mathrm{Y}_{\lambda^\downarrow}) \otimes_{H_{r,s}} 1_{\downarrow^s \uparrow^r} OS^\sharp 1_{\downarrow^{s-d}\uparrow^{r-d}} \cong \bigoplus_{\mu \in e\text{-}\mathrm{Bip}_{r-d,s-d}} (\mathrm{Y}_{\mu^\uparrow} \otimes \mathrm{Y}_{\mu^\downarrow})^{\bigoplus M^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(e,p)}$$

as a right $H_{r-d,s-d}$-module. By the opening claim, the module on the left hand side is isomorphic to

$$(\mathrm{Y}_{\lambda^\uparrow} \boxtimes \mathrm{Y}_{\lambda^\downarrow}) \otimes_{H_{r,s}} \tau\text{-}\mathrm{Hom}_{H_d}(H_s, H_r) \cong \tau\text{-}\mathrm{Hom}_{H_d}(\mathrm{Y}_{\lambda^\downarrow}, \mathrm{Y}_{\lambda^\uparrow}),$$

where we have used the self-duality of $\mathrm{Y}_{\lambda^\downarrow}$. Then we note by Frobenius reciprocity that

$$\mathrm{res}^{H_r}_{H_{r-d,d}} \mathrm{Y}_{\lambda^\uparrow} \cong \bigoplus_{\mu^\uparrow \vdash (r-d), \nu^\uparrow \vdash d} (\mathrm{Y}_{\mu^\uparrow} \boxtimes \mathrm{Y}_{\nu^\uparrow})^{\oplus [\mathrm{D}_{\mu^\uparrow} \circ \mathrm{D}_{\nu^\uparrow} : \mathrm{D}_{\lambda^\uparrow}]},$$
$$\mathrm{res}^{H_s}_{H_{s-d,d}} \mathrm{Y}_{\lambda^\downarrow} \cong \bigoplus_{\mu^\downarrow \vdash (s-d), \nu^\downarrow \vdash d} (\mathrm{Y}_{\mu^\downarrow} \boxtimes \mathrm{Y}_{\nu^\downarrow})^{\oplus [\mathrm{D}_{\mu^\downarrow} \circ \mathrm{D}_{\nu^\downarrow} : \mathrm{D}_{\lambda^\downarrow}]}.$$

Making these substitutions in $\tau\text{-}\mathrm{Hom}_{H_d}(\mathrm{Y}_{\lambda^\downarrow}, \mathrm{Y}_{\lambda^\uparrow})$, using also $\dim \mathrm{Hom}_{H_d}(\mathrm{Y}_{\nu^\downarrow}, \mathrm{Y}_{\nu^\uparrow}) = [\mathrm{Y}_{\nu^\uparrow} : \mathrm{D}_{\nu^\downarrow}]$ and self-duality of $\mathrm{Y}_{\mu^\downarrow}$, gives the conclusion. □

Corollary 5.10. *For $\boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{r,s}$, the module $\mathrm{Q}(\boldsymbol{\lambda})$ is isomorphic $\mathrm{P}(\boldsymbol{\lambda})$ plus a finite direct sum of projectives $\mathrm{P}(\boldsymbol{\mu})$ for bipartitions $\boldsymbol{\mu} \in \coprod_{d=0}^{\min(r,s)-1} e\text{-}\mathrm{Bip}_{r-d,s-d}$. Hence, the classes $\{[\mathrm{Q}(\boldsymbol{\lambda})] \mid \boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}\}$ give another basis for $K_0(\mathrm{pMod}\text{-}OS)$.*

Proof. Note that $\mathrm{Q}(\boldsymbol{\lambda})$ is projective since the functor $? \otimes_{OS^\circ} OS$ sends projectives to projectives. Also the top section of the Δ-flag of $\mathrm{Q}(\boldsymbol{\lambda})$ constructed in Theorem 5.9 is $\Delta(\boldsymbol{\lambda})$, so $\mathrm{Q}(\boldsymbol{\lambda})$ has $\mathrm{P}(\boldsymbol{\lambda})$ as an indecomposable summand. The other sections only involve $\Delta(\boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \coprod_{d=0}^{\min(r,s)-1} \mathrm{Bip}_{r-d,s-d}$, so all other summands are of the form $\mathrm{P}(\boldsymbol{\mu})$ for such $\boldsymbol{\mu}$. □

Corollary 5.11. *The projective cover $\mathrm{P}(\boldsymbol{\lambda})$ of $\mathrm{L}(\boldsymbol{\lambda})$ has a finite Δ-flag such that*

$$(\mathrm{P}(\boldsymbol{\lambda}) : \Delta(\boldsymbol{\mu})) = [\bar{\Delta}(\boldsymbol{\mu}) : \mathrm{L}(\boldsymbol{\lambda})].$$

Proof. By Corollary 5.10 and Theorem 5.9, $\mathrm{P}(\boldsymbol{\lambda})$ is a summand of $\mathrm{Q}(\boldsymbol{\lambda})$, and $\mathrm{Q}(\boldsymbol{\lambda})$ has a finite Δ-flag. Hence, $\mathrm{P}(\boldsymbol{\lambda})$ has one too due to Lemma 5.8. To deduce the BGG reciprocity formula, we use Lemma 5.7: $(\mathrm{P}(\boldsymbol{\lambda}) : \Delta(\boldsymbol{\mu})) = \dim \mathrm{Hom}_{OS}(\mathrm{P}(\boldsymbol{\lambda}), \bar{\nabla}(\boldsymbol{\mu})) = [\bar{\nabla}(\boldsymbol{\mu}) : \mathrm{L}(\boldsymbol{\lambda})]$. This equals $[\bar{\Delta}(\boldsymbol{\mu}) : \mathrm{L}(\boldsymbol{\lambda})]$ by (5.10). □

Corollary 5.12. *By Corollary 5.11, there is an embedding $\mathrm{pMod}\text{-}OS \to \Delta\mathrm{Mod}\text{-}OS$. This induces an isomorphism $K_0(\mathrm{pMod}\text{-}OS) \xrightarrow{\sim} K_0(\Delta\mathrm{Mod}\text{-}OS)$.*

Proof. The transition matrix arising from (5.12) can be inverted to express each $[\Delta(\boldsymbol{\lambda})]$ as a finite linear combination of $[\mathrm{Q}(\boldsymbol{\mu})]$'s. □

Corollary 5.13. *For $\boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{r,s}$, $\mathrm{P}(\boldsymbol{\lambda})$ has a finite filtration with sections $\tilde{\Delta}(\boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \coprod_{d=0}^{\min(r,s)} \mathrm{Bip}_{r-d,s-d}$, each appearing $[\tilde{\Delta}(\boldsymbol{\mu}) : \mathrm{L}(\boldsymbol{\lambda})]$ times.*

Proof. Recall for $\boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{r,s}$ that $\mathrm{Y}(\boldsymbol{\lambda})$ has a finite filtration with sections $\mathrm{S}(\boldsymbol{\mu})$, each appearing $[\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\lambda})]$ times. Applying the exact standardization functor, we deduce that $\Delta(\boldsymbol{\lambda})$ has a finite filtration with sections $\tilde{\Delta}(\boldsymbol{\mu})$, each appearing $[\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\lambda})]$ times.

Combined with Corollary 5.11, it follows that $\mathrm{P}(\boldsymbol{\lambda})$ has a finite filtration with sections $\tilde{\Delta}(\boldsymbol{\mu})$, each appearing with multiplicity

$$\sum_{\boldsymbol{\nu}\in e\text{-Bip}} [\bar{\Delta}(\boldsymbol{\nu}) : \mathrm{L}(\boldsymbol{\lambda})][\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\nu})].$$

Also, applying Δ to a composition series of $\mathrm{S}(\boldsymbol{\mu})$, we see that $\tilde{\Delta}(\boldsymbol{\mu})$ has a filtration with sections $\bar{\Delta}(\boldsymbol{\nu})$, each appearing $[\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\nu})]$ times. Hence, the multiplicity just displayed is equal to $[\tilde{\Delta}(\boldsymbol{\mu}) : \mathrm{L}(\boldsymbol{\lambda})]$. □

Proof of Theorem 1.7. The monoidal functor $\mathcal{OS}^\circ(z,t) \to \mathcal{OS}(z,t)$ corresponds to the induction functor $? \otimes_{OS^\circ} OS : \text{pMod-}OS^\circ \to \text{pMod-}OS$, since the latter sends eOS° to eOS for any idempotent e. So by the definition (5.11) it sends $\mathrm{Y}(\boldsymbol{\lambda})$ to $\mathrm{Q}(\boldsymbol{\lambda})$. Theorem 1.7 follows because the classes $\{[\mathrm{Q}(\boldsymbol{\lambda})] \mid \boldsymbol{\lambda} \in e\text{-Bip}\}$ form a basis for $K_0(\text{pMod-}OS)$ according to Corollary 5.10. □

For the next lemma, we return to the situation of Theorem 1.3. We want to relate the labelling of irreducible OS-modules obtained thus far with the usual labelling of irreducible $U_q(\mathfrak{gl}_n)$-modules via their highest weights. Take $\boldsymbol{\lambda} \in \mathrm{Bip}_{r,s}$. Since $e = 0$, Theorem 5.9 tells us simply that $\mathrm{Q}(\boldsymbol{\lambda})$ has a filtration with sections

$$\bigoplus_{\boldsymbol{\mu}\in \mathrm{Bip}_{r-d,s-d}} \Delta(\boldsymbol{\mu})^{\oplus M^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}} \qquad \text{where} \qquad M^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}} := M^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(0,0) = \sum_{\nu\vdash d} LR^{\lambda^\uparrow}_{\mu^\uparrow,\nu} LR^{\lambda^\downarrow}_{\mu^\downarrow,\nu} \tag{5.13}$$

for $d = 0, \dots, \min(r,s)$, and it decomposes as $\mathrm{P}(\boldsymbol{\lambda})$ plus a direct sum of projectives $\mathrm{P}(\boldsymbol{\mu})$ for various bipartitions $\boldsymbol{\mu}$ obtained from $\boldsymbol{\lambda}$ by removing the same number $d > 0$ of boxes from both $\lambda^\uparrow$ and $\lambda^\downarrow$. Recalling the Young symmetrizer (3.9), we have that $\mathrm{Y}(\boldsymbol{\lambda}) = \mathrm{S}(\boldsymbol{\lambda}) = \mathrm{D}(\boldsymbol{\lambda}) = \imath_{r,s}(e_{\lambda^\uparrow} \otimes e_{\lambda^\downarrow})OS^\circ$. Hence,

$$\mathrm{Q}(\boldsymbol{\lambda}) = \imath_{r,s}(e_{\lambda^\uparrow} \otimes e_{\lambda^\downarrow})OS. \tag{5.14}$$

Let $e_{\boldsymbol{\lambda}}$ be the projection of $\mathrm{Q}(\boldsymbol{\lambda})$ onto its unique summand that is isomorphic to $\mathrm{P}(\boldsymbol{\lambda})$. Thus, $e_{\boldsymbol{\lambda}}$ is a primitive idempotent in the quantized walled Brauer algebra $B_{r,s}$. The following recovers results of [KM1, KM2].

Lemma 5.14. *Let notation be as in Theorem 1.3 and assume $n \geq 0$. Take $\boldsymbol{\lambda} \in \mathrm{Bip}_{r,s}$ such that $h(\boldsymbol{\lambda})$, its total number of non-zero parts, is $\leq n$. Consider the idempotent $\Psi(e_{\boldsymbol{\lambda}}) \in \mathrm{End}_{U_q(\mathfrak{gl}_n)}((V^-)^{\otimes s} \otimes (V^+)^{\otimes r})$. Its image is the irreducible $U_q(\mathfrak{gl}_n)$-module $\mathrm{V}(\boldsymbol{\lambda})$ labelled by the dominant weight*

$$(\lambda^\uparrow_1 - \lambda^\downarrow_n)\varepsilon_1 + (\lambda^\uparrow_2 - \lambda^\downarrow_{n-1})\varepsilon_2 + \cdots + (\lambda^\uparrow_n - \lambda^\downarrow_1)\varepsilon_n, \tag{5.15}$$

using standard conventions for the root system of $\mathfrak{gl}_n$.

Proof. We proceed by induction on $r+s$, the case $r+s = 0$ being trivial. Since $e_{\lambda^\uparrow}$ is the Young symmetrizer, the image of $\Psi(e_{\lambda^\uparrow}) \in \mathrm{End}_{U_q(\mathfrak{gl}_n)}((V^+)^{\otimes r})$ is the irreducible polynomial representation $\mathrm{V}(\lambda^\uparrow)$ of $U_q(\mathfrak{gl}_n)$ of highest weight $\lambda^\uparrow_1\varepsilon_1 + \cdots + \lambda^\uparrow_n\varepsilon_n$. Similarly, the image of $\Psi(e_{\lambda^\downarrow})$ is the dual irreducible polynomial representation $\mathrm{V}(\lambda^\downarrow)^*$ of highest weight $-\lambda^\downarrow_n\varepsilon_1 + \cdots + \lambda^\downarrow_1\varepsilon_n$. Hence, the image of $\Psi(\imath_{r,s}(e_{\lambda^\uparrow} \otimes e_{\lambda^\downarrow}))$ is $\mathrm{V}(\lambda^\downarrow)^* \otimes \mathrm{V}(\lambda^\uparrow)$. Using characters, it is easy to see that this tensor product has a unique irreducible constituent $\mathrm{V}(\boldsymbol{\lambda})$ of highest weight (5.15), plus a sum of irreducible modules $\mathrm{V}(\boldsymbol{\mu})$ for bipartitions $\boldsymbol{\mu} \in \coprod_{d>0} \mathrm{Bip}_{r-d,s-d}$ with $h(\boldsymbol{\mu}) \leq n$. Now using induction, we deduce that $\Psi(e_{\boldsymbol{\lambda}})$ must be the projection onto $\mathrm{V}(\boldsymbol{\lambda})$. □

Remark 5.15. A helpful picture of the weight (5.15) is displayed in [Ko, Figure 2]. It is also worth noting that multiplicities $M^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}$ appearing in (5.13) are the same as the $U_q(\mathfrak{gl}_n)$-composition multiplicities $[\mathrm{V}(\lambda^\downarrow)^* \otimes \mathrm{V}(\lambda^\uparrow) : \mathrm{V}(\boldsymbol{\mu})]$ computed in [Ko, Corollary 2.3.1]. Given this, the same induction as used to prove Lemma 5.14 can be used to show that $\Delta(\boldsymbol{\lambda}) = \mathrm{P}(\boldsymbol{\lambda})$ when in the situation of the lemma. We will prove this in a different way in Corollary 6.13 below.

Remark 5.16. To get the appropriate analog of Lemma 5.14 when $n \leq 0$, one just needs to twist by the isomorphism #. Recalling at the level of the Hecke algebra that this is "tensoring with sign," one can show that # maps the primitive idempotent $e_{\boldsymbol{\lambda}}$ to a conjugate of $e_{\boldsymbol{\lambda}^{\mathrm{t}}}$, where $\boldsymbol{\lambda}^t := \big((\lambda^{\uparrow})^{\mathrm{t}}, (\lambda^{\downarrow})^{\mathrm{t}}\big)$. So, for negative n, the $U_q(\mathfrak{gl}_n)$-module $V(\boldsymbol{\lambda})$ arises as the image of $\Psi(e_{\boldsymbol{\lambda}^{\mathrm{t}}})$ (instead of $\Psi(e_{\boldsymbol{\lambda}})$).

The final result in the section justifies the description of $K_0(\dot{\mathcal{OS}}(z,t))$ made after Theorem 1.7 in the introduction; the discussion there also depends on Theorem 1.6 which will be proved in the next section, and the highest weight/standardly stratified structure which will be explained in section 7.

