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An orthogonal form for level two Hecke algebras with applications

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Abstract. This is a survey of some recent results relating Khovanov's arc algebra to category $\mathcal{O}$ for Grassmannians, the general linear supergroup, and the walled Brauer algebra. The exposition emphasizes an extension of Young's orthogonal form for level two cyclotomic Hecke algebras.

1. Introduction

This article is primarily intended as a survey of some of my recent joint results with Catharina Stroppel from [**BS1**]–[**BS5**]. In that work, we exploited the isomorphism constructed in [**BK3**] between level two cyclotomic quotients of certain affine Hecke algebras and quiver Hecke algebras to establish some remarkable connections between Khovanov's arc algebra from [**K**], the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for Grassmannians, the general linear supergroup $GL(m|n)$, and the walled Brauer algebra $B_{r,s}(\delta)$. Our results can be viewed as an application of some of the ideas emerging from the development of higher representation theory by Khovanov and Lauda [**KL**] and Rouquier [**R2**]; see the recent survey [**S2**] which adopts this point of view. We are going to approach the subject instead from a more classical direction, focussing primarily on a certain extension of Young's orthogonal form for level two Hecke algebras. This orthogonal form is really the technical heart of the paper [**BS3**] but it is quite well hidden. We will also explain its q-analogue not mentioned at all there.

The level two Hecke algebras studied here include as a special case the usual finite Iwahori-Hecke algebras of type B when the long root parameter q is generic and the short root parameter Q is chosen so that the algebra is *not* semisimple (so $Q = -q^r$ for some r). The orthogonal form allows

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many of usual problems of representation theory to be solved for these algebras in an unusually explicit way. For example it yields constructions of all the irreducible modules, hence we can compute their dimensions, and all the projective indecomposable modules, hence we can identify the endomorphism algebra of a minimal projective generator. It is a tantalizing problem to try to find something like this for cyclotomic Hecke algebras of higher levels, or at roots of unity, but at the moment this seems out of reach.

Once we have explained the orthogonal form, we discuss the main applications obtained in [**BS3**]–[**BS5**]. These applications rely also on three generalizations of Schur-Weyl duality. The first of these generalizations was developed in detail already in [**BK1**], [**BK2**], and is exploited in [**BS3**] to relate the level two Hecke algebras to category $\mathcal{O}$ for Grassmannians. The second generalized Schur-Weyl duality appears for the first time in [**BS4**], and relates the same Hecke algebras to finite dimensional representations of the complex general linear supergroup. Finally in [**BS5**] we use the Schur-Weyl duality between $GL(m|n)$ and the walled Brauer algebra arising from "mixed" tensor space to prove a conjecture suggested by Cox and De Visscher [**CD**].

In the remainder of the article, in an attempt to improve readability, we have postponed precise references to notes at the end of each section. We fix once and for all a ground field F and a parameter $\xi \in F^\times$ such that *either* ξ is not a root of unity in F, *or* $\xi = 1$ and F is of characteristic zero. For the applications beginning in section 5, we always take $F = \mathbb{C}$ and $\xi = 1$.

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2. Level one Hecke algebras

We begin by recalling briefly Young's classical orthogonal form for the symmetric group and its (not quite so classical) analogue for the corresponding Iwahori-Hecke algebra. Let H_d denote the finite Iwahori-Hecke algebra associated to the symmetric group S_d over the field F at defining parameter ξ. Thus H_d is a finite dimensional algebra of dimension $d!$, with generators $T_1, \dots, T_{d-1}$ subject to the usual braid relations plus the quadratic relations

$$(T_r + 1)(T_r - \xi) = 0 \tag{2.1}$$

for each $r = 1, \dots, d-1$. In the degenerate case $\xi = 1$ we can simply identify H_d with the group algebra FS_d of the symmetric group, so that T_r is identified with the basic transposition $s_r := (r\ \ r{+}1)$. The assumptions on F and ξ mean that H_d is a semisimple algebra. Up to isomorphism, the irreducible H_d-modules are the *Specht modules* $\{S(\lambda) \mid \lambda \vdash d\}$ parametrized by partitions of d.

Given a partition $\lambda \vdash d$, we draw the Young diagram of λ in the usual English way. A λ-*tableau* means a filling of the boxes of this Young diagram with the entries $1, \dots, d$ (each appearing exactly once). The symmetric group S_d acts on such tableaux via its natural action on the entries. Let $\mathscr{T}(\lambda)$ denote the set of all *standard* λ-tableaux, that is, the ones whose entries are strictly increasing both along rows from left to right and down columns from top to bottom. The *residue sequence* $\boldsymbol{i}^{\mathtt{T}} \in \mathbb{Z}^d$ of $\mathtt{T} \in \mathscr{T}(\lambda)$ is the sequence $(i_1, \dots, i_d)$ where i_r is the *residue* of the box of $\mathtt{T}$ containing entry r, that is, the integer $(c - b)$ if this box is in row b and column c. Of course residues are constant along diagonals; they are the numbers labelling the boundary of $\mathtt{T}$ in the following example:

$$\mathtt{T} = \begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{0} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} \\ \cline{2-5} -1 & 1 & 3 & 4 & 9 \\ \cline{2-5} -2 & 2 & 5 & 7 & \multicolumn{1}{c}{} \\ \cline{2-4} -3 & 6 & 8 & 10 & \multicolumn{1}{c}{} \\ \cline{2-4} \end{array} \qquad \leftrightarrow \qquad \boldsymbol{i}^{\mathtt{T}} = (0, -1, 1, 2, 0, -2, 1, -1, 3, 0)$$

Notice the residue sequence of a standard tableau always has the property that $|i_r - i_{r+1}| \geq 1$ for every r, and $|i_r - i_{r+1}| > 1$ if and only if the tableau $s_r \cdot \mathtt{T}$ is again standard. Moreover the original standard tableau $\mathtt{T}$ can be recovered uniquely from the sequence $\boldsymbol{i}^{\mathtt{T}}$, because for any Young diagram there is at most one "addable" box of a given residue.

Young's orthogonal form gives an explicit construction of each $S(\lambda)$ as a vector space with a distinguished basis on which the actions of the generators of H_d are given by explicit formulae. To write it down, let

$$[n] := \begin{cases} n & \text{if } \xi = 1, \\ \frac{\xi^n - 1}{\xi - 1} & \text{if } \xi \neq 1, \end{cases}$$

for any $n \in \mathbb{Z}$.

Theorem 2.1 (Young's orthogonal form). *For $\lambda \vdash d$, the irreducible H_d-module $S(\lambda)$ has a basis $\{v_{\mathtt{T}} \mid \mathtt{T} \in \mathscr{T}(\lambda)\}$ on which $T_r \in H_d$ acts by*

$$T_r v_{\mathtt{T}} := \left(\psi_r \left(1 - \tfrac{1}{[i_r - i_{r+1}]} \right) - \tfrac{1}{[i_r - i_{r+1}]} \right) v_{\mathtt{T}}, \tag{2.2}$$

where $\boldsymbol{i} := \boldsymbol{i}^{\mathtt{T}}$ and ψ_r is the endomorphism with

$$\psi_r v_{\mathtt{T}} := \begin{cases} v_{s_r \cdot \mathtt{T}} & \textit{if } s_r \cdot \mathtt{T} \in \mathscr{T}(\lambda), \\ 0 & \textit{otherwise.} \end{cases} \tag{2.3}$$

Recall also that the *Jucys-Murphy elements* in H_d are the commuting elements $1 = L_1, \dots, L_d$ defined from $L_r := \xi^{1-r} T_{r-1} \cdots T_2 T_1 T_2 \cdots T_{r-1}$ in the case $\xi \neq 1$, or the elements $0 = L_1, \dots, L_d$ defined from $L_r := \sum_{1 \leq s < r} (s\, r)$ in the case $\xi = 1$. Although not obvious from (2.2), the Jucys-Murphy elements act on Young's basis so that

$$L_r v_{\mathtt{T}} = \begin{cases} i_r v_{\mathtt{T}} & \text{if } \xi = 1, \\ \xi^{i_r} v_{\mathtt{T}} & \text{if } \xi \neq 1, \end{cases} \tag{2.4}$$

where $\boldsymbol{i}^{\mathtt{T}} = (i_1, \dots, i_d)$. Thus Young's basis consists of simultaneous eigenvectors for the Jucys-Murphy elements.

We end the section by defining some explicit but rather complicated power series $p_r(\boldsymbol{i}), q_r(\boldsymbol{i}) \in F[[y_1, \dots, y_d]]$. These will be needed at a crucial point in the next section in order to write down our extension of Young's orthogonal form. For fixed $\boldsymbol{i} \in \mathbb{Z}^d$ and $1 \le r < d$, set

$$N := \begin{cases} (y_{r+1} + i_{r+1} + 1) - (y_r + i_r) & \text{if } \xi = 1, \\ \xi^{i_r}(1 - y_r) - \xi^{i_{r+1}+1}(1 - y_{r+1}) & \text{if } \xi \neq 1, \end{cases} \tag{2.5}$$

$$D := \begin{cases} (y_{r+1} + i_{r+1}) - (y_r + i_r) & \text{if } \xi = 1, \\ \xi^{i_r}(1 - y_r) - \xi^{i_{r+1}}(1 - y_{r+1}) & \text{if } \xi \neq 1. \end{cases} \tag{2.6}$$

Note D is a unit in $F[[y_1, \dots, y_d]]$ if $i_r \neq i_{r+1}$, so it then makes sense to define

$$p_r(\boldsymbol{i}) := \begin{cases} 1 & \text{if } i_r = i_{r+1}, \\ 1 - N/D & \text{if } i_r \neq i_{r+1}, \end{cases} \tag{2.7}$$

$$q_r(\boldsymbol{i}) := \begin{cases} \xi^{-i_r} N & \text{if } i_r = i_{r+1}, \\ N/D & \text{if } |i_r - i_{r+1}| > 1, \\ N/D^2 & \text{if } i_r = i_{r+1} - 1, \\ \xi^{i_r} & \text{if } i_r = i_{r+1} + 1. \end{cases} \tag{2.8}$$

To make the connection with Theorem 2.1, we observe on setting $y_1 = \cdots = y_d = 0$ that $p_r(\boldsymbol{i})$ evaluates to $\frac{1}{[i_r - i_{r+1}]}$ assuming $i_r \neq i_{r+1}$, and $q_r(\boldsymbol{i})$ evaluates to $1 - \frac{1}{[i_r - i_{r+1}]}$ assuming $|i_r - i_{r+1}| > 1$. So taking $y_1 = \cdots = y_d = 0$ the formula (2.2) can be rewritten as

$$T_r v_{\mathtt{T}} = (\psi_r q_r(\boldsymbol{i}) - p_r(\boldsymbol{i})) v_{\mathtt{T}}, \tag{2.9}$$

for any $\mathtt{T} \in \mathscr{T}(\lambda)$ and $\boldsymbol{i} := \boldsymbol{i}^{\mathtt{T}}$.

Notes. Theorem 2.1 originates in [**Y**]. Its extension to the Iwahori-Hecke algebra was worked out by Hoefsmit in [**H**]; see also [**W**]. The account here is based closely on [**BK3**, §5] in which the endomorphisms ψ_r and y_r are interpreted as certain *Khovanov-Lauda-Rouquier generators* for H_d, satisfying the defining relations of the cyclotomic quiver Hecke algebras of [**KL**], [**R2**] attached to the infinite linear quiver A_∞ and the fundamental dominant weight Λ_0. The formulae (2.7), (2.8) are exactly [**BK3**, (3.22), (3.30)] if $\xi = 1$ and [**BK3**, (4.27), (4.36)] if $\xi \neq 1$.

