Modular Littlewood-Richardson Coefficients

Jonathan Brundan and Alexander Kleshchev

Introduction

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \geq 0$. We are interested in polynomial representations of the general linear group $GL(n) := GL_n(\mathbb{F})$ and modular representations of the symmetric group Σ_r over $\mathbb{F}$. If $p = 0$ then the classical Littlewood-Richardson coefficients describe the decomposition of all of the following modules:

(M1) the restriction of an irreducible $GL(n)$-module to a Levi subgroup of $GL(n)$;

(M2) the tensor product of irreducible $GL(n)$-modules;

(M3) the restriction of an irreducible Σ_r-module to a Young subgroup of Σ_r;

(M4) the Σ_r-module induced from an irreducible module for a Young subgroup.

However in positive characteristic, very little is known about the structure of these modules. The goal of the article is to describe three quite different sources of connections between them, hence obtaining connections between various 'modular Littlewood-Richardson coefficients'.

In the modular case, we also have tilting modules (for general linear groups) and Young modules (for symmetric groups). So we also consider:

(M1$'$) the restriction of an indecomposable tilting module to a Levi subgroup of $GL(n)$;

(M2$'$) the tensor product of indecomposable tilting modules;

(M3$'$) the restriction of a Young module to a Young subgroup of Σ_r;

(M4$'$) the Σ_r-module induced from a Young module for a Young subgroup.

Our main results, in sections 2, 3 and 4 respectively, obtain the following connections between these modules:

First, we construct a polynomial induction functor from a Levi subgroup of $GL(n)$ to $GL(n)$, which plays the same role in the $GL(n)$-setting as ordinary induction does for symmetric groups. Our main result about this induction functor (Theorem 2.7) shows that, when applied to an outer tensor product of modules for the Levi subgroup, it gives an inner tensor product of $GL(n)$-modules in a precise way. This gives a functorial connection between branching rules and tensor products via 'Frobenius reciprocity'. For example, it allows us to give a direct relationship between the spaces of high weight vectors and the socles of the modules (M1) and (M2) (see Theorem 2.8 and Theorem 2.20). As a special case, taking the Levi subgroup to be a maximal torus of $GL(n)$, we recover a character formula due to Donkin [D3] (see Corollary 2.11).

Next, we use dual pairs to explain the connection between composition multiplicities in (M1) to the tilting module decomposition multiplicities in (M2$'$), and similarly between the composition multiplicities in (M2) and the tilting decomposition multiplicities in (M1$'$). The main result here is Theorem 3.7. The proof depends on a modular version of Howe duality (as in [D3] and [AR]) allowing us to construct a dual pair associated to a Levi subgroup of $GL(n)$. As a special case, taking the Levi subgroup to be a maximal torus of $GL(n)$ again, we recover two character formulae (see Corollary 3.8 and Corollary 3.9), the first of which is a result of Mathieu-Papadopoulo [MP].

Finally, we obtain the analogues of these results for the symmetric groups. To do this, we first explain how to apply Schur functors to tensor products of $GL(n)$-modules (resp. restrictions of $GL(n)$-modules to Levi subgroups), to obtain induced modules from Young subgroups (resp. restrictions to Young subgroups) in the symmetric group setting. Using this, we then translate our earlier connections in the $GL(n)$-setting to analogous connections for symmetric groups. See Theorems 4.10, 4.11, 4.16, 4.17 and 4.19.

To give the reader a flavour of the questions to be considered in this paper we formulate some special cases of our results. If G and H are groups, M is a G-module and N is an H-module, we denote by $M \boxtimes N$ the outer tensor product (which is a $G \times H$-module). If L is another G-module then $M \otimes L$ is the inner tensor product (which is a G-module with the diagonal action). If L is irreducible we denote by $[M : L]$ its multiplicity in M, and if L is indecomposable its multiplicity as a direct summand of M will be denoted $(M : L)$. For a subgroup $G_1 < G$ we write $M \downarrow_{G_1}$ (or just $M \downarrow$ if the subgroup is clear from the context) for the restriction of M to G_1. Finally, $S^i(M)$ and $\bigwedge^i(M)$ denote the ith symmetric and exterior powers of M respectively.

The compositions (resp. partitions) with at most n non-zero parts are identified with 'polynomial' weights (resp. 'polynomial' dominant weights) for $GL(n)$ and we denote the set of all such by $\Lambda(n)$ (resp. $\Lambda^+(n)$). The partitions of r in $\Lambda^+(n)$ are denoted by $\Lambda^+(n, r)$, and we write $|\lambda| = r$ to indicate that λ is a partition of r. The partitions λ with at most n non-zero parts and such that the first part λ_1 is at most m are denoted $\Lambda^+(n \times m)$. The transpose λ^t of a partition λ is the partition whose Young diagram is the transpose of the Young diagram λ (with respect to the main diagonal). If $\lambda \in \Lambda^+(n\times m)$ then $\lambda^t \in \Lambda^+(m\times n)$. For $\lambda \in \Lambda^+(n)$, we write $L_n(\lambda), \Delta_n(\lambda), \nabla_n(\lambda)$, and $T_n(\lambda)$ for the irreducible, standard (or Weyl), costandard and the indecomposable tilting modules over $GL(n)$ with highest weight λ, respectively. We always consider $GL(m)\times GL(n)$ as a subgroup of $GL(m+n)$ (embedded diagonally).

Theorem A. *Let $\mu, \nu \in \Lambda^+(m+n)$ be partitions such that μ has at most m non-zero parts, and ν has at most n non-zero parts. Set $\bar\mu = (\mu_1, \dots, \mu_m) \in \Lambda^+(m)$ and $\bar\nu = (\nu_1, \dots, \nu_n) \in \Lambda^+(n)$. Let M be any polynomial $GL(m+n)$-module. Then,*

(i) $\mathrm{Hom}_{GL(m+n)}(M, \nabla_{m+n}(\mu) \otimes \nabla_{m+n}(\nu)) \cong \mathrm{Hom}_{GL(m)\times GL(n)}(M\downarrow, \nabla_m(\bar\mu) \boxtimes \nabla_n(\bar\nu))$.

If in addition $|\mu| \le m$ and $|\nu| \le n$, then

(ii) $\mathrm{Hom}_{GL(m+n)}(M, L_{m+n}(\mu) \otimes L_{m+n}(\nu)) \cong \mathrm{Hom}_{GL(m)\times GL(n)}(M\downarrow, L_m(\bar\mu) \boxtimes L_n(\bar\nu))$.

If we take $M = \Delta_{n+m}(\lambda)$ for some $\lambda \in \Lambda^+(n+m)$, the numbers appearing in Theorem A(i) are – by standard properties of ∇-filtrations – the usual Littlewood-Richardson coefficients. If instead we take $M = L_{n+m}(\lambda)$, we obtain our first candidate for ***modular Littlewood-Richardson coefficients***. The other candidates are the numbers appearing in (i) and (ii) of the next theorem. We note that the $GL(m)\times GL(n)$-module $T_m(\mu)\boxtimes T_n(\nu)$ appearing in Theorem B is an indecomposable tilting module.

Theorem B. *Let $\lambda \in \Lambda^+((m+n)\times k)$, $\mu \in \Lambda^+(m \times k)$ and $\nu \in \Lambda^+(n \times k)$. Then,*

(i) $(T_{m+n}(\lambda)\downarrow_{GL(m)\times GL(n)} : T_m(\mu)\boxtimes T_n(\nu)) = [L_k(\mu^t)\otimes L_k(\nu^t) : L_k(\lambda^t)]$.

(ii) $[L_{m+n}(\lambda)\downarrow_{GL(m)\times GL(n)} : L_m(\mu)\boxtimes L_n(\nu)] = (T_k(\mu^t)\otimes T_k(\nu^t) : T_k(\lambda^t))$.

Now we state the character formulae mentioned earlier.

Theorem C. *Let $\lambda \in \Lambda^+(n \times m)$, $\mu = (\mu_1, \dots, \mu_n) \in \Lambda(n \times m)$. Let V and W be the natural $GL(n)$- and $GL(m)$-modules respectively. Then,*

(i) (Donkin) $\dim L_n(\lambda)_\mu = (S^{\mu_1}(V)\otimes \cdots \otimes S^{\mu_n}(V) : Q_n(\lambda))$, *where $Q_n(\lambda)$ is the the injective hull of $L_n(\lambda)$ in the category of polynomial $GL(n)$-modules.*

(ii) (Mathieu-Papadopoulo) $\dim L_n(\lambda)_\mu = (\bigwedge^{\mu_1}(W)\otimes\cdots\otimes\bigwedge^{\mu_n}(W) : T_m(\lambda^t))$.

(iii) $\dim T_n(\lambda)_\mu = [\bigwedge^{\mu_1}(W)\otimes\cdots\otimes\bigwedge^{\mu_n}(W) : L_m(\lambda^t)]$.

Now, let Σ_n be the symmetric group on n letters. The irreducible $\mathbb{F}\Sigma_n$ module corresponding to a *p-regular* partition λ of n is denoted by D^λ as in [J1]. Let Y^μ be the Young module corresponding to (an arbitrary) partition μ of n (see [J2]).

Theorem D. *Let λ, μ and ν be partitions of $m+n$, m, and n, respectively. Then,*

(i) $(Y^\lambda \downarrow_{\Sigma_m\times\Sigma_n} : Y^\mu \boxtimes Y^\nu) = [L_{m+n}(\mu)\otimes L_{m+n}(\nu) : L_{m+n}(\lambda)]$*;*

(ii) $((Y^\mu\boxtimes Y^\nu)\uparrow^{\Sigma_{m+n}} : Y^\lambda) = [L_{m+n}(\lambda)\downarrow_{GL(m)\times GL(n)} : L_m(\mu)\boxtimes L_n(\nu)]$.

Moreover, if λ, μ and ν are all p-regular, then

(iii) $[D^\lambda\downarrow_{\Sigma_m\times\Sigma_n} : D^\mu\boxtimes D^\nu] = (T_{m+n}(\mu)\otimes T_{m+n}(\nu) : T_{m+n}(\lambda))$*;*

(iv) $[(D^\mu\boxtimes D^\nu)\uparrow^{\Sigma_{m+n}} : D^\lambda] = (T_{m+n}(\lambda)\downarrow_{GL(m)\times GL(n)} : T_m(\mu)\boxtimes T_n(\nu))$.

The main results of the article will be applied in [BK], where we use our earlier work [K2, K3, K4, B1, B2] on modular branching rules from $GL(n)$ to $GL(n-1)$ to obtain new results about versions of 'translation functors' in general linear and symmetric groups. In particular, we will compute certain modular Littlewood-Richardson coefficients exactly.

Acknowledgement. The material on dual pairs in section 3 of the paper was explained to us by O. Mathieu. In particular, he communicated to us Lemma 3.1 and Theorem D(iii). We are very grateful to Prof. Mathieu for the illuminating discussion.

1 Preliminaries

We introduce some notation for rational and polynomial representations of $GL(n)$ and its Levi subgroups. We begin at the level of rational representations, following Jantzen [J]. A $GL(n)$-module always means a rational left $\mathbb{F}GL(n)$-module as in [J], and we denote the category of all $GL(n)$-modules by $GL(n)$-Rat. More generally, given any algebraic group G over $\mathbb{F}$, a G-module means a rational $\mathbb{F}G$-module and G-Rat denotes the category of all such G-modules.

Let $T(n)$ denote the fixed maximal torus of $GL(n)$ consisting of all diagonal invertible matrices, and let $B^+(n)$ denote the positive Borel subgroup consisting of all upper triangular invertible matrices. The ***character group*** of $T(n)$ is the free abelian group with generators $\varepsilon_1, \dots, \varepsilon_n$, where ε_i denotes the standard character defined by $\varepsilon_i(\mathrm{diag}(t_1, \dots, t_n)) = t_i$ for all $t_1, \dots, t_n \in \mathbb{F}^\times$. The ***root system*** of $GL(n)$ (relative to $T(n)$) is the set $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n, i \neq j\}$, and the root $\varepsilon_i - \varepsilon_j$ is ***positive*** if $i < j$.

We let $X(n)$ denote all n-tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ of integers, and call the elements of $X(n)$ ***weights***. We identify $X(n)$ with the character group of $T(n)$, by letting $(\lambda_1, \dots, \lambda_n) \in X(n)$ correspond to the character $\sum_i \lambda_i \varepsilon_i$. We have the usual ***dominance order*** on $X(n)$, defined by $\lambda > \mu$ if $(\lambda - \mu)$ is a sum of positive roots. A weight $(\lambda_1, \dots, \lambda_n) \in X(n)$ is ***dominant*** (relative to $B^+(n)$) precisely when $\lambda_1 \geq \dots \geq \lambda_n$, and we let $X^+(n)$ denote all such dominant weights. Say $\lambda \in X^+(n)$ is ***p-restricted*** if $\lambda_i - \lambda_{i+1} < p$ for $i = 1, \dots, n-1$.

For $\lambda \in X^+(n)$, we have the $GL(n)$-modules $L_n(\lambda), \Delta_n(\lambda)$ (denoted $V(\lambda)$ in [J]) and $\nabla_n(\lambda)$ (denoted $H^0(\lambda)$ in [J]) which are the irreducible, standard and costandard modules of highest weight λ respectively. We also denote the natural $GL(n)$-module $\Delta_n(\varepsilon_1)$ by V, with standard basis $e_1, \dots, e_n$. Let $f_1, \dots, f_n$ denote the dual basis for $V^* \cong \Delta_n(-\varepsilon_n)$.

Let $\Lambda(n) \subset X(n)$ denote all n-tuples $(\alpha_1, \dots, \alpha_n)$ satisfying $\alpha_i \geq 0$ for $i = 1, \dots, n$, and $\Lambda^+(n) := \Lambda(n) \cap X^+(n)$. Let $\Lambda(n, r) \subset \Lambda(n)$ denote all n-tuples $(\alpha_1, \dots, \alpha_n)$ satisfying $|\alpha| := \alpha_1 + \dots + \alpha_n = r$, and $\Lambda^+(n, r) := \Lambda(n, r) \cap X^+(n)$. We call elements of $\Lambda(n, r)$ ***compositions*** of r (with at most n non-zero parts), and elements of $\Lambda^+(n, r)$ ***partitions*** of r (with at most n non-zero parts).

Fix $a \geq 1$ and $\nu = (n_1, \dots, n_a) \in \Lambda(a, n)$ (a composition of n with at most a non-zero parts). Let $GL(\nu) = GL(n_1) \times \dots \times GL(n_a)$ denote the ***standard Levi subgroup*** of $GL(n)$ consisting of all invertible block diagonal matrices with block sizes $n_1, \dots, n_a$. Of course, if $\nu = (n)$ then $GL(\nu) = GL(n)$ while, at the other extreme, if $\nu = (1, \dots, 1)$ then $GL(\nu) = T(n)$.