Lemma 5.17. *For $\boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{r,s}$ and $\boldsymbol{\nu} \in \mathrm{Bip}_{r-d,s-d}$, we have that*

$$\sum_{\boldsymbol{\mu} \in \mathrm{Bip}_{r,s}} [\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\lambda})] M^{\boldsymbol{\mu}}_{\boldsymbol{\nu}} = \sum_{\boldsymbol{\mu} \in e\text{-}\mathrm{Bip}_{r-d,s-d}} M^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(e,p)[\mathrm{S}(\boldsymbol{\nu}) : \mathrm{D}(\boldsymbol{\mu})].$$

Proof. We have that

$$\begin{aligned}
&\sum_{\boldsymbol{\mu} \in e\text{-}\mathrm{Bip}_{r-d,s-d}} M^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(e,p)[\mathrm{S}(\boldsymbol{\nu}) : \mathrm{D}(\boldsymbol{\mu})] \\
&= \sum_{\substack{\boldsymbol{\mu} \in e\text{-}\mathrm{Bip}_{r-d,s-d} \\ \boldsymbol{\kappa} \in e\text{-}\mathrm{Bip}_{d,d}, \gamma \vdash d}} \big[\mathrm{D}_{\mu^{\uparrow}} \circ \mathrm{D}_{\kappa^{\uparrow}} : \mathrm{D}_{\lambda^{\uparrow}}\big] \big[\mathrm{S}_{\nu^{\uparrow}} : \mathrm{D}_{\mu^{\uparrow}}\big] \big[\mathrm{S}_{\gamma} : \mathrm{D}_{\kappa^{\uparrow}}\big] \times \\
&\qquad\qquad \big[\mathrm{D}_{\mu^{\downarrow}} \circ \mathrm{D}_{\kappa^{\downarrow}} : \mathrm{D}_{\lambda^{\downarrow}}\big] \big[\mathrm{S}_{\nu^{\downarrow}} : \mathrm{D}_{\mu^{\downarrow}}\big] \big[\mathrm{S}_{\gamma} : \mathrm{D}_{\kappa^{\downarrow}}\big] \\
&= \sum_{\boldsymbol{\kappa} \in e\text{-}\mathrm{Bip}_{d,d}, \gamma \vdash d} \big[\mathrm{S}_{\nu^{\uparrow}} \circ \mathrm{D}_{\kappa^{\uparrow}} : \mathrm{D}_{\lambda^{\uparrow}}\big] \big[\mathrm{S}_{\gamma} : \mathrm{D}_{\kappa^{\uparrow}}\big] \big[\mathrm{S}_{\nu^{\downarrow}} \circ \mathrm{D}_{\kappa^{\downarrow}} : \mathrm{D}_{\lambda^{\downarrow}}\big] \big[\mathrm{S}_{\gamma} : \mathrm{D}_{\kappa^{\downarrow}}\big] \\
&= \sum_{\gamma \vdash d} \big[\mathrm{S}_{\nu^{\uparrow}} \circ \mathrm{S}_{\gamma} : \mathrm{D}_{\lambda^{\uparrow}}\big] \big[\mathrm{S}_{\nu^{\downarrow}} \circ \mathrm{S}_{\gamma} : \mathrm{D}_{\lambda^{\downarrow}}\big] \\
&= \sum_{\boldsymbol{\mu} \in \mathrm{Bip}_{r,s}, \gamma \vdash d} \big[\mathrm{S}_{\nu^{\uparrow}} \circ \mathrm{S}_{\gamma} : \mathrm{S}_{\mu^{\uparrow}}\big] \big[\mathrm{S}_{\mu^{\uparrow}} : \mathrm{D}_{\lambda^{\uparrow}}\big] \big[\mathrm{S}_{\nu^{\downarrow}} \circ \mathrm{S}_{\gamma} : \mathrm{S}_{\mu^{\downarrow}}\big] \big[\mathrm{S}_{\mu^{\downarrow}} : \mathrm{D}_{\lambda^{\downarrow}}\big] \\
&= \sum_{\boldsymbol{\mu} \in \mathrm{Bip}_{r,s}} [\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\lambda})] M^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}.
\end{aligned}$$

□

Theorem 5.18. *For any choices of q and t, the ring $K_0(\mathrm{pMod}\text{-}OS)$ may be identified with a subring of $\mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$ so that (1.14) and (1.15) hold.*

Proof. When $e = 0$, Lemma 5.2 and the well-known representation theory of Hecke algebras imply that the rings $K_0(\mathrm{pMod}\text{-}OS^{\circ})$ and $\mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$ may be identified so that $[\mathrm{S}(\boldsymbol{\lambda})] \leftrightarrow \chi_{\lambda^{\uparrow}} \otimes \chi_{\lambda^{\downarrow}}$. For general e, using also Brauer reciprocity for the Hecke algebra, we may identify $K_0(\mathrm{pMod}\text{-}OS^{\circ})$ with a subring of $\mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$ so that

$$[\mathrm{Y}(\boldsymbol{\lambda})] \leftrightarrow \sum_{\boldsymbol{\mu} \in \mathrm{Bip}_{r,s}} [\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\lambda})] \chi_{\mu^{\uparrow}} \otimes \chi_{\mu^{\downarrow}}$$

for $\boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{r,s}$ and $r, s \geq 0$. In view of Theorem 1.7 (and its proof), we deduce that $K_0(\mathrm{pMod}\text{-}OS)$ is identified with a subring of $\mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$ so that

$$[\mathrm{Q}(\boldsymbol{\lambda})] \leftrightarrow \sum_{\boldsymbol{\mu} \in \mathrm{Bip}_{r,s}} [\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\lambda})] \chi_{\mu^{\uparrow}} \otimes \chi_{\mu^{\downarrow}}$$

for $\boldsymbol{\lambda} \in e\text{-Bip}_{r,s}$ and $r, s \geq 0$. Now recall the definition (1.9) and (5.13). Setting $N_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}} = M_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}} := 0$ whenever $\boldsymbol{\lambda} \in \text{Bip}_{r,s}$ and $\boldsymbol{\mu} \notin \coprod_{d=0}^{\min(r,s)} \text{Bip}_{r-d,s-d}$, the matrix $(N_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}})_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\text{Bip}}$ is inverse to the matrix $(M_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}})_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\text{Bip}}$ by [Ko, Theorem 2.3]. So

$$[\mathrm{Q}(\boldsymbol{\lambda})] \leftrightarrow \sum_{\substack{\boldsymbol{\mu}\in\text{Bip}_{r,s}\\ 0\leq d\leq \min(r,s)\\ \boldsymbol{\nu}\in\text{Bip}_{r-d,s-d}}} [\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\lambda})] M_{\boldsymbol{\nu}}^{\boldsymbol{\mu}} \chi_{\boldsymbol{\nu}}$$

for $\boldsymbol{\lambda} \in e\text{-Bip}_{r,s}$ and $r, s \geq 0$. By Lemma 5.17, this gives

$$[\mathrm{Q}(\boldsymbol{\lambda})] \leftrightarrow \sum_{\substack{0\leq d\leq \min(r,s)\\ \boldsymbol{\lambda}\in e\text{-Bip}_{r-d,s-d}\\ \boldsymbol{\nu}\in\text{Bip}_{r-d,s-d}}} M_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}}(e,p) [\mathrm{S}(\boldsymbol{\nu}) : \mathrm{D}(\boldsymbol{\mu})] \chi_{\boldsymbol{\nu}}.$$

Now use Corollary 5.12 to identify $K_0(\text{pMod-}OS) = K_0(\Delta\text{Mod-}OS)$. Comparing with (5.12), we deduce that

$$[\Delta(\boldsymbol{\lambda})] \leftrightarrow \sum_{\boldsymbol{\nu}\in\text{Bip}_{r,s}} [\mathrm{S}(\boldsymbol{\nu}) : \mathrm{D}(\boldsymbol{\lambda})] \chi_{\boldsymbol{\nu}}$$

for $\boldsymbol{\lambda} \in e\text{-Bip}_{r,s}$. This establishes (1.14). To get (1.15) too, use Corollary 5.11. □

6. Branching rules and characters

We continue with the setup of the previous section. In this section, we introduce a biadjoint pair of endofunctors E and F of Mod-OS, which lift the endofunctors $\uparrow\otimes ?$ and $\downarrow\otimes ?$ of $\mathcal{OS}(z,t)$. We will use the Jucys-Murphy elements from section 4 to decompose these endofunctors into direct sums of refined functors E_i and F_i, which we study by comparing them to some well-known induction and restriction functors on Mod-OS°.

To start with, let us recall some standard facts about induction and restriction for the Iwahori-Hecke algebra H_r. Let

$$\text{ind}_{r-1}^r : \text{Mod-}H_{r-1} \to \text{Mod-}H_r, \qquad \text{res}_{r-1}^r : \text{Mod-}H_r \to \text{Mod-}H_{r-1} \tag{6.1}$$

be the usual induction and restriction functors with respect to the natural embedding $H_{r-1} \hookrightarrow H_r, S_i \mapsto S_i$. So ind_{r-1}^r is defined by tensoring over H_{r-1} with H_r viewed as an (H_{r-1}, H_r)-bimodule and res_{r-1}^r is defined by tensoring over H_r with H_r viewed as an (H_r, H_{r-1})-bimodule; equivalently, res_{r-1}^r is the functor $\text{Hom}_{H_r}(H_r, ?)$. Adjointness of tensor and hom implies that induction is left adjoint to the restriction functor res_{r-1}^r. It is also right adjoint; cf. [DJ1, Theorem 2.6]. The *Jucys-Murphy element*

$$L_r := S_{r-1} \cdots S_2 S_1 S_1 S_2 \cdots S_{r-1} \in H_r \tag{6.2}$$

centralizes H_{r-1}, so left multiplication by it defines an endomorphism of the (H_{r-1}, H_r)-bimodule H_r. For any $i \in \mathbb{k}$, let $i\text{-ind}_{r-1}^r$ be the *i-induction functor* defined by tensoring with the summand of this bimodule that arises as the generalized i-eigenspace of this endomorphism. Let $i\text{-res}_{r-1}^r$ be the biadjoint *i-restriction functor*; explicitly, $i\text{-res}_{r-1}^r M$ may be realized as the generalized i-eigenspace of L_r on $\text{res}_{r-1}^r M$.

The following "classical" branching rules[3] describe the effect of these functors on the Specht module S_λ. In formulating the result, we identify partition λ with its *Young diagram*, that is, the set $\{(i,j) \mid i \geq 1, 1 \leq j \leq \lambda_r\}$, and define the *content* of the *node*

[3]Probably the best way to prove them is by applying the "Schur functor" to an analogous result for quantum GL_n.

$\mathsf{A} = (i,j) \in \mathbb{N}\times\mathbb{N}$ from $\mathrm{cont}(\mathsf{A}) := q^{2(j-i)} \in \Bbbk$. For example, here is the Young diagram $\lambda = (5,3,2)$ with its nodes labeled by their contents:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & q^2 & q^4 & q^6 & q^8 \\ \hline q^{-2} & 1 & q^2 \\ \cline{1-3} q^{-4} & q^{-2} \\ \cline{1-2}\end{array}\;.$$

Let I_1 be the set of all possible contents of nodes of partitions. More generally, for any $c \in \Bbbk^\times$, we let

$$I_c := \{cq^{2n} \mid n \in \mathbb{Z}\} \subseteq \Bbbk^\times. \tag{6.3}$$

Lemma 6.1. *The following hold for each $i \in \Bbbk^\times$:*

(1) For $\lambda \vdash (r-1)$, the H_r-module $i\text{-}\mathrm{ind}_{r-1}^r \mathrm{S}_\lambda$ has a multiplicity-free filtration with sections S_μ for $\mu \vdash r$ obtained by adding a node of content i to the Young diagram of λ.

(2) For $\lambda \vdash r$, the H_r-module $i\text{-}\mathrm{res}_{r-1}^r \mathrm{S}_\lambda$ has multiplicity-free filtration with sections $\cong \mathrm{S}_\mu$ for $\mu \vdash (r-1)$ obtained by removing a node of content i from the Young diagram of λ.

In both cases, the filtration should be ordered according to the usual dominance ordering on the partitions labelling the sections, most dominant at the bottom. Hence:

$$\mathrm{ind}_{r-1}^r = \bigoplus_{i \in I_1} i\text{-}\mathrm{ind}_{r-1}^r, \qquad \mathrm{res}_{r-1}^r = \bigoplus_{i\in I_1} i\text{-}\mathrm{res}_{r-1}^r\,. \tag{6.4}$$

The results just explained extend immediately to $H_{r,s} = H_r \otimes H_s$. For these algebras, there are two commuting i-induction functors $i\text{-}\mathrm{ind}_{r-1,s}^{r,s}$ and $i\text{-}\mathrm{ind}_{r,s-1}^{r,s}$, defined by tensoring with the bimodules that arise by taking the generalized i-eigenspaces of the endomorphisms of $H_{r,s}$ defined by left multiplication by $L_r \otimes 1$ or $1 \otimes L_s$, respectively. The biadjoint i-restriction functors are denoted $i\text{-}\mathrm{res}_{r-1,s}^{r,s}$ and $i\text{-}\mathrm{res}_{r,s-1}^{r,s}$. Lemma 6.1 extends in an obvious way to describe the effect of these functors on the modules $\mathrm{S}_{\lambda^\uparrow} \boxtimes \mathrm{S}_{\lambda^\downarrow}$.

The next step is to use the Morita equivalences from Lemma 5.2 to transport the branching rules for $H_{r,s}$ just described to the algebra OS°. Let

$$\imath_\downarrow^\circ : OS^\circ \to OS^\circ, \qquad f \mapsto \downarrow \otimes f, \tag{6.5}$$

$$\imath_\uparrow^\circ : OS^\circ \to OS^\circ, \qquad f \mapsto \uparrow \otimes f \tag{6.6}$$

be the algebra homomorphisms associated to the functors $\downarrow\otimes - : \mathcal{OS}^\circ(z,t) \to \mathcal{OS}^\circ(z,t)$ and $\uparrow\otimes - : \mathcal{OS}^\circ(z,t) \to \mathcal{OS}^\circ(z,t)$. These are not *locally unital* algebra homomorphisms: they send the idempotent 1_a to $1_{\uparrow\mathsf{a}}$ and to $1_{\downarrow\mathsf{a}}$, respectively. Then let

$$_\uparrow OS^\circ := \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle} 1_{\uparrow\mathsf{a}} OS^\circ 1_\mathsf{b}, \qquad OS^\circ_\uparrow := \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle} 1_\mathsf{a} OS^\circ 1_{\uparrow\mathsf{b}}, \tag{6.7}$$

$$_\downarrow OS^\circ := \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle} 1_{\downarrow\mathsf{a}} OS^\circ 1_\mathsf{b}, \qquad OS^\circ_\downarrow := \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle} 1_\mathsf{a} OS^\circ 1_{\downarrow\mathsf{b}}, \tag{6.8}$$

which we view as (OS°, OS°)-bimodules with left and right actions of $a, b \in OS^\circ$ on f defined by $a \cdot f \cdot b := \imath_\uparrow^\circ(a) f b, a f \imath_\uparrow^\circ(b), \imath_\downarrow^\circ(a) f b$ and $a f \imath_\downarrow^\circ(b)$, respectively. Tensoring with these bimodules give us four endofunctors of Mod-OS°:

$$E^\uparrow := ? \otimes_{OS^\circ} {}_\uparrow OS^\circ : \text{Mod-}OS^\circ \to \text{Mod-}OS^\circ, \tag{6.9}$$

$$F^\uparrow := ? \otimes_{OS^\circ} OS^\circ_\uparrow : \text{Mod-}OS^\circ \to \text{Mod-}OS^\circ, \tag{6.10}$$

$$F^{\downarrow} := ? \otimes_{OS^\circ} {}_{\downarrow}OS^\circ : \text{Mod-}OS^\circ \to \text{Mod-}OS^\circ, \tag{6.11}$$

$$E^{\downarrow} := ? \otimes_{OS^\circ} OS^\circ_{\downarrow} : \text{Mod-}OS^\circ \to \text{Mod-}OS^\circ. \tag{6.12}$$

The functors $E^{\uparrow}$ and $F^{\downarrow}$ send $OS^\circ_{r,s}$-modules to $OS^\circ_{r+1,s}$- and $OS^\circ_{r,s+1}$-modules, respectively; they will be called *induction functors*. The functors $F^{\uparrow}$ and $E^{\downarrow}$ send $OS^\circ_{r,s}$-modules to $OS^\circ_{r-1,s}$- and $OS^\circ_{r,s-1}$-modules, respectively; they will be called *restriction functors*. This terminology is justified by the following lemma.

Lemma 6.2. *The following diagrams commute up to natural isomorphisms:*

$$\begin{tikzcd}
\text{Mod-}H_{r+1,s} \arrow[r, "\Upsilon_{r+1,s}"] & \text{Mod-}OS^\circ_{r+1,s} \\
\text{Mod-}H_{r,s} \arrow[u, "\mathrm{ind}^{r+1,s}_{r,s}"] \arrow[r, "\Upsilon_{r,s}"'] \arrow[ur, phantom, "\searrow\alpha"] & \text{Mod-}OS^\circ_{r,s}, \arrow[u, "E^{\uparrow}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{Mod-}H_{r+1,s} \arrow[r, "\Upsilon_{r+1,s}"] \arrow[d, "\mathrm{res}^{r+1,s}_{r,s}"] \arrow[dr, phantom, "\beta\nearrow"] & \text{Mod-}OS^\circ_{r+1,s} \arrow[d, "F^{\uparrow}"] \\
\text{Mod-}H_{r,s} \arrow[r, "\Upsilon_{r,s}"'] & \text{Mod-}OS^\circ_{r,s},
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Mod-}H_{r,s+1} \arrow[r, "\Upsilon_{r,s+1}"] & \text{Mod-}OS^\circ_{r,s+1} \\
\text{Mod-}H_{r,s} \arrow[u, "\mathrm{ind}^{r,s+1}_{r,s}"] \arrow[r, "\Upsilon_{r,s}"'] \arrow[ur, phantom, "\searrow\gamma"] & \text{Mod-}OS^\circ_{r,s}, \arrow[u, "F^{\downarrow}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{Mod-}H_{r,s+1} \arrow[r, "\Upsilon_{r,s+1}"] \arrow[d, "\mathrm{res}^{r,s+1}_{r,s}"] \arrow[dr, phantom, "\delta\nearrow"] & \text{Mod-}OS^\circ_{r,s+1} \arrow[d, "E^{\downarrow}"] \\
\text{Mod-}H_{r,s} \arrow[r, "\Upsilon_{r,s}"'] & \text{Mod-}OS^\circ_{r,s}.
\end{tikzcd}$$

Hence, the functors $E^{\uparrow}$ and $F^{\uparrow}$ are biadjoint, as are the functors $E^{\downarrow}$ and $F^{\downarrow}$.