3. Level two Hecke algebras

Continuing to work over the ground field F, let $\widehat{H}_d$ be the affine Hecke algebra on generators $\{X_1^{\pm 1}, \dots, X_d^{\pm 1}\} \cup \{T_1, \dots, T_{d-1}\}$ if $\xi \neq 1$, or its degenerate analogue on generators $\{x_1, \dots, x_d\} \cup \{s_1, \dots, s_{d-1}\}$ if $\xi = 1$; in the latter case it is convenient to set $X_r := x_r$ and $T_r := s_r$. The relations are as follows: the X_r's commute, the T_r's satisfy the defining relations of the finite Iwahori-Hecke algebra H_d from (2.1), $T_r X_s = X_s T_r$ if $s \neq r, r+1$, and finally

$$\begin{cases} T_r X_r T_r = \xi X_{r+1} & \text{if } \xi \neq 1, \\ s_r x_{r+1} = x_r s_r + 1 & \text{if } \xi = 1. \end{cases}$$

The finite Iwahori-Hecke algebra H_d is a subalgebra of $\widehat{H}_d$ in the obvious way. Moreover it is also a quotient algebra in many different ways: for each $r \in \mathbb{Z}$ there is an *evaluation homomorphism*

$$\mathrm{ev}_r : \widehat{H}_d \twoheadrightarrow H_d \tag{3.1}$$

which is the identity on the subalgebra H_d and maps $X_1 \mapsto \xi^r$ if $\xi \neq 1$ or $x_1 \mapsto r$ if $\xi = 1$. The Jucys-Murphy element $L_r \in H_d$ from (2.4) is $\mathrm{ev}_0(X_r)$.

More generally, we can consider the quotient of $\widehat{H}_d$ by the two-sided ideal generated by a monic polynomial of degree k in X_1. This gives a finite dimensional algebra of dimension $k^d d!$ known as an *Ariki-Koike algebra* of level k or a *cyclotomic Hecke algebra* of type $G(k,1,d)$. The original finite Iwahori-Hecke algebra H_d corresponds to level one. In the remainder of the article we are interested in the level two case. So we fix integers $p, q \in \mathbb{Z}$ and set

$$H_d^{p,q} := \begin{cases} \widehat{H}_d \big/ \langle\, (X_1 - \xi^p)(X_1 - \xi^q)\, \rangle & \text{if } \xi \neq 1, \\ \widehat{H}_d \big/ \langle\, (x_1 - p)(x_1 - q)\, \rangle & \text{if } \xi = 1, \end{cases} \tag{3.2}$$

of dimension $2^d d!$. We use the same notation for the generators $T_1, \dots, T_{d-1}$ of $\widehat{H}_d$ and for their canonical images in $H_d^{p,q}$, and denote the canonical images of $X_1, \dots, X_d$ by $L_1, \dots, L_d$.

The algebra $H_d^{p,q}$ is semisimple if and only if $d \leq |q - p|$, in which case its representation theory is just as easy as the level one case discussed in the previous section. It turns out that the representation theory of $H_d^{p,q}$ is still very manageable even when it is not semisimple. In fact it provides delightful "baby model" for the representation theory of arbitrary cyclotomic Hecke algebras. We still have *Specht modules* $S(\lambda)$ but they are no longer irreducible; they are parametrized now by *bipartitions* $\lambda \Vdash d$, which are ordered pairs $\lambda = (\lambda^L, \lambda^R)$ of partitions $\lambda^L \vdash a$ and $\lambda^R \vdash b$ such that $d = a + b$. For such a bipartition λ, the corresponding Specht module is

$$S(\lambda) := \widehat{H}_d \otimes_{\widehat{H}_a \otimes \widehat{H}_b} \left(\mathrm{ev}_p^* S(\lambda^L) \boxtimes \mathrm{ev}_q^* S(\lambda^R) \right), \tag{3.3}$$

where $\widehat{H}_a \otimes \widehat{H}_b$ is the parabolic subalgebra of $\widehat{H}_d$, and $\mathrm{ev}_p^* S(\lambda^L) \boxtimes \mathrm{ev}_q^* S(\lambda^R)$ denotes the $\widehat{H}_a \otimes \widehat{H}_b$-module arising as the outer tensor product of level one Specht modules $S(\lambda^L)$ and $S(\lambda^R)$ viewed as modules over $\widehat{H}_a$ and $\widehat{H}_b$, respectively, via the evaluation homomorphisms $\mathrm{ev}_p : \widehat{H}_a \twoheadrightarrow H_a$ and $\mathrm{ev}_q : \widehat{H}_b \twoheadrightarrow H_b$ as in (3.1). This induced module is *a priori* an $\widehat{H}_d$-module, but one can check from (3.2) that it factors through to the quotient $H_d^{p,q}$, hence $S(\lambda)$ is a well-defined $H_d^{p,q}$-module.

We say that a bipartition $\lambda \Vdash d$ is *restricted* if the appropriate one of the following conditions holds for each $i \geq 1$:

$$\begin{cases} \lambda_i^L \leq \lambda_i^R + q - p & \text{if } p \leq q, \\ \lambda_{i+p-q}^L \leq \lambda_i^R & \text{if } p \geq q, \end{cases} \tag{3.4}$$

where $\lambda_1^L \geq \lambda_2^L \geq \cdots$ are the parts of λ^L and $\lambda_1^R \geq \lambda_2^R \geq \cdots$ are the parts of λ^R. The following theorem gives a classification of the irreducible $H_d^{p,q}$-modules.

Theorem 3.1. *If $\lambda \Vdash d$ is restricted, then the Specht module $S(\lambda)$ has a unique irreducible quotient denoted $D(\lambda)$, and the modules*

$$\{D(\lambda) \mid \textit{for all restricted } \lambda \Vdash d\} \tag{3.5}$$

give a complete set of pairwise inequivalent irreducible $H_d^{p,q}$-modules.

Next we want to describe the *composition multiplicities* $[S(\lambda) : D(\mu)]$ of Specht modules, all of which turn out to be either zero or one. First we need some combinatorics. By a *weight diagram* we mean a horizontal number line with vertices at all integers labelled by one of the symbols $\circ, \vee, \wedge$ and $\times$; we require moreover that it is impossible to find a vertex labelled $\vee$ to the left of a vertex labelled $\wedge$ outside of some finite subset of the vertices. We always identify bipartitions with particular weight diagrams so that $\lambda \Vdash d$ corresponds to the weight diagram obtained by putting the symbol $\vee$ at all the vertices indexed by the set

$$\{p + \lambda_1^L, p + \lambda_2^L - 1, p + \lambda_3^L - 2, \dots\},$$

the symbol $\wedge$ at all the vertices indexed by the set

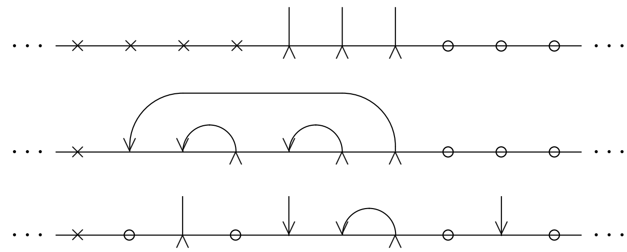
$$\{q + \lambda_1^R, q + \lambda_2^R - 1, q + \lambda_3^R - 2, \dots\},$$

and interpreting vertices labelled both $\vee$ and $\wedge$ as the label $\times$ and vertices labelled neither $\vee$ nor $\wedge$ as the label $\circ$. Of course this depends implicitly on the fixed choices of p and q. Here are some examples:

$$\begin{array}{rcl}
 & & p \\
(\varnothing, \varnothing) = & \cdots \ \times \ \times \ \times \ \times \ \wedge \ \wedge \ \wedge \ \circ \ \circ \ \circ \ \cdots, & q = p+3, \\
((1), (3^2 2)) = & \cdots \ \times \ \vee \ \vee \ \wedge \ \vee \ \wedge \ \wedge \ \circ \ \circ \ \circ \ \cdots, & q = p, \\
((53^2), (41)) = & \cdots \ \times \ \circ \ \wedge \ \circ \ \vee \ \vee \ \wedge \ \circ \ \vee \ \circ \ \cdots, & q = p-1.
\end{array}$$

These examples have infinitely many vertices labelled $\times$ to the left and infinitely many vertices labelled $\circ$ to the right, as do *all* weight diagrams arising from bipartitions. Later in the article, we will meet other sorts of weight diagrams which are not of this form.

Given a weight diagram λ, a λ-*cap diagram* is a diagram obtained by attaching caps $\cap$ and rays up to infinity $|$ to all the vertices of λ labelled $\vee$ or $\wedge$, so that there are no crossings of caps and/or rays, the labels at the ends of caps are either $\vee\!\frown\!\wedge$ ("counter-clockwise") or $\wedge\!\frown\!\vee$ ("clockwise"), all rays labelled $\wedge$ are strictly to the left of all rays labelled $\vee$, and the total number of caps is finite. Here are some examples:



The *weight* $\mathrm{wt}(\mathtt{A})$ of a λ-cap diagram $\mathtt{A}$ is the weight diagram obtained from λ by switching the labels at the ends of all the clockwise caps of $\mathtt{A}$. Observe in particular that there is always a unique λ-cap diagram of weight λ.

We say that a λ-cap diagram is *restricted* if all its rays are labelled in the same way; in the above examples, the first two are restricted but the third is not. We say that the weight diagram λ itself is restricted if the unique λ-cap diagram of weight λ is restricted. In the case that λ is the weight diagram arising from a bipartition of d, λ is restricted as a weight diagram if and only if it is a restricted bipartition in the sense of (3.4); e.g. in our running example the first two bipartitions are restricted, but the third is not.

Finally define a reflexive and anti-symmetric relation $\supset$ on weight diagrams by declaring that $\lambda \supset \mu$ if there exists a (necessarily unique) λ-cap diagram of weight μ. For fixed μ it is easy to find all λ such that $\lambda \supset \mu$: they are all the weight diagrams that can be obtained from μ by switching the labels at the ends of some subset of the caps in the unique μ-cap diagram of weight μ. It follows easily that the transitive closure of the relation $\supset$ is the same as the *Bruhat order* $\geq$ on weight diagrams generated by the elementary relation $\cdots \wedge \cdots \vee \cdots \geq \cdots \vee \cdots \wedge \cdots$. On the other hand, for fixed λ it is trickier to find all μ such that $\lambda \supset \mu$. For example if $\lambda = \wedge \vee \wedge \vee$ (and all other vertices are labelled $\circ$ or $\times$) there are five λ-cap diagrams hence five weights μ with $\lambda \supset \mu$, namely, $\wedge \vee \wedge \vee$, $\vee \wedge \wedge \vee$, $\wedge \vee \vee \wedge$, $\vee \vee \wedge \wedge$ and $\vee \wedge \vee \wedge$. It is no coincidence here that the third Catalan number $C_3 = 5$: consider $\lambda = \wedge \vee \wedge \vee \wedge \vee$ to get C_4 and so on.

THEOREM 3.2. *For $\lambda, \mu \Vdash d$ with μ restricted, we have that*

$$[S(\lambda) : D(\mu)] = \begin{cases} 1 & \text{if } \lambda \supset \mu, \\ 0 & \text{otherwise.} \end{cases}$$

It is already clear from this that we are in a rather unusual situation. In fact, much more is possible: there is a remarkable extension of Young's orthogonal form for the algebra $H_d^{p,q}$ giving an explicit construction of another family of $H_d^{p,q}$-modules denoted $\{Y(\lambda) \mid \lambda \Vdash d\}$. As usual, we need some more combinatorial preparation. For $\lambda \Vdash d$, the *Young diagram* of λ means the ordered pair of the Young diagrams of λ^L and λ^R. A λ-*tableau* $\mathtt{T} = (\mathtt{T}^L, \mathtt{T}^R)$ means a filling of the boxes of this diagram by the numbers $1, \dots, d$ (each appearing exactly once), and as in the previous section the symmetric group S_d acts on λ-tableaux by its action on the entries. We let $\mathscr{T}(\lambda)$ denote the set of all *standard* λ-tableaux, that is, the $\mathtt{T} = (\mathtt{T}^L, \mathtt{T}^R)$ such that the entries of both $\mathtt{T}^L$ and $\mathtt{T}^R$ increase strictly along rows and down columns. The *residue* of the box in the bth row and cth column of the Young diagram of λ^L (resp. λ^R) is $p+b-c$ (resp. $q+b-c$). Then the *residue sequence* $\boldsymbol{i}^{\mathtt{T}} \in \mathbb{Z}^d$ of $\mathtt{T} \in \mathscr{T}(\lambda)$ is defined just like in the previous section. For example:

$$\mathtt{T} = \left(\begin{array}{|c|c|c|}\hline 2 & 5 & 6 \\ \hline 3 & 8 \\ \cline{1-2}\end{array} \;,\; \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 7 \\ \cline{1-1}\end{array} \right) \leftrightarrow \boldsymbol{i}^{\mathtt{T}} = (q, p, p-1, q+1, p+1, p+2, q-1, p)$$