The torus $T(n)$ is also a maximal torus of $GL(\nu)$. Given a weight $\lambda \in X(n)$ which is dominant with respect to the Borel subgroup $B^+(\nu) := GL(\nu) \cap B^+(n)$ of $GL(\nu)$, we denote the corresponding irreducible, standard and costandard $GL(\nu)$-modules of highest weight λ by $L_\nu(\lambda), \Delta_\nu(\lambda)$ and $\nabla_\nu(\lambda)$ respectively. Let

$$X(\nu) := X(n_1) \times \dots \times X(n_a).$$

The dominance order on $X(\nu)$ is defined as the product of the orders on $X(n_1), \dots, X(n_a)$. The bijective map $X(n) \to X(\nu)$ defined by $\lambda \mapsto (\lambda^{(1)}, \dots, \lambda^{(a)})$, where for $j = 1, \dots, a$, $\lambda^{(j)} := (\lambda_{n_1 + \dots + n_{j-1} + 1}, \dots, \lambda_{n_1 + \dots + n_j})$, induces a bijection between the $B^+(\nu)$-dominant weights in $X(n)$ and the set

$$X^+(\nu) := X^+(n_1) \times \dots \times X^+(n_a).$$

Moreover, for $\lambda \in X^+(\nu)$, the $GL(\nu)$-modules $L_\nu(\lambda), \Delta_\nu(\lambda)$ and $\nabla_\nu(\lambda)$ correspond under this bijection to the $GL(n_1) \times \dots \times GL(n_a)$-modules $L_{n_1}(\lambda^{(1)}) \boxtimes \dots \boxtimes L_{n_a}(\lambda^{(a)}), \nabla_{n_1}(\lambda^{(1)}) \boxtimes \dots \boxtimes \nabla_{n_a}(\lambda^{(a)})$ and $\Delta_{n_1}(\lambda^{(1)}) \boxtimes \dots \boxtimes \Delta_{n_a}(\lambda^{(a)})$ respectively, where $\boxtimes$ denotes outer tensor product.

Following [V], we say that the $GL(\nu)$-module M has a ***∇-filtration*** (resp. a ***Δ-filtration***) if it has an ascending filtration $0 = M_0 < M_1 < \dots$ with $\bigcup_{i \geq 0} M_i = M$ such that each factor M_i / M_{i-1} is isomorphic to a (possibly infinite) direct sum of copies of $\nabla_\nu(\lambda_i)$ (resp $\Delta_\nu(\lambda_i)$) for some $\lambda_i \in X^+(\nu)$.

We say a $GL(\nu)$-module is ***tilting*** if it has both a Δ-filtration and a ∇-filtration. Donkin [D3] has shown that, for each $\lambda \in X^+(\nu)$, there a unique (finite dimensional) indecomposable tilting module $T_\nu(\lambda)$ of highest weight λ, satisfying $[T_\nu(\lambda) : L_\nu(\lambda)] = 1$ and $[T_\nu(\lambda) : L_\nu(\mu)] = 0$ for $\mu \not\leq \lambda$.

For a left exact additive functor $\mathcal{F}$ from an abelian category with enough injectives to another abelian category, $R^i\mathcal{F}$ denotes its ith (right) derived functor. The category $GL(\nu)$-Rat contains enough injectives, and $\mathrm{Ext}^i_{GL(\nu)}(M, -)$ denotes $R^i \mathrm{Hom}_{GL(\nu)}(M, -)$. We recall the cohomological criterion for ∇-filtrations (see [J, II.4.16(b)] and the proof of [V, Theorem 3.2.7]):

1.1. *A module $M \in GL(\nu)$-Rat has a ∇-filtration if and only if $\mathrm{Ext}^1_{GL(\nu)}(\Delta(\lambda), M) = 0$ for all $\lambda \in X^+(\nu)$.*

Note that the last few definitions, and (1.1), include $GL(n)$ as a special case, taking $\nu = (n)$.

Following [J, I.2.7], there are two commuting left actions of $GL(n)$ on its coordinate ring $\mathbb{F}[GL(n)]$, the ***left regular*** and ***right regular*** actions, which we define for $g, g' \in GL(n), f \in \mathbb{F}[GL(n)]$ by $(g \cdot_l f)(g') = f(g^{-1}g')$ and $(g \cdot_r f)(g') = f(g'g)$ respectively. We stress that both $\cdot_l$ and $\cdot_r$ are left actions, in spite of their ambigious names! Regarding $GL(\nu)$ as a closed subgroup of $GL(n)$, we have the usual restriction functor $\mathrm{res}^{GL(n)}_{GL(\nu)} : GL(n)\text{-Rat} \to GL(\nu)\text{-Rat}$. For $M \in GL(\nu)$-Mod, we define the ***induced module*** $\mathrm{ind}^{GL(n)}_{GL(\nu)} M$ to be the set of $GL(\nu)$-fixed points $(M \otimes \mathbb{F}[GL(n)])^{GL(\nu)}$ where the $GL(n)$-action on the induced module comes from the right regular action of $GL(n)$ on $\mathbb{F}[GL(n)]$ and the trivial action on M, and the action of $GL(\nu)$ on $M \otimes \mathbb{F}[GL(n)]$ under which we are taking fixed points comes from the given action on M and the left regular action on $\mathbb{F}[GL(n)]$.

This gives a functor $\mathrm{ind}^{GL(n)}_{GL(\nu)} : GL(\nu)\text{-Rat} \to GL(n)\text{-Rat}$ which (precisely as in [J, I.3.4]) is right adjoint to the exact functor $\mathrm{res}^{GL(n)}_{GL(\nu)}$, so sends injectives to injectives. Note that this is not quite the same definition as in [J, I.3.3] – the roles of the left and right regular actions are swapped there. This is important for us in relating our results to Schur algebras in the most natural way. Our induction functor is naturally isomorphic to the one in [J, I.3.3] (for example, because both definitions give functors which are right adjoint to $\mathrm{res}^{GL(n)}_{GL(\nu)}$).

Now, by [R], the quotient $GL(n)/GL(\nu)$ is an affine variety, so [J, I.5.13(a)] implies:

1.2. *The functor $\mathrm{ind}^{GL(n)}_{GL(\nu)}$ is exact.*

Hence [J, I.4.6] gives us generalized Frobenius reciprocity:

1.3. *For $M \in GL(n)$-Rat, $N \in GL(\nu)$-Rat and $i \geq 0$,*

$$\mathrm{Ext}^i_{GL(n)}(M, \mathrm{ind}^{GL(n)}_{GL(\nu)} N) \cong \mathrm{Ext}^i_{GL(\nu)}(\mathrm{res}^{GL(n)}_{GL(\nu)} M, N)$$

We recall the Donkin-Mathieu theorem [W, D1, M]:

1.4. (i) *If $M, N \in GL(\nu)$-Rat are modules with ∇-filtrations (resp. tilting modules), then $M \otimes N$ has a ∇-filtration (resp. is tilting);*

(ii) *If $M \in GL(n)$-Rat has a ∇-filtration (resp. is tilting), then $\mathrm{res}^{GL(n)}_{GL(\nu)} M$ has a ∇-filtration (resp. is tilting).*

In particular, (1.4)(ii) combined with (1.3) and (1.1) shows:

1.5. *The functor* $\mathrm{ind}_{GL(\nu)}^{GL(n)}$ *sends modules with* ∇*-filtrations to modules with* ∇*-filtrations.*

We also note:

1.6. *For* $\lambda \in X^+(\nu)$, $T_\nu(\lambda) \cong T_{n_1}(\lambda^{(1)}) \boxtimes \cdots \boxtimes T_{n_a}(\lambda^{(a)})$.

To prove this, note that the outer tensor product is certainly a tilting module and it has the correct highest weight. So it suffices to show that the outer tensor product of two indecomposable tilting modules is indecomposable. This follows from the following general fact for which we were unable to find a reference:

1.7. **Lemma.** *Let* A *and* B *be two* $\mathbb{F}$*-algebras and let* M *and* N *be finite dimensional indecomposables for* A *and* B *respectively. Then,* $M \boxtimes N$ *is an indecomposable* $A \otimes B$*-module.*

Proof. Note that as M and N are indecomposable, the identity is a primitive idempotent in each of the finite dimensional algebras $\mathrm{End}_A(M)$ and $\mathrm{End}_B(N)$. Consequently (by the commutant correspondence described in section 3) $\mathrm{End}_A(M)$ and $\mathrm{End}_B(N)$ each possess a unique irreducible module, of dimension 1. So by [CR, (10.38)(iii)] and the fact that $\mathbb{F}$ is algebraically closed, $\mathrm{End}_A(M) \otimes \mathrm{End}_B(N)$ possesses a unique irreducible module, also of dimension 1. Hence, as $\mathrm{End}_{A\otimes B}(M \boxtimes N) \cong \mathrm{End}_A(M) \otimes \mathrm{End}_B(N)$ by [CR, (10.37)], the identity is a primitive idempotent of $\mathrm{End}_{A\otimes B}(M \boxtimes N)$, so $M \boxtimes N$ is indecomposable. □

Now we specialize to polynomial representations of $GL(n)$, following Green [G] and the appendix to Donkin's monograph [D5]. We note however that Green and Donkin consider only finite dimensional modules whereas we include infinite dimensional (locally finite) modules. All the results from [G, D5] that we need are valid in this slightly more general setting (without significant alterations to the proofs).

The coordinate ring $\mathbb{F}[GL(n)]$ of $GL(n)$ has the structure of a Hopf algebra, with comultiplication, counit and antipode coming from multiplication, unit and inversion in $GL(n)$. Given a coalgebra A over $\mathbb{F}$, Comod-A denotes the category of all right A-comodules (a right A-comodule means an $\mathbb{F}$-module M with structure map $\mu : M \to M \otimes A$ as in [S]). Every $GL(n)$-module is naturally a right $\mathbb{F}[GL(n)]$-comodule, and this gives an equivalence of categories between $GL(n)$-Rat and Comod-$\mathbb{F}[GL(n)]$.

Let $A(n)$ denote the subalgebra of $\mathbb{F}[GL(n)]$ generated by the functions $\{c_{ij} | 1 \leq i, j \leq n\}$, where c_{ij} picks out the ij-entry of a matrix $g \in GL(n)$. Then, $A(n)$ is a sub-bialgebra of $\mathbb{F}[GL(n)]$, isomorphic to the free polynomial algebra $\mathbb{F}[c_{ij} \,|\, 1 \leq i, j \leq n]$, and $\mathbb{F}[GL(n)]$ is the localization of $A(n)$ at the determinant function. For $r \geq 0$, the subspace $A(n, r) < A(n)$ consisting of all homogeneous polynomials of degree r in the c_{ij} is a sub-coalgebra of $A(n)$.

Let M be a left $GL(n)$-module with structure map $\mu : M \to M \otimes \mathbb{F}[GL(n)]$. We say M is a ***polynomial module*** (resp. a polynomial module of ***degree r***) if the image of μ lies in $M \otimes A(n)$ (resp. $M \otimes A(n, r)$). We denote the category of all left polynomial $GL(n)$-modules (resp. polynomial modules of degree r) by $M_{\mathbb{F}}(n)$ (resp. $M_{\mathbb{F}}(n, r)$). The category $M_{\mathbb{F}}(n)$ (resp. $M_{\mathbb{F}}(n, r)$) is equivalent to Comod-$A(n)$ (resp. Comod-$A(n, r)$). Moreover, by [G, 2.2c], any polynomial module is a direct sum of polynomial modules of various degrees, and $M_{\mathbb{F}}(n) \cong \bigoplus_{r \geq 0} M_{\mathbb{F}}(n,r)$.

The dual space $S(n,r) := A(n,r)^*$ of the coalgebra $A(n,r)$ inherits a natural algebra structure from the counit and comultiplication in $A(n,r)$; $S(n,r)$ is the ***Schur algebra*** of [G]. Given any algebra S over $\mathbb{F}$, S-Mod denotes the category of all (possibly infinite dimensional) left S-modules. By [G, §1] there is an equivalence of categories between $S(n,r)$-Mod and Comod-$A(n,r)$, hence between $M_{\mathbb{F}}(n,r)$ and $S(n,r)$-Mod. This can be described more directly using the surjective homomorphism $e : \mathbb{F}GL(n) \to S(n,r)$ of [G, 2.4], defined for $g \in GL(n)$ by letting $e(g) \in S(n,r) = A(n,r)^*$ be the unique element satisfying $e(g)(a) = a(g)$ for all $a \in A(n,r)$. We can regard any $S(n,r)$-module as an $\mathbb{F}GL(n)$-module via this surjection e, and Green shows that this gives the equivalence of categories between $M_{\mathbb{F}}(n,r)$ and $S(n,r)$-Mod.

A rational $GL(n)$-module M is polynomial (resp. polynomial of degree r) if and only if all weights of M are elements of $\Lambda(n)$ (resp. $\Lambda(n,r)$). Hence, the modules $L_n(\lambda), \Delta_n(\lambda), \nabla_n(\lambda)$ and $T_n(\lambda)$ lie in $M_{\mathbb{F}}(n)$ (resp. $M_{\mathbb{F}}(n,r)$) if and only if $\lambda \in \Lambda^+(n)$ (resp. $\Lambda^+(n,r)$). Noting that $M_{\mathbb{F}}(n)$ has enough injectives (which look like direct sums of injectives for various $S(n,r)$), we let $Q_n(\lambda)$ denote the injective hull of $L_n(\lambda)$ in the category $M_{\mathbb{F}}(n)$, for $\lambda \in \Lambda^+(n)$.

Now, the category $M_{\mathbb{F}}(n)$ is a full subcategory of $GL(n)$-Rat. So, given $M, N \in M_{\mathbb{F}}(n)$, we can compute $\mathrm{Ext}^i(M,N)$ either by computing $\mathrm{Ext}^i_{M_{\mathbb{F}}(n)}(M,N)$ in the category $M_{\mathbb{F}}(n)$, or by regarding M and N as elements of $GL(n)$-Rat and computing $\mathrm{Ext}^i_{GL(n)}(M,N)$ in the category $GL(n)$-Rat. By [D2, Theorem 2.1f], these two calculations of $\mathrm{Ext}^i(M,N)$ give the same answer:

1.8. *Given modules $M, N \in M_{\mathbb{F}}(n)$,* $\mathrm{Ext}^i_{M_{\mathbb{F}}(n)}(M,N) \cong \mathrm{Ext}^i_{GL(n)}(M,N)$.

Define the ***polynomial truncation*** functor $\mathrm{Pol} : GL(n)\text{-Rat} \to M_{\mathbb{F}}(n)$ by letting $\mathrm{Pol}\, M$ equal the largest polynomial submodule of M for $M \in GL(n)$-Rat, and by restriction on morphisms. Note Pol is a special case of Donkin's truncation functor O_π from [D2, 2.1] (take $\pi := \Lambda^+(n)$, a saturated subset of $X^+(n)$). We record the following basic property of Pol [D2, 2.1b]:

1.9. *For a module $M \in GL(n)$-Rat with a ∇-filtration,* $(R^i\,\mathrm{Pol})M = 0$ *for $i > 0$.*

Note that Pol is obviously right adjoint to the inclusion functor $M_{\mathbb{F}}(n) \to GL(n)$-Rat (which is exact). So, Pol is left exact and sends injectives to injectives.

1.10. **Lemma.** *Take $M \in M_{\mathbb{F}}(n)$ and $N \in GL(n)$-Rat, and suppose that N has a ∇-filtration. Then,* $\mathrm{Ext}^i_{GL(n)}(M,N) \cong \mathrm{Ext}^i_{GL(n)}(M, \mathrm{Pol}\, N)$.