Proof. First we construct the isomorphism $\alpha : \Upsilon_{r+1,s} \circ \mathrm{ind}^{r+1,s}_{r,s} \xrightarrow{\sim} E^{\uparrow} \circ \Upsilon_{r,s}$. The northwest functor is defined by tensoring over $H_{r,s}$ with the $(H_{r,s}, OS^\circ)$-bimodule

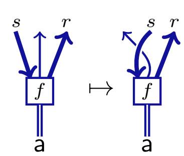
$$H_{r+1,s} \otimes_{H_{r+1,s}} 1_{\downarrow^s\uparrow^{r+1}} OS^\circ \cong 1_{\downarrow^s\uparrow^{r+1}} OS^\circ.$$

The southeast functor is defined by tensoring with

$$1_{\downarrow^s\uparrow^r} OS^\circ \otimes_{OS^\circ} {}_{\uparrow}OS^\circ \cong 1_{\uparrow\downarrow^s\uparrow^r} OS^\circ.$$

The following gives an isomorphism between these two bimodules:

$$1_{\downarrow^s\uparrow^{r+1}} OS^\circ \xrightarrow{\sim} 1_{\uparrow\downarrow^s\uparrow^r} OS^\circ,$$



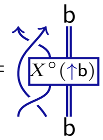
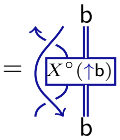
for any $f \in 1_{\downarrow^s\uparrow^{r+1}} OS^\circ 1_{\mathsf{a}}$. This establishes the existence of α.

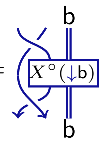
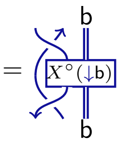
To deduce the existence of β, we claim that $F^{\uparrow}$ is right adjoint to $E^{\uparrow}$. To see this, $F^{\uparrow}M = M \otimes_{OS^\circ} OS^\circ_{\uparrow} = \bigoplus_{\mathsf{a} \in \langle \uparrow,\downarrow\rangle} M1_{\uparrow\mathsf{a}} \cong \bigoplus_{\mathsf{a} \in \langle \uparrow,\downarrow\rangle} \mathrm{Hom}_{OS^\circ}(1_{\uparrow\mathsf{a}} OS^\circ, M)$. So, by adjointness of tensor and hom, $F^{\uparrow}$ is right adjoint to $\bigoplus_{\mathsf{a} \in \langle \uparrow,\downarrow\rangle} ? \otimes_{OS^\circ} 1_{\uparrow\mathsf{a}} OS^\circ = ? \otimes_{OS^\circ} {}_{\uparrow}OS^\circ = E^{\uparrow}$. Since $\mathrm{res}^{r+1,s}_{r,s}$ is right adjoint to $\mathrm{ind}^{r+1,s}_{r,s}$, and the horizontal functors in our diagrams are equivalences of categories, we can now deduce the existence of the desired isomorphism β using the previous paragraph and unicity of right adjoints.

The construction of γ is very similar to that of α. In fact, it is even easier since both of the $(H_{r,s}, OS^\circ)$-bimodules being considered turn out to be the same bimodule $1_{\downarrow^{s+1}\uparrow^r} OS^\circ$, so we can take γ to be induced by the identity map. Then we get δ from γ as in the previous paragraph. $\square$

Now we need versions of Jucys-Murphy elements for OS°, which extend the Jucys-Murphy elements of $H_{r,s}$. For $\varnothing \neq \mathsf{b} \in \langle \uparrow,\downarrow\rangle$, define $X^\circ(\mathsf{b}) \in 1_{\mathsf{b}} OS^\circ 1_{\mathsf{b}}$ by setting

$X^\circ(\uparrow) := 1_\uparrow$, $X^\circ(\downarrow) := t^{-2}1_\downarrow$, and then recursively defining

$X^\circ(\uparrow\uparrow\mathsf{b}) :=$  , $\qquad X^\circ(\uparrow\downarrow\mathsf{b}) :=$  , (6.13)

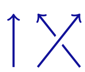
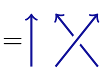
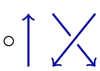
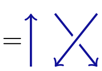
$X^\circ(\downarrow\downarrow\mathsf{b}) :=$  , $\qquad X^\circ(\downarrow\uparrow\mathsf{b}) :=$  , (6.14)

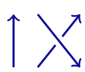
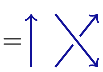
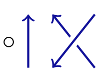
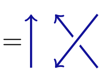
for any $\mathsf{a} \in \langle\uparrow,\downarrow\rangle$; cf. (4.11)–(4.12).

Let ${}_\uparrow X^\circ : {}_\uparrow OS^\circ \to {}_\uparrow OS^\circ$ and $X^\circ_\uparrow : OS^\circ_\uparrow \to OS^\circ_\uparrow$ be the linear endomorphisms defined on $1_{\uparrow\mathsf{a}}OS^\circ$ or $OS^\circ 1_{\uparrow\mathsf{a}}$ by left or right multiplication by $X^\circ(\uparrow\mathsf{a})$, respectively. Similarly, replacing $\uparrow$ by $\downarrow$ everywhere, we define linear endomorphisms ${}_\downarrow X^\circ$ and $X^\circ_\downarrow$ of ${}_\downarrow OS^\circ$ and $OS^\circ_\downarrow$.

Lemma 6.3. *All of the linear endomorphisms ${}_\uparrow X^\circ, X^\circ_\uparrow, {}_\downarrow X^\circ$ and $X^\circ_\downarrow$ are (OS°, OS°)-bimodule endomorphisms.*

Proof. We just explain the argument for ${}_\uparrow X^\circ$. It is obvious that this defines a right OS°-module homomorphism. To see that it also commutes with the left action of OS°, it suffices to check that it commutes with left multiplication by any element of OS° defined by a crossing of a neighboring pairs of strands (excluding the leftmost strand). This quickly reduces by induction to checking the following four identities:

$X^\circ(\uparrow\uparrow\uparrow) \circ$  $=$  $\circ X^\circ(\uparrow\uparrow\uparrow)$, $\qquad X^\circ(\uparrow\downarrow\downarrow) \circ$  $=$  $\circ X^\circ(\uparrow\downarrow\downarrow)$,

$X^\circ(\uparrow\downarrow\uparrow) \circ$  $=$  $\circ X^\circ(\uparrow\uparrow\downarrow)$, $\qquad X^\circ(\uparrow\uparrow\downarrow) \circ$  $=$  $\circ X^\circ(\uparrow\downarrow\uparrow)$.

These are all straightforward on drawing the diagrams for these X°'s explicity. $\square$

For $i \in \Bbbk^\times$, let $E_i^\uparrow$ be the subfunctor of E that is defined by tensoring with the bimodule that is the generalized i-eigenspace of ${}_\uparrow X^\circ : {}_\uparrow OS^\circ \to {}_\uparrow OS^\circ$. Define $F_i^\uparrow, F_i^\downarrow$ and $E_i^\downarrow$ similarly using the endomorphisms $X^\circ_\uparrow, {}_\downarrow X^\circ$ and $X^\circ_\downarrow$.

Lemma 6.4. *The following diagrams commute up to natural isomorphisms:*

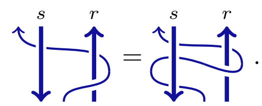
$$\begin{tikzcd}
\text{Mod-}H_{r+1,s} \arrow[r, "\Upsilon_{r+1,s}"] & \text{Mod-}OS^\circ_{r+1,s} \\
\text{Mod-}H_{r,s} \arrow[u, "i\text{-ind}_{r,s}^{r+1,s}"] \arrow[r, "\Upsilon_{r,s}"'] & \text{Mod-}OS^\circ_{r,s}, \arrow[u, "E_i^\uparrow"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{Mod-}H_{r+1,s} \arrow[r, "\Upsilon_{r+1,s}"] \arrow[d, "i\text{-res}_{r,s}^{r+1,s}"] & \text{Mod-}OS^\circ_{r+1,s} \arrow[d, "F_i^\uparrow"] \\
\text{Mod-}H_{r,s} \arrow[r, "\Upsilon_{r,s}"'] & \text{Mod-}OS^\circ_{r,s},
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Mod-}H_{r,s+1} \arrow[r, "\Upsilon_{r,s+1}"] & \text{Mod-}OS^\circ_{r,s+1} \\
\text{Mod-}H_{r,s} \arrow[u, "t^{-2}i^{-1}\text{-ind}_{r,s}^{r,s+1}"] \arrow[r, "\Upsilon_{r,s}"'] & \text{Mod-}OS^\circ_{r,s}, \arrow[u, "F_i^\downarrow"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{Mod-}H_{r,s+1} \arrow[r, "\Upsilon_{r,s+1}"] \arrow[d, "t^{-2}i^{-1}\text{-res}_{r,s}^{r,s+1}"] & \text{Mod-}OS^\circ_{r,s+1} \arrow[d, "E_i^\downarrow"] \\
\text{Mod-}H_{r,s} \arrow[r, "\Upsilon_{r,s}"'] & \text{Mod-}OS^\circ_{r,s}.
\end{tikzcd}$$

Hence, the functors $E_i^\uparrow$ and $F_i^\uparrow$ are biadjoint, as are the functors $E_i^\downarrow$ and $F_i^\downarrow$. Moreover:

$$E^\uparrow = \bigoplus_{i\in I_1} E_i^\uparrow, \quad F^\uparrow = \bigoplus_{i\in I_1} F_i^\uparrow, \quad F^\downarrow = \bigoplus_{i\in I_{t^{-2}}} F_i^\downarrow, \quad E^\downarrow = \bigoplus_{i\in I_{t^{-2}}} E_i^\downarrow. \tag{6.15}$$

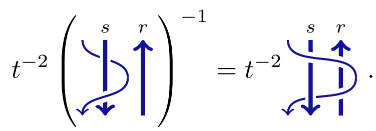
Proof. Consider the first diagram. In the proof of Lemma 6.2, the isomorphism of functors α was induced by an explicit bimodule isomorphism $1_{\downarrow^s\uparrow^{r+1}}OS^\circ \to 1_{\uparrow\downarrow^s\uparrow^r}OS^\circ$. This isomorphism intertwines the endomorphism of $1_{\downarrow^s\uparrow^{r+1}}OS^\circ$ defined by left multiplication by $\imath_{r+1,s}(L_{r+1}\otimes 1)$ with the endomorphism of $1_{\uparrow\downarrow^s\uparrow^r}OS^\circ$ defined by left multiplication by $X^\circ(\uparrow\downarrow^s\uparrow^r)$; the appropriate picture needed to see this is as follows:



Consequently, this bimodule isomorphism restricts to an isomorphism between the appropriate summands of these bimodules, showing that the restriction of α gives the desired natural transformation.

The second diagram follows from the first by unicity of adjoints on observing that $F_i^\uparrow$ is right adjoint to $E_i^\uparrow$, which follows from the explicit construction of the adjunction in the second paragraph of the proof of Lemma 6.2.

The third diagram is established in the same way as the first diagram. One needs to check that the endomorphisms of $1_{\downarrow^{s+1}\uparrow^r}OS^\circ$ defined by left multiplication by $t^{-2}\imath_{r,s+1}(1\otimes L_{s+1})^{-1}$ and by $X^\circ(\downarrow^{s+1}\uparrow^r)$ are equal, which is clear from the following picture:



The fourth diagram follows by adjunction as before.

The final statement of the lemma follows using these diagrams plus facts we have already discussed about the induction and restriction functors for $H_{r,s}$. $\square$

We assemble the results so far into the following theorem, which describes all of the branching rules for the functors $E_i^\uparrow$, $F_i^\uparrow$, $F_i^\downarrow$ and $E_i^\downarrow$.

Lemma 6.5. *The following hold for $i\in\Bbbk^\times$ and $\boldsymbol{\lambda}\in\mathrm{Bip}_{r,s}$.*

(1) $E_i^\uparrow\mathrm{S}(\boldsymbol{\lambda})$ has a multiplicity-free filtration with sections $\mathrm{S}(\boldsymbol{\mu})$ for $\boldsymbol{\mu}\in\mathrm{Bip}_{r+1,s}$ obtained by adding a node of content i to the Young diagram of $\lambda^\uparrow$.

(2) $F_i^\uparrow\mathrm{S}(\boldsymbol{\lambda})$ has a multiplicity-free filtration with sections $\mathrm{S}(\boldsymbol{\mu})$ for $\boldsymbol{\mu}\in\mathrm{Bip}_{r-1,s}$ obtained by removing a node of content i from the Young diagram of $\lambda^\uparrow$.

(3) $F_i^\downarrow\mathrm{S}(\boldsymbol{\lambda})$ has a multiplicity-free filtration with sections $\mathrm{S}(\boldsymbol{\mu})$ for $\boldsymbol{\mu}\in\mathrm{Bip}_{r,s+1}$ obtained by adding a node of content $t^{-2}i^{-1}$ to the Young diagram of $\lambda^\downarrow$.

(4) $E_i^\downarrow\mathrm{S}(\boldsymbol{\lambda})$ has a multiplicity-free filtration with sections $\mathrm{S}(\boldsymbol{\mu})$ for $\boldsymbol{\mu}\in\mathrm{Bip}_{r,s-1}$ obtained by removing a node of content $t^{-2}i^{-1}$ from the Young diagram of $\lambda^\downarrow$.

In all cases, the filtrations should be ordered according to the usual dominance ordering on the partitions labelling the sections, most dominant at the bottom.

Proof. This follows from Lemmas 6.4 and 6.1. $\square$

Now we turn our attention at last to OS itself. Mimicking the definitions made above for OS°, we write $\imath_\downarrow : OS\to OS$ and $\imath_\uparrow : OS\to OS$ for the algebra homomorphisms associated to the functors $\downarrow\otimes - : \mathcal{OS}(z,t)\to\mathcal{OS}(z,t)$ and $\uparrow\otimes - : \mathcal{OS}(z,t)\to\mathcal{OS}(z,t)$. Then let

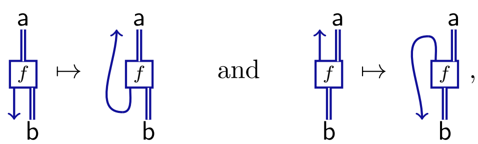
$$ {}_\uparrow OS := \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle} 1_{\uparrow\mathsf{a}}OS1_{\mathsf{b}}, \qquad OS_\uparrow := \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle} 1_{\mathsf{a}}OS1_{\uparrow\mathsf{b}}, \tag{6.16}$$

$$ {}_\downarrow OS := \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle} 1_{\downarrow\mathsf{a}}OS1_{\mathsf{b}}, \qquad OS_\downarrow := \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\uparrow,\downarrow\rangle} 1_{\mathsf{a}}OS1_{\downarrow\mathsf{b}}, \tag{6.17}$$

which we view as (OS, OS)-bimodules with left and right actions of $a, b \in OS$ on f defined by $a \cdot f \cdot b := \imath_{\uparrow}(a) f b, a f \imath_{\uparrow}(b), \imath_{\downarrow}(a) f b$ and $a f \imath_{\downarrow}(b)$, respectively. A key difference to the situation for OS° emerges right away:

Lemma 6.6. *We have that* $OS_{\uparrow} \cong {}_{\downarrow}OS$ *and* $OS_{\downarrow} \cong {}_{\uparrow}OS$ *as* (OS, OS)*-bimodules.*

Proof. The mutually inverse bimodule isomorphisms $OS_{\downarrow} \to {}_{\uparrow}OS$ and ${}_{\uparrow}OS \to OS_{\downarrow}$ are defined on diagrams by the maps



respectively. The isomorphism $OS_{\uparrow} \cong {}_{\downarrow}OS$ is constructed similarly. $\square$

This means that we only need to define *two* functors:

$$E := ? \otimes_{OS} {}_{\uparrow}OS \cong ? \otimes_{OS} OS_{\downarrow} : \text{Mod-}OS \to \text{Mod-}OS, \tag{6.18}$$

$$F := ? \otimes_{OS} {}_{\downarrow}OS \cong ? \otimes_{OS} OS_{\uparrow} : \text{Mod-}OS \to \text{Mod-}OS. \tag{6.19}$$

Note that

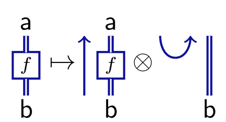
$$E(1_{\mathsf{a}}OS) = 1_{\mathsf{a}}OS \otimes_{OS} ({}_{\uparrow}OS) = 1_{\mathsf{a}}({}_{\uparrow}OS) = 1_{\uparrow\mathsf{a}}OS, \tag{6.20}$$

and similarly $F(1_{\mathsf{a}}OS) = 1_{\downarrow\mathsf{a}}OS$. By adjointness of tensor and hom, the functor E has a canonical right adjoint

$$\bigoplus_{\mathsf{a} \in \langle \uparrow, \downarrow \rangle} \operatorname{Hom}_{OS}(1_{\uparrow\mathsf{a}}OS, ?) \cong ? \otimes_{OS} OS_{\uparrow}, \tag{6.21}$$

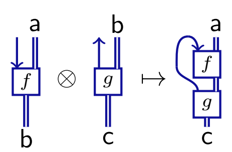
with the isomorphism here sending f in the ath summand to $f(1_{\uparrow\mathsf{a}}) \otimes 1_{\uparrow\mathsf{a}}$. In view of Lemma 6.6, the functor $? \otimes_{OS} OS_{\uparrow}$ on the right hand side of (6.21) is isomorphic to F. Thus, we see that (E, F) is an adjoint pair. Explicitly, the unit $\mathrm{Id} \to FE$ of this adjunction is defined by the bimodule homomorphism

$$OS \to {}_{\uparrow}OS \otimes_{OS} {}_{\downarrow}OS, \tag{6.22}$$



and the counit $EF \to \mathrm{Id}$ is defined by

$${}_{\downarrow}OS \otimes_{OS} {}_{\uparrow}OS \to OS, \tag{6.23}$$



Reversing the roles of $\uparrow$ and $\downarrow$ in this argument, we get another canonical adjunction making (F, E) is an adjoint pair. So E and F are biadjoint, hence, they are exact, and preserve locally finite-dimensional, finitely generated, finitely cogenerated, projective and injective objects; cf. [BD, Theorem 2.11].