Now for $\lambda \Vdash d$, we can define the *Young module* $Y(\lambda)$ to be the vector space on basis $\{v_{\mathtt{T}}^{\lambda} \mid \mathtt{T} \in \bigcup_{\mu \supset \lambda} \mathscr{T}(\mu)\}$. To define the action of $H_d^{p,q}$, take $\mathtt{T} \in \mathscr{T}(\mu)$ for some $\mu \supset \lambda$ and set $\boldsymbol{i} := \boldsymbol{i}^{\mathtt{T}}$. At the end of the section we will define endomorphisms y_r and ψ_r of the vector space $Y(\lambda)$ such that

$$y_r v_{\mathtt{T}}^{\lambda} \in \left\langle v_{\mathtt{S}}^{\lambda} \;\middle|\; \mathtt{S} \in \textstyle\bigcup_{\mu \leq \nu \supset \lambda} \mathscr{T}(\nu), \boldsymbol{i}^{\mathtt{S}} = \boldsymbol{i} \right\rangle, \tag{3.6}$$

$$\psi_r v_{\mathtt{T}}^{\lambda} \in \left\langle v_{\mathtt{S}}^{\lambda} \;\middle|\; \mathtt{S} \in \textstyle\bigcup_{\mu \leq \nu \supset \lambda} \mathscr{T}(\nu), \boldsymbol{i}^{\mathtt{S}} = s_r \cdot \boldsymbol{i} \right\rangle. \tag{3.7}$$

Moreover we will have that $y_r^2 = 0$, hence it makes sense to view the power series $p_r(\boldsymbol{i})$ and $q_r(\boldsymbol{i})$ from (2.7)–(2.8) as endomorphisms of $Y(\lambda)$. Then the generators of $H_d^{p,q}$ act by the formulae

$$T_r v_{\mathtt{T}}^{\lambda} := (\psi_r q_r(\boldsymbol{i}) - p_r(\boldsymbol{i})) v_{\mathtt{T}}^{\lambda}, \tag{3.8}$$

$$L_r v_{\mathtt{T}}^{\lambda} := \begin{cases} (y_r + i_r) v_{\mathtt{T}}^{\lambda} & \text{if } \xi = 1, \\ \xi^{i_r}(1 - y_r) v_{\mathtt{T}}^{\lambda} & \text{if } \xi \neq 1, \end{cases} \tag{3.9}$$

which should be compared with (2.9) and (2.4) in the level one case. The following theorem justifies the terminology "Young module."

THEOREM 3.3. *The endomorphisms (3.8)–(3.9) satisfy the defining relations of $H_d^{p,q}$, so make $Y := \bigoplus_{\lambda \Vdash d} Y(\lambda)$ into an $H_d^{p,q}$-module. Let*

$$K_d^{p,q} := \operatorname{End}_{H_d^{p,q}}(Y)^{\mathrm{op}}$$

and $e_\lambda \in K_d^{p,q}$ be the projection of Y onto the summand $Y(\lambda)$. Then $K_d^{p,q}$ is a basic quasi-hereditary algebra with weight poset $\{\lambda \Vdash d\}$ partially ordered by $\geq$, and projective indecomposable modules $P(\lambda) := K_d^{p,q} e_\lambda$, standard modules $V(\lambda)$ and irreducible modules $L(\lambda)$ for $\lambda \Vdash d$. Moreover:

1. *The left $K_d^{p,q}$-module $T := \operatorname{Hom}_{K_d^{p,q}}(Y, K_d^{p,q})$ is a projective-injective generator for the category $K_d^{p,q}$-mod of finite dimensional left $K_d^{p,q}$-modules, i.e. it is both projective and injective and every finite dimensional projective-injective $K_d^{p,q}$-module is isomorphic to a summand of a direct sum of copies of T.*

(2) *The following double centralizer property holds: the natural right action of $H_d^{p,q}$ on T induces an algebra isomorphism*

$$H_d^{p,q} \xrightarrow{\sim} \mathrm{End}_{K_d^{p,q}}(T)^{\mathrm{op}}.$$

(3) *The exact Schur functor*

$$\pi := \mathrm{Hom}_{K_d^{p,q}}(T, ?) : K_d^{p,q}\text{-mod} \to H_d^{p,q}\text{-mod}$$

is fully faithful on projective objects, i.e. $K_d^{p,q}$ is a quasi-hereditary cover of $H_d^{p,q}$.

(4) *For each $\lambda \Vdash d$, we have that $\pi P(\lambda) \cong Y(\lambda)$ and $\pi V(\lambda) \cong S(\lambda)$. Moreover if λ is restricted then $\pi L(\lambda) \cong D(\lambda)$, hence $Y(\lambda)$ is the projective cover of $D(\lambda)$; if λ is not restricted then $\pi L(\lambda) = \mathbf{0}$.*

By the general theory of quasi-hereditary algebras, the projective indecomposable module $P(\lambda)$ in Theorem 3.3 has a filtration whose sections are standard modules with $V(\lambda)$ appearing at the top. Applying the Schur functor π from Theorem 3.3(3), we get a filtration of the Young module $Y(\lambda)$ whose sections are Specht modules with $S(\lambda)$ at the top. The next theorem explains how to see this filtration explicitly in terms of the orthogonal basis; cf. (3.6)–(3.7).

Theorem 3.4. *For any $\lambda \Vdash d$, the Specht module $S(\lambda)$ is isomorphic to the quotient of $Y(\lambda)$ by the submodule $\left\langle v_{\mathtt{T}}^{\lambda} \,\middle|\, \mathtt{T} \in \bigcup_{\lambda \neq \mu \supset \lambda} \mathscr{T}(\mu) \right\rangle$. Hence $S(\lambda)$ has a distinguished basis $\{v_{\mathtt{T}} \,|\, \mathtt{T} \in \mathscr{T}(\lambda)\}$ arising from the images of the elements $\{v_{\mathtt{T}}^{\lambda} \mid \mathscr{T}(\lambda)\}$, on which the actions of the generators of $H_d^{p,q}$ can be computed explicitly via (3.8)–(3.9). Moreover if we let $\mu_1, \dots, \mu_n$ be all the $\mu \supset \lambda$ ordered so that $\mu_i \geq \mu_j \Rightarrow i \leq j$ and set $M_j := \left\langle v_{\mathtt{T}}^{\lambda} \,\middle|\, \mathtt{T} \in \bigcup_{i \leq j} \mathscr{T}(\mu_i) \right\rangle$, we get a filtration*

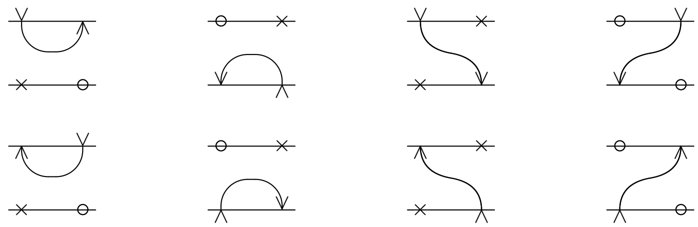
$$\mathbf{0} = M_0 \subset M_1 \subset \cdots \subset M_n = Y(\lambda)$$

such that the map $M_j/M_{j-1} \to S(\mu_j)$ sending $v_{\mathtt{T}}^{\lambda} + M_{j-1} \mapsto v_{\mathtt{T}}$ for $\mathtt{T} \in \mathscr{T}(\mu_j)$ is an $H_d^{p,q}$-module isomorphism.

The basis $\{v_{\mathtt{T}} \mid \mathtt{T} \in \mathscr{T}(\lambda)\}$ for $S(\lambda)$ arising from Theorem 3.4 is very special. For example if λ is restricted, it contains a basis for the kernel of the homomorphism $S(\lambda) \twoheadrightarrow D(\lambda)$, so that we also get induced an equally explicit basis for the irreducible module $D(\lambda)$. In order to explain this precisely, and also for use when we define the endomorphisms $y_r, \psi_r \in \mathrm{End}_F(Y(\lambda))$ at the end of the section, we need one more combinatorial excursion.

Suppose we are given a standard λ-tableau $\mathtt{T}$ for some $\lambda \Vdash d$. We are going to represent $\mathtt{T}$ by a new sort of diagram which we call a *stretched λ-cup diagram.* To make the translation, let $\boldsymbol{i} := \boldsymbol{i}^{\mathtt{T}}$ and $\varnothing = \lambda_0, \lambda_1, \dots, \lambda_{d-1}, \lambda_d = \lambda$ be the sequence of bipartitions such that λ_i is the shape of the standard tableau obtained from $\mathtt{T}$ by removing all the boxes containing the entries $\geq (i+1)$. Stack the weight diagrams of the bipartitions $\lambda_0, \lambda_1, \dots, \lambda_d$ in order from bottom to top, and observe that the weight diagrams λ_{r-1} and

λ_r only differ at vertices i_r and i_r+1. For each $r = 1, \dots, d$, insert vertical line segments connecting all vertices $< i_r$ or $> (i_r+1)$ that are labelled $\vee$ or $\wedge$ in λ_{r-1} and λ_r. Then connect the remaining vertices i_r and i_r+1 of λ_{r-1} and λ_r as in the appropriate one of the following pictures:



See Table 1 for some examples.

Ignoring the weight diagrams themselves, the stretched cup diagram of any $\mathtt{T} \in \mathscr{T}(\lambda)$ decomposes into various connected components: *circles* in the interior of the diagram, *boundary cups* whose endpoints are vertices on the top number line, and *line segments* which stretch between the bottom and top number lines. The top weight diagram λ gives an orientation to each of the boundary cups, either counter-clockwise or clockwise. We define the *weight* $\mathrm{wt}(\mathtt{T})$ to be the bipartition whose weight diagram is obtained from λ by switching the labels at the ends of each of the clockwise boundary cups. Also for $i \geq 0$ let $\mathscr{T}_i(\lambda)$ denote the set of all $\mathtt{T} \in \mathscr{T}(\lambda)$ such that the corresponding stretched cup diagram has exactly i clockwise boundary cups. In particular,

$$\mathscr{T}_0(\lambda) = \{\mathtt{T} \in \mathscr{T}(\lambda) \mid \mathrm{wt}(\mathtt{T}) = \lambda\}, \tag{3.10}$$

which is non-empty if and only if λ is restricted.

Theorem 3.5. *Given a restricted $\lambda \Vdash d$, the irreducible module $D(\lambda)$ is isomorphic to the quotient of $S(\lambda)$ by the submodule $\left\langle v_{\mathtt{T}} \,\middle|\, \mathtt{T} \in \bigcup_{i\geq 1} \mathscr{T}_i(\lambda) \right\rangle$. Hence $D(\lambda)$ has a distinguished basis $\{\bar{v}_{\mathtt{T}} | \mathtt{T} \in \mathscr{T}_0(\lambda)\}$ arising from the images of the elements $\{v_{\mathtt{T}} \mid \mathtt{T} \in \mathscr{T}_0(\lambda)\}$, on which the actions of the generators of $H_d^{p,q}$ can be computed explicitly via (3.8)–(3.9). Moreover given an arbitrary $\lambda \Vdash d$, let $N_j := \left\langle v_{\mathtt{T}} \,\middle|\, \mathtt{T} \in \bigcup_{i\geq j} \mathscr{T}_i(\lambda) \right\rangle$. Then*

$$S(\lambda) = N_0 \supseteq N_1 \supseteq \cdots$$

is a filtration of $S(\lambda)$ such that $N_j/N_{j+1} \cong \bigoplus_\mu D(\mu)$, direct sum over all $\mu \Vdash d$ such that there is a λ-cap diagram of weight μ with exactly j clockwise caps; the explicit isomorphism here sends $v_{\mathtt{T}} + N_{j+1} \in N_j/N_{j+1}$ for $\mathtt{T} \in \mathscr{T}_j(\lambda)$ to $\bar{v}_{\bar{\mathtt{T}}} \in D(\mathrm{wt}(\mathtt{T}))$ where $\bar{\mathtt{T}}$ is the standard tableau whose stretched cup diagram is obtained from that of $\mathtt{T}$ by reversing the labels on all clockwise boundary cups.