Proof. By the adjoint functor property, there is an isomorphism of functors between $\mathrm{Hom}_{GL(n)\text{-Rat}}(M,-)$ and $\mathrm{Hom}_{M_{\mathbb{F}}(n)}(M,-) \circ \mathrm{Pol}$. Moreover, Pol sends injectives to injectives and $\mathrm{Hom}_{M_{\mathbb{F}}(n)}(M,-)$ is left exact, so for each $M \in M_{\mathbb{F}}(n)$, there is a Grothendieck spectral sequence with

$$E_2^{i,j} = \mathrm{Ext}^i_{M_{\mathbb{F}}(n)}(M, (R^j\,\mathrm{Pol})N) \Rightarrow \mathrm{Ext}^{i+j}_{GL(n)}(M,N).$$

If N has a ∇-filtration, then $(R^j \operatorname{Pol})N = 0$ for $j > 0$ by (1.9), so this degenerates to give the isomorphism

$$\operatorname{Ext}^i_{M_{\mathbb{F}}(n)}(M, \operatorname{Pol} N) \cong \operatorname{Ext}^i_{GL(n)}(M, N).$$

Now the lemma follows by (1.8). $\square$

Combining (1.1) and Lemma 1.10, we deduce:

1.11. *The functor* Pol *sends modules with ∇-filtrations to modules with ∇-filtrations.*

We finally extend the notion of polynomial representations from $GL(n)$ to the standard Levi subgroup $GL(\nu) < GL(n)$, for fixed $\nu \in \Lambda(a, n)$. The coordinate ring of $GL(\nu)$ is isomorphic to $\mathbb{F}[GL(n_1)] \otimes \cdots \otimes \mathbb{F}[GL(n_a)]$, which is the localization of $A(\nu) := A(n_1) \otimes \cdots \otimes A(n_a)$ at determinant. Now, $A(\nu)$ can be regarded as a free polynomial algebra in indeterminates c_{ij}, where $1 \leq i, j \leq n$ and both i and j lie in the same one of the following intervals: $[1, n_1], [n_1 + 1, n_1 + n_2], \ldots, [n_1 + \cdots + n_{a-1} + 1, n]$. If we let $A(\nu, r)$ denote the sub-coalgebra of $A(\nu)$ consisting of all homogeneous polynomials of degree r, then we have the coalgebra isomorphism

$$A(\nu, r) \cong \bigoplus_{\rho \in \Lambda(a,r)} A(n_1, \rho_1) \otimes \cdots \otimes A(n_a, \rho_a).$$

Define the category $M_{\mathbb{F}}(\nu)$ (resp. $M_{\mathbb{F}}(\nu, r)$) of ***polynomial $GL(\nu)$-modules*** (resp. of ***degree r***) to be all rational $GL(\nu)$-modules M for which the image of the structure map $\mu : M \to M \otimes \mathbb{F}[GL(\nu)]$ lies in $M \otimes A(\nu)$ (resp. $M \otimes A(\nu, r)$). Copying Green's construction, the category $M_{\mathbb{F}}(\nu, r)$ is equivalent to the category $S(\nu, r)$-Mod of left modules for the algebra $S(\nu, r) := A(\nu, r)^*$. Obviously,

$$S(\nu, r) \cong \bigoplus_{\rho \in \Lambda(a,r)} S(n_1, \rho_1) \otimes \cdots \otimes S(n_a, \rho_a). \tag{1.12}$$

Let $J(\nu)$ be the ideal of $A(n)$ generated by all c_{ij} with i and j *not* both lying in the same one of the intervals $[1, n_1], [n_1 + 1, n_1 + n_2], \ldots, [n_1 + \cdots + n_{a-1} + 1, n]$. It is graded by degree as $\bigoplus_{r \geq 0} J(\nu, r)$.

1.13. **Lemma.** *$S(\nu, r)$ is isomorphic as an algebra to the annihilator $J(\nu, r)^\circ$ of $J(\nu, r)$ in $S(n, r)$.*

Proof. The inclusion $GL(\nu) \hookrightarrow GL(n)$ induces a surjection $\mathbb{F}[GL(n)] \to \mathbb{F}[GL(\nu)]$ of the coordinate rings. Since $\mathbb{F}[GL(n)]$ and $\mathbb{F}[GL(\nu)]$ are the localizations of $A(n)$ and $A(\nu)$ respectively at determinant, this restricts to a surjection $A(n) \to A(\nu)$. So we can regard $A(\nu)$ as the bialgebra quotient $A(n)/J(\nu)$, and so $A(\nu, r) \cong A(n, r)/J(\nu, r)$. Hence, $S(\nu, r) = A(\nu, r)^*$ is isomorphic as an algebra to the annihilator $J(\nu, r)^\circ$. $\square$

1.14. **Lemma.** *If $e : \mathbb{F}GL(n) \to S(n, r)$ is the natural surjection, $e(\mathbb{F}GL(\nu)) \cong S(\nu, r)$.*

Proof. By definition of the map e, the annihilator $e(\mathbb{F}GL(\nu))^\circ < A(n,r)$ of $e(\mathbb{F}GL(\nu)) < S(n,r)$ is $\{f \in A(n,r) \mid f(g) = 0 \text{ for all } g \in GL(\nu)\}$, which is precisely $J(\nu,r)$. Taking annihilators, we deduce $e(\mathbb{F}GL(\nu)) = J(\nu,r)^\circ \cong S(\nu,r)$, by Lemma 1.13. □

Because of this lemma, we call $S(\nu,r)$ a ***standard Levi subalgebra*** of $S(n,r)$ (see [D4, Section 2] and [D5, 4.6] for the quantum analogue). We can also describe an explicit basis for $S(\nu,r)$ in terms of Green's basis for $S(n,r)$. Let $I(n,r)$ denote the set of all functions $\{1,\dots,r\} \to \{1,\dots,n\}$. We regard a function $\boldsymbol{i} \in I(n,r)$ as an r-tuple $\boldsymbol{i} = (i_1,\dots,i_r)$ where $i_k := \boldsymbol{i}(k)$. Define a right action of Σ_r on $I(n,r)$ by letting $\boldsymbol{i}\pi$ be the composition of functions $\boldsymbol{i} \circ \pi$, for $\boldsymbol{i} \in I(n,r), \pi \in \Sigma_r$. The set

$$I^2(n,r) := \{(\boldsymbol{i},\boldsymbol{j}) \in I(n,r) \times I(n,r) \mid j_1 \leq \cdots \leq j_r \text{ and } i_k \leq i_{k+1} \text{ whenever } j_k = j_{k+1}\}$$

is a set of orbit representatives for the diagonal action of Σ_r on $I(n,r) \times I(n,r)$. Then $\{c_{\boldsymbol{i},\boldsymbol{j}} \mid (\boldsymbol{i},\boldsymbol{j}) \in I^2(n,r)\}$ is a basis for $A(n,r)$. We let $\{\xi_{\boldsymbol{i},\boldsymbol{j}} \mid (\boldsymbol{i},\boldsymbol{j}) \in I^2(n,r)\}$ denote the dual basis for $S(n,r)$. With this notation, Lemma 1.13 also implies:

1.15. *The set of all $\xi_{\boldsymbol{i},\boldsymbol{j}}$ with $(\boldsymbol{i},\boldsymbol{j}) \in I^2(n,r)$ such that i_k and j_k belong to the same one of the intervals $[1,n_1],[n_1+1,n_1+n_2],\dots,[n_1+\cdots+n_{a-1}+1,n]$, $k = 1,2\dots,r$, is a basis of $S(\nu,r)$.*

Let $\Lambda^+(\nu,r)$ denote the set of all $\lambda = (\lambda^{(1)},\dots,\lambda^{(a)}) \in X^+(\nu)$ with $0 \leq \lambda_i^{(j)}$ for all i,j and $|\lambda^{(1)}| + \cdots + |\lambda^{(a)}| = r$. Then, by (1.12), the modules $L_\nu(\lambda), \Delta_\nu(\lambda), \nabla_\nu(\lambda)$ and $T_\nu(\lambda)$ for $\lambda \in X^+(\nu)$ lie in $M_{\mathbb{F}}(\nu,r)$ if and only if $\lambda \in \Lambda^+(\nu,r)$.

2 Polynomial induction functors

In this section, we introduce a 'polynomial induction' functor from Levi subgroups, and use it to obtain our first functorial connection between tensor products and branching rules.

Fix $\nu = (n_1,\dots,n_a) \in \Lambda(a,n)$ for some a, and let $GL(\nu) < GL(n)$ be the standard Levi subgroup. By definition, the algebra $A(\nu)$ (resp. $A(\nu,r)$) is the image of $A(n)$ (resp. $A(n,r)$) under the quotient map $\mathbb{F}[GL(n)] \to \mathbb{F}[GL(\nu)]$. Hence, the restriction of a polynomial module (resp. a polynomial module of degree r) from $GL(n)$ to $GL(\nu)$ is again a polynomial module (resp. a polynomial module of degree r). So we have the exact restriction functor

$$R^n_\nu : M_{\mathbb{F}}(n) \to M_{\mathbb{F}}(\nu),$$

which sends $M_{\mathbb{F}}(n,r)$ into $M_{\mathbb{F}}(\nu,r)$.

Now, the left and right regular actions of $GL(n)$ on $\mathbb{F}[GL(n)]$ stabilize $A(n)$ and $A(n,r)$. So, for $M \in M_{\mathbb{F}}(\nu)$, we can define $GL(n)$-modules $(M \otimes A(n))^{GL(\nu)}$ and $(M \otimes A(n,r))^{GL(\nu)}$, in the same way as the induced module $\mathrm{ind}_{GL(\nu)}^{GL(n)} = (M \otimes \mathbb{F}[GL(n)])^{GL(\nu)}$ was defined in section 1. Note that both $(M \otimes A(n))^{GL(\nu)}$ and $(M \otimes A(n,r))^{GL(\nu)}$ are submodules of $\mathrm{ind}_{GL(\nu)}^{GL(n)}$. Define the functor

$$I^n_\nu : M_{\mathbb{F}}(\nu) \to M_{\mathbb{F}}(n)$$

by letting $I^n_\nu M := (M \otimes A(n))^{GL(\nu)}$, with the obvious definition on morphisms.

We regard $S(n,r)$ as a left $GL(n)$-module by letting $g.s$ be the unique element of $S(n,r)$ such that $(g.s)(a) = s(g^{-1} \cdot_l a)$ for $s \in S(n,r)$, $a \in A(n,r) = S(n,r)^*$, $g \in GL(n)$. Similarly, we regard $S(n,r)$ as a right $GL(n)$-module by defining $(s.g)(a) = s(g \cdot_r a)$, for $s \in S(n,r), g \in GL(n)$. Note that $g.s = e(g)s$ and $s.g = se(g)$, where $e : \mathbb{F}GL(n) \to S(n,r)$ is the natural surjection. Now, $\mathrm{Hom}_{GL(\nu)}(S(n,r), M)$ is naturally a left $GL(n)$-module, with action $(gf)(s) = f(sg)$ for $g \in GL(n), s \in S(n,r)$ and $f \in \mathrm{Hom}_{GL(\nu)}(S(n,r), M)$.

2.1. **Lemma.** *For $M \in M_{\mathbb{F}}(\nu)$ and $r \geq 0$, the left $GL(n)$-modules $(M \otimes A(n,r))^{GL(\nu)}$ and $\mathrm{Hom}_{GL(\nu)}(S(n,r), M)$ are naturally isomorphic.*

Proof. Identify $M \otimes A(n,r)$ with $\mathrm{Hom}_{\mathbb{F}}(S(n,r), M)$ by letting a generator $m \otimes a \in M \otimes A(n,r)$ correspond to the homomorphism $s \mapsto s(a)m$ $(s \in S(n,r))$. □

Take $M \in M_{\mathbb{F}}(\nu, r)$. By the isomorphism $A(n) \cong \bigoplus_{s \geq 0} A(n,s)$ and Lemma 2.1,

$$I_\nu^n M \cong \bigoplus_{s \geq 0} \mathrm{Hom}_{GL(\nu)}(S(n,s), M).$$

Now $S(n,s)$ is by definition a polynomial $GL(\nu)$-module of degree s. But M is of degree r and there are no $GL(\nu)$-homomorphisms between polynomial modules of different degrees. Hence,

$$I_\nu^n M \cong \mathrm{Hom}_{GL(\nu)}(S(n,r), M) \cong (M \otimes A(n,r))^{GL(\nu)}.$$

So I_ν^n sends $M_{\mathbb{F}}(\nu, r)$ into $M_{\mathbb{F}}(n,r)$. Moreover, it sends finite dimensional modules to finite dimensional modules (unlike $\mathrm{ind}_{GL(\nu)}^{GL(n)}$).

2.2. **Lemma.** (i) *The functor I_ν^n is right adjoint to R_ν^n. Hence, I_ν^n is left exact and sends injectives in $M_{\mathbb{F}}(\nu)$ to injectives in $M_{\mathbb{F}}(n)$.*

(ii) *The functors $I_\nu^n : M_{\mathbb{F}}(\nu) \to M_{\mathbb{F}}(n)$ and $\mathrm{Pol} \circ \mathrm{ind}_{GL(\nu)}^{GL(n)} : M_{\mathbb{F}}(\nu) \to M_{\mathbb{F}}(n)$ are isomorphic.*

(iii) *Let $M \in M_{\mathbb{F}}(n)$ and $N \in M_{\mathbb{F}}(\nu)$. If N has a ∇-filtration, then*

$$\mathrm{Ext}^i_{GL(\nu)}(R_\nu^n M, N) \cong \mathrm{Ext}^i_{GL(n)}(M, I_\nu^n N)$$

for $i \geq 0$.

(iv) *I_ν^n sends modules with ∇-filtrations to modules with ∇-filtrations.*

Proof. For (i), it suffices to prove that R_ν^n and I_ν^n are adjoint when restricted to $M_{\mathbb{F}}(n,r)$ and $M_{\mathbb{F}}(\nu,r)$ respectively, for all $r \geq 1$. The functor $I_\nu^n : M_{\mathbb{F}}(\nu,r) \to M_{\mathbb{F}}(n,r)$ is isomorphic to the functor $\mathrm{Hom}_{S(\nu,r)}(S(n,r), -)$ by Lemma 2.1, and it is a well-known fact about finite dimensional algebras that this is right adjoint to restriction between $S(n,r)$-Mod and $S(\nu,r)$-Mod. To prove (ii), we simply note that the functor $\mathrm{Pol} \circ \mathrm{ind}_{GL(\nu)}^{GL(n)}$ is also right adjoint to R_ν^n, which follows as $\mathrm{ind}_{GL(\nu)}^{GL(n)}$ is right adjoint to $\mathrm{res}_{GL(\nu)}^{GL(n)}$ and Pol is right adjoint to the inclusion functor $M_{\mathbb{F}}(n) \to GL(n)$-Mod. Now (iii) follows easily from (ii), (1.5), Lemma 1.10 and (1.3), while (iv) follows from (ii), (1.5) and (1.11). □

Now fix $m \leq n$, and let $\nu = (m, 1, \dots, 1)$, a composition of n. Then, $GL(m)$ is a normal subgroup of the Levi subgroup $GL(\nu) \leq GL(n)$, and we have an exact inflation functor $\mathrm{Infl}_m^\nu : GL(m)\text{-Rat} \to GL(\nu)\text{-Rat}$ which sends each $M_{\mathbb{F}}(m, r)$ into $M_{\mathbb{F}}(\nu, r)$. Let $H(m) \leq GL(\nu)$ be the $(n-m)$-dimensional torus such that $GL(\nu) = GL(m) \times H(m)$. Then, the fixed point functor $M \mapsto M^{H(m)}$ is right adjoint to Infl_ν^m (see eg [J, I.6.4]). Now define functors $I_m^n : M_{\mathbb{F}}(m) \to M_{\mathbb{F}}(n)$ and $R_m^n : M_{\mathbb{F}}(n) \to M_{\mathbb{F}}(m)$ by the compositions

$$I_m^n := I_\nu^n \circ \mathrm{Infl}_m^\nu, \qquad R_m^n := (-)^{H(m)} \circ R_\nu^n.$$

We note that I_m^n is right adjoint to R_m^n, hence left exact. The exact functor R_m^n just takes the 0-weight space of a $GL(n)$-module with respect to the torus $H(m)$ so, as special cases of [J, II.2.11] and (since R_m^n commutes with contravariant duality) [J, II.5.21], we know:

2.3. *Fix $\lambda \in \Lambda^+(n)$ with $\lambda_{m+1} = \cdots = \lambda_n = 0$ (so λ has at most m non-zero rows). Let $\bar{\lambda} = (\lambda_1, \dots, \lambda_m) \in \Lambda^+(m)$. Then, $R_m^n \Delta_n(\lambda) \cong \Delta_m(\bar{\lambda})$ and $R_m^n L_n(\lambda) \cong L_m(\bar{\lambda})$.*