Recalling (4.10), let ${}_{\uparrow}X : {}_{\uparrow}OS \to {}_{\uparrow}OS$ be the bimodule endomorphism defined on $1_{\uparrow\mathsf{b}}OS$ by left multiplication by $X(\uparrow\mathsf{b})$, and let $X_{\downarrow} : OS_{\downarrow} \to OS_{\downarrow}$ be defined on $1_{\mathsf{b}}OS_{\downarrow\mathsf{b}}$ by right multiplication by $X(\downarrow\mathsf{b})$. These are intertwined by the isomorphism from Lemma 6.6; this depends on (4.4). Similarly, switching $\uparrow$ with $\downarrow$ everywhere, we get endomorphisms ${}_{\downarrow}X : {}_{\downarrow}OS \to {}_{\downarrow}OS$ and $X_{\uparrow} : OS_{\uparrow} \to OS_{\uparrow}$, which are again intertwined by the isomorphism from Lemma 6.6.

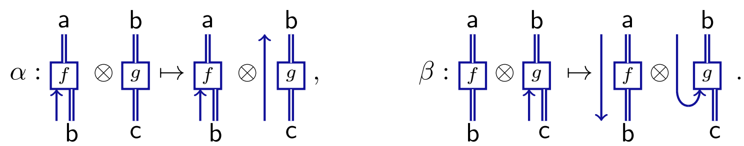
Lemma 6.7. *There are short exact sequence of (OS°, OS)-bimodules*

$$0 \longrightarrow OS^\circ_\uparrow \otimes_{OS^\sharp} OS \xrightarrow{\alpha} OS^\circ \otimes_{OS^\sharp} OS_\uparrow \xrightarrow{\beta} {}_\downarrow OS^\circ \otimes_{OS^\sharp} OS \longrightarrow 0, \tag{6.24}$$

$$0 \longrightarrow OS^\circ_\downarrow \otimes_{OS^\sharp} OS \xrightarrow{\alpha} OS^\circ \otimes_{OS^\sharp} OS_\downarrow \xrightarrow{\beta} {}_\uparrow OS^\circ \otimes_{OS^\sharp} OS \longrightarrow 0. \tag{6.25}$$

The maps α and β in the first sequence satisfy $\alpha \circ (X^\circ_\uparrow \otimes \mathrm{id}) = (\mathrm{id} \otimes X_\uparrow) \circ \alpha$ and $\beta \circ (\mathrm{id} \otimes X_\uparrow) = ({}_\downarrow X^\circ \otimes \mathrm{id}) \circ \beta$. The maps in the second sequence have analogous properties.

Proof. We just go through the details for the first short exact sequence. The bimodule homomorphisms α and β are defined on pure tensors as follows:



(6.26)

It is straightforward to see these are well-defined bimodule homomorphisms. Also $\beta \circ \alpha = 0$. Indeed, if we apply $\beta \circ \alpha$ to a pure tensor as above, we produce a pure tensor of the form $f \otimes g$ such that the strand of g starting in the top left corner is a rightward cup. This cup commutes past the tensor to give zero since we are viewing ${}_\downarrow OS^\circ$ as a right $OS^\sharp$-module by inflation.

To show that the sequence is exact, we pick bases. Recall that $\langle \uparrow, \downarrow \rangle_{r,s}$ denotes words which have exactly r letters $\uparrow$ and s letters $\downarrow$. For $\mathsf{a}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle_{r,s}$, let $A_{\mathsf{a},\mathsf{b}}$ be a basis for $1_\mathsf{a} OS^\circ 1_\mathsf{b}$ consisting of reduced lifts of matchings. Similarly, for $\mathsf{b} \in \langle \uparrow, \downarrow \rangle_{r,s}$ and $\mathsf{c} \in \langle \uparrow, \downarrow \rangle_{r+t,s+t}$ for $t \geq 0$, let $B_{\mathsf{b},\mathsf{c}}$ be a basis for $1_\mathsf{b} OS^- 1_\mathsf{c}$ consisting of reduced lifts of matchings. By Lemma 5.1, we see that

$$P := \left\{ f \otimes g \;\middle|\; \begin{array}{l} f \in A_{\mathsf{a},\uparrow\mathsf{b}}, g \in B_{\mathsf{b},\mathsf{c}} \text{ for } r,s,t \geq 0 \text{ and} \\ \mathsf{a} \in \langle \uparrow, \downarrow \rangle_{r+1,s}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle_{r,s}, \mathsf{c} \in \langle \uparrow, \downarrow \rangle_{r+t,s+t}, \end{array} \right\},$$

$$Q := \left\{ f \otimes g \;\middle|\; \begin{array}{l} f \in A_{\mathsf{a},\mathsf{b}}, g \in B_{\mathsf{b},\uparrow\mathsf{c}} \text{ for } r,s,t \geq 0 \text{ with } s+t \geq 1 \text{ and} \\ \mathsf{a}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle_{r,s}, \mathsf{c} \in \langle \uparrow, \downarrow \rangle_{r+t,s+t-1}, \end{array} \right\},$$

$$R := \left\{ f \otimes g \;\middle|\; \begin{array}{l} f \in A_{\downarrow\mathsf{a},\mathsf{b}}, g \in B_{\mathsf{b},\mathsf{c}} \text{ for } r,s,t \geq 0 \text{ and} \\ \mathsf{a} \in \langle \uparrow, \downarrow \rangle_{r,s}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle_{r,s+1}, \mathsf{c} \in \langle \uparrow, \downarrow \rangle_{r+t,s+t+1}, \end{array} \right\}$$

are bases for $OS^\circ_\uparrow \otimes_{OS^\sharp} OS$, $OS^\circ \otimes_{OS^\sharp} OS_\uparrow$ and ${}_\downarrow OS^\circ \otimes_{OS^\sharp} OS$, respectively. Then we partition the set Q as $Q_1 \sqcup Q_2$ so that Q_1 consists of all $f \otimes g \in Q$ such that the reduced lift g has a propagating upward strand on its left edge, and Q_2 consists of all remaining elements of Q. Note for each $f \otimes g \in Q_2$ that the strand of g starting in the bottom left corner is a rightward cap. Then it is clear that the map α maps P bijectively onto Q_1 and β maps Q_2 bijectively onto R. This completes the proof.

Now we check that $\alpha \circ (X^\circ_\uparrow \otimes \mathrm{id}) = (\mathrm{id} \otimes X_\uparrow) \circ \alpha$. Take $f \otimes g \in OS^\circ_\uparrow \otimes_{OS^\sharp} OS$. We must show that $(f \circ X^\circ(\uparrow\mathsf{b})) \otimes (\uparrow g) = f \otimes ((\uparrow g) \circ X(\uparrow\mathsf{c}))$ for $f \in 1_\mathsf{a} OS^\circ 1_{\uparrow\mathsf{b}}$ and $g \in 1_\mathsf{b} OS 1_\mathsf{c}$. We can move $X^\circ(\uparrow\mathsf{b})$ over the first tensor product and commute $X(\uparrow\mathsf{c})$ with $\uparrow g$, to reduce to checking that $1_{\uparrow\mathsf{b}} \otimes X(\uparrow\mathsf{b}) = 1_{\uparrow\mathsf{b}} \otimes X^\circ(\uparrow\mathsf{b})$. The morphism $X^\circ(\uparrow\mathsf{b})$ can be transformed into $X(\uparrow\mathsf{b})$ by using the quadratic relation to switch some positive crossings to negative crossings. This produces some error terms which involve caps at the top of the picture, which become zero when commuted back over the tensor product. (This argument can be made more formal by using induction on the length of the word b, using the recursions (4.11) and (6.13).)

The proof that $\beta \circ (\mathrm{id} \otimes X_\uparrow) = ({}_\downarrow X^\circ \otimes \mathrm{id}) \circ \beta$ is similar; one needs to use also (4.3) and Lemma 6.3. $\square$

Finally, we refine the functors E and F. For $i \in \Bbbk^\times$, let E_i be the subfunctor of E that is defined by tensoring with the bimodule that is the generalized i-eigenspace of ${}_\uparrow X : {}_\uparrow OS \to {}_\uparrow OS$; equivalently, it may be defined by tensoring with the generalized i-eigenspace of $X_\downarrow : OS_\downarrow \to OS_\downarrow$. Similarly, switching $\uparrow$ and $\downarrow$ everywhere, defines a subfunctor F_i of F. Let

$$I := I_1 \cup I_{t^{-2}} = \left\{ q^{2n}, t^{-2}q^{-2n} \mid n \in \mathbb{Z} \right\} \subset \Bbbk^\times. \tag{6.27}$$

Lemma 6.8. *There are short exact sequences of functors for each $i \in \Bbbk^\times$:*

$$0 \longrightarrow \Delta \circ F_i^\uparrow \longrightarrow F_i \circ \Delta \longrightarrow \Delta \circ F_i^\downarrow \longrightarrow 0, \tag{6.28}$$

$$0 \longrightarrow \Delta \circ E_i^\downarrow \longrightarrow E_i \circ \Delta \longrightarrow \Delta \circ E_i^\uparrow \longrightarrow 0. \tag{6.29}$$

Moreover, the functors E_i and F_i are biadjoint, and we have that

$$E = \bigoplus_{i \in I} E_i, \qquad F = \bigoplus_{i \in I} F_i. \tag{6.30}$$

Proof. Note to start with that although ${}_\downarrow OS$ is not finite-dimensional (or even a direct sum of finite-dimensional bimodules as was the case for ${}_\downarrow OS^\circ$), it is locally finite-dimensional in the sense that it is the direct sum of the finite-dimensional vector spaces $1_{\downarrow \mathsf{a}} OS 1_{\mathsf{b}}$ for $\mathsf{a}, \mathsf{b} \in \langle \uparrow, \downarrow \rangle$. The endomorphism ${}_\downarrow X$ leaves each of these finite-dimensional vector spaces invariant. This is enough to see that each generalized i-eigenspace of ${}_\downarrow X$ is a summand of the bimodule ${}_\downarrow OS$. However, until we have proved (6.30), there may also be summands arising from generalized eigenspaces corresponding to eigenvalues of ${}_\downarrow X$ not in $I \subset \Bbbk^\times$, and there could also be summands arising from non-linear irreducible factors of the characteristic polynomial. Similar remarks apply to F_i.

To define an adjunction making (E_i, F_i) into an adjoint pair, we project the adunction for (E, F) onto the summands E_i and F_i. To see that this does the job, one needs to use the explicit forms for the unit and counit of the adjunction (E, F) given in (6.22)–(6.23). The key point is that ${}_\uparrow X \otimes \mathrm{id} = \mathrm{id} \otimes {}_\downarrow X$ as an endomorphism of ${}_\uparrow OS \otimes_{OS} {}_\downarrow OS$ and ${}_\downarrow X \otimes \mathrm{id} = \mathrm{id} \otimes {}_\uparrow X$ as an endomorphism of ${}_\downarrow OS \otimes_{OS} {}_\uparrow OS$. A similar argument produces an adjunction (F_i, E_i) the other way around. It then follows that E_i and F_i are both exact; one can also see this since they are summands of the exact functors E and$F$.

The short exact sequences from Lemma 6.7 may be viewed equivalently as short exact sequences of functors

$$0 \longrightarrow \Delta \circ F^\uparrow \longrightarrow F \circ \Delta \longrightarrow \Delta \circ F^\downarrow \longrightarrow 0,$$

$$0 \longrightarrow \Delta \circ E^\downarrow \longrightarrow E \circ \Delta \longrightarrow \Delta \circ E^\uparrow \longrightarrow 0.$$

Similarly, using the final assertion of the lemma, we get (6.28)–(6.29) from Lemma 6.7 on passing to the appropriate generalized eigenspaces.

Finally, we must establish (6.30). The short exact sequences of functors obtained in the previous paragraph plus (6.15) imply that (6.30) holds on any standard module $\Delta(\boldsymbol{\lambda})$. By exactness and Corollary 5.11, it follows that it also holds on any indecomposable projective module. Hence, it is true on any module. $\square$

With these branching rules in hand, we can proceed to the definition of the formal character of a locally finite-dimensional OS-module.

First, we must refine the idempotents 1_{a} for $\mathsf{a} \in \langle \uparrow, \downarrow \rangle$. Let $\langle \uparrow, \downarrow \rangle_I$ be the set of words in the letters $\{\uparrow_i, \downarrow_i \mid i \in I\}$. Thus, an element of $\langle \uparrow, \downarrow \rangle_I$ has the form $\mathsf{a}_{\boldsymbol{i}} = (\mathsf{a}_n)_{i_n} \cdots (\mathsf{a}_1)_{i_1}$ for words $\mathsf{a} = \mathsf{a}_n \cdots \mathsf{a}_1 \in \langle \uparrow, \downarrow \rangle$ and $\boldsymbol{i} = i_n \cdots i_1 \in \langle I \rangle$. Take a word $\mathsf{a}_{\boldsymbol{i}} \in \langle \uparrow, \downarrow \rangle_I$ of length n. Let X_i be the Jucys-Murphy element in $1_{\mathsf{a}} OS 1_{\mathsf{a}}$ that is defined by a dot on the ith strand from the right, so that $X_1, \ldots, X_n$ generate a commutative subalgebra of the finite-dimensional algebra $1_{\mathsf{a}} OS 1_{\mathsf{a}}$. It follows that there is an idempotent $1_{\mathsf{a}_{\boldsymbol{i}}} \in 1_{\mathsf{a}} OS 1_{\mathsf{a}}$ which projects any $1_{\mathsf{a}} OS 1_{\mathsf{a}}$-module onto the simultaneous generalized eigenspaces of $X_1, \dots, X_n$ corresponding to eigenvalues $i_1, \dots, i_n$, respectively. For a given a, all but finitely many $1_{\mathsf{a}_{\boldsymbol{i}}}$ are zero.

Now define the *formal character* of a locally finite-dimensional OS-module M by

$$\operatorname{ch} M := \sum_{\mathsf{a}_{\boldsymbol{i}} \in \langle \uparrow, \downarrow \rangle_I} (\dim M 1_{\mathsf{a}_{\boldsymbol{i}}}) \mathsf{a}_{\boldsymbol{i}}, \tag{6.31}$$

which is an element of the ring of (possibly infinite) $\mathbb{Z}$-linear combinations of elements of the monoid $\langle \uparrow, \downarrow \rangle_I$. From the proof of the following lemma plus (6.30), one sees that $1_{\mathsf{a}} = \sum_{\boldsymbol{i}} 1_{\mathsf{a}_{\boldsymbol{i}}}$. Note also that ch is additive on short exact sequences.

Lemma 6.9. $\operatorname{ch} M = \sum_{i \in I} \downarrow_i (\operatorname{ch} E_i M) + \sum_{i \in I} \uparrow_i (\operatorname{ch} F_i M)$.

Proof. Take $\mathsf{a}_{\boldsymbol{i}} \in \langle \uparrow, \downarrow \rangle_I$ and suppose that $\mathsf{a} = \uparrow \mathsf{b}, \boldsymbol{i} = i\boldsymbol{j}$. We claim that $\dim M 1_{\mathsf{a}_{\boldsymbol{i}}} = \dim (F_i M) 1_{\mathsf{b}_{\boldsymbol{j}}}$. The lemma follows from this together with the analogous statement argument with $\uparrow$ replaced with $\downarrow$ and F_i replaced with E_i. To prove the claim, using (6.20), we have that

$$\begin{aligned} M 1_{\mathsf{a}} \cong \operatorname{Hom}_{OS}(1_{\mathsf{a}} OS, M) &\cong \operatorname{Hom}_{OS}(E(1_{\mathsf{b}} OS), M) \\ &\cong \operatorname{Hom}_{OS}(1_{\mathsf{b}} OS, FM) \cong (FM) 1_{\mathsf{b}}. \end{aligned}$$

Under this isomorphism, the generalized i-eigenspace of $\overset{\bullet}{\uparrow} \otimes 1_{\mathsf{b}}$ corresponds to the summand $(F_i M) 1_{\mathsf{b}}$. The result follows. $\square$

Lemma 6.10. *The characters* $\left\{ \operatorname{ch} \mathrm{L}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in e\text{-Bip} \right\}$ *of the irreducible OS-modules are linearly independent.*

Proof. Take $\boldsymbol{\lambda} \in e\text{-Bip}_{r,s}$. As $\mathrm{L}(\boldsymbol{\lambda})$ is the shortest word module of type $\boldsymbol{\lambda}$, its formal character is a sum A_λ of words of the form $\downarrow_{i_{r+s}} \cdots \downarrow_{i_{r+1}} \uparrow_{i_r} \cdots \uparrow_{i_1}$, plus a sum B_λ of words that are obtained from the ones in A_λ by properly shuffling the $\downarrow$'s and $\uparrow$'s, plus a sum C_λ of strictly longer words. By unitriangularity, it suffices to show that the "leading terms" A_λ are linearly independent for fixed r, s and all $\boldsymbol{\lambda} \in e\text{-Bip}_{r,s}$. But A_λ is just the product of the formal characters of $\mathrm{D}_{\lambda^\downarrow}$ and $\mathrm{D}_{\lambda^\uparrow}$ in the usual sense of the Hecke algebras H_s and H_r. So these words are linearly independent by the well-known linear independence of irreducible characters for the Hecke algebra[4]. $\square$

Now we define the (t-shifted) *bipartition graph* to be the I-colored directed graph with vertices Bip and an edge $\boldsymbol{\lambda} \xrightarrow{i} \boldsymbol{\mu}$ if one of the following holds:

- $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by adding a node of content i to $\lambda^\uparrow$;
- $\boldsymbol{\lambda}$ is obtained from $\boldsymbol{\mu}$ by adding a node of content $t^{-2} i^{-1}$ to $\mu^\downarrow$.