It just remains to explain the definitions of $y_r, \psi_r \in \mathrm{End}_F(Y(\lambda))$. Continue with $\lambda \Vdash d$. Fix $\mathtt{T} \in \mathscr{T}(\mu)$ for some $\mu \supset \lambda$, hence a basis vector $v_{\mathtt{T}}^{\lambda} \in Y(\lambda)$. Let $\boldsymbol{i} := \boldsymbol{i}^{\mathtt{T}}$. To start with we take care of some awkward signs:

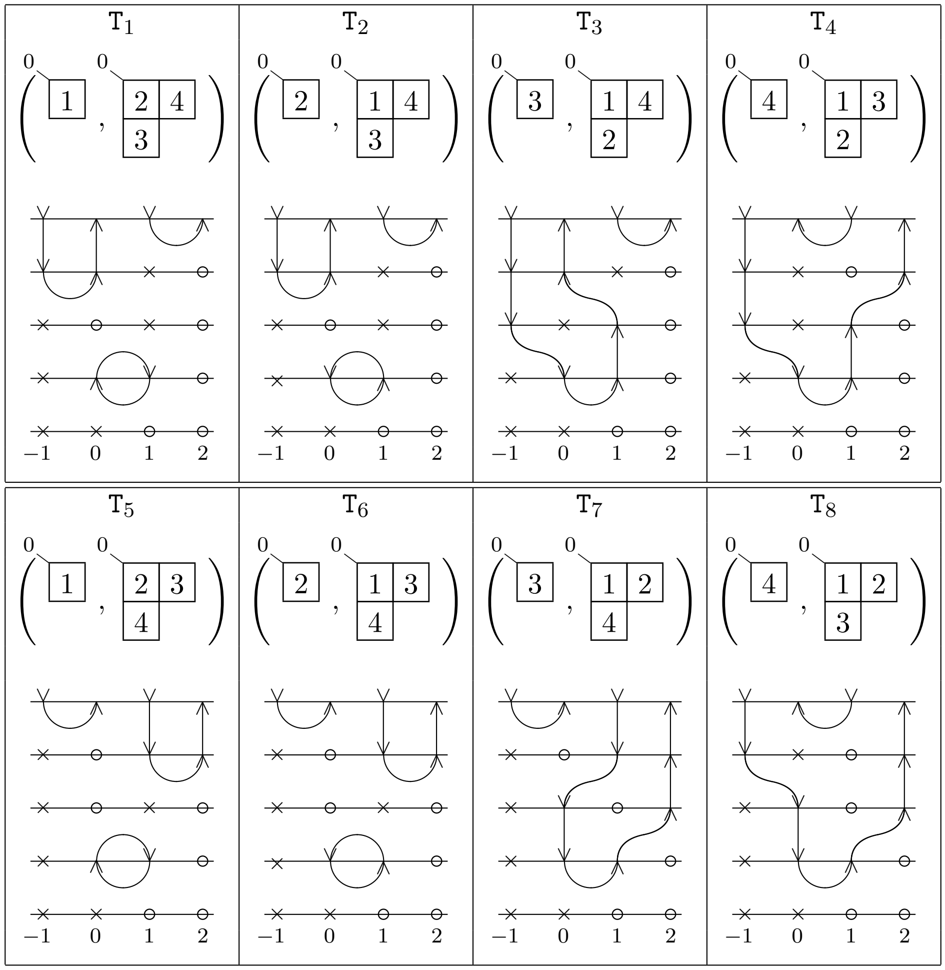


TABLE 1. For $\lambda = ((1),(21))$ and $p = q = 0$, this table displays the stretched λ-cup diagrams corresponding to the eight standard λ-tableaux, which are denoted $\mathtt{T}_1, \dots, \mathtt{T}_8$.

we will actually define $\bar{y}_r, \bar{\psi}_r \in \mathrm{End}_F(Y(\lambda))$, and then y_r and ψ_r are related to these by the formulae

$$y_r v_{\mathtt{T}}^{\lambda} = \sigma_r(\boldsymbol{i}) \bar{y}_r v_{\mathtt{T}}^{\lambda}, \qquad \psi_r v_{\mathtt{T}}^{\lambda} = \begin{cases} -\sigma_r(\boldsymbol{i}) \bar{\psi}_r v_{\mathtt{T}}^{\lambda} & \text{if } i_{r+1} \in \{i_r, i_r+1\}, \\ \bar{\psi}_r v_{\mathtt{T}} & \text{otherwise,} \end{cases} \tag{3.11}$$

where $\sigma_r(\boldsymbol{i}) := (-1)^{\min(p,i_r)+\min(q,i_r)+\delta_{i_1,i_r}+\cdots+\delta_{i_{r-1},i_r}}$.

To calculate $\bar{y}_r v_{\mathtt{T}}^{\lambda}$, let $\binom{\lambda}{\mathtt{T}}$ denote the composite diagram obtained by gluing the stretched μ-cup diagram corresponding to $\mathtt{T}$ under the unique μ-cap diagram of weight λ. We refer to the horizontal strips between the number lines in this diagram as its *layers*, and index them by $1, \dots, d$ in order from bottom to top. There is a unique connected component in the diagram $\binom{\lambda}{\mathtt{T}}$ which is non-trivial in the rth layer, i.e. its intersection with the rth layer involves something other than vertical line segments. If this connected component is a counter-clockwise circle, we reverse all the labels $\vee$ or $\wedge$ on the component to get a new diagram of the form $\binom{\lambda}{\mathtt{S}}$ for a unique standard tableau $\mathtt{S}$, then set $\bar{y}_r v^\lambda_{\mathtt{T}} := v^\lambda_{\mathtt{S}}$; otherwise we simply set $\bar{y}_r v^\lambda_{\mathtt{T}} := 0$. For example, in the notation of Table 1 taking $\lambda = ((1),(21))$, we have $y_1 v^\lambda_{\mathtt{T}_2} = v^\lambda_{\mathtt{T}_1}$, and $y_4 v^\lambda_{\mathtt{T}_4} = -v^\lambda_{\mathtt{S}}$ where $\mathtt{S} = (\mathtt{T}_4^R, \mathtt{T}_4^L)$.

To calculate $\bar{\psi}_r v^\lambda_{\mathtt{T}}$, there are three cases. The easiest is when $|i_r - i_{r+1}| > 1$, when $s_r \cdot \mathtt{T}$ is again a standard tableau as in the level one case and we set

$$\psi_r v^\lambda_{\mathtt{T}} = \bar{\psi}_r v^\lambda_{\mathtt{T}} := v^\lambda_{s_r \cdot \mathtt{T}}. \tag{3.12}$$

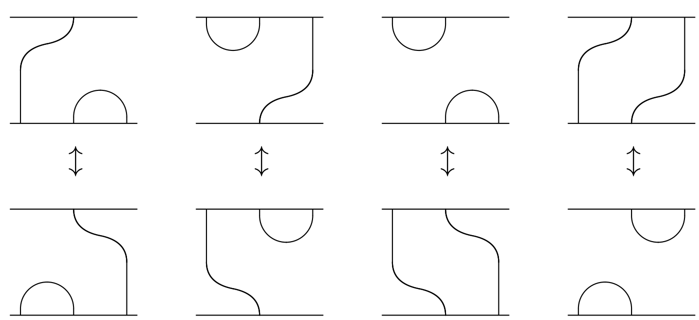
In terms of $\binom{\lambda}{\mathtt{T}}$, this corresponds to sliding the parts of the diagram that are non-trivial in layers r and $(r+1)$ past each other. For example in the notation from Table 1 again we have that $\psi_2 v^\lambda_{\mathtt{T}_4} = v^\lambda_{\mathtt{T}_8}$ and $\psi_3 v^\lambda_{\mathtt{T}_2} = v^\lambda_{\mathtt{T}_6}$.

Next suppose that $i_r = i_{r+1}$. Then the diagram $\binom{\lambda}{\mathtt{T}}$ has a small circle in layers r and $r+1$. If this circle is counter-clockwise we set $\bar{\psi}_r v^\lambda_{\mathtt{T}} := 0$; otherwise the circle is clockwise and we let $\bar{\psi}_r v^\lambda_{\mathtt{T}} := v^\lambda_{\mathtt{S}}$ where $\binom{\lambda}{\mathtt{S}}$ is obtained from $\binom{\lambda}{\mathtt{T}}$ by reversing the labels on this circle. For example $\psi_1 v^\lambda_{\mathtt{T}_1} = -v^\lambda_{\mathtt{T}_2}$ and $\psi_1 v^\lambda_{\mathtt{T}_2} = 0$.

Finally suppose that $|i_r - i_{r+1}| = 1$. If there are no standard tableaux with residue sequence $s_r \cdot \boldsymbol{i}$, we simply set $\bar{\psi}_r v^\lambda_{\mathtt{T}} := 0$. If there is at least one such standard tableau, the part of the diagram $\binom{\lambda}{\mathtt{T}}$ that is non-trivial in layers r and $(r+1)$ matches one of the following eight configurations:

$i_{r+1} = i_r - 1 :$

$i_{r+1} = i_r + 1 :$



The part of the diagram just displayed can belong to either one or two connected components in the larger diagram $\binom{\lambda}{\mathtt{T}}$. In the former case we define the type to be $1, x$ or y according to whether the connected component is a counter-clockwise circle, a clockwise circle or a line segment; in the latter case we define the type to be $1 \otimes 1, 1 \otimes x, 1 \otimes y, x \otimes x$, $x \otimes y$ or $y \otimes y$ according to whether there are two counter-clockwise circles, one counter-clockwise and one clockwise circle, one counter-clockwise circle and one line segment, two clockwise circles, one clockwise circle and one line segment, or two line segments. Erase all the labels from the one or two components, transform the rth and $(r+1)$th layers as indicated by the correspondence $\updownarrow$ in the above diagrams, then finally reintroduce labels into the two or one components created by this transformation to obtain some new diagrams of the form $\binom{\lambda}{\mathtt{S}}$ for standard tableaux $\mathtt{S}$; then $\bar{\psi}_r v^\lambda_{\mathtt{T}}$ is defined to be the sum of the corresponding basis vectors $v^\lambda_{\mathtt{S}}$. The rules to reintroduce labels in the final step here depends on the initial type as follows:

$$1 \mapsto 1 \otimes x + x \otimes 1, \qquad x \mapsto x \otimes x, \qquad y \mapsto x \otimes y, \tag{3.13}$$

$$1 \otimes 1 \mapsto 1,\ 1 \otimes x \mapsto x,\ 1 \otimes y \mapsto y,\ x \otimes x \mapsto 0,\ x \otimes y \mapsto 0,\ y \otimes y \mapsto 0, \tag{3.14}$$

where again 1 represents a counter-clockwise circle, x a clockwise circle and y a line segment. The first rule in (3.13) means that we get two diagrams in which the two components are oriented counter-clockwise and clockwise in the first and vice versa in the second; the last three rules in (3.14) mean that we get zero; the other five rules are interpreted similarly. For example $\psi_1 v^\lambda_{\mathtt{T}_7} = 0$, $\psi_2 v^\lambda_{\mathtt{T}_2} = v^\lambda_{\mathtt{T}_3}$ and $\psi_3 v^\lambda_{\mathtt{T}_8} = y_3 v^\lambda_{\mathtt{T}_7} - y_4 v^\lambda_{\mathtt{T}_7}$.

Notes. In the case that $\xi \neq 1$, the algebra $H^{p,q}_d$ can obviously be identified with the finite Iwahori-Hecke algebra of type B_d at long root parameter ξ and short root parameter $-\xi^{q-p}$, the generator usually denoted T_0 in that Iwahori-Hecke algebra being $-\xi^{-p} L_1$. Furthermore by the main result of [**BK3**] the algebra $H^{p,q}_d$ is isomorphic in both the non-degenerate and the degenerate cases to the cyclotomic quiver Hecke algebra of [**KL**], [**R2**] for the quiver A_∞ and the level two weight $\Lambda_p + \Lambda_q$. The semisimplicity criterion for $H^{p,q}_d$ is due to Dipper and James [**DJ**, Theorem 5.5]. In all the semisimple cases an analogue of Young's orthogonal form was worked out by Hoefsmit in [**H**]. The construction of Specht modules as induced modules originates in work of Vazirani. Theorem 3.1 is essentially the level two case of [**V**, Theorem 3.4] if $p \geq q$; it can be proved by similar techniques when $p < q$.

