

GRADED DECOMPOSITION NUMBERS FOR CYCLOTOMIC HECKE ALGEBRAS

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ABSTRACT. In recent joint work with Wang, we have constructed graded Specht modules for cyclotomic Hecke algebras. In this article, we prove a graded version of the Lascoux-Leclerc-Thibon conjecture, describing the decomposition numbers of graded Specht modules over a field of characteristic zero.

1. INTRODUCTION

Since the classic work of Bernstein and Zelevinsky [BZ], the representation theory of the affine Hecke algebra H_d associated to the symmetric group Σ_d has been a fundamental topic in representation theory from many points of view. For brevity in this introduction, we discuss only the situation when H_d is defined over the ground field \mathbb{C} at parameter $1 \neq \xi \in \mathbb{C}^\times$ that is a primitive e th root of unity. In [A1], building on powerful geometric results of Kazhdan and Lusztig [KL] and Ginzburg [CG, Chapter 8], Ariki established a remarkable connection between the representation theory of certain finite dimensional quotients of H_d , known as cyclotomic Hecke algebras, and the canonical bases of integrable highest weight modules over the affine Lie algebra $\widehat{\mathfrak{sl}}_e(\mathbb{C})$. In a special case, this connection had been suggested earlier by Lascoux, Leclerc and Thibon [LLT]. Similar results were announced by Grojnowski following [G1], but the proofs were never published.

To recall some of these results in a little more detail, let Λ be a dominant integral weight of level l for $\widehat{\mathfrak{sl}}_e(\mathbb{C})$. Let $V(\Lambda)_{\mathbb{C}}$ be the corresponding integrable highest weight module and fix a highest weight vector $v_\Lambda \in V(\Lambda)_{\mathbb{C}}$. To the weight Λ we associate cyclotomic Hecke algebras H_d^Λ for each $d \geq 0$; see §4.2. Letting $\underline{\text{Proj}}(H_d^\Lambda)$ denote the category of finitely generated projective H_d^Λ -modules and writing $[\underline{\text{Proj}}(H_d^\Lambda)]_{\mathbb{C}}$ for its complexified Grothendieck group, Ariki showed that there is a unique \mathbb{C} -linear isomorphism

$$\underline{\delta} : V(\Lambda)_{\mathbb{C}} \xrightarrow{\sim} \bigoplus_{d \geq 0} [\underline{\text{Proj}}(H_d^\Lambda)]_{\mathbb{C}}$$

such that v_Λ maps to the class of the regular H_0^Λ -module, and the actions of the Chevalley generators $e_i, f_i \in \widehat{\mathfrak{sl}}_e(\mathbb{C})$ correspond to certain exact *i-restriction* and *i-induction functors* on the Hecke algebra side.

Now the key result obtained by Ariki in [A1, Theorem 4.4] can be formulated as follows: the isomorphism $\underline{\delta}$ maps the Kashiwara-Lusztig canonical basis for $V(\Lambda)_{\mathbb{C}}$ to the basis of the Grothendieck group arising from the isomorphism

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classes of projective indecomposable modules. Ariki then applied this theorem to compute the decomposition numbers of *Specht modules*, for which some foundational results in levels $l > 1$ were developed subsequently by Dipper, James and Mathas [DJM]. In level one this gave a proof of the Lascoux-Leclerc-Thibon conjecture from [LLT] concerning decomposition numbers of the Iwahori-Hecke algebra of type A at an l th root of unity over \mathbb{C} ; moreover it generalized the conjecture to higher levels.

Recently, there have been some exciting new developments thanks to works of Khovanov and Lauda [KL1, KL2] and Rouquier [R2], who have independently introduced a new family of algebras attached to Cartan matrices. For the rest of the introduction, we let R_d denote the Khovanov-Lauda-Rouquier algebra of degree d attached to the Cartan matrix of type $\widehat{\mathfrak{sl}}_e$; we mean the direct sum over all $\alpha \in Q_+$ of height d of the algebras R_α defined by generators and relations in §2.3 below. Unlike the affine Hecke algebra H_d , the algebra R_d is \mathbb{Z} -graded in a canonical way.

In [KL1, §3.4], Khovanov and Lauda also introduced certain “cyclotomic” finite dimensional graded quotients R_d^Λ of R_d (see §4.1), and conjectured a result which can be viewed as a graded version of Ariki’s categorification theorem as formulated above. Remarkably, the Khovanov-Lauda categorification conjecture makes equally good sense in any type. One of the main results of this article proves the conjecture in the $\widehat{\mathfrak{sl}}_e$ -case. To do this, we exploit an explicit algebra isomorphism $\rho : R_d^\Lambda \xrightarrow{\sim} H_d^\Lambda$ constructed in [BK4], which allows us to lift existing results about H_d^Λ to R_d^Λ , incorporating additional information about gradings as we go.

To give a little more detail, the algebra R_d^Λ is graded, so it makes sense to consider the category $\text{Proj}(R_d^\Lambda)$ of finitely generated projective *graded* R_d^Λ -modules. The Grothendieck group $[\text{Proj}(R_d^\Lambda)]$ is a $\mathbb{Z}[q, q^{-1}]$ -module, with multiplication by q corresponding to shifting the grading on a module up by one. Let $[\text{Proj}(R_d^\Lambda)]_{\mathbb{Q}(q)} := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Proj}(R_d^\Lambda)]$. Let $V(\Lambda)$ be the integrable highest weight module for the quantized enveloping algebra $U_q(\widehat{\mathfrak{sl}}_e)$ over the field $\mathbb{Q}(q)$, with highest weight vector v_Λ . Combining results from [G2, KL1, CR, R2], we show to start with that there is a unique $\mathbb{Q}(q)$ -linear isomorphism

$$\delta : V(\Lambda) \xrightarrow{\sim} \bigoplus_{d \geq 0} [\text{Proj}(R_d^\Lambda)]_{\mathbb{Q}(q)}$$

such that v_Λ maps to the class of the regular R_0^Λ -module and the actions of the Chevalley generators $E_i, F_i \in U_q(\widehat{\mathfrak{sl}}_e)$ correspond to graded analogues of the i -restriction and i -induction functors from before; see §4.4.

Moreover, we show that the isomorphism δ maps the canonical basis for $V(\Lambda)$ to the basis of the Grothendieck group arising from the isomorphism classes of indecomposable projective graded modules that are self-dual with respect to a certain duality \otimes ; see §4.5. Our proof of this relies ultimately on Ariki’s original categorification theorem from [A1].

In joint work with Wang [BKW], we have also defined graded versions of Specht modules for the algebras R_d^Λ . Another of our main results gives an explicit formula for the decomposition numbers of graded Specht modules. This

should be regarded as a graded version of the Lascoux-Leclerc-Thibon conjecture (generalized to higher levels). It shows that the decomposition numbers of graded Specht modules are obtained by expanding the “standard monomials” in $V(\Lambda)$ in terms of the dual-canonical basis; see §5.5 and §3.8 for details.

The results of this article fit naturally into the general framework of 2-representations of 2-Kac-Moody algebras developed by Rouquier in [R2]; see also [KL3]. While writing up this work, we have learnt of an announcement by Rouquier indicating that he has found a direct geometric proof of the Khovanov-Lauda categorification conjecture that is valid in arbitrary type, although details are not yet available. More recently still, Varagnolo and Vasserot have released a preprint in which they prove the Khovanov-Lauda categorification conjecture at the affine level in arbitrary simply-laced type; see [VV3]. We point out however that these results do not immediately imply the graded version of the Lascoux-Leclerc-Thibon conjecture proved here, since for that one needs to deal with Specht modules over the cyclotomic quotients.

We end the introduction with a brief guide to the rest of the article, indicating some of the other things to be found here. Section 2 is primarily devoted to recalling the definition of the algebras R_d in type $\widehat{\mathfrak{sl}}_e$, and then reviewing some of the foundational results proved about them in [KL1].

In section 3 we review the construction of the irreducible highest weight module $V(\Lambda)$ over $U_q(\widehat{\mathfrak{sl}}_e)$ as a summand of Fock space. At the same time, we construct various bases for these modules, paralleling the setup of [BK5, §2] closely. This part of the story is surprisingly lengthy as there are some subtle combinatorial issues surrounding the triangularity of the standard monomials in $V(\Lambda)$; see §3.9. Unlike almost all of the literature in the subject, our approach emphasizes the dual-canonical basis rather than the canonical basis.

In section 4 we consider the cyclotomic quotients R_d^Λ of R_d introduced originally in [KL1, §3.4]. We use the isomorphism between R_d^Λ and H_d^Λ from [BK4] to quickly deduce the classification of irreducible graded R_d^Λ -modules from Grojnowski’s classification of irreducible H_d^Λ in terms of crystal graphs from [G2]; see §4.8. At the same time we lift various branching rules to the graded setting. Then we prove the first key categorification theorem, which identifies $V(\Lambda)$ with the direct sum $\bigoplus_{d \geq 0} [\text{Proj}(R_d^\Lambda)]_{\mathbb{Q}(q)}$ as above; see §4.10. As an application, we compute the graded dimension of R_d^Λ ; see §4.11. We stress that this part of the development makes sense over any ground field, and does not depend on any results from geometric representation theory.

In section 5 we lift Ariki’s results to the graded setting to prove simultaneously the graded version of the Lascoux-Leclerc-Thibon conjecture and the Khovanov-Lauda conjecture; see §5.5. In the course of this we encounter some non-trivial issues related to the parametrization of irreducible modules: there are two relevant parametrizations, one arising from the crystal graph and the other arising from Specht module theory; see §5.4 for the latter. The identification of the two parametrizations is addressed in Ariki’s work, but we give a self-contained treatment here in order to keep track of gradings. We also discuss the situation over fields of positive characteristic, introducing graded analogues of James’ adjustment matrices; see §5.6.

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2. REVIEW OF RESULTS OF KHOVANOV AND LAUDA

Fix an algebraically closed field F and an integer e such that either $e = 0$ or $e \geq 2$. Always q denotes an indeterminate.

2.1. Cartan integers, weights and roots. Let Γ be the quiver with vertex set $I := \mathbb{Z}/e\mathbb{Z}$, and a directed edge from i to j if $i \neq j = i + 1$ in I . Thus Γ is the quiver of type A_∞ if $e = 0$ or $A_{e-1}^{(1)}$ if $e > 0$, with a specific orientation:

$$A_\infty : \quad \cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

$$A_{e-1}^{(1)} : \quad 0 \rightleftarrows 1 \quad \begin{array}{ccc} & 0 & \\ \nearrow & & \searrow \\ 2 & & 1 \\ \longleftarrow & & \longrightarrow \end{array} \quad \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \uparrow & & \downarrow \\ 3 & \longleftarrow & 2 \end{array} \quad \begin{array}{ccc} & 0 & \\ \nearrow & & \searrow \\ 4 & & 1 \\ \nwarrow & & \swarrow \\ & 3 \longleftarrow 2 & \end{array} \quad \cdots$$

The corresponding (symmetric) Cartan matrix $(a_{i,j})_{i,j \in I}$ is defined by

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \\ -1 & \text{if } i \rightarrow j \text{ or } i \leftarrow j, \\ -2 & \text{if } i \rightleftarrows j. \end{cases} \quad (2.1)$$

Here the symbols $i \rightarrow j$ and $j \leftarrow i$ both indicate that $i \neq j = i + 1 \neq i - 1$, $i \rightleftarrows j$ indicates that $i \neq j = i + 1 = i - 1$, and $i \neq j$ indicates that $i \neq j \neq i \pm 1$.

Following [K], let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of the Cartan matrix $(a_{i,j})_{i,j \in I}$, so we have the simple roots $\{\alpha_i \mid i \in I\}$, the fundamental dominant weights $\{\Lambda_i \mid i \in I\}$, and the normalized invariant form (\cdot, \cdot) such that

$$(\alpha_i, \alpha_j) = a_{i,j}, \quad (\Lambda_i, \alpha_j) = \delta_{i,j} \quad (i, j \in I).$$

Let $Q_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ denote the positive part of the corresponding root lattice. For $\alpha \in Q_+$, we write $\text{ht}(\alpha)$ for the sum of its coefficients when expanded in terms of the α_i 's.

2.2. The algebra \mathbf{f} . Let \mathbf{f} denote Lusztig's algebra from [Lus, §1.2] attached to the Cartan matrix (2.1) over the field $\mathbb{Q}(q)$. We adopt the same conventions as [KL1, §3.1], so our q is Lusztig's v^{-1} . To be more precise, denote

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! := [n][n-1] \cdots [1], \quad \begin{bmatrix} n \\ m \end{bmatrix} := \frac{[n]!}{[n-m]![m]}.$$

Then \mathbf{f} is the $\mathbb{Q}(q)$ -algebra on generators θ_i ($i \in I$) subject to the quantum Serre relations

$$(\text{ad}_q \theta_i)^{1-a_{j,i}}(\theta_j) = 0 \quad (2.2)$$

where

$$(\text{ad}_q x)^n(y) := \sum_{m=0}^n (-1)^m \binom{n}{m} x^{n-m} y x^m. \quad (2.3)$$

There is a Q_+ -grading $\mathbf{f} = \bigoplus_{\alpha \in Q_+} \mathbf{f}_\alpha$ such that θ_i is of degree α_i . The algebra \mathbf{f} possesses a bar-involution $- : \mathbf{f} \rightarrow \mathbf{f}$ that is anti-linear with respect to the field automorphism sending q to q^{-1} , such that $\overline{\theta_i} = \theta_i$ for each $i \in I$.

If we equip $\mathbf{f} \otimes \mathbf{f}$ with algebra structure via the rule

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = q^{-(\alpha, \beta)} x_1 y_1 \otimes x_2 y_2$$

for $x_2 \in \mathbf{f}_\alpha$ and $y_1 \in \mathbf{f}_\beta$, there is a Q_+ -graded comultiplication $m^* : \mathbf{f} \rightarrow \mathbf{f} \otimes \mathbf{f}$, which is the unique algebra homomorphism such that $\theta_i \mapsto \theta_i \otimes 1 + 1 \otimes \theta_i$ for each $i \in I$. For $\alpha, \beta \in Q_+$, we let

$$m_{\alpha, \beta} : \mathbf{f}_\alpha \otimes \mathbf{f}_\beta \rightarrow \mathbf{f}_{\alpha+\beta}, \quad m_{\alpha, \beta}^* : \mathbf{f}_{\alpha+\beta} \rightarrow \mathbf{f}_\alpha \otimes \mathbf{f}_\beta$$

denote the multiplication and comultiplication maps induced on individual weight spaces, so $m = \sum m_{\alpha, \beta}$ is the multiplication on \mathbf{f} and $m^* = \sum m_{\alpha, \beta}^*$.

Finally let $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$ and $\mathcal{A}\mathbf{f}$ be the \mathcal{A} -subalgebra of \mathbf{f} generated by the quantum divided powers $\theta_i^{(n)} := \theta_i^n / [n]!$. The bar involution induces an involution of $\mathcal{A}\mathbf{f}$, and also the map m^* restricts to a well-defined comultiplication $m^* : \mathcal{A}\mathbf{f} \rightarrow \mathcal{A}\mathbf{f} \otimes \mathcal{A}\mathbf{f}$.

2.3. The algebra R_α . The symmetric group Σ_d acts on the left on the set I^d by place permutation. The orbits are the sets

$$I^\alpha := \{\mathbf{i} = (i_1, \dots, i_d) \in I^d \mid \alpha_{i_1} + \dots + \alpha_{i_d} = \alpha\}$$

for each $\alpha \in Q_+$ with $\text{ht}(\alpha) = d$. As usual, we let s_1, \dots, s_{d-1} denote the basic transpositions in Σ_d .

For $\alpha \in Q_+$ of height d , let R_α denote the associative, unital F -algebra on generators $\{e(\mathbf{i}) \mid \mathbf{i} \in I^\alpha\} \cup \{y_1, \dots, y_d\} \cup \{\psi_1, \dots, \psi_{d-1}\}$ subject only to the following relations for $\mathbf{i}, \mathbf{j} \in I^\alpha$ and all admissible r, s :

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i}, \mathbf{j}} e(\mathbf{i}); \quad \sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i}) = 1; \quad (2.4)$$

$$y_r e(\mathbf{i}) = e(\mathbf{i}) y_r; \quad \psi_r e(\mathbf{i}) = e(s_r \mathbf{i}) \psi_r; \quad (2.5)$$

$$y_r y_s = y_s y_r; \quad (2.6)$$

$$\psi_r y_s = y_s \psi_r \quad \text{if } s \neq r, r+1; \quad (2.7)$$

$$\psi_r \psi_s = \psi_s \psi_r \quad \text{if } |r-s| > 1; \quad (2.8)$$

$$\psi_r y_{r+1} e(\mathbf{i}) = \begin{cases} (y_r \psi_r + 1) e(\mathbf{i}) & \text{if } i_r = i_{r+1}, \\ y_r \psi_r e(\mathbf{i}) & \text{if } i_r \neq i_{r+1}; \end{cases} \quad (2.9)$$

$$y_{r+1} \psi_r e(\mathbf{i}) = \begin{cases} (\psi_r y_r + 1) e(\mathbf{i}) & \text{if } i_r = i_{r+1}, \\ \psi_r y_r e(\mathbf{i}) & \text{if } i_r \neq i_{r+1}; \end{cases} \quad (2.10)$$

$$\psi_r^2 e(\mathbf{i}) = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ e(\mathbf{i}) & \text{if } i_r \neq i_{r+1}, \\ (y_{r+1} - y_r) e(\mathbf{i}) & \text{if } i_r \rightarrow i_{r+1}, \\ (y_r - y_{r+1}) e(\mathbf{i}) & \text{if } i_r \leftarrow i_{r+1}, \\ (y_{r+1} - y_r)(y_r - y_{r+1}) e(\mathbf{i}) & \text{if } i_r \rightleftarrows i_{r+1}; \end{cases} \quad (2.11)$$

$$\psi_r \psi_{r+1} \psi_r e(\mathbf{i}) = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1)e(\mathbf{i}) & \text{if } i_{r+2} = i_r \rightarrow i_{r+1}, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1)e(\mathbf{i}) & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 2y_{r+1} \\ \quad + y_r + y_{r+2})e(\mathbf{i}) & \text{if } i_{r+2} = i_r \rightleftharpoons i_{r+1}, \\ \psi_{r+1} \psi_r \psi_{r+1} e(\mathbf{i}) & \text{otherwise.} \end{cases} \quad (2.12)$$

There is a unique \mathbb{Z} -grading on R_α such that each $e(\mathbf{i})$ is of degree 0, each y_r is of degree 2, and each $\psi_r e(\mathbf{i})$ is of degree $-a_{i_r, i_{r+1}}$.