On the other hand, if $\lambda \in \Lambda^+(n)$ has $\lambda_{m+1} > 0$, then all weights μ of $\nabla_n(\lambda)$ also satisfy $\mu_{m+1} > 0$. Hence:

2.4. *For $\lambda \in \Lambda^+(n)$ with $\lambda_{m+1} > 0$, we have $R_m^n \Delta_n(\lambda) = R_m^n L_n(\lambda) = 0$.*

Moreover by Lemma 2.2(iv), I_m^n sends modules with ∇-filtrations to modules with ∇-filtrations. Hence, for $\mu \in \Lambda^+(m)$, $I_m^n \nabla_m(\mu)$ has a ∇-filtration, and on calculating the factors occuring in a ∇-filtration of $I_m^n \nabla_m(\mu)$ using [J, II.4.16(a)], together with the isomorphism $\mathrm{Hom}_{GL(n)}(\Delta_n(\lambda), I_m^n \nabla_m(\mu)) \cong \mathrm{Hom}_{GL(m)}(R_m^n \Delta_n(\lambda), \nabla_m(\mu))$ and (2.3),(2.4), one deduces:

2.5. *Let λ and $\bar{\lambda}$ be as in (2.3). Then, $I_m^n \nabla_m(\bar{\lambda}) \cong \nabla_n(\lambda)$.*

Now, let λ and $\bar{\lambda}$ be as in (2.3). Since I_m^n sends injectives to injectives, $I_m^n Q_m(\bar{\lambda})$ is injective. Moreover, for $\mu \in \Lambda^+(n)$, $\mathrm{Hom}_{GL(n)}(L_n(\mu), I_m^n Q_m(\bar{\lambda})) \cong \mathrm{Hom}_{GL(m)}(R_m^n L_n(\mu), Q_m(\bar{\lambda}))$. If $\mu_{m+1} = 0$ this is zero by (2.3) unless $\bar{\mu} = \bar{\lambda}$, when it is one dimensional. If $\mu_{m+1} > 0$, then $R_m^n L_n(\mu) = 0$ by (2.4), so again the Hom space is zero. This shows that $I_m^n Q_m(\bar{\lambda})$ is injective with simple socle $L_n(\lambda)$. Hence:

2.6. *Let λ and $\bar{\lambda}$ be as in (2.3). Then, $I_m^n Q_m(\bar{\lambda}) = Q_n(\lambda)$.*

It is not in general true that $I_m^n L_m(\bar{\lambda}) \cong L_n(\lambda)$ (but see (2.19)). For instance, consider $I_1^n L_1(a\varepsilon_1)$ which equals $\nabla_n(a\varepsilon_1)$ by (2.5), which is not necessarily irreducible. This example also shows that Lemma 2.2(iv) is false if we replace ∇-filtrations with Δ-filtrations.

Our main reason for introducing the functors I_m^n and I_ν^n is the following result describing polynomial induction applied to an outer tensor product:

2.7. **Theorem.** *Let $\nu \in \Lambda(a, n)$. Take any modules $M_1, \dots, M_a$ with $M_i \in M_{\mathbb{F}}(n_i)$, so that $M_1 \boxtimes \cdots \boxtimes M_a \in M_{\mathbb{F}}(\nu)$. Then*

$$I_\nu^n(M_1 \boxtimes \cdots \boxtimes M_a) \cong (I_{n_1}^n M_1) \otimes \cdots \otimes (I_{n_a}^n M_a).$$

Proof. Let $M = M_1 \boxtimes \cdots \boxtimes M_a$. We can identify $A(n)$ with the symmetric algebra $S(V^* \otimes V)$, letting the generator $c_{ij} \in A(n,1)$ correspond to $f_i \otimes e_j \in V^* \otimes V$. Having done this, it is routine to check that the left regular (resp. right regular) action of $GL(n)$ on $A(n)$ corresponds to the action of $GL(n)$ on $S(V^* \otimes V)$ coming from the action on V^* (resp. V). By the definition of I_ν^n and these remarks,

$$I_\nu^n M \cong [M \otimes S(V^* \otimes V)]^{GL(\nu)}$$

where the $GL(n)$-action on $I_\nu^n M$ comes just from its action on V and the $GL(\nu)$-action under which we are taking fixed points comes from the action on M and V^*. As a $GL(\nu)$-module, $V^* \cong V_1^* \oplus \cdots \oplus V_a^*$, where V_i is the natural module for the ith factor $GL(n_i)$ of $GL(\nu)$ (and $GL(n_j)$ acts trivially on V_i for $j \neq i$). So, as a $GL(\nu) \times GL(n)$-module,

$$\begin{aligned} M \otimes S(V^* \otimes V) &\cong M \otimes S((V_1^* \otimes V) \oplus \cdots \oplus (V_a^* \otimes V)) \\ &\cong M \otimes S(V_1^* \otimes V) \otimes \cdots \otimes S(V_a^* \otimes V) \\ &\cong [M_1 \otimes S(V_1^* \otimes V)] \otimes \cdots \otimes [M_a \otimes S(V_a^* \otimes V)], \end{aligned}$$

where the $GL(n_i)$-action comes from action on M_i and V_i^*, and the $GL(n)$-action comes from the action on V only. Taking fixed points, we deduce

$$I_\nu^n(M_1 \boxtimes \cdots \boxtimes M_a) \cong [M_1 \otimes S(V_1^* \otimes V)]^{GL(n_1)} \otimes \cdots \otimes [M_a \otimes S(V_a^* \otimes V)]^{GL(n_a)}.$$

So it remains to show that if $m \leq n$ and N is a polynomial $GL(m)$-module, then $I_m^n N \cong [N \otimes S(W^* \otimes V)]^{GL(m)}$ where W denotes the natural $GL(m)$-module. But by definition, $I_m^n N \cong [\mathrm{Infl}_m^n N \otimes S(V^* \otimes V)]^{GL(m) \times H(m)}$. Noting that $S(V^* \otimes V)^{H(m)} \cong S(W^* \otimes V)$ as a $GL(m) \times GL(n)$-module, the result follows on taking $H(m)$-fixed points. □

2.8. **Theorem.** *Let $\nu \in \Lambda(a,n)$ and $\mu^{(1)}, \dots, \mu^{(a)} \in \Lambda^+(n)$ be partitions such that $\mu^{(i)}$ has at most n_i non-zero rows for each i. Let $\bar{\mu}^{(i)} = (\mu_1^{(i)}, \dots, \mu_{n_i}^{(i)}) \in \Lambda^+(n_i)$. For any $M \in M_{\mathbb{F}}(n)$,*

$$\mathrm{Hom}_{GL(n)}(M, \nabla_n(\mu^{(1)}) \otimes \cdots \otimes \nabla_n(\mu^{(a)})) \cong \mathrm{Hom}_{GL(\nu)}(R_\nu^n M, \nabla_{n_1}(\bar{\mu}^{(1)}) \boxtimes \cdots \boxtimes \nabla_{n_a}(\bar{\mu}^{(a)})).$$

Proof. This is just the fact that I_ν^n is right adjoint to R_ν^n, together with Theorem 2.7 and (2.5). □

We record an elementary lemma:

2.9. **Lemma.** *Let M be a $GL(n-1)$-module all of whose weights are of the form μ with $\mu_1 + \cdots + \mu_{n-1} = s$ and N be a $GL(n)$-module all of whose weights are of the form ν with $n_1 + \cdots + n_n = r$. Then,* $\mathrm{Hom}_{GL(n-1)}(M, N) \cong \mathrm{Hom}_{GL(n-1,1)}(M \boxtimes \Delta_1((r-s)\varepsilon_1), N)$.

Proof. The image of M under any $GL(n-1)$-homomorphism lies in the subspace of N consisting of the sum of all weight spaces N_γ with $\gamma_1 + \cdots + \gamma_{n-1} = s$, hence $\gamma_n = (r-s)$. Now the conclusion is obvious. □

We now state some corollaries of Theorem 2.8. The first will be used in [BK].

2.10. **Corollary.** *Fix $\lambda, \mu \in \Lambda^+(n)$ with $\mu_n = 0$. Then,*

$$\mathrm{Hom}_{GL(n)}(L_n(\lambda), \nabla_n(\mu) \otimes S^\ell(V)) \cong \mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\bar{\mu}), L_n(\lambda) \downarrow_{GL(n-1)})$$

where $\ell = |\lambda| - |\mu|$.

Proof. By Theorem 2.8 and contravariant duality, the left hand side is isomorphic to $\mathrm{Hom}_{GL(n-1,1)}(\Delta_{n-1}(\bar{\mu}) \boxtimes \Delta_1(\ell\varepsilon_1), L_n(\lambda) \downarrow_{GL(n-1,1)})$. Now apply Lemma 2.9. $\square$

The second corollary is a result of Donkin [D3, Lemma 3.4(i)]:

2.11. **Corollary (Donkin).** *Let $\lambda \in \Lambda^+(n)$ and $\mu \in \Lambda(n)$. Then,*

$$\dim L_n(\lambda)_\mu = (S^{\mu_1}(V) \otimes \cdots \otimes S^{\mu_n}(V) : Q_n(\lambda)),$$

the multiplicity of $Q_n(\lambda)$ as summand of the injective module $S^{\mu_1}(V) \otimes \cdots \otimes S^{\mu_n}(V)$.

Proof. We first need to observe that $S^{\mu_1}(V) \otimes \cdots \otimes S^{\mu_n}(V)$ is indeed injective in $M_{\mathbb{F}}(n)$, which is proved in [D3, Lemma 3.4(i)] (it is a summand of $A(n)$). Now, taking $GL(\nu) = T(n)$, $\dim L(\lambda)_\mu = \dim \mathrm{Hom}_{GL(n)}(L_n(\lambda), S^{\mu_1}(V) \otimes \cdots \otimes S^{\mu_a}(V))$ by Theorem 2.8. The result follows immediately as $Q_n(\lambda)$ has simple socle $L_n(\lambda)$. $\square$

2.12. **Corollary.** *For $r \leq n$, $V^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda^+(n,r)} Q_n(\lambda)^{d_\lambda}$ where $d_\lambda := \dim L_n(\lambda)_\omega$ and $\omega = (1, \dots, 1, 0, \dots, 0) \in \Lambda(n, r)$. In particular, $V^{\otimes r}$ has p-restricted socle.*

Proof. The first statement is immediate from Corollary 2.11. To deduce the second statement, note that $Q_n(\lambda)$ has simple socle $L_n(\lambda)$, while by [G, 6.4b], d_λ is non-zero if and only if λ is p-restricted. $\square$

In the remainder of the section, we will study the functors R^n_m and I^n_m in more detail, by relating them to Green's functor $d_{n,m}$ from [G, 6.5]. For later use, we first recall some standard properties of Schur functors, then specialize to the case in hand. The basic references here are [G, Chapter 6] and [JS], but as we work with the right adjoint 'induction' functor whereas [G, JS] work with the left adjoint, we have given some of the proofs.

Suppose that S is a finite dimensional algebra over a field $\mathbb{F}$, and $e \in S$ is an idempotent. The ***Schur functor*** R_e is the functor $R_e : S\text{-Mod} \to eSe\text{-Mod}$ defined on modules by $M \mapsto eM$, and by restriction on morphisms. Equivalently, $R_e(M)$ is the module $\mathrm{Hom}_S(Se, M)$, regarded as an eSe-module by the action $(s\theta)(s') = \theta(s's)$ for $s \in eSe, s' \in Se, \theta \in \mathrm{Hom}_S(Se, M)$. By [G, 6.2a], the functor R_e is exact. We recall [G, 6.2g]:

2.13. *Let $\{L(\lambda) \mid \lambda \in X\}$ be a full set of non-isomorphic irreducible modules in S-Mod, indexed by a set X. Let $X' = \{\lambda \in X \mid R_e L(\lambda) \neq 0\}$. Then, $\{R_e L(\lambda) \mid \lambda \in X'\}$ is a full set of non-isomorphic irreducible modules in eSe-Mod.*

Given $N \in eSe\text{-Mod}$, we let $I_e(N) := \mathrm{Hom}_{eSe}(eS, N)$, regarded as an element of S-Mod by defining the S-action to be $(s\theta)(s') = \theta(s's)$, for $s \in S, s' \in eS, \theta \in I_e(N)$. With the obvious definition on morphisms, this gives a functor $I_e : eSe\text{-Mod} \to S\text{-Mod}$ which is right adjoint to R_e. In other words:

2.14. *For $M \in S$-Mod, $N \in eSe$-Mod,* $\mathrm{Hom}_S(M, I_e N) \cong \mathrm{Hom}_{eSe}(R_e M, N)$.

Hence, I_e is left exact and sends injective eSe-modules to injective S-modules. Note $R_e(I_e(N)) = e\,\mathrm{Hom}_{eSe}(eS, N) \cong \mathrm{Hom}_{eSe}(eSe, N) \cong N$, so I_e is a 'right inverse' to R_e:

2.15. *For $N \in eSe$-Mod, $R_e(I_e(N)) \cong N$.*

For a module $M \in S$-Mod, we let $M_{(e)}$ denote the largest submodule of M annihilated by e, and $M^{(e)}$ denote the smallest submodule such that $M/M^{(e)}$ is annihilated by e. We say a module M is ***e-restricted*** if $M^{(e)} = M$ and $M_{(e)} = 0$. Observe that $M_{(e)} = 0$ if and only if the socle of M is e-restricted, while $M^{(e)} = M$ if and only if the head of M is e-restricted. Moreover, [JS, Lemma 2.1] proves that:

2.16. *For $M \in S$-Mod, $M^{(e)} = SeM$.*

Now, take $N \in S$-Mod and let $\hat{N} := I_e(R_e(N))$. Since $N \cong \mathrm{Hom}_S(S, N)$ and $\hat{N} = \mathrm{Hom}_{eSe}(eS, eN)$, there is a natural restriction map $\mathrm{res} : N \to \hat{N}$.

2.17. **Lemma.** (i) *For $N \in S$-Mod and $\hat{N} := I_e(R_e(N))$, the kernel of the natural restriction map* $\mathrm{res} : N \to \hat{N}$ *equals $N_{(e)}$, while the image of* res *contains $\hat{N}^{(e)}$.*

(ii) *Suppose $N \in S$-Mod has e-restricted socle and $M \in S$-Mod has e-restricted head. Then,* $\mathrm{Hom}_{eSe}(R_e M, R_e N) \cong \mathrm{Hom}_S(M, N)$.

(iii) *If $SeS = S$, the functors R_e and I_e induce an equivalence of categories between S-Mod and eSe-Mod.*

Proof. (i) The image under res of $n \in N$ is the homomorphism $\mathrm{res}(n) \in \mathrm{Hom}_{eSe}(eS, eN)$ where $\mathrm{res}(n)(es) = esn$ for all $es \in eS$. So, $\mathrm{res}(n) = 0$ if and only if $eSn = 0$, which is if and only if $n \in N_{(e)}$. Hence, $\ker \mathrm{res} = N_{(e)}$. For the second statement, it suffices by (2.16) to show that $e\hat{N} \subset \mathrm{res}(N)$. But, $e\hat{N} = e\,\mathrm{Hom}_{eSe}(eS, eN) = \mathrm{res}(eN) \subset \mathrm{res}(N)$ as required. (This is an analogue of [JS, 2.11(ii)].)

(ii) By (2.14), $\mathrm{Hom}_{eSe}(R_e M, R_e N) \cong \mathrm{Hom}_S(M, \hat{N})$ where $\hat{N} = I_e(R_e(N))$. By (i) and the fact that $N_{(e)} = 0$, $\hat{N}$ is an extension of N by a module annihilated by e. But M has e-restricted head, so any homomorphism from M to $\hat{N}$ must have image wholly contained in N, and $\mathrm{Hom}_S(M, \hat{N}) \cong \mathrm{Hom}_S(M, N)$.