The cyclotomic quiver Hecke algebra just mentioned is naturally $\mathbb{Z}$-graded. The Young module $Y(\lambda)$ can be interpreted as graded module over this graded algebra, with $\mathbb{Z}$-grading defined so that the basis vector $v^\lambda_{\mathtt{T}}$ is of degree equal to the number of $\wedge$'s or the number of $\vee$'s in the weight diagram of λ, whichever is smaller, plus the total number of clockwise circles minus the total number of counter-clockwise circles in the diagram $\binom{\lambda}{\mathtt{T}}$. The grading on $Y(\lambda)$ induces gradings on the quotients $S(\lambda)$ and (assuming λ is restricted) $D(\lambda)$, so that the basis elements $v_{\mathtt{T}}$ and $\bar{v}_{\mathtt{T}}$ constructed in Theorems 3.4 and 3.5 are homogeneous of degree equal to

$$\deg(\mathtt{T}) := \#(\text{clockwise cups}) - \#(\text{counter-clockwise caps}) \tag{3.15}$$

in the stretched cup diagram associated to $\mathtt{T}$. In fact with this grading $S(\lambda)$ is isomorphic to the *graded Specht module* of [**BKW**], and the definition (3.15) agrees with [**BKW**, (3.5)]. The filtrations in Theorems 3.4 and 3.5 are filtrations of graded modules; the section M_j/M_{j-1} in Theorem 3.4 is actually isomorphic as a graded module to $S(\mu_j)\langle d_j\rangle$ (the graded Specht module $S(\mu_j)$ shifted up in degree by d_j) where d_j is the number of clockwise caps in the unique μ_j-cap diagram of weight λ; the section N_j/N_{j+1} in Theorem 3.5 is isomorphic as a graded module to $\bigoplus_\mu D(\mu)\langle j\rangle$ summing over μ as in the statement of the theorem.

Over fields of characteristic 0, Theorem 3.2 can be deduced from Ariki's categorification theorem [**A**], [**BK2**]; the necessary combinatorics of canonical bases was worked out by Leclerc and Miyachi [**LM**] in terms of the combinatorics of Lusztig's symbols. Our formulation using cap diagrams is equivalent to this. The observation that Theorem 3.2 is also valid over fields of positive characteristic was first observed by Ariki and Mathas [**AM**, Corollary 3.7].

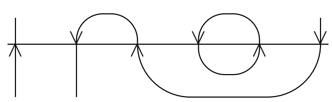
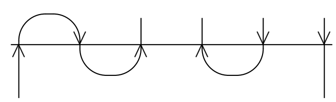
In [**BS3**], we explained a different approach starting from the explicit construction of the algebra $K_d^{p,q}$ (to be explained in section 4 below), and also an alternative construction of the left $K_d^{p,q}$-module T from Theorem 3.3 in terms of certain special projective functors F_i; although we worked over the ground field $\mathbb{C}$ the relevant arguments in [**BS3**] can be carried out over arbitrary fields with only minor modifications as noted in [**BS3**, Remark 8.7]. The precise references needed to extract Theorem 3.3 from [**BS1**]–[**BS3**] are as follows: (1) follows from the alternative definition of T from [**BS3**, (6.1)] plus [**BS3**, Lemma 6.1]; (2) follows from [**BS3**, Corollary 8.6] plus the definition [**BS3**, (6.2)]; (3) and (4) apart from the isomorphisms $\pi P(\lambda) \cong Y(\lambda)$ and $\pi V(\lambda) \cong S(\lambda)$ follow from [**BS3**, Lemma 8.13]. The isomorphism $\pi P(\lambda) \cong Y(\lambda)$ and the explicit construction of $Y(\lambda)$ via the orthogonal form described above can be deduced from [**BS3**, Lemma 6.6]; to get the precise formulae (3.8)–(3.9) one needs also to use the isomorphism theorem from [**BK3**]. The isomorphism $\pi V(\lambda) \cong S(\lambda)$ is established in the degenerate case in [**BS3**, Lemma 9.3 and Corollary 9.6]; it can be deduced in the non-degenerate case too by a base change argument involving the construction of [**BKW**]. Theorem 3.4 is a consequence of [**BS1**, Theorem 5.1] on applying the Schur functor. Similarly Theorem 3.5 is a consequence of [**BS1**, Theorem 5.2]; it obviously implies Theorem 3.2 too.

The notion of quasi-hereditary cover mentioned in Theorem 3.3(3) was introduced by Rouquier in [**R1**, §4.2]; quasi-hereditary algebras of course go back to the seminal work of Cline, Parshall and Scott [**CPS**]. The quasi-hereditary cover $K_d^{p,q}$ of $H_d^{p,q}$ is Morita equivalent to another well known quasi-hereditary cover of $H_d^{p,q}$, namely, the (level two) *cyclotomic Schur algebra* of Dipper, James and Mathas from [**DJM**]; see also [**AMR**, §6] which described the degenerate analogues of these algebras too. This Morita equivalence is a consequence of the double centralizer property. The key point is that our Young modules are the same as the images of the projective indecomposable modules of the cyclotomic Schur algebra under its Schur functor, as can be proved by an argument involving the special projective functors F_i analogous to the proof of [**BS3**, Lemma 8.16(ii)]; one just needs to know in the cyclotomic Schur algebra setting that F_i commutes with the Schur functor just like in [**BS3**, Lemma 8.13(iii)]. For a diagrammatic description of this algebra in the spirit of Khovanov and Lauda, and a remarkable generalization to other quivers, see the recent preprint of Webster [**W**].

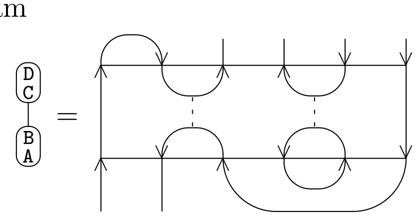
4. Khovanov's arc algebra

Let $\sim$ be the equivalence relation on the set of weight diagrams defined by $\lambda \sim \mu$ if μ is obtained from λ by permuting some of the labels $\vee$ and $\wedge$. Let Λ be any (not necessarily finite) set of weight diagrams closed under $\sim$. We are going to recall the definition of an algebra K_Λ, which is a generalization of Khovanov's arc algebra. Then we will relate this algebra for particular Λ to the algebra $K_d^{p,q}$ from the previous section.

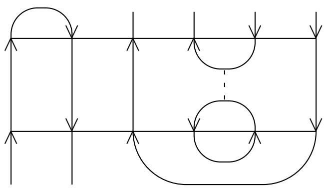
We introduced already the notion of a λ-cap diagram for any weight diagram λ. There is an entirely analogous notion of a λ-*cup diagram*, attaching cups $\cup$ and rays down to infinity $|$ below the number line following the same rules as before. The *weight* $\mathrm{wt}(\mathtt{A})$ of a λ-cup diagram is defined in the same way as for cap diagrams. A λ-*circle diagram* means a composite diagram of the form $\binom{\mathtt{B}}{\mathtt{A}}$ obtained by gluing a λ-cup diagram $\mathtt{A}$ under a λ-cap diagram $\mathtt{B}$. Here are two examples (where all vertices not displayed are labelled $\circ$ or $\times$):

$\binom{\mathtt{B}}{\mathtt{A}} =$  $\binom{\mathtt{D}}{\mathtt{C}} =$ 

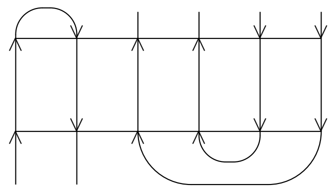
Now we can define the algebra K_Λ. As a vector space (over our fixed ground field F) K_Λ has a distinguished basis consisting of all the λ-circle diagrams $\binom{\mathtt{B}}{\mathtt{A}}$ for all $\lambda \in \Lambda$. The multiplication is defined as follows. Given two basis vectors $\binom{\mathtt{B}}{\mathtt{A}}$ and $\binom{\mathtt{D}}{\mathtt{C}}$, their product is zero unless $\mathrm{wt}(\mathtt{B}) = \mathrm{wt}(\mathtt{C})$. Assuming $\mathrm{wt}(\mathtt{B}) = \mathrm{wt}(\mathtt{C})$, all the caps and rays in $\mathtt{B}$ are in the same positions as the cups and rays in $\mathtt{C}$. We draw the diagram $\binom{\mathtt{B}}{\mathtt{A}}$ under the diagram $\binom{\mathtt{D}}{\mathtt{C}}$ and stitch corresponding rays together to obtain a new composite diagram with a symmetric middle section. For example if $\mathtt{A}, \mathtt{B}, \mathtt{C}$ and $\mathtt{D}$ are as above we get the diagram



Then we iterate a certain *surgery procedure* to be explained in the next paragraph in order to smooth out all the cup-cap pairs in the symmetric middle section of the diagram (indicated by dotted lines in the above example). This produces some new diagrams in which the middle section involves only vertical line segments. Finally we collapse the middle sections in these new diagrams to obtain some circle diagrams, and define the desired product to be the sum of the corresponding basis vectors in K_Λ. In the above example applying the left then right surgeries produces the diagrams

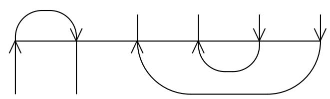


then



Hence we have that

$$\binom{\mathtt{B}}{\mathtt{A}}\binom{\mathtt{D}}{\mathtt{C}} =$$



The surgery procedure is similar to the procedure for computing $\bar{\psi}_r$ explained in the last paragraph of the previous section, and goes as follows. The cup-cap pair to be smoothed either belong to one or two connected components in the larger diagram. We record a type $1, x, y, 1 \otimes 1, 1 \otimes x, 1 \otimes y, x \otimes x, x \otimes y, y \otimes y$ according to whether these one or two components are counter-clockwise circles (1), clockwise circles (x) or line segments (y). Then erase the labels on the one or two components and smooth out the cup-cap pair to get two vertical lines. Finally reintroduce the labels according to the same rules (3.13)–(3.14) as before with one modification (to take account of a configuration which did not arise before): in the case $y \otimes y$ if it happens that both the components to start with are lines stretching from infinity at the bottom to infinity at the top, with one oriented upwards and the other oriented downwards, then we replace the rule $y \otimes y \mapsto 0$ with the rule $y \otimes y \mapsto y \otimes y$. The first surgery in the above example is exactly this situation.

The algebra K_Λ has a $\mathbb{Z}_{\geq 0}$-grading defined by declaring that the basis vector $\binom{\mathtt{B}}{\mathtt{A}}$ is of degree equal to the total number of clockwise cups and cups in the circle diagram. The mutually orthogonal idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ defined by setting $e_\lambda := \binom{\mathtt{B}}{\mathtt{A}}$ where $\mathtt{A}$ and $\mathtt{B}$ are the unique λ-cup and λ-cap diagrams of weight λ, respectively, give a basis for the degree zero component of K_Λ. Hence K_Λ is a basic algebra and its degree zero component is a (possibly infinite) direct sum of copies of F. We have moreover for arbitrary $\binom{\mathtt{B}}{\mathtt{A}}$ that

$$e_\lambda \binom{\mathtt{B}}{\mathtt{A}} = \begin{cases} \binom{\mathtt{B}}{\mathtt{A}} & \text{if } \lambda = \mathrm{wt}(\mathtt{A}), \\ 0 & \text{otherwise,} \end{cases} \qquad \binom{\mathtt{B}}{\mathtt{A}} e_\lambda = \begin{cases} \binom{\mathtt{B}}{\mathtt{A}} & \text{if } \mathrm{wt}(\mathtt{B}) = \lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Hence $K_\Lambda = \bigoplus_{\lambda,\mu \in \Lambda} e_\lambda K_\Lambda e_\mu$, so that K_Λ is a *locally unital algebra*. It is a unital algebra with identity element $1 = \sum_{\lambda \in \Lambda} e_\lambda$ if and only if $|\Lambda| < \infty$. When talking about modules over K_Λ, we always mean modules M that are locally unital in the sense that $M = \bigoplus_{\lambda \in \Lambda} e_\lambda M$.