A small piece of this graph is displayed in Figure 2.

By a *path* $\gamma : \boldsymbol{\lambda} \rightsquigarrow \boldsymbol{\mu}$ we mean an undirected path in the bipartition graph starting at $\boldsymbol{\lambda}$ and ending at $\boldsymbol{\mu}$. The *type* of such a path γ is $\operatorname{type}(\gamma) := \mathsf{a}_{\boldsymbol{i}} \in \langle \uparrow, \downarrow \rangle_I$ where $\boldsymbol{i} = i_n \cdots i_1$ records the colors on the edges of the path $\boldsymbol{\lambda} \overset{i_1}{\text{---}} \cdots \overset{i_n}{\text{---}} \boldsymbol{\mu}$ and $\mathsf{a} = \mathsf{a}_n \cdots \mathsf{a}_1 \in \langle \uparrow, \downarrow \rangle$ is defined so $\mathsf{a}_m = \uparrow$ or $\downarrow$ according to whether the mth edge is traversed forwards or backwards according to its direction. For example, the path

$$(\square, \varnothing) \xleftarrow{t^{-2}} (\square, \square) \xleftarrow{1} (\varnothing, \square) \xrightarrow{t^{-2}} \boldsymbol{\varnothing} \xleftarrow{t^{-2}} (\varnothing, \square)$$

is of type $\downarrow_{t^{-2}} \uparrow_{t^{-2}} \downarrow_1 \downarrow_{t^{-2}}$.

Theorem 6.11. *For* $\boldsymbol{\lambda} \in \mathrm{Bip}$*, we have that* $\operatorname{ch} \tilde{\Delta}(\boldsymbol{\lambda}) = \sum_{\gamma : \boldsymbol{\varnothing} \rightsquigarrow \boldsymbol{\lambda}} \operatorname{type}(\gamma)$.

[4] This may be proved in the same way as is explained for the symmetric group in [Kl, Lemma 11.2.5].

Proof. Note the infinite sum in the theorem makes sense since there are only finitely many paths of any given length. From (6.28)–(6.29) and Lemma 6.5, we get some explicit $\tilde{\Delta}$-filtrations of $E_i\tilde{\Delta}(\boldsymbol{\lambda})$ and $F_i\tilde{\Delta}(\boldsymbol{\lambda})$ with sections $\tilde{\Delta}(\boldsymbol{\mu})$ for each $\boldsymbol{\mu} \overset{i}{\leftarrow} \boldsymbol{\lambda}$ or $\boldsymbol{\mu} \overset{i}{\rightarrow} \boldsymbol{\lambda}$, respectively. Applying Lemma 6.9, we deduce that

$$\operatorname{ch}\tilde{\Delta}(\boldsymbol{\lambda}) = \sum_{i\in I}\Big(\sum_{\boldsymbol{\mu}\overset{i}{\leftarrow}\boldsymbol{\lambda}} \downarrow_i \operatorname{ch}\tilde{\Delta}(\boldsymbol{\mu}) + \sum_{\boldsymbol{\mu}\overset{i}{\rightarrow}\boldsymbol{\lambda}} \uparrow_i \operatorname{ch}\tilde{\Delta}(\boldsymbol{\mu})\Big).$$

Now use induction on path length. □

Corollary 6.12. *Take $\boldsymbol{\lambda} \in$ Bip and $\boldsymbol{\mu} \in e$-Bip. If $\mathrm{L}(\boldsymbol{\mu})$ is a composition factor of $\tilde{\Delta}(\boldsymbol{\lambda})$ then there is a path $\gamma : \boldsymbol{\varnothing} \rightsquigarrow \boldsymbol{\lambda}$ and a minimal length path $\delta : \boldsymbol{\varnothing} \rightsquigarrow \boldsymbol{\mu}$ such that* $\operatorname{type}(\gamma) = \operatorname{type}(\delta)$.

Proof. Pick any word $\mathsf{a}_{\boldsymbol{i}} \in \langle\uparrow,\downarrow\rangle_I$ that appears with non-zero coefficient in the formal character of the shortest word space of $\mathrm{L}(\boldsymbol{\mu})$. Since $[\widetilde{\Delta}(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})]$ and $[\widetilde{\Delta}(\boldsymbol{\mu}) : \mathrm{L}(\boldsymbol{\mu})]$ are both non-zero, $\mathsf{a}_{\boldsymbol{i}}$ also has non-zero coefficients in $\operatorname{ch}\widetilde{\Delta}(\boldsymbol{\lambda})$ and $\operatorname{ch}\widetilde{\Delta}(\boldsymbol{\mu})$. So Theorem 6.11 implies that there are paths $\gamma : \boldsymbol{\varnothing} \rightsquigarrow \boldsymbol{\lambda}$ and $\delta : \boldsymbol{\varnothing} \rightsquigarrow \boldsymbol{\mu}$ of the same type $\mathsf{a}_{\boldsymbol{i}}$. Moreover, δ is of minimal length amongst all paths $\boldsymbol{\varnothing} \rightsquigarrow \boldsymbol{\mu}$. □

Corollary 6.13. *Suppose that $\boldsymbol{\mu} \in e\text{-}\mathrm{Bip}_{r,s}$. If* either $t \notin \{\pm q^n \mid n \in \mathbb{Z}\}$, or $e = 0$, *$t = q^n$ for $n \in \mathbb{N}$ and $h(\boldsymbol{\mu}) \leq n$, then we have that $\mathrm{P}(\boldsymbol{\mu}) = \Delta(\boldsymbol{\mu})$.*

Proof. In view of Corollary 5.11, it suffices to show that $[\bar{\Delta}(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})] = \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ for all $\boldsymbol{\lambda} \in e$-Bip. Since $\bar{\Delta}(\boldsymbol{\lambda})$ and $\mathrm{L}(\boldsymbol{\mu})$ have the same shortest word spaces, this follows if we can show for $\boldsymbol{\lambda} \in e$-Bip that $[\bar{\Delta}(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})] \neq 0 \Rightarrow \boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{r,s}$. So suppose that $[\bar{\Delta}(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})] \neq 0$. Corollary 6.12 implies that there is a path $\gamma : \boldsymbol{\varnothing} \rightsquigarrow \boldsymbol{\lambda}$ of the same type as a minimal length path $\delta : \boldsymbol{\varnothing} \rightsquigarrow \boldsymbol{\mu}$. Being of minimal length means that δ is some permutation of the word $\uparrow_{i_{r+s}} \cdots \uparrow_{i_{r+1}} \downarrow_{i_r} \cdots \downarrow_{i_1}$, where $i_1, \dots, i_r$ are the $\uparrow$-contents of the nodes of $\mu^{\uparrow}$ and $i_{r+1}, \dots, i_{r+s}$ are the $\downarrow$-contents of the nodes of $\mu^{\downarrow}$.

If $t \notin \{\pm q^n \mid n \in \mathbb{Z}\}$, then the set of possible $\uparrow$-contents of nodes of partitions is disjoint from the set of possible $\downarrow$-contents. So any path of type δ starting at $\boldsymbol{\varnothing}$ necessarily ends at an element of $\mathrm{Bip}_{r,s}$. We deduce that $\boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{r,s}$ as required. Instead, suppose that $e = 0, t = q^n$ for $n \in \mathbb{N}$, and $h(\boldsymbol{\mu}) \leq |n|$. Recalling that $h(\boldsymbol{\mu})$ is

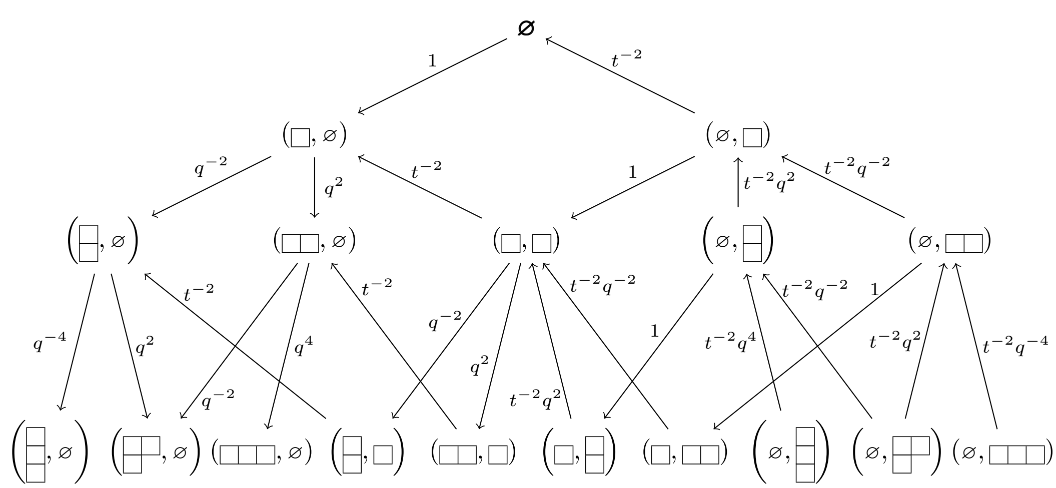


FIGURE 2. Bipartition graph up to word length 3

the total number of non-zero parts in both $\mu^{\uparrow}$ and $\mu^{\downarrow}$, these assumptions imply that $i_1, \dots, i_{r+s}$ are all distinct, and there is a unique path of type δ starting at $\boldsymbol{\varnothing}$. This shows that $\gamma = \delta$, hence, $\boldsymbol{\lambda} = \boldsymbol{\mu}$. □

Proof of Theorem 1.6. Corollary 6.13 shows that $\mathrm{P}(\boldsymbol{\lambda}) = \Delta(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in e\text{-Bip}$. So the standardization functor Δ sends the indecomposable projectives $\{\mathrm{Y}(\boldsymbol{\lambda}) \,|\, \boldsymbol{\lambda} \in e\text{-Bip}\}$ in Mod-OS° to the indecomposable projectives $\{\mathrm{P}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in e\text{-Bip}\}$ in Mod-OS. Since this functor is also exact, it follows that it is an equivalence of categories. □

Proof of Theorem 1.5. Suppose that q is not a root of unity and $t \notin \{\pm q^n \,|\, n \in \mathbb{Z}\}$. The first assumption means that $e = 0$, so OS° is semisimple. Hence, OS, or equivalently $\dot{\mathcal{OS}}(z,t)$, is semisimple thanks to Theorem 1.6. The parametrization of indecomposable objects in $\dot{\mathcal{OS}}(z,t)$ follows from Theorem 5.3: up to isomorphism they correspond to the irreducible projective modules $\{\Delta(\boldsymbol{\lambda}) \,|\, \boldsymbol{\lambda} \in \mathrm{Bip}\}$. Moreover, Theorem 5.18 shows in this case that $K_0(\dot{\mathcal{OS}}(z,t))$ may be identified with $\mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$ so that $[\Delta(\boldsymbol{\lambda})] \leftrightarrow \chi_{\boldsymbol{\lambda}}$.

It remains to show that OS is not semisimple for all other parameter choices. If q is a root of unity and $t \notin \{\pm q^n \mid n \in \mathbb{Z}\}$, this follows from Theorem 1.6, since Hecke algebras are not semisimple at roots of unity. Finally, suppose that q is arbitrary but $t = \pm q^n$ for some $n \in \mathbb{Z}$. Using the isomorphisms (2.11)–(2.12), we may as well assume that $t = q^n$ for $n \in \mathbb{N}$. Example 5.4 shows that $\bar{\Delta}(((n), \varnothing))$ is reducible, since it has a composition factor isomorphic to $\mathrm{L}(((n+1),(1)))$. Since $\bar{\Delta}(((n), \varnothing))$ is a finitely generated module with irreducible head, it is therefore not completely reducible, and OS is not semisimple. □

7. Categorical action

Recall that $q \in \Bbbk^{\times}$ is either not a root of unity (in which case $e = 0$), or that q^2 is a primitive eth root of unity for some $e > 1$. We are going to show that Mod-OS has the structure of a tensor product categorification in the general sense of Losev and Webster [LW]. This is most interesting when $t \in \{\pm q^n \mid n \in \mathbb{Z}\}$ (so that the $\mathfrak{g}$-module $V(-\Lambda_0|\Lambda_{t^{-2}})$ is reducible), but there is no need to impose this assumption.

The set I from (6.27) will now be used to index the simple roots of a symmetric Kac-Moody algebra $\mathfrak{g}$ (over ground field $\mathbb{C}$), namely, the Kac-Moody algebra with Cartan matrix $(c_{i,j})_{i,j \in I}$ defined by (1.16). The Lie algebra $\mathfrak{g}$ is generated by its Cartan subalgebra $\mathfrak{h}$ and Chevalley generators $\{e_i, f_i \,|\, i \in I\}$ subject to the Serre relations. Let

$$P := \left\{\Lambda \in \mathfrak{h}^* \,\middle|\, \langle h_i, \Lambda \rangle \in \mathbb{Z} \text{ for all } i \in I\right\}.$$

The simple roots are $\{\alpha_i \mid i \in I\}$, and we have that $\langle h_i, \alpha_j \rangle = c_{i,j}$ where $h_i := [e_i, f_i]$. The fundamental dominant weights are $\{\Lambda_i | i \in I\}$. For $i \in I$, let $V(\Lambda_i)$ (resp. $V(-\Lambda_i)$) denote the integrable highest (resp. lowest) weight module of highest weight Λ_i (resp. lowest weight $-\Lambda_i$).

Let $\mathfrak{g}^{\uparrow} = \{x^{\uparrow} | x \in \mathfrak{g}\}$ and $\mathfrak{g}^{\downarrow} = \{x^{\downarrow} | x \in \mathfrak{g}\}$ be two copies of $\mathfrak{g}$ with Cartan subalgebras $\mathfrak{h}^{\uparrow}$ and $\mathfrak{h}^{\downarrow}$, respectively. There is a Lie algebra homomorphism

$$\Delta : \mathfrak{g} \to \mathfrak{g}^{\uparrow} \oplus \mathfrak{g}^{\downarrow}, \qquad x \mapsto x^{\uparrow} + x^{\downarrow}. \tag{7.1}$$

Identifying $U(\mathfrak{g}^{\uparrow} \oplus \mathfrak{g}^{\downarrow})$ with $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, this homomorphism amounts to the usual comultiplication on $U(\mathfrak{g})$. Let $\mathcal{F}$ be the $\mathbb{C}$-vector space with basis $\{v_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathrm{Bip}\}$. The following makes $\mathcal{F}$ into a $\mathfrak{g}^{\uparrow} \oplus \mathfrak{g}^{\downarrow}$-module:

- For $i \in I^{\uparrow}$ we let $e_i^{\uparrow} v_{\boldsymbol{\lambda}}$ (resp. $f_i^{\uparrow} v_{\boldsymbol{\lambda}}$) be the vector $\sum_{\boldsymbol{\mu}} v_{\boldsymbol{\mu}}$ summing over all bipartitions $\boldsymbol{\mu}$ obtained from $\boldsymbol{\lambda}$ by adding (resp. removing) a node of $\uparrow$-content i to (resp. from) $\lambda^{\uparrow}$.