The algebra R_α is one of the algebras introduced by Khovanov and Lauda in [KL1, KL2] (except for $e = 2$), and was discovered independently by Rouquier in [R2] (in full generality).

2.4. Graded algebras and modules. Let H be a \mathbb{Z} -graded F -algebra. Let $\text{Mod}(H)$ denote the abelian category of all graded left H -modules, denoting (degree-preserving) homomorphisms in this category by Hom . Let $\text{Rep}(H)$ denote the abelian subcategory of all finite dimensional graded left H -modules and $\text{Proj}(H)$ denote the additive subcategory of all finitely generated projective graded left H -modules. Denote the corresponding Grothendieck groups by $[\text{Rep}(H)]$ and $[\text{Proj}(H)]$, respectively. We view these as \mathcal{A} -modules via $q^m[M] := [M\langle m \rangle]$, where $M\langle m \rangle$ denotes the module obtained by shifting the grading up by m :

$$M\langle m \rangle_n = M_{n-m}. \quad (2.13)$$

Given $f = \sum_{n \in \mathbb{Z}} f_n q^n \in \mathbb{Z}[[q, q^{-1}]]$ and a graded module M , we allow ourselves to write simply $f \cdot M$ for $\bigoplus_{n \in \mathbb{Z}} M\langle n \rangle^{\oplus f_n}$.

For $n \in \mathbb{Z}$, we let

$$\text{Hom}_H(M, N)_n := \text{Hom}_H(M\langle n \rangle, N) = \text{Hom}_H(M, N\langle -n \rangle)$$

denote the space of all homomorphisms that are homogeneous of degree n , i.e. they map M_i into N_{i+n} for each $i \in \mathbb{Z}$. Set

$$\text{HOM}_H(M, N) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_H(M, N)_n, \quad \text{END}_H(M) := \text{HOM}_H(M, M).$$

There is a canonical pairing, the *Cartan pairing*,

$$\langle \cdot, \cdot \rangle : \text{Proj}(H) \times \text{Rep}(H) \rightarrow \mathcal{A}, \quad \langle [P], [M] \rangle := \text{qdim HOM}_H(P, M), \quad (2.14)$$

where $\text{qdim } V$ denotes $\sum_{n \in \mathbb{Z}} q^n \dim V_n$ for any finite dimensional graded vector space V . Note the Cartan pairing is *sesquilinear* (anti-linear in the first argument, linear in the second).

Occasionally, we will need to forget the grading on H and work with ordinary ungraded H -modules. To avoid confusion in the ungraded setting, we denote the category of all left H -modules (resp. finite dimensional left H -modules, resp. finitely generated projective left H -modules) by $\underline{\text{Mod}}(H)$ (resp. $\underline{\text{Rep}}(H)$, resp. $\underline{\text{Proj}}(H)$). We denote homomorphisms in these categories by $\underline{\text{Hom}}$. Let $[\underline{\text{Rep}}(H)]$ (resp. $[\underline{\text{Proj}}(H)]$) denote the Grothendieck group of $\underline{\text{Rep}}(H)$ (resp. $\underline{\text{Proj}}(H)$). Given a graded module M , we write \underline{M} for the ungraded module obtained from it by forgetting the grading. For $M, N \in \text{Rep}(H)$, we have that

$$\underline{\text{Hom}}_H(\underline{M}, \underline{N}) = \underline{\text{HOM}}_H(M, N). \quad (2.15)$$

The following lemmas summarize some standard facts:

Lemma 2.1 ([NV, Theorem 4.4.6, Remark 4.4.8]). *If M is any finitely generated graded H -module, the radical of \underline{M} is a graded submodule of M .*

Lemma 2.2 ([NV, Theorem 4.4.4(v)]). *If $L \in \text{Rep}(H)$ is irreducible then $\underline{L} \in \underline{\text{Rep}}(H)$ is irreducible too.*

Lemma 2.3 ([NV, Theorem 9.6.8], [BGS, Lemma 2.5.3]). *Assume that H is finite dimensional. If $K \in \underline{\text{Rep}}(H)$ is irreducible, then there exists an irreducible $L \in \text{Rep}(H)$ such that $\underline{L} \cong K$. Moreover, L is unique up to isomorphism and grading shift.*

Given $M, L \in \text{Rep}(H)$ with L irreducible, we write $[M : L]_q$ for the q -composition multiplicity, i.e. $[M : L]_q := \sum_{n \in \mathbb{Z}} a_n q^n$ where a_n is the multiplicity of $L\langle n \rangle$ in a graded composition series of M . In view of Lemma 2.2, we recover the ordinary composition multiplicity $[\underline{M} : \underline{L}]$ from $[M : L]_q$ on setting q to 1.

2.5. Induction and restriction. Given $\alpha, \beta \in Q_+$, we set

$$R_{\alpha, \beta} := R_\alpha \otimes R_\beta,$$

viewed as an algebra in the usual way. We denote the “outer” tensor product of an R_α -module M and an R_β -module N by $M \boxtimes N$.

There is an obvious embedding of $R_{\alpha, \beta}$ into the algebra $R_{\alpha+\beta}$ mapping $e(\mathbf{i}) \otimes e(\mathbf{j})$ to $e(\mathbf{ij})$, where \mathbf{ij} denotes the concatenation of the two sequences. It is not a unital algebra homomorphism. We denote the image of the identity element of $R_{\alpha, \beta}$ under this map by

$$e_{\alpha, \beta} = \sum_{\mathbf{i} \in I^\alpha, \mathbf{j} \in I^\beta} e(\mathbf{ij}).$$

Let $\text{Ind}_{\alpha, \beta}^{\alpha+\beta}$ and $\text{Res}_{\alpha, \beta}^{\alpha+\beta}$ denote the corresponding induction and restriction functors, so

$$\text{Ind}_{\alpha, \beta}^{\alpha+\beta} := R_{\alpha+\beta} e_{\alpha, \beta} \otimes_{R_{\alpha, \beta}} ? : \text{Mod}(R_{\alpha, \beta}) \rightarrow \text{Mod}(R_{\alpha+\beta}), \quad (2.16)$$

$$\text{Res}_{\alpha, \beta}^{\alpha+\beta} := e_{\alpha, \beta} R_{\alpha+\beta} \otimes_{R_{\alpha+\beta}} ? : \text{Mod}(R_{\alpha+\beta}) \rightarrow \text{Mod}(R_{\alpha, \beta}). \quad (2.17)$$

Note $\text{Res}_{\alpha, \beta}^{\alpha+\beta}$ is just left multiplication by the idempotent $e_{\alpha, \beta}$, so it is exact and sends finite dimensional modules to finite dimensional modules. By [KL1, Proposition 2.16], $e_{\alpha, \beta} R_{\alpha+\beta}$ is a graded free left $R_{\alpha, \beta}$ -module of finite rank, so $\text{Res}_{\alpha, \beta}^{\alpha+\beta}$ sends finitely generated projective modules to finitely generated projective modules. The functor $\text{Ind}_{\alpha, \beta}^{\alpha+\beta}$ is left adjoint to $\text{Res}_{\alpha, \beta}^{\alpha+\beta}$, so it also sends finitely generated projective modules to finitely generated projective modules. Finally $R_{\alpha+\beta} e_{\alpha, \beta}$ is a graded free right $R_{\alpha, \beta}$ -module of finite rank, so $\text{Ind}_{\alpha, \beta}^{\alpha+\beta}$ sends finite dimensional modules to finite dimensional modules too.

We will often appeal without mention to the following general facts about the representation theory of R_α , all of which are noted in [KL1, §2.5].

Lemma 2.4. *The algebra R_α has finitely many isomorphism classes of irreducible graded modules (up to degree shift), all of which are finite dimensional. Every irreducible $L \in \text{Rep}(R_\alpha)$ has a unique (up to isomorphism) projective cover $P \in \text{Proj}(R_\alpha)$ with irreducible head isomorphic to L , and every indecomposable projective graded R_α -module is of this form.*

Proof. All irreducible modules are finite dimensional because R_α is finitely generated over its center; see [KL1, Corollary 2.10]. Every irreducible graded R_α -module is a quotient of a finite dimensional module of the form $\text{Ind}_{\alpha_{i_1}, \dots, \alpha_{i_d}}^\alpha L$ for some $\mathbf{i} \in I^\alpha$ and some irreducible graded $R_{\alpha_{i_1}, \dots, \alpha_{i_d}}$ -module L . Since I^α is finite and there is only one irreducible graded $R_{\alpha_{i_1}, \dots, \alpha_{i_d}}$ -module up to degree shift, we deduce that there are only finitely many irreducible graded R_α -modules up to degree shift. These statements imply in particular that the graded Jacobson radical $J(R_\alpha)$ of R_α is of finite codimension. Noting also that $J(R_\alpha)_0 \subseteq J((R_\alpha)_0)$ and that $(R_\alpha)_0$ is finite dimensional, the rest of the lemma follows by arguments involving lifting homogeneous idempotents from the finite dimensional semisimple graded algebra $R_\alpha/J(R_\alpha)$. \square

2.6. The functors θ_i and θ_i^* . For $i \in I$, let $P(i)$ denote the regular representation of R_{α_i} . Define a functor

$$\theta_i := \text{Ind}_{\alpha, \alpha_i}^{\alpha + \alpha_i} (? \boxtimes P(i)) : \text{Mod}(R_\alpha) \rightarrow \text{Mod}(R_{\alpha + \alpha_i}). \quad (2.18)$$

This functor is exact, and it maps finitely generated projective modules to finitely generated projective modules, so restricts to a functor $\theta_i : \text{Proj}(R_\alpha) \rightarrow \text{Proj}(R_{\alpha + \alpha_i})$.

The functor θ_i possesses a right adjoint

$$\theta_i^* := \text{Hom}_{R'_{\alpha_i}}(P(i), ?) : \text{Mod}(R_{\alpha + \alpha_i}) \rightarrow \text{Mod}(R_\alpha), \quad (2.19)$$

where R'_{α_i} denotes the subalgebra $1 \otimes R_{\alpha_i}$ of R_{α, α_i} . Equivalently, θ_i^* is defined by multiplication by the idempotent e_{α, α_i} followed by restriction to the subalgebra $R_\alpha = R_\alpha \otimes 1$ of R_{α, α_i} . The functor θ_i^* is exact, and it restricts to define a functor $\theta_i^* : \text{Rep}(R_{\alpha + \alpha_i}) \rightarrow \text{Rep}(R_\alpha)$.

2.7. Dualities. The algebra R_α possesses a graded anti-automorphism

$$* : R_\alpha \rightarrow R_\alpha \quad (2.20)$$

which is the identity on generators.

Using this we introduce a duality denoted \circledast on $\text{Rep}(R_\alpha)$, mapping a module M to $M^{\circledast} := \text{Hom}_F(M, F)$ with the action defined by $(xf)(m) = f(mx^*)$. This duality commutes with θ_i^* , i.e. there is an isomorphism of functors

$$\circledast \circ \theta_i^* \cong \theta_i^* \circ \circledast : \text{Rep}(R_{\alpha + \alpha_i}) \rightarrow \text{Rep}(R_\alpha). \quad (2.21)$$

There is another duality denoted $\#$ on $\text{Proj}(R_\alpha)$ mapping a projective module P to $P^\# := \text{Hom}_{R_\alpha}(P, R_\alpha)$ with the action defined by $(xf)(p) = f(p)x^*$. This commutes with the functor θ_i , i.e.

$$\# \circ \theta_i \cong \theta_i \circ \# : \text{Proj}(R_\alpha) \rightarrow \text{Proj}(R_{\alpha + \alpha_i}). \quad (2.22)$$

Recalling (2.14), the following lemma makes a connection between the dualities \circledast and $\#$.

Lemma 2.5. *For $P \in \text{Proj}(R_\alpha)$ and $M \in \text{Rep}(R_\alpha)$, we have that*

$$\langle [P^\#], [M] \rangle = \overline{\langle [P], [M^{\circledast}] \rangle}.$$

Proof. Let P^{op} denote P viewed as a right R_α -module with action $px := x^*p$ for $p \in P, x \in R_\alpha$. We have that

$$\begin{aligned} \langle [P^\#], [M] \rangle &= \text{qdim } \text{HOM}_{R_\alpha}(P^\#, M) = \text{qdim } \text{HOM}_{R_\alpha}(P^\#, R_\alpha) \otimes_{R_\alpha} M \\ &= \text{qdim } P^{\text{op}} \otimes_{R_\alpha} M = \overline{\text{qdim } \text{HOM}_F(P^{\text{op}} \otimes_{R_\alpha} M, F)} \\ &= \overline{\text{qdim } \text{HOM}_{R_\alpha}(P^{\text{op}}, \text{HOM}_F(M, F))} = \overline{\text{qdim } \text{HOM}_{R_\alpha}(P, M^\otimes)} \\ &= \overline{\langle [P], [M^\otimes] \rangle}, \end{aligned}$$

using the observation that $(P^\#)^\# \cong P$. \square

Corollary 2.6. *Let $L \in \text{Rep}(R_\alpha)$ be an irreducible module with projective cover $P \in \text{Proj}(R_\alpha)$. Then the projective cover of L^\otimes is isomorphic to $P^\#$. In particular, if $L \cong L^\otimes$ then $P \cong P^\#$.*

2.8. Divided powers. As explained in detail in [KL1, §2.2], in the case $\alpha = n\alpha_i$ for some $i \in I$, the algebra R_α is isomorphic to the nil-Hecke algebra NH_n . It has a canonical representation on the polynomial algebra $F[y_1, \dots, y_n]$ such that each y_r acts as multiplication by y_r and each ψ_r acts as the *divided difference operator*

$$\partial_r : f \mapsto \frac{s_r f - f}{y_r - y_{r+1}}.$$

Let $P(i^{(n)})$ denote the polynomial representation of $R_{n\alpha_i}$ viewed as a graded $R_{n\alpha_i}$ -module with grading defined by

$$\deg(y_1^{m_1} \cdots y_n^{m_n}) := 2m_1 + \cdots + 2m_n - \frac{1}{2}n(n-1).$$

Denoting the left regular $R_{n\alpha_i}$ -module by $P(i^n)$, it is noted in [KL1, §2.2] that

$$P(i^n) \cong [n]! \cdot P(i^{(n)}). \quad (2.23)$$

In particular, $P(i^{(n)})$ is projective.

Now we can generalize the definition of the functors θ_i and θ_i^* : for $i \in I$ and $n \geq 1$, set

$$\theta_i^{(n)} := \text{Ind}_{\alpha, n\alpha_i}^{\alpha+n\alpha_i} (? \boxtimes P(i^{(n)})) : \text{Mod}(R_\alpha) \rightarrow \text{Mod}(R_{\alpha+n\alpha_i}), \quad (2.24)$$

$$(\theta_i^*)^{(n)} := \text{HOM}_{R'_{n\alpha_i}}(P(i^{(n)}), ?) : \text{Mod}(R_{\alpha+n\alpha_i}) \rightarrow \text{Mod}(R_\alpha), \quad (2.25)$$

where $R'_{n\alpha_i} := 1 \otimes R_{n\alpha_i} \subseteq R_{\alpha, n\alpha_i}$. Both functors are exact, $\theta_i^{(n)}$ sends finitely generated projective modules to finitely generated projective modules, and $(\theta_i^*)^{(n)}$ sends finite dimensional modules to finite dimensional modules. By transitivity of induction and restriction, there are isomorphisms

$$\theta_i^n \cong \text{Ind}_{\alpha, n\alpha_i}^{\alpha+n\alpha_i} (? \boxtimes P(i^n)), \quad (\theta_i^*)^n \cong \text{HOM}_{R'_{n\alpha_i}}(P(i^n), ?).$$

Hence (2.23) implies that the n th powers of θ_i and θ_i^* decompose as

$$\theta_i^n \cong [n]! \cdot \theta_i^{(n)}, \quad (\theta_i^*)^n \cong [n]! \cdot (\theta_i^*)^{(n)}. \quad (2.26)$$

2.9. The Khovanov-Lauda theorem. It is convenient to abbreviate the direct sums of all our Grothendieck groups by

$$[\mathrm{Proj}(R)] := \bigoplus_{\alpha \in Q_+} [\mathrm{Proj}(R_\alpha)], \quad [\mathrm{Rep}(R)] := \bigoplus_{\alpha \in Q_+} [\mathrm{Rep}(R_\alpha)]. \quad (2.27)$$

Also, for $\alpha, \beta \in Q_+$, we identify the Grothendieck group $[\mathrm{Proj}(R_{\alpha, \beta})]$ with $[\mathrm{Proj}(R_\alpha)] \otimes_{\mathcal{A}} [\mathrm{Proj}(R_\beta)]$ so that $[P \boxtimes Q]$ is identified with $[P] \otimes [Q]$. Finally, we observe that the exact functors $\theta_i^{(n)}$ (resp. $(\theta_i^*)^{(n)}$) induce \mathcal{A} -linear endomorphisms of $[\mathrm{Proj}(R)]$ (resp. $[\mathrm{Rep}(R)]$) which we denote by the same notation. Now we can state the following important theorem proved in [KL1, Section 3].