THEOREM 4.1. *Assume that every weight $\lambda \in \Lambda$ either has finitely many vertices labelled $\vee$ or finitely many vertices labelled $\wedge$. Then the category*

K_Λ-MOD *of finite dimensional* graded *left K_Λ-modules is a graded highest weight category with projective indecomposable modules $P(\lambda) := K_\Lambda e_\lambda$, standard modules $V(\lambda)$ and irreducible modules $L(\lambda)$; the gradings on these modules are fixed so that $L(\lambda)$ is one-dimensional concentrated in degree 0, and the canonical homomorphisms $P(\lambda) \twoheadrightarrow V(\lambda) \twoheadrightarrow L(\lambda)$ are grading-preserving. Moreover:*

(1) *For $\lambda, \mu \in \Lambda$, the graded decomposition number $[V(\lambda) : L(\mu)]_q$ is equal to q^n if $\lambda \supset \mu$, where n is the number of clockwise caps in the unique λ-cap diagram of weight μ; otherwise, $[V(\lambda) : L(\mu)]_q = 0$.*

(2) *The positively graded algebra K_Λ is standard Koszul, i.e. the irreducible modules $L(\lambda)$ and the standard modules $V(\lambda)$ have linear projective resolutions. Moreover the associated Kazhdan-Lusztig polynomials*
$$p_{\lambda,\mu}(q) := \sum_{i\geq 0} q^i \dim \mathrm{Ext}^i_{K_\Lambda}(V(\lambda), L(\mu))$$
are given explicitly by the following recurrence. First $p_{\lambda,\mu}(q) = 0$ unless $\lambda \leq \mu$, and $p_{\lambda,\lambda}(q) = 1$. Now assume that $\lambda < \mu$. Pick $i < j$ such that the ith vertex of λ is labelled $\vee$, the jth vertex is labelled $\wedge$, and all vertices in between are labelled $\circ$ or $\times$. For any weight diagram ν and $x, y \in \{\circ, \times, \vee, \wedge\}$ let $\nu[xy]$ be the weight diagram obtained from ν by relabelling vertex i by x and vertex j by y. Then
$$p_{\lambda,\mu}(q) = \begin{cases} p_{\lambda[\circ\circ],\mu[\circ\circ]}(q) + q p_{\lambda[\wedge\vee],\mu}(q) & \text{if } \mu = \mu[\vee\wedge], \\ q p_{\lambda[\wedge\vee],\mu}(q) & \text{otherwise.} \end{cases}$$

(3) *For fixed $\mu \in \Lambda$, we have that $p_{\lambda,\mu}(1) \leq 1$ for all $\lambda \in \Lambda$ if and only if it is impossible to find vertices $i < j < k < l$ whose labels in μ are $\wedge, \vee, \wedge, \vee$, respectively. In that case $L(\mu)$ possesses a BGG-type resolution*
$$\cdots \to V_1(\mu) \to V_0(\mu) \to L(\mu) \to 0$$
with $V_i(\mu) = \bigoplus_{\lambda \text{ s.t. } p_{\lambda,\mu}(q)=q^i} V(\lambda)\langle i \rangle$.

(4) *For $\lambda \in \Lambda$, we have that $P(\lambda)$ is injective if and only if λ is restricted.*

(5) *The algebra K_Λ decomposes into blocks as $K_\Lambda = \bigoplus_{\Gamma \in \Lambda/\sim} K_\Gamma$.*

The precise connection between K_Λ and the algebra in Theorem 3.3 is explained by the next theorem.

Theorem 4.2. *The endomorphism algebra*
$$K_d^{p,q} = \mathrm{End}_{H_d^{p,q}}\Big(\bigoplus_{\lambda \Vdash d} Y(\lambda)\Big)^{\mathrm{op}}$$
from Theorem 3.3 is canonically isomorphic to K_Λ for $\Lambda := \{\lambda \Vdash d\}$ (interpreting bipartitions as weight diagrams as explained in the previous section).

Under the isomorphism, $\binom{\mathtt{B}}{\mathtt{A}} \in K_\Lambda$ *corresponds to a map* $Y(\lambda) \to Y(\mu)$ *where* $\lambda := \mathrm{wt}(\mathtt{A})$ *and* $\mu := \mathrm{wt}(\mathtt{B})$*. This map is defined on* $v^\lambda_{\mathtt{T}} \in Y(\lambda)$ *by drawing the diagram* $\binom{\lambda}{\mathtt{T}}$ *(as defined in the previous section) under the diagram* $\binom{\mathtt{B}}{\mathtt{A}}$ *then iterating the surgery procedure in exactly the same way as in the definition of the multiplication of* K_Λ*, to obtain some diagrams* $\binom{\mu}{\mathtt{S}}$ *hence a sum of basis vectors* $v^\mu_{\mathtt{S}} \in Y(\mu)$*.*

Notes. Assuming $|\Lambda| < \infty$, the algebra K_Λ is the quasi-hereditary cover of the generalized Khovanov algebra H_Λ from [**BS1**], which was introduced already in [**CK**] and [**S1**, §5]. More precisely, letting Λ° denote the set $\{\lambda \in \Lambda \,|\, \lambda \text{ is restricted}\}$, the *generalized Khovanov algebra* is the (symmetric) subalgebra

$$H_\Lambda := \bigoplus_{\lambda,\mu \in \Lambda^\circ} e_\lambda K_\Lambda e_\mu$$

of K_Λ, and there is a double centralizer property implying that K_Λ-mod is a highest weight cover of H_Λ-mod in the sense of [**R1**, §4.2]; see [**BS2**, §6]. In the special case that the weights in Λ have the same number of labels $\wedge$ as $\vee$, the algebra H_Λ is exactly the original arc algebra introduced by Khovanov in [**K**]; in that case the diagrams indexing the basis for H_Λ involve only (closed) circles, no line segments, and the multiplication has an elegant formulation in terms of a certain TQFT. This interpretation is the key to proving that the multiplication as formulated above is well defined independent of the order of the surgery procedures and that it is associative; see [**BS1**]. Also in [**BS1**] we showed that the diagram bases for both K_Λ and for H_Λ are cellular bases in the sense of [**GL**]; in fact they are examples of *graded cellular algebras* as recently formalized by Hu and Mathas [**HM**, §2].

In the statement of Theorem 4.1, we have used the language of highest weight categories from [**CPS**] rather than of quasi-hereditary algebras because K_Λ is not necessarily finite dimensional. The assumption on Λ in the opening sentence of the theorem is necessary since without it the analogues of the standard modules $V(\lambda)$ have infinite length, but the remaining statements (1)–(5) of the theorem remain true without this assumption. Theorem 4.1(1) is [**BS1**, Theorem 5.2] and (5) is an easy consequence; for (2), (3) and (4) see [**BS2**, §5, §7 and §6], respectively. The recurrence relation for Kazhdan-Lusztig polynomials in Theorem 4.1(2) is the same as the recurrence for the Kazhdan-Lusztig polynomials attached to Grassmannians discovered by Lascoux and Schützenberger [**LS**, Lemme 6.6]. This coincidence is explained by Theorem 5.1 below. There is also a closed formula for these Kazhdan-Lusztig polynomials due again to Lascoux and Schützenberger; see [**BS2**, (5.3)] for an equivalent formulation in terms of cap diagrams.

Theorem 4.2 is [**BS3**, Corollary 8.15]. One consequence is that the level two Hecke algebra $H^{p,q}_d$ is itself Morita equivalent to the generalized Khovanov algebra H_Λ for $\Lambda := \{\lambda \Vdash d\}$; see [**BS3**, Theorem 6.2].

5. Category $\mathcal{O}$ for Grassmannians

Let $\mathfrak{g} := \mathfrak{gl}_{m+n}(\mathbb{C})$, $\mathfrak{t}$ be the Cartan subalgebra of diagonal matrices, and $\mathfrak{b}$ be the Borel subalgebra of upper triangular matrices. Let $\varepsilon_1, \dots, \varepsilon_{m+n}$ be the basis for $\mathfrak{t}^*$ dual to the obvious basis of $\mathfrak{t}$ consisting of the diagonal matrix units, and let $(.,.)$ be the bilinear form on $\mathfrak{t}^*$ with respect to which the ε_i's are orthonormal. For each $\lambda \in \mathfrak{t}^*$, let $L(\lambda)$ be an irreducible $\mathfrak{g}$-module of $\mathfrak{b}$-highest weight λ. Finally from now until the end of the section we let

$$\Lambda := \left\{ \lambda \in \mathfrak{t}^* \;\middle|\; \begin{array}{l} (\lambda+\rho, \varepsilon_1), \dots, (\lambda+\rho, \varepsilon_{m+n}) \in \mathbb{Z} \\ (\lambda+\rho, \varepsilon_1) > \cdots > (\lambda+\rho, \varepsilon_m) \\ (\lambda+\rho, \varepsilon_{m+1}) > \cdots > (\lambda+\rho, \varepsilon_{m+n}) \end{array} \right\} \tag{5.1}$$

where $\rho := -\varepsilon_2 - 2\varepsilon_3 - \cdots - (m+n-1)\varepsilon_{m+n}$.

We are interested in the category $\mathcal{O}(m,n)$ of all $\mathfrak{g}$-modules that are semisimple over $\mathfrak{t}$ and possess a composition series with composition factors of the form $L(\lambda)$ for $\lambda \in \Lambda$. This is the sum of all "integral" blocks of the parabolic analogue of the usual Bernstein-Gelfand-Gelfand category $\mathcal{O}$ corresponding to the standard parabolic subalgebra $\mathfrak{p}$ with Levi factor $\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$. It is a highest weight category with irreducible modules $\{L(\lambda) \mid \lambda \in \Lambda\}$, standard modules $\{V(\lambda) \mid \lambda \in \Lambda\}$ (which can be constructed explicitly as parabolic Verma modules) and projective indecomposable modules $\{P(\lambda) \mid \lambda \in \Lambda\}$.

We identify $\lambda \in \Lambda$ with a weight diagram in the sense of section 3 by putting the symbol $\vee$ at all vertices indexed by the set

$$\{(\lambda+\rho, \varepsilon_1), \dots, (\lambda+\rho, \varepsilon_m)\},$$

and the symbol $\wedge$ at all vertices indexed by the set

$$\{(\lambda+\rho, \varepsilon_{m+1}), \dots, (\lambda+\rho, \varepsilon_{m+n})\},$$

interpreting both labels as $\times$ and neither as $\circ$ as before. These weight diagrams are slightly different from the weight diagrams arising from bipartitions in section 3: there are now infinitely many $\circ$'s both to the left and the right. Viewing Λ as a set of weight diagrams in this way, it is closed under $\sim$. Let $K(m,n)$ be the arc algebra K_Λ from the previous section for this choice of Λ.

THEOREM 5.1. *Let $P := \bigoplus_{\lambda \in \Lambda} P(\lambda)$. The locally finite endomorphism*[1] *algebra $\mathrm{End}^{fin}_{\mathfrak{g}}(P)^{\mathrm{op}}$ is isomorphic to the arc algebra $K(m,n)$ so that $e_\lambda \in K(m,n)$ corresponds to the projection onto the summand $P(\lambda)$. Fixing such an isomorphism, the functor*

$$\mathrm{Hom}_{\mathfrak{g}}(P, ?) : \mathcal{O}(m,n) \to K(m,n)\text{-mod} \tag{5.2}$$

is an equivalence of categories sending $P(\lambda), V(\lambda), L(\lambda) \in \mathcal{O}(m,n)$ to (the ungraded versions of) the $K(m,n)$-modules with the same name from Theorem 4.1.

[1] A *locally finite* endomorphism means one that is zero on all but finitely many $P(\lambda)$'s.

The main idea for the proof of Theorem 5.1 is to exploit another Schur-Weyl duality which relates $\mathcal{O}(m,n)$ to the level two Hecke algebras $H_d^{p,q}$ from (3.2). To formulate this, we fix integers $p,q \in \mathbb{Z}$ so that $p-m = q-n$. Suppose $\lambda \Vdash d$ is a bipartition such that $h(\lambda^L) \leq m$ and $h(\lambda^R) \leq n$, where $h(\mu)$ denotes the *height* (number of non-zero parts) of a partition μ. Let $\bar{\lambda} \in \Lambda$ be the weight obtained by viewing λ as a weight diagram as in section 3, then changing the labels of all the vertices indexed by integers $\leq (p-m)$ from $\times$ to $\circ$. For example, if $p \leq q$ then the empty bipartition $\varnothing$ becomes the diagram

$$\bar{\varnothing} = \quad \cdots \quad \circ \; \overset{p-m}{\underset{q-n}{\circ}} \; \times \; \times \; \overset{p}{\times} \; \wedge \; \wedge \; \underset{q}{\wedge} \; \circ \; \circ \quad \cdots$$

A key point is that $L(\bar{\varnothing})$ is an irreducible projective module in $\mathcal{O}(m,n)$. Hence for $d \geq 0$ the module $L(\bar{\varnothing}) \otimes V^{\otimes d}$ is projective in $\mathcal{O}(m,n)$ too, where V is the natural $\mathfrak{g}$-module of column vectors.