- For $i \in I^{\downarrow}$ we let $e_i^{\downarrow} v_{\boldsymbol{\lambda}}$ (resp. $f_i^{\downarrow} v_{\boldsymbol{\lambda}}$) be the vector $\sum_{\boldsymbol{\mu}} v_{\boldsymbol{\mu}}$ summing over all bipartitions $\boldsymbol{\mu}$ obtained from $\boldsymbol{\lambda}$ by removing (resp. adding) a node of $\downarrow$-content i from (resp. to) $\lambda^{\downarrow}$.
- The actions of the Cartan subalgebras $\mathfrak{h}^{\uparrow}$ and $\mathfrak{h}^{\downarrow}$ are defined so that $v_{\boldsymbol{\lambda}}$ is a weight vector of the following weights for $\mathfrak{h}^{\uparrow}$ or $\mathfrak{h}^{\downarrow}$, respectively:

$$\mathrm{wt}^{\uparrow}(\boldsymbol{\lambda}) := -\Lambda_1 + \sum_{A \in \lambda^{\uparrow}} \alpha_{\mathrm{cont}(A)}, \tag{7.2}$$

$$\mathrm{wt}^{\downarrow}(\boldsymbol{\lambda}) := \Lambda_{t^{-2}} - \sum_{A \in \lambda^{\downarrow}} \alpha_{t^{-2}\,\mathrm{cont}(A)^{-1}}. \tag{7.3}$$

Let $V(-\Lambda_1|\Lambda_{t^{-2}})$ be the $\mathfrak{g}^{\uparrow} \oplus \mathfrak{g}^{\downarrow}$-submodule of $\mathcal{F}$ generated by $v_{\varnothing}$. When $e = 0$, we have that $V(-\Lambda_1|\Lambda_{t^{-2}}) = \mathcal{F}$, but it is a proper submodule otherwise. Since $v_{\varnothing}$ is a lowest weight vector for $\mathfrak{g}^{\uparrow}$ of weight $-\Lambda_1$ and a highest weight vector for $\mathfrak{g}^{\downarrow}$ of weight $\Lambda_{t^{-2}}$, $V(-\Lambda_1|\Lambda_{t^{-2}})$ is isomorphic to the irreducible $\mathfrak{g}^{\uparrow} \oplus \mathfrak{g}^{\downarrow}$-module $V(-\Lambda_1) \boxtimes V(\Lambda_{t^{-2}})$ (with $\mathfrak{g}^{\uparrow}$ acting on the first tensor factor and $\mathfrak{g}^{\downarrow}$ acting on the second).

For $\boldsymbol{\lambda} \in e\text{-Bip}_{r,s}$ and $r, s \geq 0$, we let

$$b_{\boldsymbol{\lambda}}(e,p) := \sum_{\boldsymbol{\mu} \in \mathrm{Bip}_{r,s}} [\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\lambda})] v_{\boldsymbol{\mu}} \in \mathcal{F}, \tag{7.4}$$

so called because it depends on both e and p. The following lemma is a reinterpretation of a well-known result about the representation theory of Hecke algebras. It shows in particular that the vectors $\{b_{\boldsymbol{\lambda}}(e,p) \mid \boldsymbol{\lambda} \in e\text{-Bip}\}$ give a basis for $V(-\Lambda_1|\Lambda_{t^{-2}})$.

Lemma 7.1. *There is a vector space isomorphism*

$$\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{pMod}\text{-}OS^{\circ}) \xrightarrow{\sim} V(-\Lambda_1|\Lambda_{t^{-2}}), \qquad [\mathrm{Y}(\boldsymbol{\lambda})] \mapsto b_{\boldsymbol{\lambda}}(e,p).$$

This map intertwines the endomorphisms of $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{pMod}\text{-}OS^{\circ})$ induced by the endofunctors $E_i^{\uparrow}, E_i^{\downarrow}, F_i^{\uparrow}, F_i^{\downarrow}$ from (6.9)–(6.12) with the actions of the Chevalley generators $e_i^{\uparrow}, e_i^{\downarrow}, f_i^{\uparrow}, f_i^{\downarrow}$ of $\mathfrak{g}^{\uparrow} \oplus \mathfrak{g}^{\downarrow}$ on $V(-\Lambda_1|\lambda_{t^{-2}})$.

Proof. Since the rectangular matrix $([\mathrm{S}(\boldsymbol{\mu}) : \mathrm{D}(\boldsymbol{\lambda})])$ is unitriangular, the elements $b_{\boldsymbol{\lambda}}(e,p)$ for $\boldsymbol{\lambda} \in e\text{-Bip}$ are linearly independent. So the linear map

$$f : \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{pMod}\text{-}OS^{\circ}) \to \mathcal{F}, \qquad [\mathrm{Y}(\boldsymbol{\lambda})] \mapsto b_{\boldsymbol{\lambda}}(e,p)$$

is injective. In the next paragraph, we show that f intertwines $[E_i^{\uparrow}], [E_i^{\downarrow}], [F_i^{\uparrow}], [F_i^{\downarrow}]$ with $e_i^{\uparrow}, e_i^{\downarrow}, f_i^{\uparrow}, f_i^{\downarrow}$, respectively. Actually, we prove an equivalent dual statement.

Let $K_0(\mathrm{fdMod}\text{-}OS^{\circ})$ be the Grothendieck group of the Abelian category fdMod-OS°, which has basis given by the classes $\{[\mathrm{D}(\boldsymbol{\lambda})] \mid \boldsymbol{\lambda} \in e\text{-Bip}\}$. We have the non-degenerate Cartan pairing

$$\langle \cdot, \cdot \rangle : K_0(\mathrm{pMod}\text{-}OS^{\circ}) \times K_0(\mathrm{fdMod}\text{-}OS^{\circ}) \to \mathbb{Z}$$

such that $\langle [\mathrm{Y}(\boldsymbol{\lambda})], [\mathrm{D}(\boldsymbol{\mu})] \rangle = \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in e\text{-Bip}$. Lemma 6.4 implies that the linear maps $[E_i^{\uparrow}]$ and $[E_i^{\downarrow}]$ are biadjoint to $[F_i^{\uparrow}]$ and $[F_i^{\downarrow}]$, respectively. There is also a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{F}$ defined so that $\{v_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathrm{Bip}\}$ is an orthonormal basis. Again, $e_i^{\uparrow}$ and $e_i^{\downarrow}$ are biadjoint to $f_i^{\uparrow}$ and $f_i^{\downarrow}$, respectively, as is clear from the explicit definition of their actions on the basis. Let

$$f^* : \mathcal{F} \to \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{fdMod}\text{-}OS^{\circ})$$

be the dual map to f. It sends $v_{\boldsymbol{\lambda}} \mapsto \sum_{\boldsymbol{\mu} \in e\text{-Bip}} [\mathrm{S}(\boldsymbol{\lambda}) : \mathrm{D}(\boldsymbol{\mu})][\mathrm{D}(\boldsymbol{\mu})]$, i.e., to the isomorphism class $[\mathrm{S}(\boldsymbol{\lambda})]$ of the Specht module. Now it is clear from Lemma 6.5 that f^* intertwines $f_i^{\uparrow}, f_i^{\downarrow}, e_i^{\uparrow}, e_i^{\downarrow}$ with $[F_i^{\uparrow}], [F_i^{\downarrow}], [E_i^{\uparrow}], [E_i^{\downarrow}]$, respectively.

The proof so far shows that $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{pMod}\text{-}OS^{\circ})$ has the structure of an integrable $\mathfrak{g}^{\uparrow} \oplus \mathfrak{g}^{\downarrow}$-module. It remains to show that the image of f is the submodule $V(-\Lambda_1|\Lambda_{t^{-2}})$.

This follows because f sends $[\mathrm{Y}(\varnothing)]$ to the generator $v_{\varnothing}$ of $V(-\Lambda_1|\Lambda_{t^{-2}})$, and $\mathbb{C}\otimes_{\mathbb{Z}} K_0(\text{pMod-}\mathcal{O}S^\circ)$ is actually generated as a $\mathfrak{g}^{\uparrow}\oplus\mathfrak{g}^{\downarrow}$-module by this vector. The latter assertion is a consequence of the analogous statement for the Hecke algebra, which is well known; e.g., see [BD, Corollary 4.34]. □

Remark 7.2. When $p=0$, the basis $\{b_{\boldsymbol{\lambda}}(e,p) \mid \Lambda \in e\text{-Bip}\}$ is the monomial basis consisting of pure tensors in Lusztig's canonical bases for $V(-\Lambda_1)$ and $V(\Lambda_{t^{-2}})$. This follows from [A]. When $p>0$, the decomposition numbers $[\mathrm{S}(\boldsymbol{\lambda}):\mathrm{D}(\boldsymbol{\mu})]$ are not known, so it is hard to compute this basis explicitly.

Using the homomorphism Δ from (7.1), we can instead view $\mathcal{F}$ also as a $\mathfrak{g}$-module. For this action, $v_{\boldsymbol{\lambda}}$ is of weight

$$\mathrm{wt}(\boldsymbol{\lambda}) := \mathrm{wt}^{\uparrow}(\boldsymbol{\lambda}) + \mathrm{wt}^{\downarrow}(\boldsymbol{\lambda}) \tag{7.5}$$

with respect to the Cartan subalgebra $\mathfrak{h}$. Also set

$$\mathbf{wt}(\boldsymbol{\lambda}) := (\mathrm{wt}^{\uparrow}(\boldsymbol{\lambda}), \mathrm{wt}^{\downarrow}(\boldsymbol{\lambda})) \in P\times P. \tag{7.6}$$

The cyclic $\mathfrak{g}^{\uparrow}\oplus\mathfrak{g}^{\downarrow}$-submodule $V(-\Lambda_1|\Lambda_{t^{-2}})$ of $\mathcal{F}$ becomes a $\mathfrak{g}$-submodule isomorphic to the tensor product $V(-\Lambda_1)\otimes V(\Lambda_{t^{-2}})$. A simple induction on weights shows that the vector $v_{\varnothing}$ also generates this module over $\mathfrak{g}$. However, it is not an irreducible $\mathfrak{g}$-module when $t\in\{\pm q^n \mid n\in\mathbb{Z}\}$. For the statement of the next lemma, it may be helpful to recall that $K_0(\text{pMod-}\mathcal{O}S)$ is identified with $K_0(\Delta\text{Mod-}\mathcal{O}S)$ by Corollary 5.12.

Lemma 7.3. *The functors E_i and F_i send modules with Δ-flags to modules with Δ-flags, hence, they induce endomorphisms of $K_0(\Delta\text{Mod-}\mathcal{O}S)$. Moreover, there is a vector space isomorphism*

$$\mathbb{C}\otimes_{\mathbb{Z}} K_0(\Delta\text{Mod-}\mathcal{O}S) \stackrel{\sim}{\to} V(-\Lambda_1|\Lambda_{t^{-2}}),\quad [\Delta(\boldsymbol{\lambda})]\mapsto b_{\boldsymbol{\lambda}}(e,p)$$

which intertwines these endomorphisms with the actions of the Chevalley generators e_i, f_i.

Proof. The given linear isomorphism fits into a commutative diagram

$$\begin{array}{ccc}
\mathbb{C}\otimes_{\mathbb{Z}} K_0(\text{pMod-}\mathcal{O}S^\circ) & \xrightarrow{\sim} & V(-\Lambda_1|\Lambda_{t^{-2}})\\
{\scriptstyle[\Delta]}\big\downarrow & & \big\|\\
\mathbb{C}\otimes_{\mathbb{Z}} K_0(\Delta\text{Mod-}\mathcal{O}S) & \xrightarrow{\sim} & V(-\Lambda_1|\Lambda_{t^{-2}})
\end{array}$$

where the top map is the isomorphism from Lemma 7.1. Lemma 6.8 implies that E_i and F_i preserve Δ-flags. Moreover, it shows that $[E_i]\circ[\Delta] = [\Delta]\circ[E_i^{\uparrow}] + \Delta\circ[E_i^{\downarrow}]$. Since the top map intertwines $[E_i^{\uparrow}], [E_i^{\downarrow}]$ with $e_i^{\uparrow}, e_i^{\downarrow}$, we deduce from (7.1) that the bottom map intertwines $[E_i]$ with e_i, and similarly for $[F_i]$ and f_i. □

Let $\leq$ be the usual dominance order on P: $\rho\leq\sigma$ if $\sigma-\rho$ is a sum of simple roots. Then, we introduce the *inverse dominance order* on $P\times P$ by declaring that

$$(\rho,\sigma)\leq(\rho',\sigma') \Leftrightarrow \rho+\sigma=\rho'+\sigma' \text{ and } \rho\geq\rho' \Leftrightarrow \rho+\sigma=\rho'+\sigma' \text{ and } \sigma\leq\sigma'.$$

Recalling (7.5)–(7.6), the next result is the *linkage principle.*

Theorem 7.4. *For $\boldsymbol{\lambda}\in\mathrm{Bip}$ and $\boldsymbol{\mu}\in e\text{-Bip}$, we have that*

$$[\tilde{\Delta}(\boldsymbol{\lambda}):\mathrm{L}(\boldsymbol{\mu})]\neq 0 \Rightarrow \mathbf{wt}(\boldsymbol{\mu})\leq\mathbf{wt}(\boldsymbol{\lambda}).$$

Proof. Suppose that $\boldsymbol{\mu}\in e\text{-Bip}_{r,s}$ is chosen so that $[\tilde{\Delta}(\boldsymbol{\lambda}):\mathrm{L}(\boldsymbol{\mu})]\neq 0$. By Corollary 6.12, there is a path $\gamma:\varnothing\rightsquigarrow\boldsymbol{\lambda}$ and a minimal length path $\delta:\varnothing\rightsquigarrow\boldsymbol{\mu}$ with $\mathrm{type}(\gamma)=\mathrm{type}(\delta)$. We show that the existence of such a pair of paths implies that $\mathbf{wt}(\boldsymbol{\mu})\leq\mathbf{wt}(\boldsymbol{\lambda})$ by induction on $r+s$. The base case $r+s=0$ is trivial as then $\boldsymbol{\lambda}=\boldsymbol{\mu}=\varnothing$. For the induction step, remove the last edge from each of the paths γ and

δ, to obtain shorter paths $\gamma' : \varnothing \rightsquigarrow \boldsymbol{\lambda}'$ and $\delta' : \varnothing \rightsquigarrow \boldsymbol{\mu}'$. We assume that this last edge is directed in the forward direction, i.e., $\boldsymbol{\lambda}' \xrightarrow{i} \boldsymbol{\lambda}$ and $\boldsymbol{\mu}' \xrightarrow{i} \boldsymbol{\mu}$; the argument is entirely similar if it goes backwards. By induction $\mathbf{wt}(\boldsymbol{\mu}') \leq \mathbf{wt}(\boldsymbol{\lambda}')$, i.e., $\mathrm{wt}(\boldsymbol{\mu}') = \mathrm{wt}(\boldsymbol{\lambda}')$ and $\mathrm{wt}^{\downarrow}(\boldsymbol{\mu}') \leq \mathrm{wt}^{\downarrow}(\boldsymbol{\lambda}')$. The assumption on the last edge means that $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\mu}'$ by adding a node of $\uparrow$-content i to $(\mu')^{\uparrow}$, and similarly for $\boldsymbol{\lambda}$. We deduce that $\mathrm{wt}(\boldsymbol{\mu}) = \mathrm{wt}(\boldsymbol{\mu}')+\alpha_i = \mathrm{wt}(\boldsymbol{\lambda}')+\alpha_i = \mathrm{wt}(\boldsymbol{\lambda})$ and $\mathrm{wt}^{\downarrow}(\boldsymbol{\mu}) = \mathrm{wt}^{\downarrow}(\boldsymbol{\mu}') \leq \mathrm{wt}^{\downarrow}(\boldsymbol{\lambda}') = \mathrm{wt}^{\downarrow}(\boldsymbol{\lambda})$. Hence, $\mathbf{wt}(\boldsymbol{\mu}) \leq \mathbf{wt}(\boldsymbol{\lambda})$. ☐

Corollary 7.5. *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in e\text{-Bip}$ with $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$, we have that*

$$[\bar{\Delta}(\boldsymbol{\lambda}) : \mathrm{L}(\boldsymbol{\mu})] \neq 0 \Rightarrow \mathbf{wt}(\boldsymbol{\mu}) < \mathbf{wt}(\boldsymbol{\lambda}).$$

Proof. By shortest word theory, if $\boldsymbol{\lambda} \in e\text{-Bip}_{r,s}$ we must have that $\boldsymbol{\mu} \in e\text{-Bip}_{r+d,s+d}$ for some $d > 0$. Hence, $\mathbf{wt}(\boldsymbol{\mu}) \neq \mathbf{wt}(\boldsymbol{\lambda})$. Now we are done since $\mathbf{wt}(\boldsymbol{\mu}) \leq \mathbf{wt}(\boldsymbol{\lambda})$ by Theorem 7.4. ☐

Corollary 7.6. *Suppose that $\mathrm{L}(\boldsymbol{\lambda})$ and $\mathrm{L}(\boldsymbol{\mu})$ belong to the same block of Mod-OS for some $\boldsymbol{\lambda}, \boldsymbol{\mu} \in e\text{-Bip}$. Then we have that $\mathrm{wt}(\boldsymbol{\lambda}) = \mathrm{wt}(\boldsymbol{\mu})$.*

Proof. It suffices to show that $\mathrm{Hom}_{OS}(\mathrm{P}(\boldsymbol{\lambda}), \mathrm{P}(\boldsymbol{\mu})) \neq \mathbf{0} \Rightarrow \mathrm{wt}(\boldsymbol{\lambda}) \neq \mathrm{wt}(\boldsymbol{\mu})$. To see this, we apply Corollary 5.13 to see if $[\mathrm{P}(\boldsymbol{\mu}) : \mathrm{L}(\boldsymbol{\lambda})] \neq 0$ that there exists $\boldsymbol{\nu} \in \mathrm{Bip}$ such that $[\tilde{\Delta}(\boldsymbol{\nu}) : \mathrm{L}(\boldsymbol{\lambda})][\tilde{\Delta}(\boldsymbol{\nu}) : \mathrm{L}(\boldsymbol{\mu})] \neq 0$. By Theorem 7.4, this implies that $\mathbf{wt}(\boldsymbol{\lambda}) \leq \mathbf{wt}(\boldsymbol{\nu}) \geq \mathbf{wt}(\boldsymbol{\mu})$. Hence, $\mathrm{wt}(\boldsymbol{\lambda}) = \mathrm{wt}(\boldsymbol{\nu}) = \mathrm{wt}(\boldsymbol{\mu})$. ☐

The two functions $\mathrm{wt} : \mathrm{Bip} \to P$ and $\mathbf{wt} : \mathrm{Bip} \to P \times P$ give partitions

$$\mathrm{Bip} = \coprod_{\omega \in P} \mathrm{Bip}_{\omega} = \coprod_{(\rho,\sigma) \in P \times P} \mathrm{Bip}_{\rho,\sigma} \tag{7.7}$$

where $\mathrm{Bip}_{\omega} := \mathrm{wt}^{-1}(\omega)$ and $\mathrm{Bip}_{\rho,\sigma} := \mathbf{wt}^{-1}((\rho,\sigma))$. Define $e\text{-Bip}_{\omega}$ and $e\text{-Bip}_{\rho,\sigma}$ similarly. There are corresponding block decompositions

$$\text{Mod-}OS = \prod_{\omega \in P} \text{Mod-}OS_{\omega}, \tag{7.8}$$

$$\text{Mod-}OS^{\circ} = \prod_{(\rho,\sigma) \in P \times P} \text{Mod-}OS^{\circ}_{\rho,\sigma} \tag{7.9}$$

defined by letting Mod-OS_{ω} be the Serre subcategory of Mod-OS consisting of all modules M such that $\mathrm{Hom}_{OS}(\mathrm{P}(\boldsymbol{\lambda}), M) \neq 0 \Rightarrow \mathrm{wt}(\boldsymbol{\lambda}) = \omega$, and Mod-$OS^{\circ}_{\rho,\sigma}$ be the Serre subcategory of Mod-OS° consisting of M such that $\mathrm{Hom}_{OS^{\circ}}(\mathrm{Y}(\boldsymbol{\lambda}), M) \neq 0 \Rightarrow \mathbf{wt}(\boldsymbol{\lambda}) = (\rho, \sigma)$. For Mod-$OS$, the existence of this decomposition depends on Corollary 7.6 and the general theory of blocks in locally Schurian categories discussed in [BD, (L9)–(L10)]. For Mod-OS°, this decomposition refines the one arising from the algebra decomposition $OS^{\circ} = \bigoplus_{r,s \geq 0} OS^{\circ}_{r,s}$. In view of Lemma 5.2, it is a reformulation of the usual block decomposition of the Hecke algebras [DJ2, Theorem 4.13].