Theorem 2.7 (Khovanov-Lauda). *There is a unique \mathcal{A} -module isomorphism*

$$\gamma : \mathcal{A}\mathbf{f} \xrightarrow{\sim} [\mathrm{Proj}(R)]$$

such that $1 \mapsto [R_0]$ (the class of the left regular representation of the trivial algebra R_0) and $\gamma(x\theta_i^{(n)}) = \theta_i^{(n)}(\gamma(x))$ for each $x \in \mathcal{A}\mathbf{f}$, $i \in I$ and $n \geq 1$. Under this isomorphism:

- (1) the multiplication $m_{\alpha, \beta} : \mathcal{A}\mathbf{f}_\alpha \otimes_{\mathcal{A}} \mathcal{A}\mathbf{f}_\beta \rightarrow \mathcal{A}\mathbf{f}_{\alpha+\beta}$ corresponds to the induction product induced by the exact induction functor $\mathrm{Ind}_{\alpha, \beta}^{\alpha+\beta}$;
- (2) the comultiplication $m_{\alpha, \beta}^* : \mathcal{A}\mathbf{f}_{\alpha+\beta} \rightarrow \mathcal{A}\mathbf{f}_\alpha \otimes_{\mathcal{A}} \mathcal{A}\mathbf{f}_\beta$ corresponds to the restriction coproduct induced by the exact restriction functor $\mathrm{Res}_{\alpha, \beta}^{\alpha+\beta}$;
- (3) the bar-involution on $\mathcal{A}\mathbf{f}_\alpha$ corresponds to the anti-linear involution induced by the duality $\#$.

Remark 2.8. Theorem 2.7 establishes in particular that the functors θ_i induce \mathcal{A} -linear operators on $[\mathrm{Proj}(R)]$ that satisfy the quantum Serre relations from (2.2). For a more general categorical version of this statement, see [KL2, Proposition 6] or [R2, Proposition 4.2].

2.10. q -Characters. The dual statement to Theorem 2.7, in which γ gets replaced by its graded dual $\gamma^* : [\mathrm{Rep}(R)] \hookrightarrow \mathcal{A}\mathbf{f}^*$, has a natural representation theoretic extension related to the notion of q -character. This goes back at a purely combinatorial level to work of Leclerc in [L]. We only need one basic fact from this side of the picture. Given $\alpha \in Q_+$, let $'\mathbf{f}_\alpha^*$ denote the $\mathbb{Q}(q)$ -vector on basis $\{\mathbf{i} \mid \mathbf{i} \in I^\alpha\}$ and set $'\mathbf{f}^* := \bigoplus_{\alpha \in Q_+} '\mathbf{f}_\alpha^*$. Given $M \in \mathrm{Rep}(R_\alpha)$, its q -character means the formal expression

$$\mathrm{ch}_q M := \sum_{\mathbf{i} \in I^\alpha} (\mathrm{qdim} e(\mathbf{i})M) \cdot \mathbf{i} \in '\mathbf{f}_\alpha^*. \quad (2.28)$$

The following theorem is established in [KL1] in order to prove the surjectivity of the map γ in Theorem 2.7.

Theorem 2.9 ([KL1, Theorem 3.17]). *The map*

$$\mathrm{ch}_q : [\mathrm{Rep}(R)] \rightarrow '\mathbf{f}^*, \quad [M] \mapsto \mathrm{ch}_q M$$

is injective.

3. HIGHER LEVEL FOCK SPACES

Continuing with notation from the previous section, we fix also now a tuple $(k_1, \dots, k_l) \in I^l$ for some $l \geq 0$ and set

$$\Lambda := \Lambda_{k_1} + \dots + \Lambda_{k_l}. \quad (3.1)$$

For each $m = 1, \dots, l$, we pick $\tilde{k}_m \in \mathbb{Z}$ such that

$$\tilde{k}_m \equiv k_m \pmod{e}. \quad (3.2)$$

We refer to l here as the *level* and the tuple $(\tilde{k}_1, \dots, \tilde{k}_l)$ as the *multicharge*. Almost everything depends implicitly not just on Λ but on the ordered tuple (k_1, \dots, k_l) ; the choice of multicharge plays a significant role only in §§3.11–3.12.

3.1. The quantum group. Let \mathfrak{g} be the Kac-Moody algebra corresponding to the Cartan matrix (2.1), so $\mathfrak{g} = \widehat{\mathfrak{sl}}_e(\mathbb{C})$ if $e > 0$ and $\mathfrak{g} = \mathfrak{sl}_\infty(\mathbb{C})$ if $e = 0$. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of \mathfrak{g} . So $U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$ -algebra generated by the Chevalley generators $E_i, F_i, K_i^{\pm 1}$ for $i \in I$, subject only to the usual quantum Serre relations (for all admissible $i, j \in I$):

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = 1, \quad (3.3)$$

$$K_i E_j K_i^{-1} = q^{a_{i,j}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{i,j}} F_j, \quad (3.4)$$

$$[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \quad (3.5)$$

$$(\text{ad}_q E_i)^{1-a_{j,i}}(E_j) = 0 \quad (i \neq j), \quad (3.6)$$

$$(\text{ad}_q F_i)^{1-a_{j,i}}(F_j) = 0 \quad (i \neq j), \quad (3.7)$$

recalling (2.3). We consider $U_q(\mathfrak{g})$ as a Hopf algebra with respect to the co-product given for all $i \in I$ as follows:

$$\Delta : K_i \mapsto K_i \otimes K_i, \quad E_i \mapsto E_i \otimes K_i + 1 \otimes E_i, \quad F_i \mapsto F_i \otimes 1 + K_i^{-1} \otimes F_i$$

The *bar-involution* $- : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ is the anti-linear involution such that

$$\overline{K_i} = K_i^{-1}, \quad \overline{E_i} = E_i, \quad \overline{F_i} = F_i.$$

Given a $U_q(\mathfrak{g})$ -module V , a *compatible bar-involution* on V means an anti-linear involution $- : V \rightarrow V$ such that $\overline{\overline{xv}} = \overline{x} \overline{v}$ for all $x \in U_q(\mathfrak{g})$ and $v \in V$.

Also let $\tau : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ be the anti-linear anti-automorphism defined by

$$\tau : K_i \mapsto K_i^{-1}, \quad E_i \mapsto q F_i K_i^{-1}, \quad F_i \mapsto q^{-1} K_i E_i. \quad (3.8)$$

Note τ is not an involution: its inverse τ^{-1} is given by the formulae

$$\tau^{-1} : K_i \mapsto K_i^{-1}, \quad E_i \mapsto q^{-1} F_i K_i, \quad F_i \mapsto q K_i^{-1} E_i. \quad (3.9)$$

In other words, $\tau^{-1} \circ - = - \circ \tau$.

Everything so far makes sense over the ground ring \mathscr{A} too. Let $U_q(\mathfrak{g})_{\mathscr{A}}$ denote Lusztig's \mathscr{A} -form for $U_q(\mathfrak{g})$, which is the \mathscr{A} -subalgebra generated by the quantum divided powers $E_i^{(n)} := E_i^n / [n]!$ and $F_i^{(n)} := F_i^n / [n]!$ for all $i \in I$ and $n \geq 1$. The bar-involution, the comultiplication Δ and the anti-automorphism τ descend in the obvious way to this \mathscr{A} -form.

3.2. Recollections about upper crystal bases. Let $\mathbb{A} := \mathbb{Q}[q, q^{-1}] \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{A}$. Let \mathbb{A}_0 (resp. \mathbb{A}_∞) denote the subalgebra of $\mathbb{Q}(q)$ consisting of all rational functions which are regular at zero (resp. at infinity). According to [Kas3, §2.1], a *balanced triple* in a $\mathbb{Q}(q)$ -vector space V is a triple $(V_{\mathbb{A}}, V_0, V_\infty)$ where $V_{\mathbb{A}}$ is an \mathbb{A} -submodule of V , V_0 is an \mathbb{A}_0 -submodule of V and V_∞ is an \mathbb{A}_∞ -submodule of M , such that the following two properties hold:

- (1) The natural multiplication maps $\mathbb{Q}(q) \otimes_{\mathbb{A}} V_{\mathbb{A}} \rightarrow V, \mathbb{Q}(q) \otimes_{\mathbb{A}_0} V_0 \rightarrow V$ and $\mathbb{Q}(q) \otimes_{\mathbb{A}_\infty} V_\infty \rightarrow V$ are all isomorphisms.
- (2) Setting $E := V_{\mathbb{A}} \cap V_0 \cap V_\infty$, one of the following three equivalent conditions holds:
 - (a) the natural map $E \rightarrow V_0/qV_0$ is an isomorphism;
 - (b) the natural map $E \rightarrow V_\infty/q^{-1}V_\infty$ is an isomorphism;
 - (c) the natural maps $\mathbb{A} \otimes_{\mathbb{Q}} E \rightarrow V_{\mathbb{A}}, \mathbb{A}_0 \otimes_{\mathbb{Q}} E \rightarrow V_0$ and $\mathbb{A}_\infty \otimes_{\mathbb{Q}} E \rightarrow V_\infty$ are all isomorphisms.

These isomorphisms provide a canonical way to lift any “local” basis for V_0/qV_0 (or for $V_\infty/q^{-1}V_\infty$) to a “global” basis for V .

For an integrable $U_q(\mathfrak{g})$ -module V , let \tilde{e}_i and \tilde{f}_i be Kashiwara’s upper crystal operators on V from [Kas1]; see also [Kas3, (3.1.2)]. Recall an *upper crystal lattice at $q = 0$* is a free \mathbb{A}_0 -submodule V_0 of V such that

- (1) $V \cong \mathbb{Q}(q) \otimes_{\mathbb{A}_0} V_0$;
- (2) V_0 is the direct sum of its weight spaces;
- (3) V_0 is invariant under the actions of \tilde{e}_i, \tilde{f}_i .

The notion V_∞ of an *upper crystal lattice at $q = \infty$* is defined similarly, replacing \mathbb{A}_0 with \mathbb{A}_∞ .

An *upper crystal basis at $q = 0$* is a pair (V_0, B_0) where V_0 is an upper crystal lattice at $q = 0$ and B_0 is a basis of the \mathbb{Q} -vector space V_0/qV_0 such that

- (1) each element of B_0 is a weight vector, i.e. it is the image of a weight vector in V_0 under the natural map $V_0 \rightarrow V_0/qV_0$;
- (2) writing also \tilde{e}_i, \tilde{f}_i for the \mathbb{Q} -linear endomorphisms of V_0/qV_0 induced by \tilde{e}_i, \tilde{f}_i , we have that $\tilde{e}_i B_0 \subseteq B_0 \cup \{0\}$ and $\tilde{f}_i B_0 \subseteq B_0 \cup \{0\}$;
- (3) for $b, b' \in B_0$, $b' = \tilde{f}_i b$ if and only if $\tilde{e}_i b' = b$.

If (V_0, B_0) is an upper crystal basis at $q = 0$, there is an induced structure of an *abstract crystal* on the set B_0 in the sense of [Kas4], that is, there is a canonically associated crystal datum $(B_0, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \text{wt})$. Here, for $b \in B_0$, $\text{wt}(b)$ denotes the *weight* of b , and

$$\varepsilon_i(b) := \max\{k \geq 0 \mid \tilde{e}_i^k(b) \neq 0\}, \quad \varphi_i(b) := \max\{k \geq 0 \mid \tilde{f}_i^k(b) \neq 0\},$$

for each $i \in I$. It is automatically the case that

$$(\text{wt}(b), \alpha_i) = \varphi_i(b) - \varepsilon_i(b). \tag{3.10}$$

By [Kas1, Proposition 6], upper crystal bases at $q = 0$ behave well under tensor product. More precisely, if (V_0, B_0) and (V'_0, B'_0) are upper crystal bases at $q = 0$ in integrable modules V and V' , then $(V_0 \otimes V'_0, B_0 \times B'_0)$ is an upper crystal basis at $q = 0$ in $V \otimes V'$. Moreover, there is an explicit combinatorial rule describing the crystal operators \tilde{e}_i and \tilde{f}_i on $B_0 \times B'_0$ in terms of the ones on B_0 and B'_0 . We refer the reader to [Kas1, Proposition 6] for the precise

statement of this; note though that we are using the opposite comultiplication to the one used in [Kas1, (1.5)] so the order of tensors needs to be flipped when translating this tensor product rule into our setup.

3.3. The module $V(\Lambda)$. Let $V(\Lambda)$ denote the integrable highest weight module for $U_q(\mathfrak{g})$ of highest weight Λ , where Λ is the dominant integral weight fixed in (3.1). Fix also a choice of a non-zero highest weight vector $v_\Lambda \in V(\Lambda)$. The module $V(\Lambda)$ possesses a unique compatible bar-involution $\bar{\cdot} : V(\Lambda) \rightarrow V(\Lambda)$ such that $\overline{\bar{v}} = v$.

The *contravariant form* (\cdot, \cdot) on $V(\Lambda)$ is the unique symmetric bilinear form such that

- (1) E_i and F_i are biadjoint, i.e. $(E_i v, w) = (v, F_i w)$ and $(F_i v, w) = (v, E_i w)$ for all $v, w \in V(\Lambda)$ and $i \in I$;
- (2) $(v_\Lambda, v_\Lambda) = 1$.

Actually, we usually prefer to work with a slightly different form on $V(\Lambda)$, namely, the *Shapovalov form* $\langle \cdot, \cdot \rangle$. By definition, this is the unique sesquilinear form (anti-linear in the first argument, linear in the second) on $V(\Lambda)$ such that

- (1) $\langle uv, w \rangle = \langle v, \tau(u)w \rangle$ for all $u \in U_q(\mathfrak{g})$ and $v, w \in V(\Lambda)$;
- (2) $\langle v_\Lambda, v_\Lambda \rangle = 1$.

Define the *defect* of $\alpha \in Q_+$ (relative to Λ) by setting

$$\text{def}(\alpha) := ((\Lambda, \Lambda) - (\Lambda - \alpha, \Lambda - \alpha))/2 = (\Lambda, \alpha) - (\alpha, \alpha)/2. \quad (3.11)$$

The contravariant and Shapovalov forms are closely related:

Lemma 3.1. *For vectors $v, w \in V(\Lambda)$ with v of weight $\Lambda - \alpha$, we have that*

- (1) $\langle v, w \rangle = q^{\text{def}(\alpha)} (\bar{v}, w) = \langle \bar{w}, \bar{v} \rangle$;
- (2) $(v, w) = \overline{\langle \bar{w}, \bar{v} \rangle}$;
- (3) $\langle v, w \rangle = q^{2\text{def}(\alpha)} \overline{\langle w, v \rangle}$.

Proof. Mimic the proof of [BK5, Lemma 2.6]. \square

Let $V(\Lambda)_{\mathcal{A}}$ denote the *standard \mathcal{A} -form* for $V(\Lambda)$, that is, the $U_q(\mathfrak{g})_{\mathcal{A}}$ -submodule of $V(\Lambda)$ generated by the highest weight vector v_Λ . This is free as an \mathcal{A} -module. Let $V(\Lambda)_{\mathcal{A}}^*$ denote the *costandard \mathcal{A} -form* for $V(\Lambda)$, that is, the dual lattice

$$\begin{aligned} V(\Lambda)_{\mathcal{A}}^* &= \{v \in V(\Lambda) \mid (v, w) \in \mathcal{A} \text{ for all } w \in V(\Lambda)_{\mathcal{A}}\} \\ &= \{v \in V(\Lambda) \mid \langle v, w \rangle \in \mathcal{A} \text{ for all } w \in V(\Lambda)_{\mathcal{A}}\}. \end{aligned}$$

As explained in [Kas3, §3.3], the results of [Kas2] imply that $V(\Lambda)$ has a unique upper crystal basis $(V(\Lambda)_0, B(\Lambda)_0)$ at $q = 0$ such that

- (1) the Λ -weight space of $V(\Lambda)_0$ is equal to $\mathbb{A}_0 v_\Lambda$;
- (2) $v_\Lambda + qV(\Lambda)_0 \in B(\Lambda)_0$.

We will describe an explicit combinatorial realization of the underlying abstract crystal in §3.7 below.

According to [Kas3, Lemma 4.2.1], $(V(\Lambda)_{\mathbb{A}}^*, V(\Lambda)_0, \overline{V(\Lambda)_0})$ is a balanced triple, where $V(\Lambda)_{\mathbb{A}}^* := \mathbb{Q} \otimes_{\mathbb{Z}} V(\Lambda)_{\mathcal{A}}^*$. Hence there is a canonical lift of the upper crystal basis $B(\Lambda)_0$ to a basis of $V(\Lambda)$. This is Kashiwara's *upper global crystal basis* of $V(\Lambda)$, which is Lusztig's *dual-canonical basis*. The dual basis

to the upper global crystal basis under the contravariant form (\cdot, \cdot) is the *lower global crystal basis*. This is Lusztig's *canonical basis* as in [Lus, §14.4].