(iii) For $N \in eSe$-Mod, (2.15) shows that $R_e(I_e(N)) \cong N$. The assumption $SeS = S$ is equivalent by (2.16) to S having e-restricted head as a left S-module, or all irreducible S-modules being e-restricted. Hence, for all $M \in S$-Mod, $M_{(e)} = 0$ and $M^{(e)} = M$. Given this, (i) implies that $M \cong I_e(R_e(M))$ for $M \in S$-Mod. Finally, one checks that these isomorphisms are functorial. $\square$

Now we specialize to the functors I_m^n and R_m^n defined earlier, for $n \geq m \geq 1$. On restricting I_m^n and R_m^n to the subcategories $M_{\mathbb{F}}(m, r)$ and $M_{\mathbb{F}}(n, r)$ respectively, we obtain functors $I_{m,r}^{n,r} : M_{\mathbb{F}}(m, r) \to M_{\mathbb{F}}(n, r)$ and $R_{m,r}^{n,r} : M_{\mathbb{F}}(n, r) \to M_{\mathbb{F}}(m, r)$. As $R_{m,r}^{n,r}$ amounts to picking certain weight spaces, it can be identified with the Schur functor $R_{e_{n,m}}$, where $e_{n,m} \in S(n, r)$ is the idempotent defined as in [G, 6.5] ($R_{e_{n,m}}$ is the functor denoted $d_{n,m}$ by Green). As I_m^n is right adjoint to R_m^n, $I_{m,r}^{n,r}$ is right adjoint to $R_{m,r}^{n,r}$. Now, $I_{e_{n,m}}$ is also right adjoint to $R_{m,r}^{n,r} = R_{e_{n,m}}$, so $I_{m,r}^{n,r}$ can be identified with $I_{e_{n,m}}$. Now, (2.15) and the isomorphism $M_{\mathbb{F}}(m) \cong \bigoplus_{r \geq 0} M_{\mathbb{F}}(m, r)$ imply:

2.18. *For* $M \in M_{\mathbb{F}}(m)$, $R^n_m(I^n_m(M)) \cong M$.

If $r \leq m \leq n$, then $R^{n,r}_{m,r} L_n(\lambda)$ is non-zero for *all* $\lambda \in \Lambda^+(n,r)$ by (2.3). Hence, in this case *all* $S(n,r)$-modules have $e_{n,m}$-restricted head, so $S(n,r) = S(n,r)e_{n,m}S(n,r)$ by (2.16). Now Lemma 2.17(iii) shows that $I^{n,r}_{m,r} : M_{\mathbb{F}}(m,r) \to M_{\mathbb{F}}(n,r)$ is an equivalence of categories for $r \leq m \leq n$, which is a result of Green [G, 6.5g]. Hence:

2.19. *Let* λ *and* $\bar{\lambda}$ *be as in (2.3). If* $|\lambda| \leq m \leq n$, *then* $I^n_m L_m(\bar{\lambda}) \cong L_n(\lambda)$.

Now we can prove an analogue of Theorem 2.8 for irreducible modules:

2.20. **Theorem.** *With the notation of Theorem 2.8, assume in addition* $|\mu^{(i)}| \leq n_i$ *for each* i. *Then,*

$$\mathrm{Hom}_{GL(n)}(M, L_n(\mu^{(1)}) \otimes \cdots \otimes L_n(\mu^{(a)})) \cong \mathrm{Hom}_{GL(\nu)}(R^n_\nu M, L_{n_1}(\bar{\mu}^{(1)}) \boxtimes \cdots \boxtimes L_{n_a}(\bar{\mu}^{(a)})).$$

Proof. Use the fact that I^n_ν is right adjoint to R^n_ν, together with Theorem 2.7 and (2.19). □

2.21. **Remarks.** (I) Of particular interest are the cases when $M = \Delta_n(\lambda)$ or $L_n(\lambda)$ for some $\lambda \in \Lambda^+(n)$. In the latter case, the theorem then gives a connection between the *socle* of a tensor product and the *head* of a restriction to a Levi subgroup.

(II) We end with an example to show that Theorem 2.20 is in general false if we remove the assumption $|\mu^{(i)}| \leq n_i$. Let $p = 2$. By Steinberg's tensor product theorem, $L_3(\varepsilon_1 + \varepsilon_2) \otimes L_3(2\varepsilon_1) \cong L_3(3\varepsilon_1 + \varepsilon_2)$. So,

$$\mathrm{Hom}_{GL(3)}(L_3(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3), L_3(\varepsilon_1 + \varepsilon_2) \otimes L_3(2\varepsilon_1)) = 0.$$

But $L_3(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ is just the natural $GL(3)$-module tensored with determinant, so its restriction to the Levi subgroup $GL(2,1)$ contains $L_2(\varepsilon_1 + \varepsilon_2) \boxtimes L_1(2\varepsilon_1)$ as a summand. So,

$$\mathrm{Hom}_{GL(2,1)}(L_3(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\downarrow, L_2(\varepsilon_1 + \varepsilon_2) \boxtimes L_1(2\varepsilon_1)) \neq 0.$$

3 Dual pairs and tilting modules

Next, we use dual pairs to explain the connection between tensor products and restrictions of irreducible $GL(n)$-modules to restrictions and tensor products of indecomposable tilting modules. We begin in a general setting and then specialize to Schur algebras.

Let M be a finite dimensional vector space over $\mathbb{F}$. Let A and C be subalgebras of $\mathrm{End}_{\mathbb{F}}(M)$. We say that (A, C) is a ***dual pair*** on M if

(D1) $C = \mathrm{End}_A(M)$;

(D2) $A = \mathrm{End}_C(M)$.

Assume now that only (D1) holds. Let

$$M = \bigoplus_{i \in I} T_i$$

be a decomposition of M into a direct sum of indecomposable A-modules T_i, over some index set I. Let $e_i \in C$ be the idempotent such that $e_i M = T_i$, and let $P_i := \mathrm{Hom}_A(M, T_i)$, regarded as a right C-module by defining the product θc to be the composition of functions $\theta \circ c$, for $\theta \in \mathrm{Hom}_A(M, T_i)$ and $c \in C$. Observe that P_i is isomorphic to the right ideal $e_i C$ of C. Fitting's theorem (see eg [L, 1.4]) says that $\sum_{i \in I} e_i$ is a decomposition of the identity in C into primitive idempotents, and $T_i \cong T_j$ as left A-modules if and only if $P_i \cong P_j$ as right C-modules.

Define an equivalence relation $\sim$ on I by $i \sim j$ if $T_i \cong T_j$, or equivalently if $P_i \cong P_j$. Let X denote the set of $\sim$-equivalence classes in I, and for $\lambda \in X$, let $T_A(\lambda) := T_i$ and $P_C(\lambda) := P_i$, for some arbitrary $i \in \lambda$. Let $L_C(\lambda) := \mathrm{hd}\, P_C(\lambda)$ be the head of $P_C(\lambda)$. Then, $\{P_C(\lambda) \mid \lambda \in X\}$ is a complete set of non-isomorphic PIMs in the category Mod-C of right C-modules, and $\{L_C(\lambda) \mid \lambda \in X\}$ is a complete set of non-isomorphic irreducible right C-modules. Moreover, $\dim L_C(\lambda)$ is equal to the cardinality $|\lambda|$ of the equivalence class λ, and so

$$M \cong \bigoplus_{\lambda \in X} T_A(\lambda)^{\dim L_C(\lambda)}$$

as an A-module.

Now, let $B < A$ be a subalgebra of A, and let $D \geq C$ be its commutant $\mathrm{End}_B(M)$. Performing the same construction for B and D, we can write

$$M = \bigoplus_{j \in J} T_j$$

as indecomposable B-modules, over some index set J (disjoint from I). We obtain a corresponding decomposition $\sum_{j \in J} f_j$ of the identity of D into primitive idempotents in D. Let Y denote the equivalence classes of J under $i \sim j$ if $T_i \cong T_j$, and for $\mu \in Y$, let $T_B(\mu)$ denote T_j and $P_D(\mu)$ denote $f_j D$ for some $j \in \mu$. Finally, set $L_D(\mu) := \mathrm{hd}\, P_D(\mu)$; we obtain all irreducible right D-modules in this way. Moreover,

$$M \cong \bigoplus_{\mu \in Y} T_B(\mu)^{\dim L_D(\mu)}$$

as a B-module.

We denote the restriction functors A-Mod $\to$ B-Mod and Mod-$D \to$ Mod-C by $\downarrow_B$ and $\downarrow_C$ respectively. We also use the induction functor $\uparrow^D$: Mod-$C \to$ Mod-D, defined as usual for finite dimensional algebras by $N \uparrow^D := N \otimes_C D$, for $N \in$ Mod-C. Restriction is exact, and induction is left adjoint to restriction, hence is right exact and sends projectives to projectives.

The following result is presumably known.

3.1. **Lemma.** *For $\lambda \in X, \mu \in Y$, we have*

$$(T_A(\lambda) \downarrow_B : T_B(\mu)) = (P_C(\lambda) \uparrow^D : P_D(\mu)) = [L_D(\mu) \downarrow_C : L_C(\lambda)].$$

Proof. Pick an idempotent $e \in C$ such that $T_A(\lambda) \cong eM$, so $P_C(\lambda) \cong eC$. Write $e = f_1 + \cdots + f_n$ where $f_1, \ldots, f_n$ are primitive idempotents in D. Then,

$$T_A(\lambda) \downarrow_B = \bigoplus_{i=1}^{n} f_i M$$

where $f_i M$ are indecomposable B-modules, and $f_i M \cong T_B(\mu)$ if and only if $f_i D \cong P_D(\mu)$, for $\mu \in Y$. Hence, $(T_A(\lambda) \downarrow_B : T_B(\mu)) = (eD : P_D(\mu))$.

Next, observe that the natural map $eC \otimes_C D \to eD$ is clearly onto. It is injective, because if $x := \sum ec_i \otimes d_i$ lies in the kernel, for $c_i \in C, d_i \in D$, then $y := \sum_i ec_i d_i = 0$, and $x = \sum_i ec_i \otimes d_i = \sum_i e \otimes ec_i d_i = e \otimes y = 0$. Hence, $eD \cong P_C(\lambda) \uparrow^D$, and we have proved the first equality $(T_A(\lambda) \downarrow_B : T_B(\mu)) = (P_C(\lambda) \uparrow^D : P_D(\mu))$. Finally, observe that

$$\begin{aligned}(P_C(\lambda) \uparrow^D : P_D(\mu)) &= \dim \mathrm{Hom}_D(P_C(\lambda) \uparrow^D, L_D(\mu)) \\ &= \dim \mathrm{Hom}_C(P_C(\lambda), L_D(\mu) \downarrow_C) \\ &= [L_D(\mu) \downarrow_C : L_C(\lambda)],\end{aligned}$$

which completes the proof. □

Fix $m, n \geq 1$. Let $\Lambda(n \times m)$ be the set of all weights $\alpha \in \Lambda(n)$ with $\alpha_i \leq m$ for all i and define $\Lambda^+(n \times m) := \Lambda(n \times m) \cap X^+(n)$. For $\lambda \in \Lambda^+(n \times m)$, we let λ^t denote the ***transpose partition***, which can be regarded as an element of $\Lambda^+(m \times n)$. By definition, $\lambda^t = (\mu_1, \ldots, \mu_m)$ where μ_i is equal to the number of λ_j $(1 \leq j \leq n)$ with $\lambda_j \geq i$.

We now briefly recall a special case of Donkin's construction of generalized Schur algebras from [D2]. Observe that $\Lambda^+(n \times m)$ is a saturated set of dominant weights of $X^+(n)$ in the sense of [D2]. We say a $GL(n)$-module ***belongs to*** $\Lambda^+(n \times m)$ if all composition factors are of the form $L_n(\lambda)$ for $\lambda \in \Lambda^+(n \times m)$. Regarding $\mathbb{F}[GL(n)]$ as a left $GL(n)$-module via the right regular action, we let $A(n \times m)$ be the largest submodule of $\mathbb{F}[GL(n)]$ (or equivalently, of $A(n)$) that belongs to $\Lambda^+(n \times m)$. By [D2, 1.2], $A(n \times m)$ is a subcoalgebra of $\mathbb{F}[GL(n)]$ and by [D2, 2.2c]:

3.2. $\dim A(n \times m) = \sum_{\lambda \in \Lambda^+(n \times m)} (\dim \Delta_n(\lambda))^2$.

Hence, $S(n \times m) := A(n \times m)^*$ is naturally a finite dimensional algebra. In fact, the results in [D2] establish that $S(n \times m)$ is a quasihereditary algebra with weight poset $\Lambda^+(n \times m)$, in the sense of [D5, Appendix].

Now the arguments of Green [G, §1, 2.4] (see also [D2, §3] for the generalization to hyperalgebras over $\mathbb{Z}$) show that there is a natural surjection $e : \mathbb{F}GL(n) \to S(n \times m)$, defined (as in section 1) by $e(g)(a) := a(g)$ for all $g \in GL(n)$ and $a \in A(n \times m)$, such that every $GL(n)$-module belonging to $\Lambda^+(n \times m)$ factors through e to give a well-defined $S(n \times m)$-module. Moreover, this gives an equivalence of categories between $S(n \times m)$-Mod and the category of all $GL(n)$-modules belonging to $\Lambda^+(n \times m)$.

Fix $m, n \geq 1$. Given any $GL(m)$-module M, we let $\tilde{M}$ denote the right $\mathbb{F}GL(m)$-module obtained from the left $\mathbb{F}GL(m)$-module M by twisting the action of $GL(m)$ with the antiautomorphism τ corresponding to matrix transposition.

Our results depend centrally on the following theorem of Donkin [D3] and Adamovich and Rybnikov [AR]. Actually, Donkin does not prove his version in quite this generality, while in [AR], a slightly different module M is used. Therefore, we have included a sketch of the proof, following the arguments in [D3] and [AR] closely.

3.3. **Theorem (Donkin, Adamovich-Rybnikov)**. *Let V and W denote the natural module for $GL(n)$ and $GL(m)$, respectively. Regard $M := \bigwedge(V \otimes W)$ as a $GL(n) \times GL(m)$-module, by letting $GL(n)$ act trivially on W and $GL(m)$ act trivially on V. Let A (resp. C) denote the image of $\mathbb{F}GL(n)$ (resp. $\mathbb{F}GL(m)$) in $\mathrm{End}_{\mathbb{F}}(M)$. Then,*

(i) *the set of indecomposable summands of M as a $GL(n)$-module is*

$$\{T_n(\lambda) \mid \lambda \in \Lambda^+(n \times m)\};$$

(ii) *$C \cong S(m \times n)$;*

(iii) *$C = \mathrm{End}_A(M)$;*

(iv) *for each $\lambda \in \Lambda^+(n \times m)$, there is an idempotent $e_\lambda \in C$ such that $e_\lambda M \cong T_n(\lambda)$ and $e_\lambda C$ is the projective cover in* Mod-C *of $\tilde{L}_m(\lambda^t)$.*

Proof. (i) We first decompose $M = \bigwedge(V \otimes W)$ as a $GL(n) \times T(m)$-module to deduce

$$M \cong \bigwedge((V \otimes \mathbb{F}_{\varepsilon_1}) \oplus \cdots \oplus (V \otimes \mathbb{F}_{\varepsilon_m})) \cong \bigwedge(V \otimes \mathbb{F}_{\varepsilon_1}) \otimes \cdots \otimes \bigwedge(V \otimes \mathbb{F}_{\varepsilon_m})$$

where $\mathbb{F}_{\varepsilon_i}$ denotes the 1-dimensional $T(m)$-module of weight ε_i. Consequently, for $\alpha \in X(m)$, the α-weight space M_α of M with respect to $T(m)$ is zero unless $\alpha \in \Lambda(m \times n)$, in which case

$$M_\alpha \cong \textstyle\bigwedge^{\alpha_1} V \otimes \cdots \otimes \bigwedge^{\alpha_m} V. \tag{3.4}$$

The highest weight of this $GL(n)$-module M_α is the transpose of the unique dominant weight conjugate to α under the Weyl group, so lies in $\Lambda^+(n \times m$. In particular, this shows that *all* weights of M with respect to $T(n)$ lie in $\Lambda(n \times m)$. By (1.4)(i), M is tilting as a $GL(n)$-module, so is a direct sum of tilting modules $T_n(\lambda)$ for certain λ (necessarily) in $\Lambda^+(n \times m)$. Moreover, for every $\lambda \in \Lambda^+(n \times m)$, $T_n(\lambda)$ definitely appears as a summand of M, since λ is the highest weight of the summand M_{λ^t}.