THEOREM 5.2. *The algebra $H_d^{p,q}$ acts on the right on $L(\bar{\varnothing}) \otimes V^{\otimes d}$ so that s_r flips the rth and $(r+1)$th tensors in $V^{\otimes d}$ as usual, and L_1 acts as the endomorphism $\sum_{i,j=1}^{m+n} e_{i,j} \otimes e_{j,i} \otimes 1 \otimes \cdots \otimes 1$ (where $e_{i,j}$ denotes the ij-matrix unit in $\mathfrak{g}$). This action induces a* surjective *homomorphism*

$$H_d^{p,q} \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(L(\bar{\varnothing}) \otimes V^{\otimes d})^{\mathrm{op}}, \tag{5.3}$$

which is an isomorphism if and only if $d \leq \min(m,n)$. Moreover the exact functor

$$\mathrm{Hom}_{\mathfrak{g}}(L(\bar{\varnothing}) \otimes V^{\otimes d}, ?) : \mathcal{O}(m,n) \to H_d^{p,q}\text{-mod} \tag{5.4}$$

sends $P(\bar{\lambda})$ to $Y(\lambda)$ for all $\lambda \Vdash d$ with $h(\lambda^L) \leq m, h(\lambda^R) \leq n$, and it is fully faithful on the additive subcategory of $\mathcal{O}(m,n)$ generated by these projective modules.

To explain how to deduce Theorem 5.1 from Theorem 5.2, let Y, $H_d^{p,q}$ and $K_d^{p,q}$ be as in Theorem 3.3, and $K(m,n)$ be as in Theorem 5.1. Set $P := \bigoplus_\lambda P(\bar{\lambda})$, $e := \sum_\lambda e_\lambda \in K_d^{p,q}$ and $\bar{e} := \sum_\lambda e_{\bar{\lambda}} \in K(m,n)$, all sums over $\lambda \Vdash d$ such that $h(\lambda^L) \leq m, h(\lambda^R) \leq n$. It is obvious from Theorem 4.2 and the diagrammatic definition of the algebra $K(m,n)$ that $eK_d^{p,q}e \cong \bar{e}K(m,n)\bar{e}$. Applying Theorem 5.2 we get that

$$\mathrm{End}_{\mathfrak{g}}(P)^{\mathrm{op}} \cong \mathrm{End}_{H_d^{p,q}}(Ye)^{\mathrm{op}} \cong eK_d^{p,q}e \cong \bar{e}K(m,n)\bar{e}. \tag{5.5}$$

Theorem 5.1 follows from this on observing given any $\sim$-equivalence class Γ of weights from Λ that we can choose p,q and d so that all weights in Γ are of the form $\bar{\lambda}$ for $\lambda \Vdash d$ with $h(\lambda^L) \leq m, h(\lambda^R) \leq n$.

Notes. For a detailed account of the general theory of parabolic category $\mathcal{O}$ for a semisimple Lie algebra, see [**Hum**, ch. 9].

Theorem 5.2 is proved in [**BK1**], [**BK2**], and the deduction of Theorem 5.1 following the argument just sketched can be found in detail in

[**BS3**, §8]. For the special case $m = n$, the identification of the principal block of $\mathcal{O}(m,n)$ with the corresponding block of the diagram algebra $K(m,n)$ was established earlier by Stroppel [**S1**, Theorem 5.8.1] using an explicit presentation for the endomorphism algebra of a minimal projective generator for the category of perverse sheaves on the Grassmannian found by Braden in [**Br**] (this category of perverse sheaves being equivalent to the principal block of $\mathcal{O}(m,n)$ thanks to the Beilinson-Bernstein localization theorem). The idea that there should be such an isomorphism originates in unpublished work of Braden and Khovanov.

With Theorem 5.1 in hand, all the statements of Theorem 4.1 for Λ as in (5.1) are equivalent to previously known facts about $\mathcal{O}(m,n)$. In particular, the fact that blocks of $\mathcal{O}(m,n)$ are Koszul coming from Theorem 4.1(2) is a very special case of the general results of Beilinson, Ginzburg and Soergel [**BGS**] and Backelin [**Ba**]; the present approach is more explicit and algebraic in nature. The possibility of computing the composition multiplicities of parabolic Verma modules in a geometry-free way as in Theorem 4.1(1) was first realized by Enright and Shelton in [**ES**].

For an application of Theorem 5.1 to classify the indecomposable projective functors on $\mathcal{O}(m,n)$ in the sense of [**BG**], see [**BS3**, Theorem 1.2]. The problem of classifying indecomposable projective functors on parabolic category $\mathcal{O}$ for an arbitrary parabolic of an arbitrary semisimple Lie algebra remains open in general.

6. The general linear supergroup

In this section we discuss the application to the representation theory of the general linear supergroup $G := GL(m|n)$ over the ground field $\mathbb{C}$. Using scheme-theoretic language, G can be regarded as a functor from the category of commutative superalgebras over $\mathbb{C}$ to the category of groups, mapping a commutative superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ to the group $G(A)$ of all invertible $(m+n) \times (m+n)$ matrices of the form

$$g = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \tag{6.1}$$

where a (resp. d) is an $m \times m$ (resp. $n \times n$) matrix with entries in $A_{\bar{0}}$, and b (resp. c) is an $m \times n$ (resp. $n \times m$) matrix with entries in $A_{\bar{1}}$. Let B and T be the standard choices of Borel subgroup and maximal torus: for each commutative superalgebra A, the groups $B(A)$ and $T(A)$ consist of all matrices $g \in G(A)$ that are upper triangular and diagonal, respectively. Let $\varepsilon_1, \dots, \varepsilon_{m+n}$ be the usual basis for the character group $X(T)$ of T, i.e. ε_r picks out the rth diagonal entry of a diagonal matrix. Equip $X(T)$ with a symmetric bilinear form $(.,.)$ such that $(\varepsilon_r, \varepsilon_s) = (-1)^{\bar{r}}\delta_{r,s}$, where $\bar{r} := \bar{0}$ if $1 \leq r \leq m$ and $\bar{r} := \bar{1}$ if $m+1 \leq r \leq m+n$. Let

$$\Lambda := \left\{ \lambda \in X(T) \;\middle|\; \begin{array}{c} (\lambda+\rho, \varepsilon_1) > \cdots > (\lambda+\rho, \varepsilon_m), \\ (\lambda+\rho, \varepsilon_{m+1}) < \cdots < (\lambda+\rho, \varepsilon_{m+n}) \end{array} \right\} \tag{6.2}$$

denote the set of *dominant weights*, where $\rho := -\varepsilon_2 - 2\varepsilon_3 - \cdots - (m-1)\varepsilon_m + (m-1)\varepsilon_{m+1} + (m-2)\varepsilon_{m+2} + \cdots + (m-n)\varepsilon_{m+n}$.

We are interested here in the abelian category $\mathrm{Rep}(G)$ of finite dimensional representations of G; we allow arbitrary (not necessarily even) morphisms between G-modules so that the existence of kernels and cokernels is not quite obvious. The category $\mathrm{Rep}(G)$ is a highest weight category with irreducible objects $\{L(\lambda) \mid \lambda \in \Lambda\}$, standard objects $\{V(\lambda) \mid \lambda \in \Lambda\}$ and projective indecomposables $\{P(\lambda) \mid \lambda \in \Lambda\}$. In this setting the standard objects are called *Kac modules* and they can be constructed by geometric induction from B: we have that $V(\lambda) = H^0(G/B, \mathscr{L}(\lambda)^*)^*$ where G/B is Manin's flag superscheme and $\mathscr{L}(\lambda)$ denotes the G-equivariant line bundle on G/B attached to the weight λ.

We identify $\lambda \in \Lambda$ with a weight diagram obtained by putting the symbol $\vee$ on vertices indexed by the set

$$\{(\lambda+\rho, \varepsilon_1), \ldots, (\lambda+\rho, \varepsilon_m)\},$$

and the symbol $\wedge$ on all vertices indexed by the set

$$\mathbb{Z} \setminus \{(\lambda+\rho, \varepsilon_{m+1}), \ldots, (\lambda+\rho, \varepsilon_{m+n})\},$$

writing $\times$ for both and $\circ$ for neither as usual. Unlike the situations considered in sections 3 and 5, the non-trivial $\sim$-equivalence classes in Λ are all infinite, and all but finitely many vertices[2] are labelled $\wedge$. Let $K(m|n)$ be the arc algebra K_Λ from section 4 for this new choice of Λ.

THEOREM 6.1. *Let $P := \bigoplus_{\lambda \in \Lambda} P(\lambda)$. The locally finite endomorphism algebra $\mathrm{End}_G^{fin}(P)^{\mathrm{op}}$ is isomorphic to the arc algebra $K(m|n)$ so that $e_\lambda \in K(m|n)$ corresponds to the projection onto $P(\lambda)$. Fixing such an isomorphism, the functor*

$$\mathrm{Hom}_G(P, ?) : \mathrm{Rep}(G) \to K(m|n)\text{-mod} \tag{6.3}$$

is an equivalence of categories sending $P(\lambda), V(\lambda), L(\lambda) \in \mathrm{Rep}(G)$ to the $K(m|n)$-modules with the same name.

Again the proof involves a Schur-Weyl duality, though it is more subtle than in the previous section due to the existence of infinite $\sim$-equivalence classes of weights in Λ. To formulate the key result, we fix integers $p \leq q$. Suppose $\lambda \Vdash d$ is a bipartition such that $h(\lambda^L) \leq m, w(\lambda^L) \leq n+q-p, h(\lambda^R) \leq m+q-p, w(\lambda^R) \leq n$, where $h(\mu)$ denotes height and $w(\mu)$ denotes the *width* (largest part) of a partition μ. Let $\hat{\lambda} \in \Lambda$ be the weight obtained by viewing λ as a weight diagram as in section 3, then changing the labels of all the vertices indexed by integers $\leq (p-m)$ from $\times$ to $\wedge$ and

[2]The reader concerned by the apparent lack of symmetry here should note that we have already made a choice earlier in defining the parities $\bar{r}$ $(1 \leq r \leq m+n)$. We could also have set $\bar{r} := \bar{1}$ for $1 \leq r \leq m$ and $\bar{r} := \bar{0}$ for $m+1 \leq r \leq m+n$, a path which leads to weight diagrams in which all but finitely many vertices are labelled $\vee$.

all the ones indexed by integers $> (q+n)$ from $\circ$ to $\wedge$. For example the empty bipartition $\varnothing$ becomes

$$\hat{\varnothing} = \cdots \begin{array}{ccccccccccccc} & p-m & & & p & & & q & & & q+n & & \\ \wedge & \wedge & \times & \times & \times & \wedge & \wedge & \wedge & \circ & \circ & \circ & \wedge & \wedge \end{array} \cdots$$

Again $L(\hat{\varnothing})$ is an irreducible projective module in $\mathrm{Rep}(G)$. Hence for $d \geq 0$ the module $L(\hat{\varnothing}) \otimes V^{\otimes d}$ is projective in $\mathrm{Rep}(G)$ too, where V is the natural G-module of column vectors with standard basis $v_1, \dots, v_m, v_{m+1}, \dots, v_{m+n}$ and $\mathbb{Z}_2$-grading defined by putting v_r in degree $\bar{r}$.