Lusztig's approach gives moreover that the canonical basis is a basis for $V(\Lambda)_{\mathcal{A}}$ as a free \mathcal{A} -module, and that each vector in the canonical basis is bar-invariant. Hence the dual-canonical basis is a basis for $V(\Lambda)_{\mathcal{A}}^*$ as a free \mathcal{A} -module, and by Lemma 3.1(2) each vector in the dual-canonical basis is bar-invariant too. (These statements can also be deduced without invoking Lusztig's geometric construction using the Fock space approach explained below.)

3.4. Combinatorics of multipartitions. A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers; we set $|\lambda| := \lambda_1 + \lambda_2 + \dots$. An *l -multipartition* is an ordered l -tuple of partitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$; we set $|\lambda| := |\lambda^{(1)}| + \dots + |\lambda^{(l)}|$. We let \mathcal{P} (resp. \mathcal{P}^Λ) denote the set of all partitions (resp. l -multipartitions).

The *Young diagram* of the multipartition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \mathcal{P}^\Lambda$ is

$$\{(a, b, m) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \dots, l\} \mid 1 \leq b \leq \lambda_a^{(m)}\}.$$

The elements of this set are called the *nodes* of λ . More generally, a *node* is an element of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \dots, l\}$. Usually, we identify the multipartition λ with its Young diagram and visualize it as a column vector of Young diagrams. For example, $((3, 1), (4, 2))$ is the Young diagram

$$\begin{array}{c} \square \square \square \\ \square \end{array} \quad \begin{array}{c} \square \square \square \square \\ \square \square \end{array} \quad (3.12)$$

A node $A \in \lambda$ is called *removable* (for λ) if $\lambda \setminus \{A\}$ has a shape of a multipartition. A node $B \notin \lambda$ is called *addable* (for λ) if $\lambda \cup \{B\}$ has a shape of a multipartition. We use the notation

$$\lambda_A := \lambda \setminus \{A\}, \quad \lambda^B := \lambda \cup \{B\}.$$

The *residue* $\text{res } A$ of the node $A = (a, b, m)$ is the integer

$$\text{res } A := \tilde{k}_m + b - a \in \mathbb{Z}. \quad (3.13)$$

Although this depends implicitly on the fixed choice of multicharge from (3.2), we will normally only be interested in residues modulo e : for $i \in I$ we say that A is an *i -node* if $\text{res } A \equiv i \pmod{e}$. Given $\lambda \in \mathcal{P}^\Lambda$, we define its *content* by

$$\text{cont}(\lambda) := \sum_{i \in I} n_i \alpha_i \in Q_+, \quad (3.14)$$

where n_i is the number of i -nodes $A \in \lambda$. For $\alpha \in Q_+$, let

$$\mathcal{P}_\alpha^\Lambda := \{\lambda \in \mathcal{P}^\Lambda \mid \text{cont}(\lambda) = \alpha\} \quad (3.15)$$

denote the set of all l -multipartitions of content α . Note that every $\lambda \in \mathcal{P}_\alpha^\Lambda$ has $|\lambda| = \text{ht}(\alpha)$.

3.5. Some partial orders. We now define two partial orders \trianglelefteq and \leq on \mathcal{P}^Λ . The first of these is the *dominance ordering* which is defined by $\mu \trianglelefteq \lambda$ if

$$\sum_{a=1}^{m-1} |\mu^{(a)}| + \sum_{b=1}^c \mu_b^{(m)} \leq \sum_{a=1}^{m-1} |\lambda^{(a)}| + \sum_{b=1}^c \lambda_b^{(m)} \quad (3.16)$$

for all $1 \leq m \leq l$ and $c \geq 1$, with equality for $m = l$ and $c \gg 1$. In other words, μ is obtained from λ by moving nodes down in the diagram. In case $l = 1$ this is just the usual notion of the dominance ordering on partitions.

For the second ordering we treat the cases $e = 0$ and $e > 0$ separately. Assume first that $e = 0$. Let \leq be the dominance ordering on $Q_+ = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_{\geq 0} \alpha_i$, i.e. $\alpha \leq \beta$ if $\beta - \alpha \in Q_+$. For $\lambda \in \mathcal{P}^\Lambda$, let $\text{cont}_m(\lambda) \in Q_+$ denote the content of the m th component $\lambda^{(m)}$ of λ defined in the analogous way to (3.14); in particular, $\text{cont}(\lambda) = \text{cont}_1(\lambda) + \cdots + \text{cont}_l(\lambda)$. Then declare that $\mu \leq \lambda$ if

$$\text{cont}_1(\mu) + \cdots + \text{cont}_m(\mu) \leq \text{cont}_1(\lambda) + \cdots + \text{cont}_m(\lambda) \quad (3.17)$$

for each $m = 1, \dots, l$, with equality in the case $m = l$.

Lemma 3.2. *Suppose that $e = 0$ and $\lambda, \mu \in \mathcal{P}^\Lambda$. Then $\mu \leq \lambda$ implies $\mu \trianglelefteq \lambda$.*

Proof. Consider the set Q_+^l of l -tuples $\alpha = (\alpha^{(1)}, \dots, \alpha^{(l)})$ of elements of Q_+ . The assumption that $e = 0$ implies that the map

$$\rho: \mathcal{P}^\Lambda \rightarrow Q_+^l, \quad \lambda \mapsto (\text{cont}_1(\lambda), \dots, \text{cont}_l(\lambda))$$

is injective. For any $\alpha \in Q_+^l$, $1 \leq m \leq l$ and $a \geq 1$, let

$$r_a^{(m)}(\alpha) := \#\{i \in \mathbb{Z} \mid 1 \leq a + \min(i - k_m, 0) \leq (\Lambda_i, \alpha^{(m)})\}.$$

The key point about this definition is that $r_a^{(m)}(\rho(\lambda)) = \lambda_a^{(m)}$ for $\lambda \in \mathcal{P}^\Lambda$. Define a pre-order \trianglelefteq on Q_+^l by declaring that $\beta \trianglelefteq \alpha$ if

$$\sum_{a=1}^{m-1} \text{ht}(\beta^{(a)}) + \sum_{b=1}^c r_b^{(m)}(\beta) \leq \sum_{a=1}^{m-1} \text{ht}(\alpha^{(a)}) + \sum_{b=1}^c r_b^{(m)}(\alpha)$$

for all $1 \leq m \leq l$ and $c \geq 1$, with equality for $m = l$ and $c \gg 1$. Then for $\lambda, \mu \in \mathcal{P}^\Lambda$ we have that $\mu \trianglelefteq \lambda$ if and only if $\rho(\mu) \trianglelefteq \rho(\lambda)$. Also let \preceq be the partial order on Q_+^l defined by $\beta \preceq \alpha$ if $\beta^{(1)} + \cdots + \beta^{(m)} \leq \alpha^{(1)} + \cdots + \alpha^{(m)}$ for each $m = 1, \dots, l$ with equality in the case $m = l$. We obviously have that $\mu \leq \lambda$ if and only if $\rho(\mu) \preceq \rho(\lambda)$. With these definitions, we are reduced to showing for $\alpha, \beta \in Q_+^l$ that $\beta \preceq \alpha$ implies $\beta \trianglelefteq \alpha$.

So now take $\alpha, \beta \in Q_+^l$ with $\beta \preceq \alpha$. If $\beta^{(1)} = \alpha^{(1)}$ then we get that $\beta \trianglelefteq \alpha$ by induction on l . So we may assume that $\beta^{(1)} < \alpha^{(1)}$. Choose $i \in \mathbb{Z}$ so that $\beta^{(1)} + \alpha_i \leq \alpha^{(1)}$. Let $m > 1$ be minimal so that $\alpha_i \leq \beta^{(m)}$. Then define $\gamma \in Q_+^l$ by setting $\gamma^{(1)} := \beta^{(1)} + \alpha_i$, $\gamma^{(m)} := \beta^{(m)} - \alpha_i$ and $\gamma^{(a)} := \beta^{(a)}$ for all $a \neq 1, m$. It follows that $\beta \prec \gamma \preceq \alpha$. By induction we get that $\gamma \trianglelefteq \alpha$. Since \trianglelefteq is transitive it remains to observe that $\beta \trianglelefteq \gamma$, which is a consequence of the definitions. \square

Assume instead that $e > 0$. Before we can define the partial order \leq in this case, we need to introduce an injective map

$$\mathcal{P}^\Lambda \hookrightarrow \mathcal{P}, \quad \lambda \mapsto \bar{\lambda}, \quad (3.18)$$

slightly weaker statement which is still enough for the subsequent arguments in [A2] to make sense, as we explain in detail later on.

Let $<_{\text{lex}}$ denote the *lexicographic ordering* on partitions, so for partitions $\lambda, \mu \in \mathcal{P}$ we have that $\mu <_{\text{lex}} \lambda$ if and only if $\mu_1 = \lambda_1, \dots, \mu_{a-1} = \lambda_{a-1}$ and $\mu_a < \lambda_a$ for some $a \geq 1$. We extend this notion to l -multipartitions: for $\lambda, \mu \in \mathcal{P}^\Lambda$ we have that $\mu <_{\text{lex}} \lambda$ if and only if $\mu^{(1)} = \lambda^{(1)}, \dots, \mu^{(m-1)} = \lambda^{(m-1)}$ and $\mu^{(m)} <_{\text{lex}} \lambda^{(m)}$ for some $1 \leq m \leq l$. It is obvious that this total order refines the dominance ordering on \mathcal{P}^Λ in the sense that $\mu \triangleleft \lambda$ implies $\mu <_{\text{lex}} \lambda$.

Lemma 3.3. *Assume we are given $\alpha \in Q_+$ such that $\tilde{k}_m - \tilde{k}_{m+1} \geq \text{ht}(\alpha) + e$ for $m = 1, \dots, l-1$. Then $\mu > \lambda$ implies that $\mu <_{\text{lex}} \lambda$ for all $\lambda, \mu \in \mathcal{P}_\alpha^\Lambda$.*

Proof. The lemma is vacuous in the case $e = 0$, as it never happens that $\mu > \lambda$ under the given assumptions, recalling from (3.17) that $\mu > \lambda$ implies $\text{cont}(\mu) = \text{cont}(\lambda)$ in the $e = 0$ case. Assume from now on that $e > 0$. It suffices to show for $\lambda, \mu \in \mathcal{P}_\alpha^\Lambda$ that $\mu >_{\text{lex}} \lambda$ implies $\bar{\mu} >_{\text{lex}} \bar{\lambda}$. For this, choose $1 \leq m \leq l$ and $a \geq 1$ such that $\mu^{(1)} = \lambda^{(1)}, \dots, \mu^{(m-1)} = \lambda^{(m-1)}$, $\mu_1^{(m)} = \lambda_1^{(m)}, \dots, \mu_{a-1}^{(m)} = \lambda_{a-1}^{(m)}$ and $\mu_a^{(m)} > \lambda_a^{(m)}$. The reader may find it helpful to keep in mind the examples of the corresponding abacus diagrams $A(\mu)$ and $A(\lambda)$ displayed above.

The rows $1, \dots, m-1$ of $A(\mu)$ and $A(\lambda)$ are exactly the same. Moreover, in the m th row, all the beads corresponding to the parts $\mu_b^{(m)} = \lambda_b^{(m)}$ with $b < a$ occupy the same positions in $A(\mu)$ and $A(\lambda)$. Also the bead B in $A(\mu)$ corresponding to the part $\mu_a^{(m)}$ is strictly to the right of the bead B' in $A(\lambda)$ corresponding to the part $\lambda_a^{(m)}$ (the part $\lambda_a^{(m)}$ could be 0 but it still makes sense to consider the corresponding bead).

Let B occupy the position indexed by $p \in \mathbb{Z}$. By the choice of m and a and the assumptions on $\tilde{k}_1, \dots, \tilde{k}_l$, a position indexed by an integer $> p$ is occupied in $A(\mu)$ if and only if it is occupied in $A(\lambda)$. Assume that there are t such occupied positions in $A(\mu)$ (or $A(\lambda)$). Then we have that $\bar{\mu}_s = \bar{\lambda}_s$ for $s = 1, 2, \dots, t$.

To finish the proof it suffices to show that $\bar{\mu}_{t+1} > \bar{\lambda}_{t+1}$. Note that $\bar{\mu}_{t+1}$ is equal to the number of unoccupied positions indexed by integers $< p$ in $A(\mu)$. By the assumptions on $\tilde{k}_1, \dots, \tilde{k}_l$, such positions can only exist in rows $m, m+1, \dots, l$. Moreover, the number of such positions in rows $m+1, \dots, l$ is determined just by p and the fixed numbers $\tilde{k}_{m+1}, \dots, \tilde{k}_l$.

Now, let p' be the largest integer such that $p' < p$ and the position indexed by p' is occupied in $A(\lambda)$. By the assumptions, p' may index the position occupied by B' or it may index a position in rows $1, \dots, m-1$ that is to the right of B' . Note $\bar{\lambda}_{t+1}$ is equal to the number of unoccupied positions indexed by integers $< p'$ in $A(\lambda)$. As in the previous paragraph, such unoccupied positions only exist in rows $m, m+1, \dots, l$, and the number of such positions in rows $m+1, \dots, l$ is exactly the same as before. Finally the number of unoccupied positions indexed by integers $< p'$ in row m of $A(\lambda)$ is always strictly smaller than the number of unoccupied positions indexed by integers $< p$ in row m of $A(\mu)$ because of the presence of the extra bead B' . Hence $\bar{\lambda}_{t+1} < \bar{\mu}_{t+1}$. \square

3.6. Fock space. Now we proceed to introduce the higher level Fock space $F(\Lambda)$ following the exposition in [A3]. Given nodes A and B from the diagram of a multipartition, we say that A is *row-above* (resp. *row-below*) B if A lies in a row that is strictly above (resp. below) the row containing B in the Young diagram when visualized as in (3.12). Given $\lambda \in \mathcal{P}^\Lambda$, $i \in I$, a removable i -node A and an addable i -node B , we set

$$d_i(\lambda) := \#\{\text{addable } i\text{-nodes of } \lambda\} - \#\{\text{removable } i\text{-nodes of } \lambda\}; \quad (3.19)$$

$$d_A(\lambda) := \#\{\text{addable } i\text{-nodes of } \lambda \text{ row-below } A\} \\ - \#\{\text{removable } i\text{-nodes of } \lambda \text{ row-below } A\}; \quad (3.20)$$

$$d^B(\lambda) := \#\{\text{addable } i\text{-nodes of } \lambda \text{ row-above } B\} \\ - \#\{\text{removable } i\text{-nodes of } \lambda \text{ row-above } B\}. \quad (3.21)$$

Note that $d_i(\lambda) = (\Lambda - \text{cont}(\lambda), \alpha_i)$.

Now define $F(\Lambda)$ to be the $\mathbb{Q}(q)$ -vector space on basis $\{M_\lambda \mid \lambda \in \mathcal{P}^\Lambda\}$ with $U_q(\mathfrak{g})$ -action defined by

$$E_i M_\lambda := \sum_A q^{d_A(\lambda)} M_{\lambda_A}, \quad F_i M_\lambda := \sum_B q^{-d^B(\lambda)} M_{\lambda_B}, \quad (3.22)$$

$$K_i M_\lambda := q^{d_i(\lambda)} M_\lambda, \quad (3.23)$$

where the first sum is over all removable i -nodes A for λ , and the second sum is over all addable i -nodes B for λ . When $l = 1$, this construction originates in work of Hayashi [H] and Misra and Miwa [MiMi]. When $l > 1$, $F(\Lambda)$ was first studied in [JMMO]. In that case it is simply the tensor product of l level one Fock spaces, indeed, we can identify $F(\Lambda)$ in general with the tensor product

$$F(\Lambda) = F(\Lambda_{k_1}) \otimes \cdots \otimes F(\Lambda_{k_l}), \quad (3.24)$$

on which the $U_q(\mathfrak{g})$ -structure is defined via the comultiplication Δ fixed above, so that M_λ is identified with $M_{\lambda^{(1)}} \otimes \cdots \otimes M_{\lambda^{(l)}}$ for each $\lambda \in \mathcal{P}^\Lambda$.

Let $F(\Lambda)_{\mathcal{A}}$ denote the free \mathcal{A} -submodule of $F(\Lambda)$ spanned by the M_λ 's, which is invariant under the action of $U_q(\mathfrak{g})_{\mathcal{A}}$. Also let $F(\Lambda)_0$ denote the free \mathbb{A}_0 -submodule of $F(\Lambda)$ spanned by the M_λ 's and set

$$C(\Lambda)_0 := \{M_\lambda + qF(\Lambda)_0 \mid \lambda \in \mathcal{P}^\Lambda\}.$$

The pair $(F(\Lambda)_0, C(\Lambda)_0)$ is then an upper crystal basis at $q = 0$. The proof of this statement in level one goes back to Misra and Miwa [MiMi]; the proof for higher levels is a consequence of the level one result in view of (3.24) and [Kas1, Proposition 6]. Hence we get induced the structure of abstract crystal on the underlying index set \mathcal{P}^Λ that parametrizes $C(\Lambda)_0$, with crystal datum denoted

$$(\mathcal{P}^\Lambda, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \text{wt}). \quad (3.25)$$

We give an explicit combinatorial description of this crystal in the next subsection. This explicit description in level one is a reformulation of the results in [MiMi]; in higher levels it follows from the level one description together with (3.24) and [Kas1, Proposition 6].

3.7. Crystals. The crystal datum (3.25) can be described in purely combinatorial terms as follows. First, for $\lambda \in \mathcal{P}^\Lambda$, we have that $\text{wt}(\lambda) = \Lambda - \text{cont}(\lambda)$, as follows from (3.23).