(ii) The argument of (i) applies equally well to the action of $GL(m)$ on M. In particular, M belongs to $\Lambda^+(m \times n)$ as a $GL(m)$-module, so the action of $\mathbb{F}GL(m)$ factors through the quotient $e : \mathbb{F}GL(m) \to S(m \times n)$ to give a well-defined $S(m \times n)$-module. Hence, C is equal to the image of $S(m \times n)$ in $\mathrm{End}_{\mathbb{F}}(M)$. But we have also shown that the set of indecomposable summands of M as an $S(m \times n)$-module is $\{T_m(\lambda) \mid \lambda \in \Lambda^+(m \times n)\}$. Hence, M is a full tilting module for the quasihereditary algebra $S(m \times n)$, so faithful by the argument of [AR, Proposition 4.4]. This shows that $C \cong S(m \times n)$.

(iii) As M is a (contravariantly self-dual) tilting $GL(n)$-module, [J, II.4.13] implies that

$$\dim \mathrm{End}_A(M) = \sum_{\lambda \in \Lambda^+(n \times m)} (\dim \mathrm{Hom}_A(M, \nabla_n(\lambda)))^2.$$

We know that $C \leq \mathrm{End}_A(M)$. By (ii) and (3.2), $\dim C = \sum_{\lambda \in \Lambda^+(m\times n)} (\dim \Delta_m(\lambda))^2$. So it suffices to show that $\dim \mathrm{Hom}_A(M, \nabla_n(\lambda)) = \dim \Delta_m(\lambda^t)$ for any $\lambda \in \Lambda^+(n \times m)$.

Let $X_m(\lambda) := \mathrm{Hom}_A(M, \nabla_n(\lambda))$, regarded as a $GL(m)$-module with action $(g\theta)(m) = \theta(\tau(g)m)$ for $g \in GL(m), \theta \in \mathrm{Hom}_A(M, \nabla_n(\lambda))$ and $m \in M$. By 3.4 and the observation that τ is the identity on restriction to $T(m)$,

$$\dim X_m(\lambda)_\alpha = \dim \mathrm{Hom}_{GL(n)}(M_\alpha, \nabla_n(\lambda)) = \dim \mathrm{Hom}_{GL(n)}(\textstyle\bigwedge^{\alpha_1} V \otimes \cdots \otimes \bigwedge^{\alpha_m} V, \nabla_n(\lambda))$$

for $\alpha \in \Lambda(m \times n)$. But by the Littlewood-Richardson rule (or a calculation involving symmetric functions), the right hand dimension is equal to the number of standard λ^t-tableaux of weight α, which is precisely $\dim \Delta_m(\lambda^t)_\alpha$ (see [BKS, section 2] for these well-known definitions). Hence, the character of the $GL(m)$-module $X_m(\lambda)$ is equal to the character of $\Delta_m(\lambda^t)$. In particular, their dimensions agree as required to prove that $C = \mathrm{End}_A(M)$.

For use in (iv), we claim now that $X_m(\lambda) \cong \Delta_m(\lambda^t)$. We know already that the characters agree, so it suffices to show that $X_m(\lambda)$ is a standard module for $GL(m)$, or equivalently that it is a standard module for the quasihereditary algebra C^{op}, which we have just shown is the Ringel dual of A (see [D5, Appendix A4]). But by definition, the C^{op}-module $X_m(\lambda)$ is $\mathrm{Hom}_A(M, \nabla_n(\lambda))$ which is certainly a standard module according to [D5, Appendix A4].

(iv) Given $\lambda \in \Lambda^+(n \times m)$, we can find a primitive idempotent $e_\lambda \in C$ such that $e_\lambda M \cong T_n(\lambda)$ by (i) and (iii). This means that $e_\lambda C$ is a projective indecomposable right C-module, so the projective cover of the irreducible right C-module $\tilde{L}_m(\mu)$ for some $\mu \in \Lambda^+(m \times n)$. To show that μ equals λ^t as required, it suffices to show that the $GL(m)$-module $Y_m(\lambda) := \mathrm{Hom}_{GL(n)}(M, T_n(\lambda))$ (with action defined as in (iii)) contains $L_m(\lambda^t)$ in its head.

As $T_n(\lambda)$ has a ∇-filtration and λ is its highest weight, we can find a submodule $K < T_n(\lambda)$ with a ∇-filtration such that

$$0 \to K \to T_n(\lambda) \to \nabla_n(\lambda) \to 0$$

is exact. Applying $\mathrm{Hom}_{GL(n)}(M, -)$ to this, we obtain the long exact sequence

$$\cdots \to \mathrm{Hom}_{GL(n)}(M, T_n(\lambda)) \to \mathrm{Hom}_{GL(n)}(M, \nabla_n(\lambda)) \to \mathrm{Ext}^1_{GL(n)}(M, K) \to \ldots.$$

But M has a Δ-filtration and K has a ∇-filtration, so $\mathrm{Ext}^1_{GL(n)}(M, K) = 0$ by [J, II.4.13]. This shows that $X_m(\lambda)$ is a quotient of $Y_m(\lambda)$. But we showed in (iii) that $X_m(\lambda) \cong \Delta_m(\lambda^t)$. Hence, $Y_m(\lambda)$ contains $L_m(\lambda^t)$ in its head, as required. □

We remark that the situation in the theorem is symmetric. So, with notation as in the theorem, the theorem shows in fact that (A, C) is a dual pair on M.

Now fix in addition $a \geq 1$. Applying the theorem a times to the outer tensor product, the following corollary is immediate because of [CR, 10.37] and (1.6):

3.5. **Corollary.** *For $i = 1, \ldots, a$, let n_i (resp. m_i) be a positive integer and let V_i (resp. W_i) denote the natural module for $GL(n_i)$ (resp. $GL(m_i)$). Regard*

$$M := \textstyle\bigwedge(V_1 \otimes W_1) \boxtimes \cdots \boxtimes \bigwedge(V_a \otimes W_a)$$

as a module for $(GL(n_1) \times \cdots \times GL(n_a)) \times (GL(m_1) \times \cdots \times GL(m_a))$ in the obvious way. Let $A_1 \otimes \cdots \otimes A_a$ (resp. $C_1 \otimes \cdots \otimes C_a$) denote the image of the group algebra $\mathbb{F}GL(n_1) \otimes \cdots \otimes \mathbb{F}GL(n_a)$ (resp. $\mathbb{F}GL(m_1) \otimes \cdots \otimes \mathbb{F}GL(m_a)$) in $\mathrm{End}_{\mathbb{F}}(M)$. Then,

(i) *the set of indecomposable summands of M as a $GL(n_1) \times \cdots \times GL(n_a)$-module is*

$$\{T_{n_1}(\lambda^{(1)}) \boxtimes \cdots \boxtimes T_{n_a}(\lambda^{(a)}) \mid \lambda^{(i)} \in \Lambda^+(n_i \times m_i), i = 1, \dots, a\};$$

(ii) *$C_i \cong S(m_i \times n_i)$ for $i = 1, \dots, a$;*

(iii) *$C_1 \otimes \cdots \otimes C_a = \mathrm{End}_{A_1 \otimes \cdots \otimes A_a}(M)$;*

(iv) *for $\lambda^{(i)} \in \Lambda^+(n_i \times m_i)$, $i = 1, \dots, a$, there is an idempotent*

$$e := e_{\lambda^{(1)}} \otimes \cdots \otimes e_{\lambda^{(a)}} \in C_1 \otimes \cdots \otimes C_a$$

such that $eM \cong T_{n_1}(\lambda^{(1)}) \boxtimes \cdots \boxtimes T_{n_a}(\lambda^{(a)})$ and $e(C_1 \otimes \cdots \otimes C_a)$ is the projective cover in $\text{Mod-}(C_1 \otimes \cdots \otimes C_a)$ *of $\tilde{L}_m((\lambda^{(1)})^t) \boxtimes \cdots \boxtimes \tilde{L}_m((\lambda^{(a)})^t)$.*

Again, the situation in this corollary is symmetric, so, with the notation of the corollary, it implies that $(A_1 \otimes \cdots \otimes A_a, C_1 \otimes \cdots \otimes C_a)$ is a dual pair on M.

Now we are ready for our application. Let $GL(n) \times GL(m)$ act on $M := \bigwedge(V \otimes W)$ as in Theorem 3.3, and let A (resp. C) be the image of $\mathbb{F}GL(n)$ (resp. $\mathbb{F}GL(m)$) in $\mathrm{End}_{\mathbb{F}}(M)$. So, by Theorem 3.3, (A, C) is a dual pair on M.

Next we choose $\nu = (n_1, \dots, n_a) \in \Lambda(a, n)$ for some a, and let $B < A$ be the image of $\mathbb{F}GL(\nu) < \mathbb{F}GL(n)$ in $\mathrm{End}_{\mathbb{F}}(M)$. Let V_i be the natural $GL(n_i)$-module, so $V \cong V_1 \oplus \cdots \oplus V_a$ as a $GL(\nu)$-module. Observe that as a $GL(\nu)$-module,

$$M \cong \bigwedge((V_1 \otimes W) \oplus \cdots \oplus (V_a \otimes W)) \cong \bigwedge(V_1 \otimes W) \boxtimes \cdots \boxtimes \bigwedge(V_a \otimes W). \tag{3.6}$$

Consequently, $GL(\nu) = GL(n_1) \times \cdots \times GL(n_a)$ acts on M in the same way as $GL(n_1) \times \cdots \times GL(n_a)$ in Corollary 3.5.

Let $GL(m) \times \cdots \times GL(m)$ (a times) act on M via its natural action on W in the outer tensor product of 3.6. This action obviously commutes with the action of $GL(\nu)$, and is precisely the action of $GL(m_1) \times \cdots \times GL(m_a)$ in Corollary 3.5 with $m_1 = \cdots = m_a = m$. We also note that the original action of $GL(m)$ on M is the restriction of the $GL(m) \times \cdots \times GL(m)$-action just defined if we embed $GL(m)$ into $GL(m) \times \cdots \times GL(m)$ diagonally. We let $D > C$ denote the image of the group algebra $\mathbb{F}GL(m) \otimes \cdots \otimes \mathbb{F}GL(m)$ in $\mathrm{End}_{\mathbb{F}}(M)$. Now, Corollary 3.5 immediately implies that (B, D) is a dual pair on M.

So we are now in the situation of the beginning of the section. We apply Lemma 3.1 twice to obtain our main result:

3.7. **Theorem.** *Fix $a, n, m \geq 1$, and choose $\nu \in \Lambda(a, n)$. Let $GL(\nu)$ denote the standard Levi subgroup of $GL(n)$. Choose $\lambda \in \Lambda^+(n \times m)$ and $\mu^{(i)} \in \Lambda^+(n_i \times m)$ for $i = 1, \dots, a$. Then,*

(i) $(T_n(\lambda) \downarrow_{GL(\nu)} : T_{n_1}(\mu^{(1)}) \boxtimes \cdots \boxtimes T_{n_a}(\mu^{(a)})) = [L_m((\mu^{(1)})^t) \otimes \cdots \otimes L_m((\mu^{(a)})^t) : L_m(\lambda^t)]$.

(ii) $[L_n(\lambda) \downarrow_{GL(\nu)} : L_{n_1}(\mu^{(1)}) \boxtimes \cdots \boxtimes L_{n_a}(\mu^{(a)})] = (T_m((\mu^{(1)})^t) \otimes \cdots \otimes T_m((\mu^{(a)})^t) : T_m(\lambda^t))$.

Proof. For (i), we apply Lemma 3.1 to the pairs (A, C) and (B, D), using Theorem 3.3(i) (resp. Corollary 3.5(i)) to identify the indecomposable summands of M as an A- (resp. B-) module and Theorem 3.3(iv) (resp. Corollary 3.5(iv)) to identify the corresponding irreducible C- (resp. D-) modules arising from the commutant construction. The only subtlety is in showing that the restriction from D to C of the right D-module corresponding to the outer tensor product

$$\tilde{L}_m((\mu^{(1)})^t) \boxtimes \cdots \boxtimes \tilde{L}_m((\mu^{(a)})^t)$$

(which is a right $\mathbb{F}GL(m) \otimes \cdots \otimes \mathbb{F}GL(m)$-module) is the right C-module corresponding to the right $\mathbb{F}GL(m)$-module

$$\tilde{L}_m((\mu^{(1)})^t) \otimes \cdots \otimes \tilde{L}_m((\mu^{(a)})^t).$$

This follows because the embedding of C into D is induced from the diagonal embedding of $GL(m)$ into $GL(m) \times \cdots \times GL(m)$. Given this (i) follows easily. The argument for (ii) is identical, but swapping the roles of A and D and the roles of B and C. $\square$

As special cases of Theorem 3.7(i) and (ii) (taking $GL(\nu) = T(n)$), we recover the following character formulae, the first of which is a result of Mathieu and Papadopoulo [MP]:

3.8. **Corollary (Mathieu-Papadopoulo).** *For $\lambda \in \Lambda^+(n \times m)$ and $\mu \in \Lambda(n \times m)$,*

$$\dim L_n(\lambda)_\mu = (\textstyle\bigwedge^{\mu_1}(W) \otimes \cdots \otimes \bigwedge^{\mu_n}(W) : T_m(\lambda^t))$$

where W denotes the natural $GL(m)$-module.

3.9. **Corollary.** *For $\lambda \in \Lambda^+(n \times m)$ and $\mu \in \Lambda(n \times m)$,*

$$\dim T_n(\lambda)_\mu = [\textstyle\bigwedge^{\mu_1}(W) \otimes \cdots \otimes \bigwedge^{\mu_n}(W) : L_m(\lambda^t)]$$

where W denotes the natural $GL(m)$-module.

The final corollary clarifies the connection between the arguments in [BKS] and [MP]:

3.10. **Corollary.** *For $\lambda \in \Lambda^+(n \times m)$ and $\mu \in \Lambda^+(n \times (m-1))$ with $|\lambda| \geq |\mu|$ put $\ell = |\lambda| - |\mu|$. Then*

$$(T_n(\mu) \otimes \textstyle\bigwedge^\ell(V) : T_n(\lambda)) = [L_m(\lambda^t) \downarrow_{GL(m-1)} : L_{m-1}(\mu^t)].$$

Proof. By Theorem 3.7(ii), the left hand side is equal to

$$[L_m(\lambda^t) \downarrow_{GL(m-1,1)} : L_{m-1}(\mu^t) \boxtimes L_1(\ell\varepsilon_1)].$$

Now a similar argument to Lemma 2.9 gives the conclusion. $\square$

4 Symmetric groups and Schur functors

We now obtain our final general link between modular Littlewood-Richardson coefficients, by explaining the relationship with representations of symmetric groups.

Throughout the section, we fix integers n, r and compositions $\nu = (n_1, \dots, n_a) \in \Lambda(a, n)$ and $\rho = (r_1, \dots, r_a) \in \Lambda(a, r)$ for some a such that $n \geq r$ and $n_i \geq r_i$ for $i = 1, \dots, a$.