Theorem 6.2. *The algebra $H_d^{p,q}$ acts on the right on $L(\hat{\varnothing}) \otimes V^{\otimes d}$ so that s_r flips the rth and $(r+1)$th tensors in $V^{\otimes d}$ with a sign if both vectors are odd, and L_1 acts as the endomorphism $\sum_{i,j=1}^{m+n} (-1)^{\bar{j}} e_{i,j} \otimes e_{j,i} \otimes 1 \otimes \cdots \otimes 1$ (where $e_{i,j}$ denotes the ij-matrix unit in the Lie superalgebra of G). This action induces a* surjective *homomorphism*

$$H_d^{p,q} \twoheadrightarrow \mathrm{End}_G(L(\hat{\varnothing}) \otimes V^{\otimes d})^{\mathrm{op}}, \tag{6.4}$$

which is an isomorphism if and only if $d \leq \min(m,n) + q - p$. Moreover the exact functor

$$\mathrm{Hom}_{\mathfrak{g}}(L(\hat{\varnothing}) \otimes V^{\otimes d}, ?) : \mathrm{Rep}(G) \to H_d^{p,q}\text{-mod} \tag{6.5}$$

sends $P(\hat{\lambda})$ to $Y(\lambda)$ for all restricted *$\lambda \Vdash d$ with $h(\lambda^L) \leq m, w(\lambda^L) \leq n+q-p, h(\lambda^R) \leq m+q-p, w(\lambda^R) \leq n$, and it is fully faithful on the additive subcategory of* $\mathrm{Rep}(G)$ *generated by these projective modules.*

If we mimic (5.5) with $P := \bigoplus_\lambda P(\hat{\lambda})$, $e := \sum_\lambda e_\lambda \in K_d^{p,q}$ and $\hat{e} := \sum_\lambda e_{\hat{\lambda}} \in K(m|n)$, where all sums are over restricted $\lambda \Vdash d$ such that $h(\lambda^L) \leq m, w(\lambda^L) \leq n+q-p, h(\lambda^R) \leq m+q-p, w(\lambda^R) \leq n$, we get that

$$\mathrm{End}_G(P)^{\mathrm{op}} \cong \mathrm{End}_{H_d^{p,q}}(Ye)^{\mathrm{op}} \cong eK_d^{p,q}e \cong \hat{e}K(m|n)\hat{e}. \tag{6.6}$$

Given any *finite* set Γ of weights from the same $\sim$-equivalence class in Λ, there exist $p \leq q$ and d such that all the weights in Γ are of the form $\hat{\lambda}$ for restricted $\lambda \Vdash d$ with $h(\lambda^L) \leq m, w(\lambda^L) \leq n+q-p, h(\lambda^R) \leq m+q-p, w(\lambda^R) \leq n$. So the endomorphism algebra of $\bigoplus_{\lambda \in \Gamma} P(\lambda)$ can be worked out from (6.6). This should at least make Theorem 6.1 rather plausible although this argument is no longer quite a proof.

Notes. The curious observation that $\mathrm{Rep}(G)$ (with not necessarily homogeneous morphisms) is abelian is made in [**CL**, §2.5]. In [**BS4**] we worked instead in a certain full subcategory $\mathscr{F}(m|n)$ of $\mathrm{Rep}(G)$ which is obviously abelian. Since every $M \in \mathrm{Rep}(G)$ is isomorphic via a not necessarily homogeneous isomorphism to an object in $\mathscr{F}(m|n)$ it follows that $\mathrm{Rep}(G)$ is abelian too. The fact that $\mathrm{Rep}(G)$ is a highest weight category is established in [**B**, Theorem 4.47].

Theorem 6.1 is proved in [**BS4**, Theorem 1.1] (see also [**BS4**, Lemmas 5.8–5.9]) by carefully taking a limit as $p \to -\infty$ and $q \to \infty$. Combined also with Theorem 4.1, it has several consequences for the structure of Rep(G). In particular using Theorem 4.1(2), we get that the category Rep(G) possesses a hidden Koszul grading in the spirit of [**BGS**].

From Theorem 4.1(1), we recover the following formula proved originally in [**B**] for the composition multiplicities of Kac modules:

$$[V(\lambda) : L(\mu)] = \begin{cases} 1 & \text{if } \lambda \supset \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}$$

In this setting, the Kazhdan-Lusztig polynomials from Theorem 4.1(2) were introduced originally by Serganova, motivated by the observation that

$$\operatorname{ch} L(\mu) = \sum_{\lambda \leq \mu} p_{\lambda,\mu}(-1) \operatorname{ch} K(\lambda). \tag{6.8}$$

Using geometric induction techniques, Serganova computed the characters of the finite dimensional irreducible G-modules already in [**Se**]. Recently Musson and Serganova [**MS**] have explained the connection between the alternating sum formula for the composition multiplicities $[V(\lambda) : L(\mu)]$ arising from Serganova's original work and the formula (6.7) from [**B**] in purely combinatorial terms.

For atypical λ, the character formula (6.8) involves an infinite sum so does not obviously imply a dimension formula for the irreducible G-modules, but Su and Zhang were able to make some simplifications to deduce such a result; see [**SuZ**]. Theorem 4.1(3) gives a BGG-type resolution for certain irreducible G-modules, including all *polynomial representations* for which this was established already by Cheng, Kwon and Lam [**CKL**].

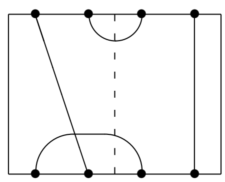
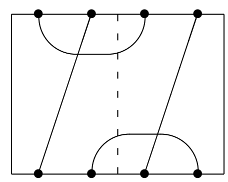
Theorem 6.2 is established in [**BS4**, §3]. The precise bound on d for (6.4) to be an isomorphism is a new observation; it follows from dimension considerations similar to [**BS4**, Theorem 3.9].

The combinatorial similarity between the formulae for composition multiplicities of Kac modules from (6.7) and of Specht modules from Theorem 3.2 was first pointed out by Leclerc and Miyachi in [**LM**], and is nicely explained by the Schur-Weyl duality in Theorem 6.2. A closely related coincidence, with a more conceptual combinatorial explanation, was pointed out by Cheng, Wang and Zhang in [**CWZ**], leading to their "super duality" conjecture. On combining Theorems 5.1 and 6.1, one can deduce a proof of this conjecture; a more direct proof was found subsequently in [**CL**].

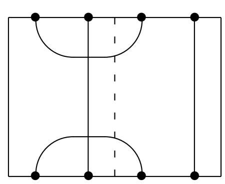
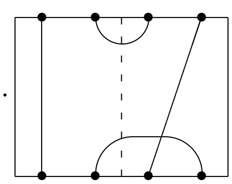
7. The walled Brauer algebra

In this section we formulate an even more recent result which explains some striking combinatorial coincidences observed recently by Cox and De Visscher. These coincidences suggest a functorial link between representations of the walled Brauer algebra $B_{r,s}(\delta)$ and of the generalised Khovanov algebra K_Λ in a situation in which weights in Λ have infinitely many vertices labelled $\wedge$ and infinitely many vertices labelled $\vee$, the one situation in which we did not know of an occurence of K_Λ "in nature" before.

Fix a parameter $\delta \in \mathbb{C}$. The walled Brauer algebra $B_{r,s}(\delta)$ is a certain subalgebra of the classical Brauer algebra $B_{r+s}(\delta)$. As a $\mathbb{C}$-vector space it has dimension $(r+s)!$, with a basis consisting of isotopy classes of diagrams drawn in a rectangle with $(r+s)$ vertices on its top and bottom edges, and a vertical wall separating the leftmost r from the rightmost s vertices. Each vertex must be connected to exactly one other vertex by a smooth curve drawn in the interior of rectangle, connected pairs of vertices on opposite edges must lie on the same side of the wall, and connected pairs of vertices on the same edge must lie on opposite sides of the wall. For example here are two basis vectors in $B_{2,2}(\delta)$:

$\alpha =$  $\beta =$ 

Multiplication is by concatenation of diagrams, so $\alpha\beta$ is obtained by putting α under β, interpreted as a basis vector by erasing closed circles in the interior of resulting diagram and multiplying by the scale factor δ each time such a circle is removed. For example, for α and β as above, we have:

$\alpha\beta =$  $\beta\alpha = \delta\cdot$ 

The algebra $B_{r,s}(\delta)$ is semisimple if $\delta \notin \{2-r-s, 3-r-s, \dots, r+s-2\}$. Introduce two more sets of bipartitions:

$$\Lambda_{r,s} := \left\{\lambda = (\lambda^L, \lambda^R) \;\middle|\; \lambda^L \vdash r-t, \lambda^R \vdash s-t, 0 \leq t \leq \min(r,s)\right\}, \tag{7.1}$$

$$\dot{\Lambda}_{r,s} := \begin{cases} \Lambda_{r,s} & \text{if } r \neq s \text{ or } \delta \neq 0 \text{ or } r = s = 0, \\ \Lambda_{r,s} \setminus \{(\varnothing, \varnothing)\} & \text{otherwise.} \end{cases} \tag{7.2}$$

The isomorphism classes of irreducible $B_{r,s}(\delta)$-modules are parametrised in a canonical way by the set $\dot{\Lambda}_{r,s}$; we write $D_{r,s}(\lambda)$ for the irreducible corresponding to $\lambda \in \dot{\Lambda}_{r,s}$.

We assume henceforth that $\delta \in \mathbb{Z}$ and identify bipartitions with certain weight diagrams, so that $\lambda = (\lambda^L, \lambda^R)$ corresponds to the weight diagram in which the vertices indexed by the set

$$\{\lambda_1^L, \lambda_2^L - 1, \lambda_3^L - 2, \dots\}$$

are labelled $\wedge$ and the vertices indexed by the set

$$\{1 - \delta - \lambda_1^R, 2 - \delta - \lambda_2^R, 3 - \delta - \lambda_3^R, \dots\}$$

are labelled $\vee$. This is a different rule from the one in section 3. For example, if $\delta = -2$ then

$$\begin{array}{rcccccccccccl} & & & & & 0 & & -\delta & & & & & \\ (\varnothing, \varnothing) = & \cdots & \wedge & \wedge & \wedge & \wedge & \circ & \circ & \vee & \vee & \vee & \vee & \cdots, \\ ((2^2 1), (32)) = & \cdots & \wedge & \circ & \wedge & \vee & \wedge & \times & \circ & \circ & \vee & \vee & \cdots. \end{array}$$

Note now there are always infinitely many vertices labelled $\wedge$ to the left and infinitely many vertices labelled $\vee$ to the right. Let Λ denote the set of all weight diagrams arising from all bipartitions in this way. Let $K(\delta)$ be the arc algebra K_Λ from section 4 for this choice of Λ, and denote its irreducible modules by $L(\lambda)$ for $\lambda \in \Lambda$.

THEOREM 7.1. *For $\delta \in \mathbb{Z}$, there is a Morita equivalence between the walled Brauer algebra $B_{r,s}(\delta)$ and the finite dimensional algebra $K_{r,s}(\delta) := e_{r,s} K(\delta) e_{r,s}$, where*

$$e_{r,s} := \sum_{\lambda \in \dot{\Lambda}_{r,s}} e_\lambda \in K(\delta).$$

Under the equivalence, the irreducible $B_{r,s}(\delta)$-module $D_{r,s}(\lambda)$ corresponds to the irreducible $K_{r,s}(\delta)$-module $L_{r,s}(\lambda) := e_{r,s} L(\lambda)$, for $\lambda \in \dot{\Lambda}_{r,s}$.

Assume at last that $\Lambda_{r,s} = \dot{\Lambda}_{r,s}$ in (7.2). This assumption ensures that $\dot{\Lambda}_{r,s}$ is an ideal in the poset $(\Lambda, \geq)$, where $\geq$ is the Bruhat order from section 3. Hence the algebra $K_{r,s}(\delta)$ is a standard Koszul algebra with weight poset $\Lambda_{r,s}$, as follows from Theorem 4.1 (though some care is needed since $K(\delta)$ itself does not satisfy the requirement from the opening sentence of that theorem). So Theorem 7.1 implies in particular that $B_{r,s}(\delta)$ is Koszul.

Notes. We refer to [**CD**] for a detailed account of the representation theory of the walled Brauer algebra in both the semisimple and non-semisimple cases. Essentially the same rule as described in this section for converting bipartitions to weight diagrams appears already in [**CD**, §4], and Theorem 7.1 is implicitly conjectured in [**CD**, Remark 9.4]. Theorem 7.1 is proved in [**BS5**], as an application of the results about $GL(m|n)$ from [**BS4**] (see the previous section), together with the Schur-Weyl duality between $GL(m|n)$ and $B_{r,s}(m-n)$ arising from their commuting actions on mixed tensor space $V^{\otimes r} \otimes (V^*)^{\otimes s}$.

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