As well as these block decompositions, we can use the inverse dominance ordering on $P \times P$ to introduce a *stratification* on Mod-OS in the sense of [LW, §2]. This is defined by letting Mod-$OS_{\leq(\rho,\sigma)}$ be the Serre subcategory of Mod-OS consisting of all M such that $\mathrm{Hom}_{OS}(\mathrm{P}(\boldsymbol{\lambda}), M) \neq 0 \Rightarrow \mathbf{wt}(\boldsymbol{\lambda}) \leq (\rho, \sigma)$. Define Mod-$OS_{<(\rho,\sigma)}$ similarly. It is important to note that the set $\coprod_{(\rho',\sigma') \geq (\rho,\sigma)} \mathrm{Bip}_{\rho',\sigma'}$ is *finite*. Indeed, if ρ is obtained from $-\Lambda_1$ by adding r simple roots and σ is obtained from $\Lambda_{t^{-2}}$ by subtracting s simple roots, then it is clear from (8.9)–(8.10) that $\mathrm{Bip}_{\rho,\sigma} \subseteq \mathrm{Bip}_{r,s}$; hence, $\coprod_{(\rho',\sigma') \geq (\rho,\sigma)} \mathrm{Bip}_{\rho',\sigma'} \subseteq \coprod_{d=0}^{\min(r,s)} \mathrm{Bip}_{r-d,s-d}$ which is finite. We say that the stratification is *upper-finite* because of this property.

For $(\rho, \sigma) \in P \times P$, let

$$\pi_{\rho,\sigma} : \text{Mod-}OS_{\leq(\rho,\sigma)} \to \text{Mod-}OS^{\circ}_{\rho,\sigma} \tag{7.10}$$

be the exact functor defined first by restriction to OS° then projection onto the block parametrized by (ρ,σ). Composing the inclusion of this block into Mod-OS° with either Δ or ∇ defines exact functors

$$\Delta_{\rho,\sigma} : \text{Mod-}OS^\circ_{\rho,\sigma} \to \text{Mod-}OS_{\leq(\rho,\sigma)}, \tag{7.11}$$
$$\nabla_{\rho,\sigma} : \text{Mod-}OS^\circ_{\rho,\sigma} \to \text{Mod-}OS_{\leq(\rho,\sigma)}. \tag{7.12}$$

These are left and right adjoint to $\pi_{\rho,\sigma}$, respectively.

Lemma 7.7. *For $(\rho,\sigma) \in P \times P$, the functor $\pi_{\rho,\sigma}$ annihilates all irreducible modules $\mathrm{L}(\boldsymbol{\lambda})$ with $\mathbf{wt}(\boldsymbol{\lambda}) < (\rho,\sigma)$. Hence, it induces an exact functor*

$$\bar{\pi}_{\rho,\sigma} : \text{Mod-}OS_{\leq(\rho,\sigma)} / \text{Mod-}OS_{<(\rho,\sigma)} \to \text{Mod-}OS^\circ_{\rho,\sigma}.$$

In fact, this induced functor is an equivalence of categories.

Proof. If $\mathrm{Bip}_{\rho,\sigma} \subseteq \mathrm{Bip}_{r,s}$ then $\mathrm{Bip}_{<(\rho,\sigma)} \subseteq \coprod_{d>0} \mathrm{Bip}_{r+d,s+d}$. Hence, for $\boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{\rho,\sigma}$, the restriction of $\mathrm{L}(\boldsymbol{\lambda})$ to OS° belongs to $\prod_{d>0} \text{Mod-}OS^\circ_{r+d,s+d}$ and its projection to $\text{Mod-}OS^\circ_{\rho,\sigma} \subseteq \text{Mod-}OS^\circ_{r,s}$ is certainly zero. Since $\pi_{\rho,\sigma}$ is also exact, we get the induced functor $\bar{\pi}_{\rho,\sigma}$ by the universal property of Serre quotients.

The irreducible objects in the Serre quotient category $\text{Mod-}OS_{\leq(\rho,\sigma)} / \text{Mod-}OS_{<(\rho,\sigma)}$ are represented by $\{\mathrm{L}(\boldsymbol{\lambda}) \,|\, \boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{\rho,\sigma}\}$. For $\boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{\rho,\sigma}$, the projective cover of $\mathrm{L}(\boldsymbol{\lambda})$ in $\text{Mod-}OS_{\leq(\rho,\sigma)}$ is the largest quotient of $\mathrm{P}(\boldsymbol{\lambda})$ which belongs to this subcategory. In view of Lemma 5.11 and Corollary 7.5, this largest quotient is $\Delta(\boldsymbol{\lambda})$. We deduce that the objects $\{\Delta(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}_{\rho,\sigma}\}$ give a complete set of pairwise inequivalent indecomposable projective objects in $\text{Mod-}OS_{\leq(\rho,\sigma)} / \text{Mod-}OS_{<(\rho,\sigma)}$.

By shortest word theory and considerations like in the first paragraph of the proof, the exact functor $\bar{\pi}_{\rho,\sigma}$ sends $\Delta(\boldsymbol{\lambda})$ to $\mathrm{Y}(\boldsymbol{\lambda})$. So it induces a bijection between isomorphism classes of indecomposable projective objects in its source and target categories. It follows that it is an equivalence. □

All of this puts us in the setup of [LW, Definition 2.1], except that our algebra OS is locally finite-dimensional rather than finite-dimensional, and our ordering is upper-finite rather than finite. The formal definition of standardly stratified category from *loc. cit.* is generalized to include this slightly more general situation in [EL, §6.2.1].

Theorem 7.8. *The category* Mod-OS *with its irreducible objects* $\{\mathrm{L}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in e\text{-}\mathrm{Bip}\}$ *and the stratification defined by the function* $\mathbf{wt} : e\text{-}\mathrm{Bip} \to P \times P$ *and the inverse dominance ordering $\leq$ is an upper-finite standardly stratified category with associated graded category* Mod-OS°*. In case $e = 0$, it is an upper-finite highest weight category.*

Proof. We have already discussed the stratification and shown that it is upper-finite. Lemma 7.7 identifies the associated graded category with Mod-OS°. Also we know already that the standardization functor $\Delta_{\rho,\sigma}$ is exact. It just remains to show that $\mathrm{P}(\boldsymbol{\lambda})$ has a finite filtration with $\Delta(\boldsymbol{\lambda})$ at the top and other sections of the form $\Delta(\boldsymbol{\mu})$ for $\boldsymbol{\mu}$ with $\mathbf{wt}(\boldsymbol{\mu}) > \mathbf{wt}(\boldsymbol{\lambda})$. This follows from Lemma 5.11 and Corollary 7.5. It is highest weight rather than standardly stratified in case $e = 0$ since then each non-zero stratum $\text{Mod-}OS^\circ_{\rho,\sigma}$ is semisimple with just one irreducible object (up to isomorphism). □

We refer the reader to [BD] for the necessary background on 2-representations of 2-Kac-Moody categories used freely in the proofs of the next two theorems. Although these notions are essentially due to Rouquier [Ro], we are applying them in a locally Schurian setting not originally considered there. In particular, the proof of the following theorem depends crucially on the (very slight) extension of Rouquier's "control by K_0" developed in [BD, Theorem 4.27].

Proof of Theorem 1.8. See [LW, Remark 3.6] for the notion of tensor product categorification. In the context of Theorem 1.8 it means the following data:

(1) A locally Schurian category $\mathcal{C}$ with isomorphism classes of irreducible objects labelled by e-Bip, i.e., the indexing set for the basis of $V(-\Lambda_1|\Lambda_{t^{-2}})$ from Remark 7.2.

(2) A nilpotent categorical action making $\mathcal{C}$ into a 2-representation of the associated Kac-Moody 2-category $\mathfrak{U}(\mathfrak{g})$.

Then we need to verify the following axioms:

(3) The category $\mathcal{C}$ is standardly stratified with respect to the function $\mathbf{wt}$: e-Bip $\to P \times P$ and the inverse dominance ordering $\leq$ on $P \times P$.

(4) For $(\rho, \sigma) \in P \times P$, the Serre quotient $\mathcal{C}_{(\rho,\sigma)} := \mathcal{C}_{\leq(\rho,\sigma)}/\mathcal{C}_{<(\rho,\sigma)}$ is equivalent to the category of modules over the (ρ, σ)-weight subcategory of the minimal categorification of the irreducible $\mathfrak{g}^{\uparrow} \oplus \mathfrak{g}^{\downarrow}$-module $V(-\Lambda_1|\Lambda_{t^{-2}})$.

(5) There is compatibility between the categorical $\mathfrak{g}$-action on $\mathcal{C}$ and the categorical $\mathfrak{g}^{\uparrow} \oplus \mathfrak{g}^{\downarrow}$-action on the associated graded category in the sense that there are short exact sequences as in (6.28)–(6.29).

We must show that $\mathcal{C} :=$ Mod-OS admits this structure. It is locally Schurian and we have parametrized the irreducibles by e-Bip above, so (1) holds. The main work still needed is to verify (2) and (4); this is done in the next two paragraphs. Then axiom (3) is Theorem 7.8, while (5) follows immediately from Lemma 6.7.

To verify (2), we use [BD, Theorem 4.27] to reduce to checking the conditions of [BD, Definition 4.25]. We need the following data:

(6) A weight decomposition of the category Mod-OS.

(7) Biadjoint endofunctors $E = \bigoplus_{i \in I} E_i$ and $F = \bigoplus_{i \in I} F_i$.

(8) Natural transformations $\underset{i}{\overset{\uparrow}{\bullet}}$: $E_i \to E_i$ and $\underset{i\quad j}{\times}$: $E_i \circ E_j \to E_j \circ E_i$ for each $i, j \in I$ inducing an action of the quiver Hecke algebra QH_r associated to $\mathfrak{g}$ on powers of E.

Then there are two additional axioms to check:

(9) The endomorphisms $[E_i]$ and $[F_i]$ make $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\text{pMod-}OS)$ into a well-defined $\mathfrak{g}$-module with ω-weight space $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\text{pMod-}OS_\omega)$.

(10) For each $i \in I$ and each finitely generated OS-module M, the endomorphism $\left(\underset{i}{\overset{\uparrow}{\bullet}}\right)_M : E_i M \to E_i M$ is nilpotent.

The weight decomposition (6) comes from (7.8). We have already constructed the functors needed for (7) in Lemma 6.8. For (8), we instead construct natural transformations $\overset{\uparrow}{\bullet} : E \to E$ and $\times : E^2 \to E^2$ inducing an action of the affine Hecke algebra AH_r on powers of E. This is good enough due to the existence of an isomorphism[5] $\widehat{AH_r} \cong \widehat{QH_r}$ between completions constructed in [BK, Ro, W1]. Recalling the definition (6.18), we define $\overset{\uparrow}{\bullet}$ by setting $\left(\overset{\uparrow}{\bullet}\right)_M := \mathrm{id} \otimes_{\uparrow} X : M \otimes_{OS} {}_{\uparrow}OS \to M \otimes_{OS} {}_{\uparrow}OS$. To define $\times$, we may identify ${}_{\uparrow}OS \otimes_{OS} {}_{\uparrow}OS$ with ${}_{\uparrow\uparrow}OS$ in the natural notation, then let $\left(\times\right)_M : M \otimes_{OS} {}_{\uparrow\uparrow}OS \to M \otimes_{OS} {}_{\uparrow\uparrow}OS$ be defined on $M1_{\mathsf{a}} \otimes {}_{\uparrow\uparrow}OS$ by left

[5]There are various versions of this isomorphism in the literature. We will not make a specific choice here since any one of them suffices for our purposes.

multiplication by $\mathrm{id}\otimes \nwarrow\!\!\!\!\diagup \;\|_{a}$. Axiom (9) follows from Lemma 7.3. For (10), note that

$$\left(\ \underset{i}{\overset{\bullet}{\uparrow}}\ \right)_M = \left(\left(\ \overset{\bullet}{\uparrow}\ \right)_M - i\,\mathrm{id}\right)\Big|_{E_iM}$$

according to the isomorphism $\widehat{QH}_r \cong \widehat{AH}_r$. It therefore suffices to show that there is a bound on the Jordan block sizes of $\mathrm{id}\otimes_{\uparrow} X : M \otimes_{OS}{}_{\uparrow}OS \to M \otimes_{OS}{}_{\uparrow}OS$ for any finitely-generated OS-module M. This follows by the local finite-dimensionality discussed in the proof of Lemma 6.8.

Finally, we need to verify (4). The categorical action of $\mathfrak{g}^{\uparrow} \oplus \mathfrak{g}^{\downarrow}$ on Mod-OS° is constructed in a similar way to the previous paragraph. The required endofuntors come from (6.9)–(6.12), the block decomposition is (7.9), and we get "control by K_0" from Lemma 7.1. In fact, due to Lemmas 5.2 and 6.2, this is just a reformulation of the familiar categorical action on modules over Hecke algebras constructed originally in [CR, §7.2]. It is a minimal categorification since $OS^{\circ}_{0,0} = \Bbbk$. □

Proof of Theorem 1.9. Theorem 1.8 implies that pMod-OS is a 2-representation of $\mathfrak{U}(\mathfrak{g})$. Thus, letting $\mathfrak{Cat}_{\Bbbk}$ be the 2-category of $\Bbbk$-linear categories, there is a strict $\Bbbk$-linear 2-functor $\mathfrak{U}(\mathfrak{g}) \to \mathfrak{Cat}_{\Bbbk}$ sending $\Lambda \in P$ (i.e., an object of $\mathfrak{U}(\mathfrak{g})$) to the block pMod-$OS_{\Lambda}$, a 1-morphism $\underline{X} : \Lambda \to \omega$ to a functor X : pMod-$OS_{\Lambda} \to$ pMod-OS_{ω}, and a 2-morphism $\underline{\eta} : \underline{X} \to \underline{Y}$ to a natural transformation $\eta : X \to Y$. Noting that $\mathrm{wt}(\boldsymbol{\varnothing}) = \Lambda_{t^{-2}} - \Lambda_1$, the universal property of $\mathcal{R}(\Lambda_{t^{-2}} - \Lambda_1)$ produces a strongly equivariant functor

$$\Theta : \mathcal{R}(\Lambda_{t^{-2}} - \Lambda_1) \to \text{pMod-}OS$$

sending an object $\underline{X} \in \mathcal{R}(\Lambda_{t^{-2}} - \Lambda_1)$ (i.e., a 1-morphism $\underline{X} : \Lambda_{t^{-2}} - \Lambda_1 \to \omega$ in $\mathfrak{U}(\mathfrak{g})$ for some $\omega \in P$) to the projective OS-module $X\Delta(\boldsymbol{\varnothing})$, and a morphism $\underline{\eta} : \underline{X} \to \underline{Y}$ (i.e., a 2-morphism in $\mathfrak{U}(\mathfrak{g})$) to the OS-module homomorphism $\eta_{\Delta(\boldsymbol{\varnothing})} : X\Delta(\boldsymbol{\varnothing}) \to Y\Delta(\boldsymbol{\varnothing})$. Because the unit object of $\mathcal{OS}(z,t)$ corresponds to the projective module $\Delta(\boldsymbol{\varnothing})$ in pMod-OS, this is the essentially the same as the functor appearing in the theorem we are trying to prove.