We denote the symmetric group on $\{1, \dots, r\}$ by Σ_r. Note that we regard the natural action of Σ_r on $\{1, \dots, r\}$ as a left action and always work with left modules for the group algebra $\mathbb{F}\Sigma_r$. First, we recall some basic facts about the representation theory of the symmetric group Σ_r, following [J1].

Given a composition $\lambda = (\lambda_1, \lambda_2, \dots)$ we write $\lambda \vDash r$ if $\lambda_1 + \lambda_2 + \dots = r$. If $\lambda \vDash r$ is actually a partition we say that λ is a partition of r and write $\lambda \vdash r$. For every $\lambda \vdash r$, there is an associated left $\mathbb{F}\Sigma_r$-module S^λ, defined as in [J1], called a ***Specht module***.

Say $\lambda \vdash r$ is ***p-regular*** if at most $(p-1)$ of the non-zero parts λ_i of λ are equal. According to [J1]:

4.1. *If λ is p-regular, then S^λ has simple head denoted D^λ, and $\{D^\lambda \mid \lambda \vdash r, \lambda \text{ p-regular}\}$ is a complete set of non-isomorphic irreducible $\mathbb{F}\Sigma_r$-modules.*

We mention some other important $\mathbb{F}\Sigma_r$-modules. For any $\lambda \vDash r$, let M^λ denote the corresponding ***permutation module***, which is the module induced from the trivial module for the Young subgroup $\Sigma_\lambda := \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \dots$ of Σ_r. Now, according to [J2], for $\lambda \vdash r$, M^λ contains a unique submodule isomorphic to S^λ, and there is a unique indecomposable summand Y^λ of M^λ containing this submodule S^λ. These modules Y^λ are the ***Young modules***. Moreover, any permutation module M^μ splits as a direct sum of Young modules.

Next, we recall Green's construction of the Schur functor $R_\xi : M_{\mathbb{F}}(n, r) \to \mathbb{F}\Sigma_r\text{-Mod}$ from [G, §6]. We write $S := S(n, r)$ for short. Fix any weight $\omega \in \Lambda(n, r)$ of the form $\omega = \varepsilon_{h_1} + \dots + \varepsilon_{h_r}$, $h_1 < h_2 < \dots < h_r$. Let $\boldsymbol{h} = \boldsymbol{h}^\omega = (h_1, \dots, h_r) \in I(n, r)$. Set $\xi = \xi^\omega := \xi_{\boldsymbol{h},\boldsymbol{h}} \in S$ (see section 1). By [G, §3], ξ is an idempotent and for any $M \in M_{\mathbb{F}}(n, r)$, ξM is precisely the ω-weight space M_ω of M. Moreover, as in [G, 6.1d]:

4.2. *The set $\{\xi_{\boldsymbol{h}\pi,\boldsymbol{h}} \mid \pi \in \Sigma_r\}$ is a basis for the algebra $\xi S \xi$. The linear map $\mathbb{F}\Sigma_r \to \xi S \xi$, defined by $\pi \mapsto \xi_{\boldsymbol{h}\pi,\boldsymbol{h}}$ for $\pi \in \Sigma_r$, is an algebra isomorphism.*

Now define the Schur functor $R_\xi : S\text{-Mod} \to \mathbb{F}\Sigma_r\text{-Mod}$ as in section two, using (4.2) to identify $\mathbb{F}\Sigma_r$ with $\xi S \xi$. Explicitly, given $M \in S\text{-Mod}$, we regard the $\xi S \xi$-module $R_\xi M$ as an $\mathbb{F}\Sigma_r$-module by letting $\pi \in \Sigma_r$ act in the same way as $\xi_{\boldsymbol{h}\pi,\boldsymbol{h}} \in \xi S \xi$.

In this section we are also going to work with a well-known equivalent, but for us more convenient, description of this functor, which we now explain. Let Perm_r denote the subgroup of $GL(n)$ consisting of all permutation matrices $g \in GL(n)$ with $g_{i,i} = 1$ whenever $i \notin \{h_1 \dots, h_r\}$. Obviously, Perm_r is isomorphic to Σ_r. More precisely:

4.3. *For $w \in \mathrm{Perm}_r$, define $\bar{w} \in \Sigma_r$ to be the unique element such that the $(h_{\bar{w}i}, h_i)$-entry of the permutation matrix w is 1 for all $i = 1, \dots, r$. This gives an isomorphism* $\mathrm{Perm}_r \to \Sigma_r$.

Given any $M \in M_{\mathbb{F}}(n,r)$, the ω-weight space M_ω of M is stable under the action of Perm_r. We now define a functor

$$F_{n,r} = F^{\omega}_{n,r} : M_{\mathbb{F}}(n,r) \to \mathbb{F}\mathrm{Perm}_r\text{-Mod}$$

on objects by letting $F_{n,r}M := M_\omega$, and by restriction on morphisms. (If we use another ω of the form $\varepsilon_{h_1} + \cdots + \varepsilon_{h_r}$ we get an isomorphic functor.)

4.4. **Lemma.** *Identify* Perm_r *and* Σ_r *using the map* $w \mapsto \bar{w}$ *from (4.3). Then,* $F_{n,r}M$ *and* $R_\xi M$ *are isomorphic as* $\mathbb{F}\Sigma_r$*-modules.*

Proof. As vector spaces, $F_{n,r}M$ and $R_\xi M$ both equal M_ω. So it suffices to show that for any $w \in \mathrm{Perm}_r$, the action of w on M_ω is the same as the action of $\bar{w} \in \Sigma_r$. Recall by definition that $\bar{w}$ acts on M as $\xi_{\boldsymbol{h}\bar{w},\boldsymbol{h}}$. The action of w on M factors through the quotient $e : \mathbb{F}GL(n) \to S$, so w acts on M_ω in the same way as the element $e(w) \in S$. Since $M_\omega = \xi M$ and ξ is idempotent, it therefore suffices to show that $\xi e(w)\xi = \xi_{\boldsymbol{h}\bar{w},\boldsymbol{h}}$. To prove this, since both lie in $\xi S \xi$ which has basis $\xi_{\boldsymbol{h}\pi,\boldsymbol{h}}$ for all $\pi \in \Sigma_r$, we just need to check that

$$(\xi e(w)\xi)(c_{\boldsymbol{h}\pi,\boldsymbol{h}}) = \xi_{\boldsymbol{h}\bar{w},\boldsymbol{h}}(c_{\boldsymbol{h}\pi,\boldsymbol{h}})$$

for all $\pi \in \Sigma_r$. The right hand side is clearly zero unless $\pi = \bar{w}$, when it is 1. On the other hand note that $(\xi e(w)\xi)(c_{\boldsymbol{h}\pi,\boldsymbol{h}}) = (e(w))(c_{\boldsymbol{h}\pi,\boldsymbol{h}}) = c_{\boldsymbol{h}\pi,\boldsymbol{h}}(w) = w_{h_{\pi 1},h_1} \dots w_{h_{\pi_r},h_r}$. This is zero unless $\pi = \bar{w}$, when it is 1, by definition of $\bar{w}$. □

Let sgn denote the 1-dimensional sign representation for $\mathbb{F}\Sigma_r$. The effect of $F_{n,r}$ on the various modules in $M_{\mathbb{F}}(n,r)$ is given by the following:

4.5. **Lemma.** *Fix* $\lambda \in \Lambda^+(n,r)$*.*

(i) $F_{n,r}\nabla_n(\lambda) \cong S^\lambda$;

(ii) $F_{n,r}\Delta_n(\lambda) \cong (S^\lambda)^* \cong S^{\lambda^t} \otimes \mathrm{sgn}$;

(iii) $F_{n,r}L_n(\lambda)$ *is zero unless* λ *is* p*-restricted, in which case* $F_{n,r}L_n(\lambda) \cong D^{\lambda^t} \otimes \mathrm{sgn}$.

(iv) *for any* $\mu \in \Lambda(n,r)$, $F_{n,r}(\bigwedge^{\mu_1}(V) \otimes \cdots \otimes \bigwedge^{\mu_n}(V)) \cong M^\mu \otimes \mathrm{sgn}$;

(v) $F_{n,r}T_n(\lambda) = Y^{\lambda^t} \otimes \mathrm{sgn}$.

Proof. (i), (ii) and (iii) are well-known, and are proved in [G, §6]. Also, (v) follows easily from (ii) and (iv), since by definition $Y^{\lambda^t} \otimes \mathrm{sgn}$ is the unique indecomposable summand of $M^{\lambda^t} \otimes \mathrm{sgn}$ containing a submodule isomorphic to $S^{\lambda^t} \otimes \mathrm{sgn}$, see [D3, (3.6)(ii)]. Finally, (iv) is proved in [D3, (3.5)(ii)] (or follows easily from Theorem 4.13). □

Recall that we have fixed $\nu = (n_1, \dots, n_a) \in \Lambda(a,n)$ and $\rho = (r_1, \dots, r_a) \in \Lambda(a,r)$ with $r_i \leq n_i$ for $i = 1, \dots, a$. Let $GL(\nu)$ be the corresponding Levi subgroup of $GL(n)$, and Σ_ρ be the corresponding Young subgroup $\Sigma_{r_1} \times \cdots \times \Sigma_{r_a}$ of Σ_r. First, we consider the effect of Schur functors on restrictions from $GL(n)$ to $GL(\nu)$.

Choose the weight ω above to be the specific weight

$$\omega = \Big(\underbrace{\overbrace{1,\dots,1}^{r_1 \text{ entries}},0,\dots,0}_{n_1 \text{ entries}},\underbrace{\overbrace{1,\dots,1}^{r_2 \text{ entries}},0,\dots,0}_{n_2 \text{ entries}},\dots,\underbrace{\overbrace{1,\dots,1}^{r_a \text{ entries}},0,\dots,0}_{n_a \text{ entries}}\Big). \tag{4.6}$$

Let $\boldsymbol{h} = \boldsymbol{h}^\omega$ and $\xi = \xi^\omega$ Then the idempotent $\xi \in S$ lies in the Levi subalgebra $L := S(\nu, r) < S$ (for example by (1.15)).

Let $\mathrm{Perm}_\rho := \mathrm{Perm}_r \cap GL(\nu)$. Then Perm_ρ is isomorphic to the Young subgroup $\Sigma_\rho < \Sigma_r$ (under the map from (4.3)).

4.7. **Lemma.** *$\xi L \xi \cong \mathbb{F}\Sigma_\rho$.*

Proof. Note that $\xi L \xi = \xi S \xi \cap L$ since $\xi \in L$. Now the result follows from (4.2) and Lemma 1.13. $\square$

Consequently, we obtain a Schur functor $\bar{R}_\xi : L\text{-Mod} \to \mathbb{F}\Sigma_\rho\text{-Mod}$. Precisely as before, we can identify this Schur functor with the functor

$$F_{\nu,\rho} : M_{\mathbb{F}}(\nu, r) \to \mathbb{F}\mathrm{Perm}_\rho\text{-Mod}$$

given on $M \in M_{\mathbb{F}}(\nu, r)$ by letting $F_{\nu,\rho} M$ denote the weight space M_ω, noting that this is stable under the action of $\mathrm{Perm}_\rho < GL(\nu)$. It is now obvious that:

4.8. *For any $M \in M_{\mathbb{F}}(n, r)$, $F_{\nu,\rho}(R^n_\nu M) \cong (F_{n,r} M) \downarrow^{\mathbb{F}\mathrm{Perm}_r}_{\mathbb{F}\mathrm{Perm}_\rho}$.*

The action of the functor $F_{\nu,\rho}$ on outer tensor products is also easily understood:

4.9. **Lemma.** *Given a $GL(\nu)$-module that is an outer tensor product $M_1 \boxtimes \cdots \boxtimes M_a$ for $M_i \in M_{\mathbb{F}}(n_i, r_i)$,*

$$F_{\nu,\rho}(M_1 \boxtimes \cdots \boxtimes M_a) \cong (F_{n_1,r_1} M_1) \boxtimes \cdots \boxtimes (F_{n_a,r_a} M_a).$$

Proof. This follows because the ω-weight space of $M_1 \boxtimes \cdots \boxtimes M_a$ is $(M_1)_{\omega^{(1)}} \boxtimes \cdots \boxtimes (M_a)_{\omega^{(a)}}$ where $\omega^{(i)} \in \Lambda(n_i, r_i)$ is the weight $(1, \dots, 1, 0, \dots, 0)$. $\square$

We now give applications of this Schur functor $F_{\nu,\rho}$. First, we have the following results on composition and tilting module multiplicities:

4.10. **Theorem.** *Fix $\lambda \vdash r$ and $\mu^{(i)} \vdash r_i$ for $i = 1, \dots, a$. Regard λ^t as an element of $\Lambda^+(n, r)$ and each $(\mu^{(i)})^t$ as elements of $\Lambda^+(n_i, r_i)$. Then,*

$$(Y^\lambda \downarrow_{\Sigma_\rho} : Y^{\mu^{(1)}} \boxtimes \cdots \boxtimes Y^{\mu^{(a)}}) = (T_n(\lambda^t) \downarrow_{GL(\nu)} : T_{n_1}((\mu^{(1)})^t) \boxtimes \cdots \boxtimes T_{n_a}((\mu^{(a)})^t)).$$

Moreover, if in addition all the partitions are p-regular, then

$$[D^\lambda \downarrow_{\Sigma_\rho} : D^{\mu^{(1)}} \boxtimes \cdots \boxtimes D^{\mu^{(a)}}] = [L_n(\lambda^t) \downarrow_{GL(\nu)} : L_{n_1}((\mu^{(1)})^t) \boxtimes \cdots \boxtimes L_{n_a}((\mu^{(a)})^t)].$$

Proof. We prove this for the irreducible modules; precisely the same argument (together with Lemma 1.7) gives the statement about tilting modules. We now identify Perm_r with Σ_r. Observe that

$$F_{\nu,\rho}(R^n_\nu L_n(\lambda^t)) \cong (F_{n,r} L_n(\lambda^t)) \downarrow_{\Sigma_\rho} \cong (D^\lambda \otimes \mathrm{sgn}) \downarrow_{\Sigma_\rho}$$

by (4.8) and Lemma 4.5(iii). By Lemma 4.5(iii) and Lemma 4.9,

$$F_{\nu,\rho}(L_{n_1}((\mu^{(1)})^t) \boxtimes \cdots \boxtimes L_{n_a}((\mu^{(a)})^t)) \cong (D^{\mu^{(1)}} \boxtimes \cdots \boxtimes D^{\mu^{(a)}}) \otimes \mathrm{sgn},$$

which is a non-zero irreducible for $\mathbb{F}\Sigma_\rho$. Since the Schur functor $F_{\nu,\rho}$ is exact, it preserves composition multiplicities, so the result follows on tensoring with sgn. $\square$

Special cases of the next more general result on homomorphisms have appeared in [JS] and [K1].