Given also $i \in I$, let A_1, \dots, A_n denote the addable and removable i -nodes of λ ordered so that A_m is row-above A_{m+1} for each $m = 1, \dots, n-1$. Consider the sequence $(\sigma_1, \dots, \sigma_n)$ where $\sigma_r = +$ if A_r is addable or $-$ if A_r is removable. If we can find $1 \leq r < s \leq n$ such that $\sigma_r = -, \sigma_s = +$ and $\sigma_{r+1} = \dots = \sigma_{s-1} = 0$ then replace σ_r and σ_s by 0. Keep doing this until we are left with a sequence $(\sigma_1, \dots, \sigma_n)$ in which no $-$ appears to the left of a $+$. This is called the *reduced i -signature* of λ .

If $(\sigma_1, \dots, \sigma_n)$ is the reduced i -signature of λ , then we have that

$$\varepsilon_i(\lambda) = \#\{r = 1, \dots, n \mid \sigma_r = -\}, \quad \varphi_i(\lambda) = \#\{r = 1, \dots, n \mid \sigma_r = +\}.$$

By (3.10) (or directly from the combinatorics) we also have that

$$(\Lambda - \text{cont}(\lambda), \alpha_i) = d_i(\lambda) = \varphi_i(\lambda) - \varepsilon_i(\lambda). \quad (3.26)$$

Finally, if $\varepsilon_i(\lambda) > 0$, we have that $\tilde{e}_i \lambda = \lambda_{A_r}$ where r indexes the leftmost $-$ in the reduced i -signature. Similarly, if $\varphi_i(\lambda) > 0$ we have that $\tilde{f}_i \lambda = \lambda^{A_r}$ where r indexes the rightmost $+$ in the reduced i -signature.

Because \emptyset is a highest weight vector in this crystal of weight Λ , we deduce from [Kas3, Theorem 3.3.1] that the subcrystal

$$(\mathcal{R}\mathcal{P}^\Lambda, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \text{wt}) \quad (3.27)$$

that is the connected component of $(\mathcal{P}^\Lambda, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \text{wt})$ generated by \emptyset gives an explicit combinatorial realization of the abstract crystal underlying the highest weight module $V(\Lambda)$. We refer to multipartitions from $\mathcal{R}\mathcal{P}^\Lambda$ as *restricted multipartitions*. Also for $\alpha \in Q_+$ set

$$\mathcal{R}\mathcal{P}_\alpha^\Lambda := \mathcal{R}\mathcal{P}^\Lambda \cap \mathcal{P}_\alpha^\Lambda.$$

These are the restricted multipartitions of content α .

Remark 3.4. The problem of finding a more explicit combinatorial description of the subset $\mathcal{R}\mathcal{P}^\Lambda$ of \mathcal{P}^Λ has received quite a lot of attention in the literature; see also Remark 3.22 below. Here are some special cases.

(1) Suppose that $e > 0$ and $l = 1$. Then $\mathcal{R}\mathcal{P}^\Lambda$ is the usual set of all *e -restricted partitions*, that is, partitions λ such that $\lambda_a - \lambda_{a+1} < e$ for $a \geq 1$.

(2) Suppose that $e = 0$ and $k_1 \geq \dots \geq k_l$. Then $\mathcal{R}\mathcal{P}^\Lambda$ consists of all l -multipartitions λ such that $\lambda_{a+k_m-k_{m+1}}^{(m)} \leq \lambda_a^{(m+1)}$ for all $m = 1, \dots, l-1$ and $a \geq 1$; see [BK5, (2.52)] or [V].

3.8. The dual-canonical basis of $V(\Lambda)$. The vector M_\emptyset is a highest weight vector of weight Λ . Moreover, the Λ -weight space of $F(\Lambda)$ is one dimensional. By complete reducibility, it follows that there is a canonical $U_q(\mathfrak{g})$ -module homomorphism

$$\pi : F(\Lambda) \twoheadrightarrow V(\Lambda), \quad M_\emptyset \mapsto v_\Lambda. \quad (3.28)$$

For any $\lambda \in \mathcal{P}^\Lambda$, we define

$$S_\lambda := \pi(M_\lambda), \quad (3.29)$$

and call this a *standard monomial* in $V(\Lambda)$. Applying π to (3.22), we get that

$$E_i S_\lambda = \sum_A q^{d_A(\lambda)} S_{\lambda_A}, \quad F_i S_\lambda = \sum_B q^{-d_B(\lambda)} S_{\lambda_B}, \quad (3.30)$$

where the first sum is over all removable i -nodes A for λ , and the second sum is over all addable i -nodes B for λ .

By [Kas3, Theorem 3.3.1], the upper crystal lattice $V(\Lambda)_0$ from §3.3 coincides with the image under π of the upper crystal lattice $F(\Lambda)_0$ from §3.6, i.e. $V(\Lambda)_0$ is the \mathbb{A}_0 -span of the standard monomials. Moreover by the definition (3.27) we have that

$$B(\Lambda)_0 = \{S_\lambda + qV(\Lambda)_0 \mid \lambda \in \mathcal{R}\mathcal{P}^\Lambda\}. \quad (3.31)$$

Thus we have given an explicit construction of $(V(\Lambda)_0, B(\Lambda)_0)$, the upper crystal basis of $V(\Lambda)$ at $q = 0$, via the Fock space $F(\Lambda)$.

Recall also from §3.3 that the dual-canonical basis of $V(\Lambda)$ is the canonical lift of $B(\Lambda)_0$ using the balanced triple $(V(\Lambda)_{\mathbb{A}}^*, V(\Lambda)_0, \overline{V(\Lambda)_0})$. In other words, in terms of our explicit parametrization, the dual-canonical basis of $V(\Lambda)$ is the basis $\{D_\lambda \mid \lambda \in \mathcal{R}\mathcal{P}^\Lambda\}$ in which D_λ is the unique vector in $V(\Lambda)_{\mathbb{A}}^* \cap V(\Lambda)_0 \cap \overline{V(\Lambda)_0}$ such that

$$D_\lambda \equiv S_\lambda \pmod{qV(\Lambda)_0} \quad (3.32)$$

for each $\lambda \in \mathcal{R}\mathcal{P}^\Lambda$. As we noted before, this is already a basis for the costandard lattice $V(\Lambda)_{\mathcal{A}}^*$ as a free \mathcal{A} -module, and each D_λ is bar-invariant.

Proposition 3.5 ([Kas3, Proposition 5.3.1]). *For $\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda$ and $i \in I$ we have that*

$$\begin{aligned} E_i D_\lambda &= [\varepsilon_i(\lambda)] D_{\tilde{e}_i \lambda} + \sum_{\substack{\mu \in \mathcal{R}\mathcal{P}_{\alpha - \alpha_i}^\Lambda \\ \varepsilon_i(\mu) < \varepsilon_i(\lambda) - 1}} x_{\lambda, \mu; i}(q) D_\mu, \\ F_i D_\lambda &= [\varphi_i(\lambda)] D_{\tilde{f}_i \lambda} + \sum_{\substack{\mu \in \mathcal{R}\mathcal{P}_{\alpha + \alpha_i}^\Lambda \\ \varphi_i(\mu) < \varphi_i(\lambda) - 1}} y_{\lambda, \mu; i}(q) D_\mu, \end{aligned}$$

for bar-invariant $x_{\lambda, \mu; i}(q) \in q^{\varepsilon_i(\lambda) - 2} \mathbb{Z}[q^{-1}]$ and $y_{\lambda, \mu; i}(q) \in q^{\varphi_i(\lambda) - 2} \mathbb{Z}[q^{-1}]$. (In these two formulae, the first term on the right hand side should be interpreted as 0 if $\varepsilon_i(\lambda) = 0$ (resp. $\varphi_i(\lambda) = 0$).

Finally for $\mu \in \mathcal{P}_\alpha^\Lambda$, consider the expansion of the standard monomial S_μ in terms of the dual-canonical basis:

$$S_\mu = \sum_{\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda} d_{\lambda, \mu}(q) D_\lambda. \quad (3.33)$$

At this point all we know about the coefficients $d_{\lambda, \mu}(q)$ is that they belong to $\delta_{\lambda, \mu} + q\mathbb{A}_0$.

Remark 3.6. We will prove eventually that $d_{\lambda, \mu}(q) = 1$ if $\lambda = \mu$, $d_{\lambda, \mu}(q) = 0$ if $\lambda \not\triangleleft \mu$, and $d_{\lambda, \mu}(q) \in q\mathbb{Z}[q]$ if $\lambda \triangleleft \mu$; see Theorem 3.9 and Corollary 5.15. Moreover we will show that $d_{\lambda, \mu}(q)$ is equal to the multiplicity $[S(\mu) : D(\lambda)]_q$ of a certain irreducible graded module $D(\lambda)$ as a composition factor of the graded Specht module $S(\mu)$ for the cyclotomic Hecke algebra associated to Λ ,

which will imply further that the coefficients of the polynomials $d_{\lambda,\mu}(q)$ are non-negative integers.

3.9. Triangularity of standard monomials. In order to establish the desired triangularity properties of the coefficients $d_{\lambda,\mu}(q)$, we need to exploit the existence of a well-behaved bar-involution on $F(\Lambda)$. Unfortunately the construction of this bar-involution in the case $e > 0$ is rather indirect, so we prefer to assume its existence first and proceed to derive the important consequences, postponing the actual construction until later on; see §3.12.

Hypothesis 3.7. *We are given an explicit compatible bar-involution on $F(\Lambda)$ and a partial order \preceq on \mathcal{P}^Λ such that*

- (1) $\overline{M_\lambda} = M_\lambda + (a \mathbb{Z}[q, q^{-1}]\text{-linear combination of } M_\mu \text{'s for } \mu \prec \lambda)$;
- (2) $\mu \prec \lambda$ implies $\mu \prec_{\text{lex}} \lambda$.

Let us explain right away how to construct such a bar-involution in the case $e = 0$; note this approach does not work for $e > 0$, the problem being the lack of integrality of Lusztig's quasi- R -matrix in the affine case. First we take the partial order \preceq in the $e = 0$ case to be the partial order \leq from (3.17), which satisfies Hypothesis 3.7(2) by Lemma 3.2. Then, to construct the bar-involution itself, we start in level one by defining the bar-involution on $F(\Lambda)$ simply to be the unique anti-linear endomorphism fixing all the monomial basis vectors M_λ . It is easy to check from (3.22) that this is a compatible bar-involution (assuming of course that $e = 0$ and $l = 1$). For higher levels, we identify $F(\Lambda)$ with the tensor product (3.24) and use Lusztig's tensor product construction from [Lus, §27.3] (adapted to our choice of comultiplication) to get an induced compatible bar-involution on $F(\Lambda)$. It is immediate from this construction, the definition (3.17) and the integrality of the quasi- R -matrix from [Lus, §24.1] that this satisfies Hypothesis 3.7(1); see [BK5, §2.3].

For the remainder of the subsection, we assume that Hypothesis 3.7 holds. Then we can introduce the *dual-canonical basis* $\{L_\lambda \mid \lambda \in \mathcal{P}^\Lambda\}$ of $F(\Lambda)$ by letting L_λ denote the unique bar-invariant vector in $F(\Lambda)$ such that

$$L_\lambda = M_\lambda + (a q\mathbb{Z}[q]\text{-linear combination of } M_\mu \text{'s with } \mu \prec \lambda) \quad (3.34)$$

for each $\lambda \in \mathcal{P}^\Lambda$. The existence and uniqueness of these vectors follows from Lusztig's lemma [Lus, Lemma 24.2.1] and the triangularity of the bar-involution from Hypothesis 3.7(1). Recall the map π and the dual-canonical basis vectors $D_\lambda \in V(\Lambda)$ from §3.8. The following theorem was established already in the case $e = 0$ in [BK5, Theorem 2.2] (via [B2, Theorem 26]); the proof given here in the general case repeats the same argument.

Proposition 3.8. *For $\lambda \in \mathcal{P}^\Lambda$, we have that $\pi(L_\lambda) = \begin{cases} D_\lambda & \text{if } \lambda \in \mathcal{R}\mathcal{P}^\Lambda \\ 0 & \text{otherwise.} \end{cases}$*

Proof. Let $F(\Lambda)_\mathbb{A} := \mathbb{Q} \otimes_{\mathbb{Z}} F(\Lambda)_{\mathcal{A}}$. In view of (3.34), this can be described alternatively as the \mathbb{A} -span of the dual-canonical basis $\{L_\lambda \mid \lambda \in \mathcal{P}^\Lambda\}$. Similarly, the upper crystal lattice $F(\Lambda)_0$ (resp. its image $\overline{F(\Lambda)}_0$ under the bar-involution) is the \mathbb{A}_0 -span (resp. the \mathbb{A}_∞ -span) of the dual-canonical basis. It follows immediately that $(F(\Lambda)_\mathbb{A}, F(\Lambda)_0, \overline{F(\Lambda)}_0)$ is a balanced triple. Moreover, our dual-canonical basis of $F(\Lambda)$ is the canonical lift of the upper crystal basis

$C(\Lambda)_0$ arising from this balanced triple. Also the image of the upper crystal lattice $F(\Lambda)_0$ at $q = 0$ under the bar-involution is an upper crystal lattice at $q = \infty$. This puts us in the setup of [Kas3, §5.2].

By [Kas3, Proposition 5.2.1], the image of $(F(\Lambda)_{\mathbb{A}}, F(\Lambda)_0, \overline{F(\Lambda)_0})$ under the map π is a balanced triple in $V(\Lambda)$. Its intersection with the Λ -weight space of $V(\Lambda)$ is $(\mathbb{A}v_{\Lambda}, \mathbb{A}_0v_{\Lambda}, \mathbb{A}_{\infty}v_{\Lambda})$, which is the same thing as for the balanced triple $(V(\Lambda)_{\mathbb{A}}^*, V(\Lambda)_0, V(\Lambda)_{\infty})$ constructed earlier. Hence by [Kas3, Proposition 5.2.2] our two balanced triples coincide:

$$\pi(F(\Lambda)_{\mathbb{A}}) = V(\Lambda)_{\mathbb{A}}^*, \quad \pi(F(\Lambda)_0) = V(\Lambda)_0, \quad \pi(\overline{F(\Lambda)_0}) = \overline{V(\Lambda)_0}.$$

As $L_{\lambda} \in F(\Lambda)_{\mathbb{A}} \cap F(\Lambda)_0 \cap \overline{F(\Lambda)_0}$ we deduce that $\pi(L_{\lambda}) \in V(\Lambda)_{\mathbb{A}}^* \cap V(\Lambda)_0 \cap \overline{V(\Lambda)_0}$ for every $\lambda \in \mathcal{P}^{\Lambda}$.

Now (3.34) implies that $M_{\lambda} \equiv L_{\lambda} \pmod{qF(\Lambda)_0}$ for every λ . Also we know that $S_{\lambda} \equiv D_{\lambda} \pmod{qV(\Lambda)_0}$ if $\lambda \in \mathcal{R}\mathcal{P}^{\Lambda}$ and $S_{\lambda} \equiv 0 \pmod{qV(\Lambda)_0}$ otherwise. As $\pi(M_{\lambda}) = S_{\lambda}$, we deduce that $\pi(L_{\lambda})$ is equal to D_{λ} plus a $q\mathbb{A}_0$ -linear combination of D_{μ} 's if $\lambda \in \mathcal{R}\mathcal{P}^{\Lambda}$, and $\pi(L_{\lambda})$ is a $q\mathbb{A}_0$ -linear combination of D_{μ} 's otherwise. But also $\pi(L_{\lambda}) \in \overline{V(\Lambda)_0}$ for every λ , so it is an \mathbb{A}_{∞} -linear combination of D_{μ} 's. Since $\mathbb{A}_{\infty} \cap q\mathbb{A}_0 = \{0\}$, we conclude that $\pi(L_{\lambda}) = D_{\lambda}$ if $\lambda \in \mathcal{R}\mathcal{P}^{\Lambda}$ and $\pi(L_{\lambda}) = 0$ otherwise. \square

Now we can define polynomials $d_{\lambda, \mu}(q) \in \mathbb{Z}[q]$ for every $\lambda, \mu \in \mathcal{P}^{\Lambda}$ from the expansion

$$M_{\mu} = \sum_{\lambda \in \mathcal{P}^{\Lambda}} d_{\lambda, \mu}(q) L_{\lambda}. \quad (3.35)$$

Applying the map π to (3.35) and using Proposition 3.8, we get that

$$S_{\mu} = \sum_{\lambda \in \mathcal{R}\mathcal{P}^{\Lambda}} d_{\lambda, \mu}(q) D_{\lambda} \quad (3.36)$$

for $\mu \in \mathcal{P}^{\Lambda}$ and $\lambda \in \mathcal{R}\mathcal{P}^{\Lambda}$. This establishes that the polynomial $d_{\lambda, \mu}(q)$ defined here agrees with the one defined earlier in (3.33) when $\lambda \in \mathcal{R}\mathcal{P}^{\Lambda}$, so our notation is consistent with the earlier notation.