In this paragraph, we check that Θ sends the 2-morphisms (1.17) to zero. For the first one, Lemmas 6.5 and 6.8 imply that $E_i\Delta(\boldsymbol{\varnothing})$ is zero (so we get done trivially) unless $i = 1$, and also $E_1\Delta(\boldsymbol{\varnothing}) \cong \Delta(((1),\varnothing))$. The relation follows in the non-trivial case $i = 1$ because $\left(\ \underset{1}{\overset{\bullet}{\uparrow}}\ \right)_{\Delta(\boldsymbol{\varnothing})}$ is a nilpotent element of $\mathrm{End}_{OS}(\Delta(((1),\varnothing))) \cong \Bbbk$. The second relation follows similarly. For the final relation, we may assume that $t = \pm 1$, and need to show that $\left(\ \underset{1}{\circlearrowright}\ \right)\Big|_{\Delta(\boldsymbol{\varnothing})} : \Delta(\boldsymbol{\varnothing}) \to \Delta(\boldsymbol{\varnothing})$ is zero. This endomorphism is the composition of two morphisms

$$\Delta(\boldsymbol{\varnothing}) \xrightarrow{f} E_1F_1\Delta(\boldsymbol{\varnothing}) \xrightarrow{g} \Delta(\boldsymbol{\varnothing})$$

(the cup and the cap). By Lemmas 6.5 and 6.8, the projective module $E_1F_1\Delta(\boldsymbol{\varnothing})$ has a two step Δ-flag with $\Delta(\boldsymbol{\varnothing})$ at the bottom and $\Delta(((1),(1)))$ at the top. By Example 5.4 with $n = 0$, we know that $[\bar{\Delta}(\boldsymbol{\varnothing}) : \mathrm{L}(((1),(1)))] \neq 0$, so deduce by BGG reciprocity that $E_1F_1\Delta(\boldsymbol{\varnothing}) = \mathrm{P}(((1),(1)))$, i.e., it is indecomposable. So the first morphism f must be a scalar multiple of an inclusion of $\Delta(\boldsymbol{\varnothing})$ into $E_1F_1\Delta(\boldsymbol{\varnothing})$, and the second morphism must contain $\Delta(\boldsymbol{\varnothing})$ in its kernel. Hence, $g \circ f = 0$ as required.

It follows that the functor Θ factors through the quotient to induce a $\Bbbk$-linear functor

$$\bar{\Theta} : \dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t^{-2}}) \to \text{pMod-}OS.$$

To show that this is an equivalence, we will show in the next two paragraphs that $\bar\Theta$ induces an isomorphism

$$\bar\Theta : \mathrm{Hom}_{\dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t-2})}(\underline{X}, \underline{Y}) \xrightarrow{\sim} \mathrm{Hom}_{OS}(X\Delta(\varnothing), Y\Delta(\varnothing)) \tag{7.13}$$

for any $\omega \in P$ and $\underline{X}, \underline{Y} : \Lambda_{t-2} - \Lambda_1 \to \omega$ obtained as compositions[6] of the generating morphisms $\underline{E} = \bigoplus_{i\in I} \underline{E}_i$ and $\underline{F} = \bigoplus_{i\in I} \underline{F}_i$ in $\dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t-2})$. Let us see how the theorem follows from this. Recall that $V(-\Lambda_1|\Lambda_{t-2})$ is generated as a $\mathfrak{g}$-module by the vector $v_\varnothing$. So, using Lemma 7.3 plus the natural positivity of the actions of $[E]$ and $[F]$ on the basis coming from indecomposable projectives, any P in pMod-OS isomorphic to a summand of $X\Delta(\varnothing)$ for some composition X of E's and F's. Let $e \in \mathrm{End}_{OS}(X\Delta(\varnothing))$ be the projection onto this summand. The inverse image of e under (7.13) gives an idempotent in $\mathrm{End}_{\dot{\mathcal{L}}(-\Lambda_1|\Lambda_{t-2})}(\underline{X})$. This defines an object of $\dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t-2})$ whose image under $\bar\Theta$ is isomorphic to P. This shows that $\bar\Theta$ is dense. It is full and faithful by (7.13).

So now we must prove (7.13). Suppose that x (resp. x') letters of $\underline{X}$ and y (resp. y') letters of $\underline{Y}$ are equal to $\underline{F}$ (resp. $\underline{E}$). We may assume further that $r := x'+y = x+y'$, since otherwise both sides of (7.13) are zero by weight considerations. We observe for each $\Lambda \in P$ that there is an isomorphism $\rho : \underline{E}\,\underline{F}1_\Lambda \cong \underline{F}\,\underline{E}1_\Lambda$ in $\dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t-2})$. To prove this, for all $i, j \in I$, the relations in $\mathfrak{U}(\mathfrak{g})$ give canonical isomorphisms $\underline{E}_i\underline{F}_j 1_\Lambda \oplus 1_\Lambda^{\oplus m_{i,j}} \cong \underline{F}_j\underline{E}_i 1_\Lambda \oplus 1_\Lambda^{\oplus n_{i,j}}$ for $m_{i,j}, n_{i,j} \in \mathbb{N}$, one of which is zero. Summing these isomorphisms over all $i, j \in I$ gives a canonical isomorphism $\underline{E}\,\underline{F}1_\Lambda \oplus 1_\Lambda^{\oplus m} \cong \underline{F}\,\underline{E} \oplus 1_\Lambda^{\oplus n}$ for some $m, n \in \mathbb{N}$. In fact, by weight considerations, we have that $m = n$. Then we use Krull-Schmidt, which holds because $\dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t-2})$ is a finite-dimensional category thanks to [BD, Corollary 4.17], to deduce that the existence of the desired (non-canonical) isomorphism $\rho : \underline{E}\,\underline{F}1_\Lambda \xrightarrow{\sim} \underline{F}\,\underline{E}1_\Lambda$. Then, using these isomorphisms plus isomorphisms coming from the adjunction 2-morphisms in $\mathfrak{U}(\mathfrak{g})$, we can construct a vector space isomorphism

$$\theta : \mathrm{Hom}_{\dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t-2})}(\underline{X}, \underline{Y}) \xrightarrow{\sim} \mathrm{Hom}_{\dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t-2})}(\underline{E}^r, \underline{E}^r)$$

in just the same way as was done in (3.12). Applying $\bar\Theta$, we get also an isomorphism ϕ making the left hand square of the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t-2})}(\underline{X}, \underline{Y}) \arrow[r, "\sim", "\theta"'] \arrow[d, "\bar\Theta"'] & \mathrm{Hom}_{\dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t-2})}(\underline{E}^r, \underline{E}^r) \arrow[d, "\bar\Theta"] & QH_r \arrow[l, "\jmath_r"'] \arrow[d, "\psi"] \\
\mathrm{Hom}_{OS}(X\Delta(\varnothing), X\Delta(\varnothing)) \arrow[r, "\sim", "\phi"'] & \mathrm{Hom}_{OS}(E^r\Delta(\varnothing), E^r\Delta(\varnothing)) & H_r. \arrow[l, "\sim"', "\imath_r"]
\end{tikzcd} \tag{7.14}$$

Using this square, we are reduced to showing that the middle vertical map is an isomorphism.

To complete the argument, we already have the isomorphism $\imath_r$ in this diagram; it comes from (1.3). Let $\jmath_r$ be the canonical homomorphism coming from the categorical action (item (8) in the proof of Theorem 1.8), then define ψ so that the right hand square commutes. We claim that $\jmath_r$ is surjective. To see this, [KL, Proposition 3.11] shows that $\mathrm{Hom}_{\dot{\mathcal{V}}(-\Lambda_1|\Lambda_{t-2})}(\underline{E}^r, \underline{E}^r)$ is generated as a right $\mathrm{End}_{\dot{\mathcal{L}}(-\Lambda_1|\Lambda_{t-2})}(1_{\Lambda_{t-2}-\Lambda_1})$-module by the image of $\jmath_r$. But $\mathrm{End}_{\dot{\mathcal{L}}(-\Lambda_1|\Lambda_{t-2})}(1_{\Lambda_{t-2}-\Lambda_1})$ is just the field $\Bbbk$ since there are enough relations in (1.17) to see that any dotted bubble is a scalar. Moreover, $\ker \jmath_r$ contains the ideal J_r of QH_r generated by $\left\{x_1^{\delta_{i_1,1}} 1_{\boldsymbol{i}} \,\middle|\, \boldsymbol{i} = (i_1, \ldots, i_r) \in I^r\right\}$ by the first relation from (1.17), so ψ induces $\bar\psi : QH_r/J_r \to H_r$. Since J_r is the cyclotomic ideal

[6]The infinite sums when $e = 0$ make sense as $\underline{E}_i 1_\Lambda$ and $\underline{F}_i 1_\Lambda$ are zero for all but finitely many $i \in I$.

of QH_r associated to the dominant weight Λ_1, we get that $\bar{\psi}$ is an isomorphism by the main result of [BK]. It follows that $\bar{\Theta}$ is an isomorphism too. $\square$

8. Modifications in the degenerate case

Assume in this section that $\Bbbk$ is a field of characteristic $p \geq 0$. As we have said already in the introduction, when $z = 0$, the category $\mathcal{OS}(z,t)$ should be replaced with the oriented Brauer category $\mathcal{OB}(\delta)$ studied in [BCNR].

Proof of Theorem 1.10. This follows by the same general argument as used to prove Theorem 1.3 (also Remark 3.4). Instead of the quantized Schur-Weyl duality used before, one uses classical Schur-Weyl duality in its "characteristic free" form established in [CP, Theorems 4.1–4.2]. $\square$

Now we discuss the degenerate analog of the results in sections 5, 6 and 7.

For section 5, we work with the locally finite-dimensional locally unital algebra

$$OB = \bigoplus_{\mathsf{a},\mathsf{b} \in \langle \uparrow,\downarrow \rangle} 1_{\mathsf{a}} OB 1_{\mathsf{b}} \qquad \text{where} \qquad 1_{\mathsf{a}} OB 1_{\mathsf{b}} = \mathrm{Hom}_{\mathcal{OB}(\delta)}(\mathsf{b},\mathsf{a}).$$

It has a triangular decomposition

$$OB \cong OB^+ \otimes_{\mathbb{K}} OB^0 \otimes_{\mathbb{K}} OB^-$$

like in Lemma 5.1. This actually becomes easier since there is no longer any need to be careful about upward strands passing underneath downward strands when defining OB^0. The subsequent arguments in section 5 then go through easily on replacing the Hecke algebra H_r with the group algebra $\Bbbk\mathfrak{S}_r$ of the symmetric group and e with p.

The results in section 6 go through too, but this needs a little more work since the definitions of the various Jucys-Murphy elements from (4.9), (6.2) and (6.13)–(6.14) need some modifications, and the details in the proofs of Lemmas 6.4 and 6.7 then need to be rechecked carefully. The affine Hecke algebra AH_r becomes the *degenerate affine Hecke algebra* dAH_r whose polynomial generators $x_1, \dots, x_r$ satisfy the relations

$$x_i x_j = x_j x_i, \qquad\qquad s_i x_{i+1} = x_i s_i + 1 \tag{8.1}$$

in place of (4.1). The unique homomorphism $dAH_r \twoheadrightarrow \Bbbk\mathfrak{S}_r$ sending $s_i \mapsto s_i$ and $x_1 \mapsto 0$ sends x_r to the *Jucys-Murphy element*

$$l_r := \sum_{i=1}^{r-1} (i\ \ r) \in \Bbbk\mathfrak{S}_r. \tag{8.2}$$

These elements are the replacements for (6.2). Then the contents of nodes of an ordinary Young diagram (which should always be interpreted as elements of the field $\Bbbk$) are as in the following example

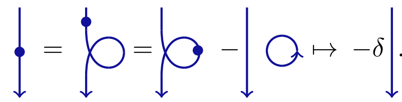
$$\begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 \\ \hline -1 & 0 & 1 & \multicolumn{2}{c}{} \\ \cline{1-3} -2 & -1 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}\ .$$

In place of (6.3), we set

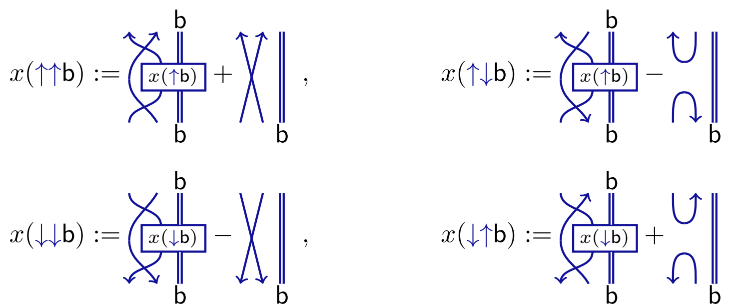
$$I_c := \{c + n \mid n \in \mathbb{Z}\} \subseteq \Bbbk \tag{8.3}$$

for $c \in \Bbbk$. The appropriate analog of Lemma 6.1 uses $I_0 \subseteq \Bbbk$ in place of $I_1 \subseteq \Bbbk^\times$. It is a classical result in the (modular) representation theory of the symmetric group.

The Jucys-Murphy elements of $\mathcal{OB}(\delta)$ are the images of corresponding elements of the *affine oriented Brauer category* $\mathcal{AOB}(\delta)$ introduced in [BCNR]. This strict $\Bbbk$-linear monoidal category is defined by by adjoining an additional generating morphism to $\mathcal{OB}(\delta)$, subject to the relation (dA) from Figure 1. The analog of Lemma 4.2 is explained in [BCNR, Theorem 3.3]: there is a $\Bbbk$-linear functor $\beta : \mathcal{AOB}(\delta) \to \mathcal{OB}(\delta)$ sending diagrams with no dots to the same diagrams in $\mathcal{OB}(\delta)$, and sending $\mapsto 0$. The following computes the image of (which is defined so that (4.3)–(4.4) hold):



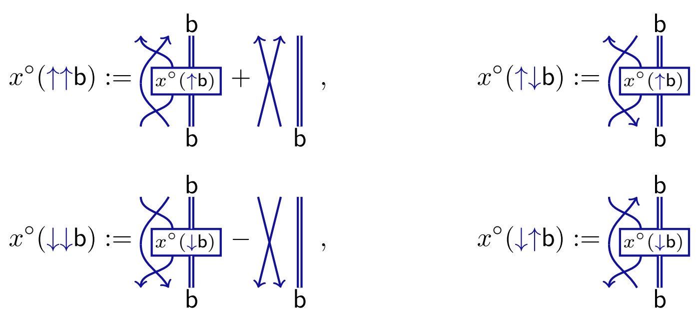
Then we define $x(\mathsf{b}) \in \mathrm{Hom}_{\mathcal{OB}(\delta)}(\mathsf{b}, \mathsf{b})$ in the same way as (4.10) for any $\varnothing \neq \mathsf{b} \in \langle\uparrow, \downarrow\rangle$. There is no longer such a nice diagrammatic interpretation of these elements like (4.9), but there is a recursive definition as in (4.11)–(4.12): we have that $x(\uparrow) = 0, x(\downarrow) = -\delta 1_\downarrow$, and



(8.4)

(8.5)

for any word b. Finally, the Jucys-Murphy elements $x^\circ(\mathsf{b})$ of $\mathcal{OB}^\circ(\delta)$, i.e., the subcategory consisting of all objects but only morphisms represented by diagrams with no cups or caps, are defined from $x^\circ(\uparrow) := 0, x^\circ(\downarrow) := -\delta 1_\downarrow$, and



(8.6)

(8.7)

We leave it to the reader to verify with these new definitions that Lemmas 6.4 and 6.7 go through; see also [Re]. In the statement of Lemma 6.4, one should replace $t^{-2}i^{-1}$ with $-i - \delta$, I_1 with I_0, and $I_{t^{-2}}$ with $I_{-\delta}$. Also the set I from (6.27) becomes

$$I := I_0 \cup I_{-\delta} = \{n, -n - \delta \mid n \in \mathbb{Z}\} \subseteq \Bbbk. \tag{8.8}$$

Adjusting the subsequent combinatorics in analogous ways, all of the other results of section 6 follow as before.

Moving on to section 7, the Lie algebra $\mathfrak{g}$ is the Kac-Moody algebra associated to the Cartan matrix $(c_{i,j})_{i,j\in I}$ defined by (1.22). The module $V(-\Lambda_1|\Lambda_{t^{-2}})$ becomes $V(-\Lambda_0|\Lambda_{-\delta})$, and the degenerate analogs of (8.9)–(8.10) are

$$\mathrm{wt}^\uparrow(\boldsymbol{\lambda}) := -\Lambda_0 + \sum_{A \in \lambda^\uparrow} \alpha_{\mathrm{cont}(A)}, \tag{8.9}$$

$$\mathrm{wt}^\downarrow(\boldsymbol{\lambda}) := \Lambda_{-\delta} - \sum_{A \in \lambda^\downarrow} \alpha_{-\mathrm{cont}(A)-\delta}, . \tag{8.10}$$

There are no other significant discrepancies.

Proof of Theorem 1.11. This is the same as the proof of Theorem 1.9 given in the previous section. □

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