4.11. **Theorem.** *Let M be a $GL(n)$-quotient of $V_n^{\otimes r}$, and for $i = 1, \dots, a$, let M_i be a module in $M_{\mathbb{F}}(n_i, r_i)$ with p-restricted socle. Then,*

$$\mathrm{Hom}_{GL(\nu)}(M \downarrow_{GL(\nu)}, M_1 \boxtimes \cdots \boxtimes M_a) \cong \mathrm{Hom}_{\mathbb{F}\Sigma_\rho}((F_{n,r}M) \downarrow_{\Sigma_\rho}, (F_{n_1,r_1}M_1) \boxtimes \cdots \boxtimes (F_{n_a,r_a}M_a)).$$

Proof. By assumption, $M \in M_{\mathbb{F}}(n, r)$. As a $GL(\nu)$-module, M splits as a direct sum

$$\bigoplus_{\rho' \in \Lambda(a,r)} M(\rho')$$

where $M(\rho')$ denotes the summand of M that is homogeneous of polynomial degree ρ'_i for $GL(n_i)$ for each $i = 1, \dots, a$. The $GL(\nu)$-module $M_1 \boxtimes \cdots \boxtimes M_a$ is of polynomial degree r_i for each $GL(n_i)$, and there are no $GL(\nu)$-homomorphisms between modules of different degrees. Consequently,

$$\mathrm{Hom}_{GL(\nu)}(M \downarrow_{GL(\nu)}, M_1 \boxtimes \cdots \boxtimes M_a) \cong \mathrm{Hom}_{GL(\nu)}(M(\rho), M_1 \boxtimes \cdots \boxtimes M_a).$$

Also note that by weights and (4.8) we have

$$(F_{n,r}M) \downarrow_{\Sigma_\rho} = F_{\nu,\rho}(R^n_\nu M) = F_{\nu,\rho}M(\rho),$$

while by Lemma 4.9,

$$(F_{n_1,r_1}M_1) \boxtimes \cdots \boxtimes (F_{n_a,r_a}M_a) = F_{\nu,\rho}(M_1 \boxtimes \cdots \boxtimes M_a).$$

So, if $L = S(\nu, r)$, we need to prove that

$$\mathrm{Hom}_L(M(\rho), M_1 \boxtimes \cdots \boxtimes M_a) \cong \mathrm{Hom}_{\mathbb{F}\Sigma_\rho}(F_{\nu,\rho}M(\rho), F_{\nu,\rho}(M_1 \boxtimes \cdots \boxtimes M_a)).$$

By Lemma 2.17(ii), the proposition follows once we have shown that $M(\rho)$ has ξ-restricted head and $M_1 \boxtimes \cdots \boxtimes M_a$ has ξ-restricted socle. By Lemma 4.5(iii) and Lemma 4.9, an L-module $M_1 \boxtimes \cdots \boxtimes M_a$ has a ξ-restricted socle if and only if each M_i has a p-restricted socle, which we have by assumption. To see that $M(\rho)$ has p-restricted head, we first decompose $V_n^{\otimes r}$ as a $GL(\nu)$-module:

$$V_n^{\otimes r} \cong \bigoplus_{\rho' \in \Lambda(r,a)} a(\rho') V_{n_1}^{\rho'_1} \boxtimes \cdots \boxtimes V_{n_a}^{\rho'_a}$$

for some multiplicities $a(\rho')$. Now, $M(\rho)$ is a submodule of the factor of this direct sum with $\rho' = \rho$, which has p-restricted head again by Corollary 2.12 (and contravariant duality). $\square$

4.12. **Remark.** By Corollary 2.12, we can take the module M in Theorem 4.11 to be $\Delta_n(\lambda)$ or $L_n(\lambda)$ for p-restricted $\lambda \in \Lambda(n,r)$, and each M_i to be $\nabla_{n_i}(\mu^{(i)})$ or $L_{n_i}(\mu^{(i)})$ for p-restricted $\mu^{(i)} \in \Lambda(n_i, r_i)$.

Now we turn our attention to tensor products. Our main result describing the Schur functor $F_{n,r}$ applied to a tensor product of $GL(n)$-modules is as follows:

4.13. **Theorem.** *Given modules $M_i \in M_{\mathbb{F}}(n, r_i)$ for $i = 1, \dots, a$,*

$$F_{n,r}(M_1 \otimes \dots \otimes M_a) \cong (F_{n,r_1} M_1 \boxtimes \dots \boxtimes F_{n,r_a} M_a) \uparrow_{\mathbb{F}\Sigma_\rho}^{\mathbb{F}\Sigma_r}.$$

Proof. For notational simplicity, we prove this in the special case that $a = 2$; the general case then follows quite easily by induction on a, using transitivity of induction and the tensor identity.

So, assume that $a = 2$. Take $M \in M_{\mathbb{F}}(n, r_1)$ and $N \in M_{\mathbb{F}}(n, r_2)$. We will show that

$$F_{n,r}(M \otimes N) \cong (F_{n,r_1} M \boxtimes F_{n,r_2} N) \uparrow_{\mathrm{Perm}_\rho}^{\mathrm{Perm}_r}.$$

Let $J = \{1, \dots, r_1, n_1 + 1, \dots, n_1 + r_2\} = \{i \mid \omega_i \neq 0\}$ (see (4.6)). Given a subset $I = \{i_1, \dots, i_k\} \subseteq J$, we write ω_I for the weight $\varepsilon_{i_1} + \dots + \varepsilon_{i_k} \in \Lambda(n)$. For a module $M \in M_{\mathbb{F}}(n)$, denote the ω_I-weight space of M by M_I. Then, as vector spaces,

$$F_{n,r}(M \otimes N) = \bigoplus_{I \subseteq J, |I| = r_1} M_I \otimes N_{J \setminus I}. \tag{4.14}$$

Taking $K = \{1, 2, \dots, r_1\}$ we see that $W := M_K \boxtimes N_{J \setminus K}$ is a $\mathbb{F}\mathrm{Perm}_\rho$-submodule of $F_{n,r}(M \otimes N)$, isomorphic to $F_{n,r_1} M \boxtimes F_{n,r_2} N$. Now it suffices to prove that $F_{n,r}(M \otimes N)$, as a vector space, equals

$$\bigoplus_{t \in T} w_t W \tag{4.15}$$

where $\{w_t \mid t \in T\}$ is a complete system of left coset representatives of Perm_ρ in Perm_r.

For $w \in \mathrm{Perm}_r$ and $I = \{i_1, \dots, i_r\} \subset J$ we write wI for $\{wi_1, \dots, wi_r\}$. Note that $wL_I = L_{wI}$. For every $I \subset J$ with $|I| = r_1$ pick $w_I \in \mathrm{Perm}_r$ such that $w_I K = I$. Then

$$\{w_I \mid I \subset [1, r],\ |I| = r_1\}$$

is a complete system of left coset representatives of Perm_ρ in Perm_r. Since

$$w_I(M_K \boxtimes N_{J \setminus K}) = M_{w_I K} \boxtimes N_{w_I(J \setminus K)} = M_I \boxtimes N_{J \setminus I},$$

(4.15) follows from (4.14). $\square$

Again, this gives direct connections between modular Littlewood-Richardson coefficients for general linear groups and the corresponding coefficients for symmetric groups. The proofs of these are essentially identical to the corresponding proofs for branching rules in Theorem 4.10 and Theorem 4.11, so we omit the details. First we have the tensor product analogue of Theorem 4.10:

4.16. **Theorem.** *Fix partitions $\lambda \vdash r$ and $\mu^{(i)} \vdash r_i$ for $i = 1, \dots, a$. Regard λ^t as an element of $\Lambda^+(n,r)$ and each $(\mu^{(i)})^t$ as elements of $\Lambda^+(n,r_i)$. Then,*

$$(Y^{\mu^{(1)}} \boxtimes \cdots \boxtimes Y^{\mu^{(a)}} \uparrow^{\Sigma_r} : Y^\lambda) = (T_n((\mu^{(1)})^t) \otimes \cdots \otimes T_n((\mu^{(a)})^t) : T_n(\lambda^t)).$$

Moreover, if all the partitions are p-regular, then

$$[D^{\mu^{(1)}} \boxtimes \cdots \boxtimes D^{\mu^{(a)}} \uparrow^{\Sigma_r} : D^\lambda] = [L_n((\mu^{(1)})^t) \otimes \cdots \otimes L_n((\mu^{(a)})^t) : L_n(\lambda^t)].$$

The analogue of Theorem 4.11 for tensor products is as follows:

4.17. **Theorem.** *Let M be a $GL(n)$-module with p-restricted head, and for $i = 1, \dots, a$, let M_i be a $GL(n)$-submodule of $V_n^{\otimes r_i}$. Then,*

$$\mathrm{Hom}_{GL(n)}(M, M_1 \otimes \cdots \otimes M_a) \cong \mathrm{Hom}_{\mathbb{F}\Sigma_r}(F_{n,r}M, ((F_{n,r_1}M_1) \boxtimes \cdots \boxtimes (F_{n,r_a}M_a)) \uparrow^{\Sigma_r}).$$

As in Remark 4.12, we can take the module M in Theorem 4.17 to be $\Delta_n(\lambda)$ or $L_n(\lambda)$ for p-restricted $\lambda \in \Lambda^+(n,r)$, while M_i can be $\nabla_n(\mu^{(i)})$ or $L_n(\mu^{(i)})$ for p-restricted $\mu^{(i)} \in \Lambda^+(n,r_i)$. In particular, this gives the following special cases of Theorem 4.17, which will be used in [BK]:

4.18. **Corollary.** *Fix p-regular partitions $\mu \vdash r$ and $\lambda \vdash (r+1)$, and take $n > r$. Regard μ^t and λ^t as elements of $\Lambda^+(n)$. Then,*

$$\begin{aligned}
\mathrm{Hom}_{\Sigma_{r+1}}(D^\lambda, D^\mu \uparrow^{\Sigma_{r+1}}) &\cong \mathrm{Hom}_{GL(n)}(L_n(\lambda^t), L_n(\mu^t) \otimes V_n),\\
\mathrm{Hom}_{\Sigma_{r+1}}(S^\lambda, D^\mu \uparrow^{\Sigma_{r+1}}) &\cong \mathrm{Hom}_{GL(n)}(\Delta_n(\lambda^t), L_n(\mu^t) \otimes V_n),\\
\mathrm{Hom}_{\Sigma_{r+1}}(S^\mu \uparrow^{\Sigma_{r+1}}, D^\lambda) &\cong \mathrm{Hom}_{GL(n)}(L_n(\lambda^t), \nabla_n(\mu^t) \otimes V_n),\\
\mathrm{Hom}_{\Sigma_{r+1}}(S^\mu \uparrow^{\Sigma_{r+1}}, (S^\lambda)^*) &\cong \mathrm{Hom}_{GL(n)}(\Delta_n(\lambda^t), \nabla_n(\mu^t) \otimes V_n).
\end{aligned}$$

Finally, observe that on combining Theorem 4.10 and Theorem 4.16 with the main result Theorem 3.7 from section 3, one obtains (under our usual assumptions $n \geq r, n_i \geq r_i$) the final connections between modular Littlewood-Richardson coefficients in the symmetric group setting:

4.19. **Theorem.** *Fix partitions $\lambda \vdash r$ and $\mu^{(i)} \vdash r_i$ for $i = 1, \dots, a$. Regard λ as an element of $\Lambda^+(n,r)$ and each $\mu^{(i)}$ as elements of $\Lambda^+(n_i,r_i)$ or $\Lambda^+(n,r_i)$. Then,*

(i) $(Y^\lambda \downarrow_{\Sigma_\rho} : Y^{\mu^{(1)}} \boxtimes \cdots \boxtimes Y^{\mu^{(a)}}) = [L_n(\mu^{(1)}) \otimes \cdots \otimes L_n(\mu^{(a)}) : L_n(\lambda)]$;

(ii) $((Y^{\mu^{(1)}} \boxtimes \cdots \boxtimes Y^{\mu^{(a)}}) \uparrow^{\Sigma_r} : Y^\lambda) = [L_n(\lambda) \downarrow_{GL(\nu)} : L_{n_1}(\mu^{(1)}) \boxtimes \cdots \boxtimes L_{n_a}(\mu^{(a)})]$.

Moreover, if all the partitions are p-regular, then

(iii) $[D^\lambda \downarrow_{\Sigma_\rho} : D^{\mu^{(1)}} \boxtimes \cdots \boxtimes D^{\mu^{(a)}}] = (T_n(\mu^{(1)}) \otimes \cdots \otimes T_n(\mu^{(a)}) : T_n(\lambda))$;

(iv) $[(D^{\mu^{(1)}} \boxtimes \cdots \boxtimes D^{\mu^{(a)}}) \uparrow^{\Sigma_r} : D^\lambda] = (T_n(\lambda) \downarrow_{GL(\nu)} : T_{n_1}(\mu^{(1)}) \boxtimes \cdots \boxtimes T_{n_a}(\mu^{(a)}))$.

References

[AR] A. Adamovich and G. Rybnikov, Tilting modules for classical groups and Howe duality in positive characteristic, *Transform. Groups* **1** (1996), 1–34.

[B1] J. Brundan, Lowering operators for $GL(n)$ and quantum $GL(n)$, *Proc. Symposia in Pure Math.* **63** (1998), 95–114.

[B2] J. Brundan, Modular branching rules and the Mullineux map for Hecke algebras of type **A**, to appear in *Proc. London Math. Soc.*.

[BK] J. Brundan and A. Kleshchev, On translation functors for general linear and symmetric groups, preprint, University of Oregon, 1998.

[BKS] J. Brundan, A. Kleshchev and I. Suprunenko, Semisimple restrictions from $GL(n)$ to $GL(n-1)$, to appear in *J. reine angew. Math.*.

[CR] C. Curtis and I. Reiner, *Methods of representation theory*, volume 1, Wiley, New York, 1981

[D1] S. Donkin, *Rational representations of algebraic groups: Tensor products and filtrations*, Lecture Notes in Math. 1140, Springer-Verlag, 1985.

[D2] S. Donkin, On Schur algebras and related algebras I, *J. Algebra* **104** (1986), 310–328.

[D3] S. Donkin, On tilting modules for algebraic groups, *Math. Z.* **212** (1993), 39–60.

[D4] S. Donkin, Standard homological properties for quantum GL_n, *J. Algebra* **181** (1996), 235–266.

[D5] S. Donkin, *The q-Schur algebra*, CUP, to appear (1997).

[G] J. A. Green, *Polynomial representations of* GL_n, Lecture Notes in Math., vol. 830, Springer-Verlag, 1980.

[J1] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Math., vol. 682, Springer-Verlag, 1978.

[J2] G. D. James, The representation theory of the symmetric groups, *Proc. Symposia in Pure Math.* **47** (1987). 111–126.

[J] J. C. Jantzen, *Representations of algebraic groups*, Academic Press, Orlando, 1987.

[JS] J. C. Jantzen and G. M. Seitz, On the representation theory of the symmetric groups, *Proc. London Math. Soc.* **65** (1992), 475–504.

[K1] A. S. Kleshchev, Branching rules for modular representations of symmetric groups I, *J. Algebra* **178** (1995), 493–511.

[K2] A. Kleshchev, Branching rules for modular representations of symmetric groups II, *J. reine angew. Math.* **459** (1995), 163–212.

[K3] A. Kleshchev, Branching rules for modular representations of symmetric groups IV, to appear in *J. Algebra*.

[K4] A. Kleshchev, On decomposition numbers and branching coefficients for symmetric and special linear groups, *Proc. London Math. Soc.* **75** (1997), 497–558.

[L] P. Landrock, *Finite Group Algebras and their Modules*, Cambridge University Press, Cambridge, 1983.

[M] O. Mathieu, Filtrations of G-modules, *Ann. Sci. Ecole Norm. Sup.* **23** (1990), 625–644.

[MP] O. Mathieu and G. Papadopoulo, A character formula for a family of simple modular representations of GL_n, preprint, 1997.

[R] R. W. Richardson, Affine coset spaces of reductive algebraic groups, *Bull. London Math. Soc.* **9** (1977), 38–41.

[S] M. Sweedler, *Hopf algebras*, Benjamin, New York, 1969.

[V] W. van der Kallen, *Lectures on Frobenius splittings and B-modules*, Springer-Verlag, Berlin, 1993.

[W] Wang J-P., *Sheaf cohomology on G/B and tensor products of Weyl modules*, J. Algebra **77** (1982), 162–185.

brundan@darkwing.uoregon.edu, klesh@math.uoregon.edu
Department of Mathematics, *University of Oregon*, *Eugene, Oregon*, *U.S.A.*