Theorem 3.9. *Given $\lambda, \mu \in \mathcal{P}_{\alpha}^{\Lambda}$ we have that $d_{\lambda, \mu}(q) = 1$ if $\lambda = \mu$, $d_{\lambda, \mu}(q) = 0$ if $\lambda \not\prec \mu$, and $d_{\lambda, \mu}(q) \in q\mathbb{Z}[q]$ if $\lambda \prec \mu$. Hence:*

- (1) *The vectors $\{S_{\lambda} \mid \lambda \in \mathcal{R}\mathcal{P}^{\Lambda}\}$ give a basis for $V(\Lambda)_{\mathcal{A}}^*$ as a free \mathcal{A} -module.*
- (2) *For $\lambda \in \mathcal{P}_{\alpha}^{\Lambda} \setminus \mathcal{R}\mathcal{P}_{\alpha}^{\Lambda}$, the standard monomial S_{λ} can be expressed as a $q\mathbb{Z}[q]$ -linear combination of S_{μ} 's for $\mu \in \mathcal{R}\mathcal{P}_{\alpha}^{\Lambda}$ with $\mu \prec \lambda$.*
- (3) *For $\lambda \in \mathcal{R}\mathcal{P}_{\alpha}^{\Lambda}$, the difference $S_{\lambda} - \overline{S_{\lambda}}$ is a $q\mathbb{Z}[q]$ -linear combination of S_{μ} 's for $\mu \in \mathcal{R}\mathcal{P}_{\alpha}^{\Lambda}$ with $\mu \prec \lambda$.*

Proof. Use (3.34), Hypothesis 3.7 and the fact that $\{D_{\lambda} \mid \lambda \in \mathcal{R}\mathcal{P}^{\Lambda}\}$ is a bar-invariant basis for $V(\Lambda)_{\mathcal{A}}^*$ as a free \mathcal{A} -module. \square

Remark 3.10. When $e = 0$, the Fock space $F(\Lambda)$ is categorified by a certain graded highest weight category arising from parabolic category \mathcal{O} attached to the finite general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. The monomial basis $\{M_{\lambda}\}$ corresponds to the standard objects in this category and the dual-canonical basis $\{L_{\lambda}\}$ corresponds to the irreducible objects. Apart from the grading (which

comes via [BGS]) this is developed in detail in [BK5]; see especially [BK5, Theorem 3.1]. When $e > 0$ we expect that the Fock space $F(\Lambda)$ should be categorified in similar fashion by the cyclotomic ξ -Schur algebras of [DJM] for ξ a primitive e th root of unity (though many questions about the grading remain open); see [Y1] for a related conjecture. We speculate that there may be a version of the theory of [BK3] establishing a Morita equivalence between the cyclotomic ξ -Schur algebras and certain blocks of quantum parabolic category \mathcal{O} at the root of unity ξ .

3.10. The quasi-canonical basis. In this subsection we continue to assume that Hypothesis 3.7 holds. Introduce a new basis $\{P_\lambda \mid \lambda \in \mathcal{P}^\Lambda\}$ for $F(\Lambda)$, which we call the *quasi-canonical basis*, by setting

$$P_\lambda := \sum_{\mu \in \mathcal{P}^\Lambda} d_{\lambda,\mu}(q) M_\mu. \quad (3.37)$$

So we have simply transposed the transition matrix appearing in (3.35), i.e. we are mimicking BGG reciprocity at a combinatorial level. Let $(p_{\lambda,\mu}(-q))_{\lambda,\mu \in \mathcal{P}^\Lambda}$ be the inverse of the unitriangular matrix $(d_{\lambda,\mu}(q))_{\lambda,\mu \in \mathcal{P}^\Lambda}$, so that

$$M_\lambda = \sum_{\mu \in \mathcal{P}^\Lambda} p_{\lambda,\mu}(-q) P_\mu, \quad L_\mu = \sum_{\lambda \in \mathcal{P}^\Lambda} p_{\lambda,\mu}(-q) M_\lambda \quad (3.38)$$

by (3.35) and (3.37). Also define a sesquilinear form $\langle \cdot, \cdot \rangle$ on $F(\Lambda)$, which we call the *Shapovalov form*, by declaring that

$$\langle M_\lambda, \overline{M_\mu} \rangle := \delta_{\lambda,\mu} \quad (3.39)$$

for all $\lambda, \mu \in \mathcal{P}^\Lambda$. Note that $\langle v, w \rangle = \langle \overline{w}, \overline{v} \rangle$ for all $v, w \in F(\Lambda)$. Moreover using (3.39) it is routine to check that

$$\langle xu, v \rangle = \langle u, \tau(x)v \rangle \quad (3.40)$$

for all $x \in U_q(\mathfrak{g})$ and $u, v \in F(\Lambda)$; see also [BK5, (2.41)].

Lemma 3.11. *For $\lambda, \mu \in \mathcal{P}^\Lambda$, we have that $\langle P_\lambda, L_\mu \rangle = \delta_{\lambda,\mu}$.*

Proof. Since L_μ is bar-invariant, we have from (3.37), (3.38) and (3.40) that

$$\begin{aligned} \langle P_\lambda, L_\mu \rangle &= \left\langle \sum_{\sigma \in \mathcal{P}^\Lambda} d_{\lambda,\sigma}(q) M_\sigma, \sum_{\tau \in \mathcal{P}^\Lambda} p_{\tau,\mu}(-q^{-1}) \overline{M_\tau} \right\rangle \\ &= \sum_{\sigma, \tau \in \mathcal{P}^\Lambda} d_{\lambda,\sigma}(q^{-1}) p_{\tau,\mu}(-q^{-1}) \langle M_\sigma, \overline{M_\tau} \rangle \\ &= \sum_{\sigma \in \mathcal{P}^\Lambda} d_{\lambda,\sigma}(q^{-1}) p_{\sigma,\mu}(-q^{-1}) = \delta_{\lambda,\mu}. \end{aligned}$$

This proves the lemma. \square

Next recall the definition of the Shapovalov form $\langle \cdot, \cdot \rangle$ on $V(\Lambda)$ from §3.3. We introduce a new basis $\{Y_\lambda \mid \lambda \in \mathcal{R}\mathcal{P}^\Lambda\}$ for $V(\Lambda)$ by declaring that

$$\langle Y_\lambda, D_\mu \rangle = \delta_{\lambda,\mu} \quad (3.41)$$

for all $\lambda, \mu \in \mathcal{R}\mathcal{P}^\Lambda$. This is actually a basis for the standard lattice $V(\Lambda)_{\mathcal{A}}$ as a free \mathcal{A} -module. We call it the *quasi-canonical basis* of $V(\Lambda)$. The precise

relationship between the quasi-canonical and the usual canonical basis of $V(\Lambda)$ is explained by the following lemma.

Lemma 3.12. *The canonical basis for $V(\Lambda)$ is $\bigcup_{\alpha \in Q_+} \{q^{-\text{def}(\alpha)} Y_\lambda \mid \lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda\}$. In particular, we have that $\overline{Y}_\lambda = q^{-2\text{def}(\alpha)} Y_\lambda$ for each $\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda$.*

Proof. Recall that the canonical basis for $V(\Lambda)$ is the dual basis to the dual-canonical basis with respect to the contravariant form (\cdot, \cdot) . Moreover vectors from the canonical basis are bar-invariant. Using these two things, the lemma follows from the definition (3.41) and Lemma 3.1. \square

Since the vector $M_\emptyset \in F(\Lambda)$ is a non-zero highest weight vector of weight Λ , there is a canonical embedding

$$\pi^* : V(\Lambda) \hookrightarrow F(\Lambda), \quad v_\Lambda \mapsto M_\emptyset. \quad (3.42)$$

The Shapovalov form on $V(\Lambda)$ is actually the restriction of the Shapovalov form on $F(\Lambda)$ via this embedding, that is, we have that

$$\langle v, v' \rangle = \langle \pi^*(v), \pi^*(v') \rangle \quad (3.43)$$

for all $v, v' \in V(\Lambda)$. This holds because it is true when $v = v' = v_\Lambda$, and both forms have the property (3.40). Note also for π as in (3.28) that

$$\pi \circ \pi^* = \text{id}_{V(\Lambda)}. \quad (3.44)$$

This holds because it is true on the highest weight vector v_Λ . The following lemma shows that π^* is adjoint to π with respect to the Shapovalov forms.

Lemma 3.13. *We have that $\langle v, \pi(w) \rangle = \langle \pi^*(v), w \rangle$ for all $v \in V(\Lambda)$ and $w \in F(\Lambda)$.*

Proof. Consider the orthogonal complement $(\ker \pi)^\perp$ to $\ker \pi$ with respect to the form $\langle \cdot, \cdot \rangle$. By Proposition 3.8, $\ker \pi$ has basis $\{L_\mu \mid \mu \in \mathcal{P}^\Lambda \setminus \mathcal{R}\mathcal{P}^\Lambda\}$. Hence by Lemma 3.11, the vector $\pi^*(v_\Lambda) = M_\emptyset = P_\emptyset$ belongs to $(\ker \pi)^\perp$. Moreover $(\ker \pi)^\perp$ is a $U_q(\mathfrak{g})$ -submodule of $F(\Lambda)$ thanks to (3.40). As v_Λ generates $V(\Lambda)$, we deduce that the image of π^* lies in $(\ker \pi)^\perp$. Now take any $v \in V(\Lambda)$ and $w \in F(\Lambda)$. By (3.44), we can write $w = \pi^*(v') + z$ for some $v' \in V(\Lambda)$ and $z \in \ker \pi$. Using (3.43) and the observation just made, we get that

$$\langle v, \pi(w) \rangle = \langle v, v' \rangle = \langle \pi^*(v), \pi^*(v') \rangle = \langle \pi^*(v), \pi^*(v') + z \rangle = \langle \pi^*(v), w \rangle,$$

as required. \square

In the case $e = 0$, the following theorem was established in [BK5, §2.6].

Theorem 3.14. *We have that $\pi^*(Y_\lambda) = P_\lambda$ and $\pi(P_\lambda) = Y_\lambda$ for any $\lambda \in \mathcal{R}\mathcal{P}^\Lambda$. Hence*

$$Y_\lambda = \sum_{\mu \in \mathcal{P}^\Lambda} d_{\lambda, \mu}(q) S_\mu \quad (3.45)$$

for all $\lambda \in \mathcal{R}\mathcal{P}^\Lambda$.

Proof. Applying Lemma 3.13, Proposition 3.8 and the definition (3.41), we have for $\lambda \in \mathcal{R}\mathcal{P}^\Lambda$ and any $\mu \in \mathcal{P}^\Lambda$ that

$$\langle \pi^*(Y_\lambda), L_\mu \rangle = \langle Y_\lambda, \pi(L_\mu) \rangle = \langle Y_\lambda, D_\mu \rangle = \delta_{\lambda, \mu}.$$

This establishes that $\pi^*(Y_\lambda) = P_\lambda$ thanks to Lemma 3.11. Combined with (3.44) we deduce that $Y_\lambda = \pi(P_\lambda)$. The final statement follows by applying π to the definition (3.37). \square

Remark 3.15. More generally, we can define vectors $Y_\lambda \in V(\Lambda)$ for any $\lambda \in \mathcal{P}^\Lambda$ by setting $Y_\lambda := \pi(P_\lambda) = \sum_{\mu \in \mathcal{P}^\Lambda} d_{\lambda,\mu}(q) S_\mu$. These are expected to correspond to *Young modules* at the categorical level; see [BK5, Theorem 4.6] where this is justified in the case $e = 0$.

Remark 3.16. Most of the rest of the literature in this subject works with the *canonical basis* $\{T_\lambda \mid \lambda \in \mathcal{P}^\Lambda\}$ for $F(\Lambda)$ rather than the quasi-canonical basis introduced here, where $T_\lambda \in F(\Lambda)$ is the unique bar-invariant vector with

$$\overline{T}_\lambda = M_\lambda + (\text{a } q^{-1}\mathbb{Z}[q^{-1}]\text{-linear combination of } M_\mu\text{'s with } \mu \prec \lambda). \quad (3.46)$$

In the categorification mentioned in Remark 3.10, the canonical basis $\{T_\lambda\}$ should correspond to the indecomposable tilting modules, whereas the quasi-canonical basis $\{P_\lambda\}$ corresponds to the indecomposable projectives; see [BK5, Theorem 3.1] in the $e = 0$ case and [VV1, Theorem 11] (combined with [D, Proposition 4.1.5(ii)]) for $e > 0, l = 1$. Actually there is a close connection between the quasi-canonical and canonical bases, as follows. Let

$$\Lambda^t := \Lambda_{-k_l} + \cdots + \Lambda_{-k_1} \quad (3.47)$$

and define the Fock space $F(\Lambda^t)$ as above but replacing (k_1, \dots, k_l) everywhere with $(-k_l, \dots, -k_1)$. For an l -multipartition λ , set

$$\lambda^t := ((\lambda^{(l)})^t, \dots, (\lambda^{(1)})^t), \quad (3.48)$$

where $(\lambda^{(m)})^t$ denotes the usual transpose of a partition. Then the anti-linear vector space isomorphism

$$t : F(\Lambda) \xrightarrow{\sim} F(\Lambda^t), \quad M_\lambda \mapsto M_{\lambda^t} \quad (3.49)$$

has the property that $(P_\lambda)^t = T_{\lambda^t}$ for each $\lambda \in \mathcal{P}^\Lambda$. We omit the proof since we do not need this result here. The isomorphism t corresponds to *Ringel duality* at the categorical level; see [M].

Remark 3.17. The canonical basis $\{T_\lambda \mid \lambda \in \mathcal{P}^\Lambda\}$ from Remark 3.16 is a lower global crystal basis in the sense of Kashiwara [Kas3]. The underlying lower crystal operators \tilde{e}'_i and \tilde{f}'_i induce another structure

$$(\mathcal{P}^\Lambda, \tilde{e}'_i, \tilde{f}'_i, \varepsilon'_i, \varphi'_i, \text{wt}) \quad (3.50)$$

of abstract crystal on the index set \mathcal{P}^Λ that is different from the one in (3.25). It can be described explicitly in exactly the same way as in §3.7, except that at the beginning we list the addable and removable i -nodes of λ as A_1, \dots, A_n so that A_m is row-below A_{m+1} for each $m = 1, \dots, n - 1$ (the reverse order to the one used before). This follows because Kashiwara's combinatorial tensor product rule for lower crystal bases at $q = \infty$ from [Kas2, Theorem 1] is the opposite of the one for upper crystal bases at $q = 0$ from [Kas1, Proposition 6]. Equivalently, by direct comparison of the combinatorics, it is the case that

$$\tilde{f}'_i \lambda = (\tilde{f}'_{-i}(\lambda^t))^t \quad (3.51)$$

for each $i \in I$ and $\lambda \in \mathcal{P}^\Lambda$, where $\lambda^t \in \mathcal{P}^{\Lambda^t}$ is as in (3.48) and \tilde{f}_{-i} is the upper crystal operator on \mathcal{P}^{Λ^t} defined exactly as in §3.7 but computing residues via the multicharge $(-\tilde{k}_l, \dots, -\tilde{k}_1)$ instead of $(\tilde{k}_1, \dots, \tilde{k}_l)$.

Remark 3.18. Let $(\mathcal{R}\mathcal{P}^\Lambda)' := (\mathcal{R}\mathcal{P}^{\Lambda^t})^t$, where $\mathcal{R}\mathcal{P}^{\Lambda^t}$ is the set from (3.27) but defined from $(-\tilde{k}_l, \dots, -\tilde{k}_1)$ instead of $(\tilde{k}_1, \dots, \tilde{k}_l)$. We refer to elements of $(\mathcal{R}\mathcal{P}^\Lambda)'$ as *regular multipartitions*. In view of (3.51), this is the vertex set of the connected component of the crystal (3.50) generated by the empty multipartition \emptyset , which gives another realization

$$((\mathcal{R}\mathcal{P}^\Lambda)', \tilde{e}'_i, \tilde{f}'_i, \varepsilon'_i, \varphi'_i, \text{wt}) \quad (3.52)$$

of the abstract crystal attached to the module $V(\Lambda)$ different to the one in (3.27). There is a canonical bijection

$$\mathcal{R}\mathcal{P}^\Lambda \rightarrow (\mathcal{R}\mathcal{P}^\Lambda)', \quad \lambda \mapsto \lambda' \quad (3.53)$$

such that $\emptyset' = \emptyset$ and $\tilde{f}'_i(\lambda') = (\tilde{f}_i\lambda)'$ for all $\lambda \in \mathcal{R}\mathcal{P}^\Lambda$ and $i \in I$. For example, in the special case that $e > 0$ and $l = 1$, the set $(\mathcal{R}\mathcal{P}^\Lambda)'$ is the usual set of *e-regular partitions*, that is, partitions λ that do not have e or more non-zero repeated parts. The map $\lambda \mapsto \lambda'$ in this case is the composition first of the map $\lambda \mapsto \lambda^t$ followed by the Mullineux involution on e -regular partitions; see [K1, K2, FK].

Remark 3.19. In view of Lemma 3.12 and Theorem 3.14, the vectors $q^{-\text{def}(\alpha)}P_\lambda$ for $\lambda \in \mathcal{R}\mathcal{P}^\Lambda_\alpha$ must coincide with some of the canonical basis elements of $F(\Lambda)$ from Remark 3.16. We can make this precise using Remark 3.18: we have that

$$P_\lambda = q^{\text{def}(\alpha)}T_{\lambda'} \quad (3.54)$$

for each $\lambda \in \mathcal{R}\mathcal{P}^\Lambda_\alpha$, where λ' is as in (3.53). Arguing exactly as in the proof of [BK5, Corollary 2.8], it follows easily for each $\lambda \in \mathcal{R}\mathcal{P}^\Lambda_\alpha$ and $\mu \in \mathcal{P}^\Lambda_\alpha$ that

- (1) $d_{\lambda, \mu}(q) = 0$ unless $\lambda \preceq \mu \preceq \lambda'$;
- (2) $d_{\lambda, \mu}(q) \in q\mathbb{Z}[q] \cap q^{\text{def}(\alpha)-1}\mathbb{Z}[q^{-1}]$ if $\lambda \prec \mu \prec \lambda'$;
- (3) $d_{\lambda, \lambda}(q) = 1$ and $d_{\lambda, \lambda'}(q) = q^{\text{def}(\alpha)}$.

3.11. Twisted Fock space. We now turn our attention to the problem of constructing a bar-involution on $F(\Lambda)$ as in Hypothesis 3.7 when $e > 0$. In preparation for this, we need to recall the twisted version of Fock space, whose construction in higher levels is due to Takemura and Uglov [TU]. Our exposition follows [BK5, §2.5] in the case $e = 0$ and [U] in the case $e > 0$ (noting our q is equal to q^{-1} there).

We first introduce a different ordering on the nodes of a multipartition. Say that a node $A = (a, b, m)$ is *residue-above* node $B = (c, d, n)$ (or B is *residue-below* A) if either $\text{res } A > \text{res } B$ or $\text{res } A = \text{res } B$ and $m > n$. The following lemma relates this ordering on nodes to the one used earlier.

Lemma 3.20. *Let $\lambda \in \mathcal{P}^\Lambda_\alpha$ and $i \in I$.*

- (1) *Assume $\tilde{k}_m - \tilde{k}_{m+1} \geq \text{ht}(\alpha)$ for all $m = 1, \dots, l-1$. Let A be a removable i -node for λ and B be either an addable or a removable i -node for λ . Then B is row-below A if and only if B is residue-below A .*

- (2) Assume $\tilde{k}_m - \tilde{k}_{m+1} \geq \text{ht}(\alpha) + 1$ for all $m = 1, \dots, l-1$. Let A be either an addable or a removable i -node for λ and B be an addable i -node for λ . Then A is row-above B if and only if A is residue-above B .

Given $\lambda \in \mathcal{P}^\Lambda$, $i \in I$, a removable i -node A and an addable i -node B , we set

$$\begin{aligned} \tilde{d}_A(\lambda) &:= \#\{\text{addable } i\text{-nodes of } \lambda \text{ residue-below } A\} \\ &\quad - \#\{\text{removable } i\text{-nodes of } \lambda \text{ residue-below } A\}; \end{aligned} \quad (3.55)$$

$$\begin{aligned} \tilde{d}^B(\lambda) &:= \#\{\text{addable } i\text{-nodes of } \lambda \text{ residue-above } B\} \\ &\quad - \#\{\text{removable } i\text{-nodes of } \lambda \text{ residue-above } B\}. \end{aligned} \quad (3.56)$$

By Lemma 3.20, we have that $\tilde{d}_A(\lambda) = d_A(\lambda)$ under the hypotheses of part (1) of the lemma, and $\tilde{d}^B(\lambda) = d^B(\lambda)$ under the hypotheses of part (2).

Now we can define the twisted Fock space $\tilde{F}(\Lambda)$ to be the $\mathbb{Q}(q)$ -vector space on basis $\{\tilde{M}_\lambda \mid \lambda \in \mathcal{P}^\Lambda\}$. We make $\tilde{F}(\Lambda)$ into a $U_q(\mathfrak{g})$ -module by defining

$$E_i \tilde{M}_\lambda := \sum_A q^{\tilde{d}_A(\lambda)} \tilde{M}_{\lambda_A}, \quad F_i \tilde{M}_\lambda := \sum_B q^{-\tilde{d}^B(\lambda)} \tilde{M}_{\lambda_B}, \quad (3.57)$$

$$K_i \tilde{M}_\lambda := q^{d_i(\lambda)} \tilde{M}_\lambda, \quad (3.58)$$

where the first sum is over all removable i -nodes A for λ , and the second sum is over all addable i -nodes B for λ . These are almost the same as the formulae (3.22)–(3.23), but we have replaced $d_A(\lambda)$ and $d^B(\lambda)$ from before with $\tilde{d}_A(\lambda)$ and $\tilde{d}^B(\lambda)$ defined using the new ordering on nodes.

If $l = 1$ we have simply that $\tilde{d}_A(\lambda) = d_A(\lambda)$ and $\tilde{d}^B(\lambda) = d^B(\lambda)$ for all addable nodes A and removable nodes B , so in this case we can simply *identify* $\tilde{F}(\Lambda)$ with $F(\Lambda)$ by identifying $\tilde{M}(\lambda)$ with $M(\lambda)$ for each $\lambda \in \mathcal{P}^\Lambda$. In particular this shows that the formulae (3.57)–(3.58) give a well-defined action of $U_q(\mathfrak{g})$ on $\tilde{F}(\Lambda)$ in the level one case, since we already knew that for $F(\Lambda)$. For a proof that this action is well defined for arbitrary level and $e > 0$, we refer to [U, Theorem 2.1]. When $e = 0$ there is a different approach noted in [BK5, §2.5]: in that case we can simply identify

$$\tilde{F}(\Lambda) = F(\Lambda_{k_l}) \otimes \cdots \otimes F(\Lambda_{k_1}) \quad (3.59)$$

by identifying \tilde{M}_λ with $M_{\lambda^{(l)}} \otimes \cdots \otimes M_{\lambda^{(1)}}$ for each $\lambda \in \mathcal{P}^\Lambda$. The formulae (3.57)–(3.58) describe the natural action of $U_q(\mathfrak{g})$ on this tensor product, so they give a well-defined $U_q(\mathfrak{g})$ -action in the $e = 0$ case too.

Recalling the partial order \geq from §3.5, the twisted Fock space $\tilde{F}(\Lambda)$ possesses a canonical compatible bar-involution with the property that

$$\overline{\tilde{M}_\lambda} = \tilde{M}_\lambda + (\text{a } \mathbb{Z}[q, q^{-1}]\text{-linear combination of } \tilde{M}_\mu\text{'s for } \mu > \lambda) \quad (3.60)$$

for any $\lambda \in \mathcal{P}^\Lambda$. The existence of this bar-involution is established in [LT1, LT2] in the case that $l = 1$ and $e > 0$ by reinterpreting $\tilde{F}(\Lambda)$ in that case as a semi-infinite wedge as in [S]. The construction of the bar-involution in the level one case from [LT1, LT2] was extended to higher levels in the $e > 0$ case by Uglov; see [U, Proposition 4.11]. In the case $e = 0$, it is clear from (3.59) and (3.24) that the space $\tilde{F}(\Lambda)$ is just the same as the space $F(\Lambda)$ but reversing the order

of the underlying sequence k_1, \dots, k_l . So we get the compatible bar-involution in this case from the same construction as explained at the beginning of §3.9; see also [BK5, §2.5].

At this point one can repeat almost word-for-word the development from §§3.6–3.10, replacing the Fock space $F(\Lambda)$ with the twisted Fock space $\tilde{F}(\Lambda)$ and using the known bar-involution from (3.60) in place of the hypothesized bar-involution from Hypothesis 3.7. All we actually need from this here is the definition of the *dual-canonical basis* $\{\tilde{L}_\lambda \mid \lambda \in \mathcal{P}^\Lambda\}$ of $\tilde{F}(\Lambda)$, which is defined by letting \tilde{L}_λ denote the unique bar-invariant vector such that

$$\tilde{L}_\lambda = \tilde{M}_\lambda + (\text{a } q\mathbb{Z}[q]\text{-linear combination of } \tilde{L}_\mu\text{'s with } \mu > \lambda). \quad (3.61)$$

From this, we obtain polynomials $\tilde{d}_{\lambda,\mu}(q) \in \mathbb{Z}[q]$ such that

$$\tilde{M}_\mu = \sum_{\lambda \in \mathcal{P}^\Lambda} \tilde{d}_{\lambda,\mu}(q) \tilde{L}_\lambda. \quad (3.62)$$

These have the property that $\tilde{d}_{\lambda,\mu}(q) = 0$ unless $\lambda \geq \mu$, $\tilde{d}_{\lambda,\mu}(q) = 1$ if $\lambda = \mu$, and $\tilde{d}_{\lambda,\mu}(q) \in q\mathbb{Z}[q]$ if $\lambda > \mu$.

Remark 3.21. The dual-canonical basis $\{\tilde{L}_\lambda \mid \lambda \in \mathcal{P}^\Lambda\}$ is an upper global crystal basis in the sense of [Kas3]. It leads to yet another abstract crystal structure on the index set \mathcal{P}^Λ , which can be described combinatorially by the same method as in §3.7, except that one needs to start by listing the addable and removable i -nodes of $\lambda \in \mathcal{P}^\Lambda$ as A_1, \dots, A_n so that A_m is residue-above A_{m+1} for each $m = 1, \dots, n-1$. Let $\widetilde{\mathcal{R}\mathcal{P}^\Lambda}$ denote the vertex set of the connected component of this crystal generated by \emptyset . This provides another realization of the abstract crystal associated to $V(\Lambda)$. In the case $e = 0$, the set $\widetilde{\mathcal{R}\mathcal{P}^\Lambda}$ happens to be the same as the set $(\mathcal{R}\mathcal{P}^\Lambda)'$ introduced in Remark 3.18.

Remark 3.22. There is a special case in which the set $\widetilde{\mathcal{R}\mathcal{P}^\Lambda}$ from Remark 3.21 has an elementary description. Suppose that $\tilde{k}_1 \geq \dots \geq \tilde{k}_l$ and either $e = 0$ or $\tilde{k}_l > \tilde{k}_1 - e$. Then $\widetilde{\mathcal{R}\mathcal{P}^\Lambda}$ consists of all l -multipartitions λ such that

- (1) $\lambda_a^{(m)} + \tilde{k}_m - \tilde{k}_{m+1} \geq \lambda_a^{(m+1)}$ for each $a \geq 1$ and $m = 1, \dots, l-1$;
- (2) if $e > 0$ then $\lambda_a^{(l)} + e + \tilde{k}_l - \tilde{k}_1 \geq \lambda_a^{(1)}$ for each $a \geq 1$;
- (3) it is impossible find nodes $\{A_i \mid i \in I\}$ from the bottoms of columns of the same length in λ such that $\text{res } A_i \equiv i \pmod{e}$ for each $i \in I$.

This follows from [BK5, (2.53)] in the case $e = 0$, and it is a reformulation of a result from [FLOTW] in the case $e > 0$.

Remark 3.23. In the $e = 0$ case, the twisted Fock space $\tilde{F}(\Lambda)$ can be categorified by means of the opposite parabolic category \mathcal{O} to the one mentioned in Remark 3.10. By Arkhipov-Soergel reciprocity, this categorification of $\tilde{F}(\Lambda)$ is the Ringel dual of the categorification of $F(\Lambda)$; see [BK5, §4.3]. When $e > 0$ there should also be a highest weight category categorifying the twisted Fock space $\tilde{F}(\Lambda)$ arising from rational Cherednik algebras, although the picture here is not yet complete; see [R1, §6.8] and also [VV2, §8] which develops another

approach in terms of affine parabolic category \mathcal{O} . Under a stability hypothesis similar to the one in Proposition 3.24 below, this category is known to be equivalent to the one mentioned in Remark 3.10 arising from cyclotomic ξ -Schur algebras; see [R1, Theorem 6.8].

3.12. Construction of the bar-involution. In this subsection we assume that $e > 0$ and explain how to construct a bar-involution on $F(\Lambda)$ as in Hypothesis 3.7. To do this we exploit the following stability result of Yvonne.

Proposition 3.24 ([Y2, Theorem 5.2]). *Let k_1, \dots, k_l be fixed as in (3.1). For each $\alpha \in Q_+$, there exists an integer $N_\alpha \gg 0$ such that the transition matrix $(\tilde{d}_{\lambda,\mu}(q))_{\lambda,\mu \in \mathcal{P}_\alpha^\Lambda}$ is the same matrix for every multicharge $(\tilde{k}_1, \dots, \tilde{k}_l) \in \mathbb{Z}^l$ with $\tilde{k}_1 \equiv k_1, \dots, \tilde{k}_l \equiv k_l \pmod{e}$ and $\tilde{k}_m - \tilde{k}_{m+1} \geq N_\alpha$ for $m = 1, \dots, l-1$.*

Remark 3.25. Conjecturally, one can take $N_\alpha := \text{ht}(\alpha)$; see [Y2, Remark 5.3].

Now define $d_{\lambda,\mu}(q) \in \mathbb{Z}[q]$ for any $\lambda, \mu \in \mathcal{P}^\Lambda$ as follows. If $\text{cont}(\lambda) \neq \text{cont}(\mu)$ we set $d_{\lambda,\mu}(q) := 0$. Otherwise, if $\text{cont}(\lambda) = \text{cont}(\mu) = \alpha$ for some $\alpha \in Q_+$, pick $(\tilde{k}_1, \dots, \tilde{k}_l) \in \mathbb{Z}^l$ so that $\tilde{k}_1 \equiv k_1, \dots, \tilde{k}_l \equiv k_l \pmod{e}$ and $\tilde{k}_m - \tilde{k}_{m+1} \geq N_\alpha$ for $m = 1, \dots, l-1$, and then set $d_{\lambda,\mu}(q) := \tilde{d}_{\lambda,\mu}(q)$, i.e. the polynomial defined as in (3.62) for this choice of multicharge. Proposition 3.24 implies that $d_{\lambda,\mu}(q)$ is well defined independent of the particular choice of $(\tilde{k}_1, \dots, \tilde{k}_l)$. Finally let $(p_{\lambda,\mu}(-q))_{\lambda,\mu \in \mathcal{P}^\Lambda}$ be the inverse of the matrix $(d_{\lambda,\mu}(q))_{\lambda,\mu \in \mathcal{P}^\Lambda}$ and define an anti-linear endomorphism $-$ of $F(\Lambda)$ by setting

$$\overline{M}_\mu := \sum_{\kappa, \lambda \in \mathcal{P}^\Lambda} p_{\kappa,\lambda}(-q) d_{\lambda,\mu}(q^{-1}) M_\kappa \quad (3.63)$$

for each $\mu \in \mathcal{P}^\Lambda$. The following theorem shows that this is a compatible bar-involution on $F(\Lambda)$ as in Hypothesis 3.7 (taking the order \preceq there to be \leq_{lex}).

Theorem 3.26. *The map (3.63) is a compatible bar-involution on $F(\Lambda)$ with the following property for every $\lambda \in \mathcal{P}^\Lambda$:*

$$\overline{M}_\lambda = M_\lambda + (a \mathbb{Z}[q, q^{-1}]\text{-linear combination of } M_\mu \text{'s for } \mu <_{\text{lex}} \lambda). \quad (3.64)$$

Proof. Fix some $d \geq 0$ and let $F(\Lambda)_{\leq d}$ denote the subspace of $F(\Lambda)$ spanned by all M_λ 's for $\lambda \in \mathcal{P}^\Lambda$ with $|\lambda| \leq d$. We claim that the restriction of $-$ to $F(\Lambda)_{\leq d}$ is an involution with the property (3.64) for all λ with $|\lambda| \leq d$, and moreover that $\overline{E_i v} = E_i \overline{v}$ and $\overline{F_i v} = F_i \overline{v}$ for all $v \in F(\Lambda)_{\leq (d-1)}$ and $i \in I$. The theorem follows from claim by letting $d \rightarrow \infty$.

To prove the claim, set $N := \max\{d + e, N_\alpha \mid \alpha \in Q_+, \text{ht}(\alpha) \leq d\}$. Choose the multicharge so that $\tilde{k}_1 \equiv k_1, \dots, \tilde{k}_l \equiv k_l \pmod{e}$ and $\tilde{k}_m - \tilde{k}_{m+1} \geq N$ for $m = 1, \dots, l$. Defining $\tilde{F}(\Lambda)$ with this choice of multicharge, we define a vector space isomorphism

$$\iota : F(\Lambda) \xrightarrow{\sim} \tilde{F}(\Lambda), \quad M_\lambda \mapsto \widetilde{M}_\lambda.$$

Using Lemma 3.20, (3.22) and (3.57), we see that $\iota(E_i v) = E_i \iota(v)$ and $\iota(F_i v) = F_i \iota(v)$ for all $v \in F(\Lambda)_{\leq (d-1)}$. By the definition (3.63), the bar-involution on $F(\Lambda)$ is the unique anti-linear map fixing the vectors $\sum_\lambda p_{\lambda,\mu}(-q) M_\lambda$ for all $\mu \in \mathcal{P}^\Lambda$. Moreover for $\lambda, \mu \in \mathcal{P}^\Lambda$ with $|\mu| \leq d$, we have that $d_{\lambda,\mu}(q) = \tilde{d}_{\lambda,\mu}(q)$,

so recalling (3.62) we see that ι maps $\sum_{\lambda} p_{\lambda, \mu}(-q)M_{\lambda}$ to the bar-invariant vector \widetilde{L}_{μ} for $|\mu| \leq d$. This shows that $\iota(\bar{v}) = \overline{\iota(v)}$ for all $v \in F(\Lambda)_{\leq d}$. Putting these things together gives that $-$ is an involution on $F(\Lambda)_{\leq d}$ such that $\overline{E_i v} = E_i \bar{v}$ and $\overline{F_i v} = F_i \bar{v}$ for all $v \in F(\Lambda)_{\leq (d-1)}$ and $i \in I$. Moreover from (3.61) we get that

$$\overline{M_{\lambda}} = M_{\lambda} + (\text{a } \mathbb{Z}[q, q^{-1}]\text{-linear combination of } M_{\mu}\text{'s for } \mu > \lambda)$$

for $|\lambda| \leq d$. Finally an application of Lemma 3.3 gives (3.64) for $|\lambda| \leq d$. This establishes the claim. \square

Remark 3.27. It would be interesting to find a direct construction of the bar-involution on $F(\Lambda)$ in the $e > 0$ case by-passing twisted Fock space; such a construction should also produce a natural choice for the partial order \preceq .

4. GRADED BRANCHING RULES AND CATEGORIFICATION OF $V(\Lambda)$

Continue with F denoting an algebraically closed field, but assume also now that $\xi \in F^{\times}$ is an invertible element and take the integer e to be the smallest positive integer such that $1 + \xi + \dots + \xi^{e-1} = 0$, setting $e := 0$ if no such integer exists. All other notation is the same as in the previous sections for this choice of e ; in particular, we have fixed Λ as in (3.1).

4.1. **The algebra R_{α}^{Λ} .** Following [KL1, §3.4], we let

$$R_{\alpha}^{\Lambda} := R_{\alpha} / \langle y_1^{(\Lambda, \alpha_{i_1})} e(\mathbf{i}) \mid \mathbf{i} \in I^{\alpha} \rangle. \quad (4.1)$$

We use the same notation for elements of R_{α}^{Λ} as in R_{α} , relying on context to distinguish which we mean.

For any $i \in I$, the embedding $R_{\alpha} = R_{\alpha} \otimes 1 \hookrightarrow R_{\alpha, \alpha_i} \hookrightarrow R_{\alpha + \alpha_i}$ factors through the quotients to induce a (not necessarily injective) graded algebra homomorphism

$$\iota_{\alpha, \alpha_i} : R_{\alpha}^{\Lambda} \rightarrow R_{\alpha + \alpha_i}^{\Lambda}. \quad (4.2)$$

This maps the identity element of R_{α}^{Λ} to the idempotent $e_{\alpha, \alpha_i} \in R_{\alpha + \alpha_i}^{\Lambda}$.

4.2. **The algebra H_{α}^{Λ} .** Let H_d denote the affine Hecke algebra associated to the symmetric group Σ_d on generators $\{X_1^{\pm 1}, \dots, X_d^{\pm 1}\} \cup \{T_1, \dots, T_{d-1}\}$ if $\xi \neq 1$, or its degenerate analogue on generators $\{x_1, \dots, x_d\} \cup \{s_1, \dots, s_{d-1}\}$ if $\xi = 1$. For the full relations, which are quite standard, we refer the reader to [BK4], noting here just that

$$\begin{cases} T_r^2 = (\xi - 1)T_r + \xi, & T_r X_r T_r = \xi X_{r+1} & \text{if } \xi \neq 1, \\ s_r^2 = 1, & s_r x_{r+1} = x_r s_r + 1 & \text{if } \xi = 1. \end{cases}$$

Then we consider the cyclotomic quotient

$$H_d^{\Lambda} := \begin{cases} H_d / \langle \prod_{i \in I} (X_1 - \xi^i)^{(\Lambda, \alpha_i)} \rangle & \text{if } \xi \neq 1, \\ H_d / \langle \prod_{i \in I} (x_1 - i)^{(\Lambda, \alpha_i)} \rangle & \text{if } \xi = 1. \end{cases} \quad (4.3)$$

We refer to this algebra simply as the *cyclotomic Hecke algebra* if $\xi \neq 1$ and the *degenerate cyclotomic Hecke algebra* if $\xi = 1$.

There is a natural system $\{e(\mathbf{i}) \mid \mathbf{i} \in I^d\}$ of mutually orthogonal idempotents in H_d^Λ called *weight idempotents*; see [BK4]. These are characterized uniquely by the property that $e(\mathbf{i})M = M_{\mathbf{i}}$ for any finite dimensional H_d^Λ -module M , where

$$M_{\mathbf{i}} := \begin{cases} \{v \in M \mid (X_r - \xi^{i_r})^N v = 0 \text{ for } N \gg 0\} & \text{if } \xi \neq 1, \\ \{v \in M \mid (x_r - i_r)^N v = 0 \text{ for } N \gg 0\} & \text{if } \xi = 1. \end{cases} \quad (4.4)$$

Note all but finitely many of the $e(\mathbf{i})$'s are zero, and their sum is the identity element in H_d^Λ .

Given $\alpha \in Q_+$ of height d , we set

$$e_\alpha := \sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i}) \in H_d^\Lambda. \quad (4.5)$$

As a consequence of [LM] or [B3, Theorem 1], e_α is either zero or it is a primitive central idempotent in H_d^Λ . Hence the algebra

$$H_\alpha^\Lambda := e_\alpha H_d^\Lambda \quad (4.6)$$

is either zero or it is a single *block* of the algebra H_d^Λ , and we have that

$$H_d^\Lambda = \bigoplus_{\alpha \in Q_+, \text{ht}(\alpha)=d} H_\alpha^\Lambda \quad (4.7)$$

as a direct sum of algebras. For $h \in H_d^\Lambda$, we still write h for the projection $e_\alpha h \in H_\alpha^\Lambda$.

The natural embedding of H_d into H_{d+1} factors through the quotients to induce an embedding of H_d^Λ into H_{d+1}^Λ . Composing this on the right with the inclusion $H_\alpha^\Lambda \hookrightarrow H_d^\Lambda$ and then on the left with multiplication by the idempotent e_{α, α_i} , we obtain a non-unital algebra homomorphism

$$\iota_{\alpha, \alpha_i} : H_\alpha^\Lambda \rightarrow H_{\alpha + \alpha_i}^\Lambda. \quad (4.8)$$

Just like in (4.2), this maps the identity element of H_α^Λ to the idempotent

$$e_{\alpha, \alpha_i} := \sum_{\mathbf{i} \in I^{\alpha + \alpha_i}, i_{d+1} = i} e(\mathbf{i}). \quad (4.9)$$

4.3. The isomorphism theorem. According to the main theorem of [BK4], the cyclotomic algebras R_α^Λ and H_α^Λ are isomorphic. Although not used explicitly here, we note that a closely related result for the affine algebras has been obtained independently by Rouquier in [R2, §3.2.6].

Theorem 4.1 ([BK4]). *For $\alpha \in Q_+$ of height d , there is an algebra isomorphism $\rho : R_\alpha^\Lambda \xrightarrow{\sim} H_\alpha^\Lambda$ such that*

$$\begin{aligned} e(\mathbf{i}) &\mapsto e(\mathbf{i}), \\ y_r e(\mathbf{i}) &\mapsto \begin{cases} (1 - \xi^{-i_r} X_r) e(\mathbf{i}) & \text{if } \xi \neq 1, \\ (x_r - i_r) e(\mathbf{i}) & \text{if } \xi = 1, \end{cases} \end{aligned}$$

for each $r = 1, \dots, d$ and $\mathbf{i} \in I^\alpha$. Moreover, the following diagram commutes for all $\alpha \in Q_+$ and $i \in I$:

$$\begin{array}{ccc} R_\alpha^\Lambda & \xrightarrow{\iota_{\alpha, \alpha_i}} & R_{\alpha+\alpha_i}^\Lambda \\ \rho \downarrow & & \downarrow \rho \\ H_\alpha^\Lambda & \xrightarrow{\iota_{\alpha, \alpha_i}} & H_{\alpha+\alpha_i}^\Lambda. \end{array} \quad (4.10)$$

Remark 4.2. In [BK4], one can also find formulae for the images of the generators $\psi_r e(\mathbf{i})$, but we do not need to know these explicitly here.

Henceforth, we will simply *identify* the algebras R_α^Λ and H_α^Λ via the isomorphism ρ from Theorem 4.1. Despite the fact that $R_\alpha^\Lambda = H_\alpha^\Lambda$ now, we will usually talk in terms of R_α^Λ when discussing *graded* representation theory and H_α^Λ when discussing *ungraded* representation theory. For instance, as in §2.4, for a graded R_α^Λ -module M we write \underline{M} for the ungraded H_α^Λ -module obtained from M by forgetting the grading.

4.4. i -Induction and i -restriction. For $i \in I$ and $\alpha \in Q_+$, let e_i and f_i be the functors

$$e_i := e_{\alpha, \alpha_i} H_{\alpha+\alpha_i}^\Lambda \otimes_{H_{\alpha+\alpha_i}^\Lambda} ? : \underline{\text{Mod}}(H_{\alpha+\alpha_i}^\Lambda) \rightarrow \underline{\text{Mod}}(H_\alpha^\Lambda), \quad (4.11)$$

$$f_i := H_{\alpha+\alpha_i}^\Lambda e_{\alpha, \alpha_i} \otimes_{H_\alpha^\Lambda} ? : \underline{\text{Mod}}(H_\alpha^\Lambda) \rightarrow \underline{\text{Mod}}(H_{\alpha+\alpha_i}^\Lambda), \quad (4.12)$$

viewing $e_{\alpha, \alpha_i} H_{\alpha+\alpha_i}^\Lambda$ (resp. $H_{\alpha+\alpha_i}^\Lambda e_{\alpha, \alpha_i}$) as a left (resp. right) H_α^Λ -module via the homomorphism (4.8). The functor e_i is particularly simple to understand: it is just multiplication by the idempotent e_{α, α_i} followed by restriction to H_α^Λ via the homomorphism ι_{α, α_i} . We use the same notation e_i and f_i for the direct sums of these functors over all $\alpha \in Q_+$. They are exactly Robinson's *i-restriction* and *i-induction* functors as in [A3, §13.6] or [K3, (8.4), (8.6)]. In particular, it is known that e_i and f_i are biadjoint, hence both are exact and send projectives to projectives. They obviously both send finite dimensional (resp. finitely generated) modules to finite dimensional (resp. finitely generated) modules.

Similarly define functors E_i and F_i by setting

$$E_i := e_{\alpha, \alpha_i} R_{\alpha+\alpha_i}^\Lambda \otimes_{R_{\alpha+\alpha_i}^\Lambda} ? : \text{Mod}(R_{\alpha+\alpha_i}^\Lambda) \rightarrow \text{Mod}(R_\alpha^\Lambda), \quad (4.13)$$

$$F_i := R_{\alpha+\alpha_i}^\Lambda e_{\alpha, \alpha_i} \otimes_{R_\alpha^\Lambda} ? \langle 1 - (\Lambda - \alpha, \alpha_i) \rangle : \text{Mod}(R_\alpha^\Lambda) \rightarrow \text{Mod}(R_{\alpha+\alpha_i}^\Lambda), \quad (4.14)$$

interpreting the tensor products via (4.2), then taking the direct sums over all $\alpha \in Q_+$. By (4.10), these are graded versions of e_i and f_i in the sense that

$$\underline{E_i(M)} \cong e_i(\underline{M}), \quad \underline{F_i(M)} \cong f_i(\underline{M}) \quad (4.15)$$

for any graded R_α^Λ -module M . In particular, we deduce from this that E_i and F_i are both exact, and send finite dimensional (resp. finitely generated projective) modules to finite dimensional (resp. finitely generated projective) modules, since we already know that for e_i and f_i .

Also define a functor K_i by letting

$$K_i : \text{Mod}(R_\alpha^\Lambda) \rightarrow \text{Mod}(R_\alpha^\Lambda) \quad (4.16)$$

denote the degree shift functor $M \mapsto M\langle(\Lambda - \alpha, \alpha_i)\rangle$. If we use this functor to cancel out the degree shifts in (4.14), we see that

$$F_i K_i \langle -1 \rangle \cong R_{\alpha + \alpha_i}^\Lambda e_{\alpha, \alpha_i} \otimes_{R_\alpha} ? \quad (4.17)$$

Combining this with adjointness of tensor and hom we deduce:

Lemma 4.3. *There is a canonical adjunction making $(F_i K_i \langle -1 \rangle, E_i)$ into an adjoint pair.*

There is an equivalent way to describe the functors E_i and F_i which relates them to the functors θ_i^* and θ_i from (2.18)–(2.19). To formulate this, we first introduce the *inflation* and *truncation* functors

$$\text{infl} : \text{Mod}(R_\alpha^\Lambda) \rightarrow \text{Mod}(R_\alpha) \quad \text{pr} : \text{Mod}(R_\alpha) \rightarrow \text{Mod}(R_\alpha^\Lambda). \quad (4.18)$$

So for $M \in \text{Mod}(R_\alpha^\Lambda)$, we write $\text{infl } M$ for its pull-back through the natural surjection $R_\alpha \twoheadrightarrow R_\alpha^\Lambda$, and for $N \in \text{Mod}(R_\alpha)$ we write $\text{pr } N$ for $R_\alpha^\Lambda \otimes_{R_\alpha} N$, which is the largest graded quotient of N that factors through to R_α^Λ . Note pr depends implicitly on the fixed choice of Λ , but we omit it from our notation since this should be clear from context. We obviously have that

$$\text{pr} \circ \text{infl} = \text{Id}. \quad (4.19)$$

Observe also that (pr, infl) is an adjoint pair in a canonical way. Hence, pr sends projective modules to projective modules. It follows easily that infl and pr restrict to functors

$$\text{infl} : \text{Rep}(R_\alpha^\Lambda) \rightarrow \text{Rep}(R_\alpha), \quad \text{pr} : \text{Proj}(R_\alpha) \rightarrow \text{Proj}(R_\alpha^\Lambda). \quad (4.20)$$

Lemma 4.4. *There are canonical isomorphisms of functors $E_i \cong \text{pr} \circ \theta_i^* \circ \text{infl}$ and $F_i K_i \langle -1 \rangle \cong \text{pr} \circ \theta_i \circ \text{infl}$.*

Proof. For E_i , note that both $\text{infl} \circ E_i$ and $\theta_i^* \circ \text{infl}$ are defined on $M \in \text{Mod}(R_{\alpha + \alpha_i}^\Lambda)$ by multiplying by the idempotent e_{α, α_i} . Hence $\text{infl} \circ E_i \cong \theta_i^* \circ \text{infl}$. Using also (4.19) this implies that $E_i \cong \text{pr} \circ \theta_i^* \circ \text{infl}$.

For F_i , there is a canonical adjunction making $(\text{pr} \circ \theta_i, \theta_i^* \circ \text{infl})$ into an adjoint pair. Hence for $M \in \text{Mod}(R_\alpha^\Lambda)$ and $N \in \text{Mod}(R_{\alpha + \alpha_i}^\Lambda)$ we have natural isomorphisms

$$\begin{aligned} \text{Hom}_{R_{\alpha + \alpha_i}^\Lambda}(\text{pr } \theta_i \text{ infl}(M), N) &\cong \text{Hom}_{R_\alpha}(\text{infl } M, \theta_i^* \text{ infl}(N)) \\ &\cong \text{Hom}_{R_\alpha}(\text{infl } M, \text{infl } E_i(N)) = \text{Hom}_{R_\alpha^\Lambda}(M, E_i N). \end{aligned}$$

This establishes that $\text{pr} \circ \theta_i \circ \text{infl}$ is left adjoint to E_i . Hence $\text{pr} \circ \theta_i \circ \text{infl} \cong F_i K_i \langle -1 \rangle$ by Lemma 4.3 and unicity of adjoints. \square

4.5. Cyclotomic duality. The anti-automorphism $*$: $R_\alpha \rightarrow R_\alpha$ from (2.20) descends to the quotient R_α^Λ , yielding a graded anti-automorphism $*$: $R_\alpha^\Lambda \rightarrow R_\alpha^\Lambda$. Using this we can define a duality \otimes on $\text{Rep}(R_\alpha^\Lambda)$ in the same way as the duality \otimes was defined on $\text{Rep}(R_\alpha)$ in §2.7. It is then clear that \otimes commutes with inflation, i.e.

$$\text{infl} \circ \otimes \cong \otimes \circ \text{infl} \quad (4.21)$$

as functors from $\text{Rep}(R_\alpha^\Lambda)$ to $\text{Rep}(R_\alpha)$. The following lemma follows by the same argument as (2.21):

Lemma 4.5. *There is an isomorphism $\circledast \circ E_i \cong E_i \circ \circledast$.*

In view of the next lemma, the duality \circledast on $\text{Rep}(R_\alpha^\Lambda)$ restricts to give a well-defined duality \circledast on the subcategory $\text{Proj}(R_\alpha^\Lambda)$ too.

Lemma 4.6. *For $P \in \text{Proj}(R_\alpha^\Lambda)$, the dual P^{\circledast} is a graded projective module.*

Proof. It suffices to show that $\underline{P}^{\circledast}$ is a projective H_α^Λ -module. This follows because H_α^Λ is a symmetric algebra, so its injective modules are projective; see [MM] or [BK3, Theorem A.2] in the degenerate case. \square

Although not used explicitly here, we note for completeness that there is another duality $\#$ on $\text{Rep}(R_\alpha^\Lambda)$ mapping M to $M^\# := \text{Hom}_{R_\alpha^\Lambda}(M, R_\alpha^\Lambda)$ with action defined by $(xf)(p) = f(p)x^*$. The fact that this is exact (hence a duality) follows because R_α^Λ is injective by Lemma 4.6. The duality $\#$ obviously restricts to a well-defined duality $\#$ on $\text{Proj}(R_\alpha^\Lambda)$. Recalling the duality $\#$ on $\text{Proj}(R_\alpha)$ from §2.7, it is also clear that $\#$ commutes with truncation, i.e.

$$\text{pr} \circ \# \cong \# \circ \text{pr} \tag{4.22}$$

as functors from $\text{Proj}(R_\alpha)$ to $\text{Proj}(R_\alpha^\Lambda)$.

Remark 4.7. We conjecture that R_α^Λ is a graded symmetric algebra in the sense that it possesses a homogeneous symmetrizing form $\tau : R_\alpha^\Lambda \rightarrow F$ of degree $-2 \text{def}(\alpha)$. By general principles this would imply that there is an isomorphism of functors $\# \cong \langle 2 \text{def}(\alpha) \rangle \circ \circledast$; see e.g. [R, Theorem 3.1]. Given this, one could deduce that F_i commutes with \circledast (because an argument similar to the proof of (2.22) shows already that $F_i K_i \langle -1 \rangle$ commutes with $\#$). The latter statement can be proved indirectly by appealing to the formalism of Remark 4.19 below and [R2, Theorem 5.16].

4.6. Cyclotomic divided powers. Lemma 4.4 also makes it clear how to define divided powers $E_i^{(n)}$ and $F_i^{(n)}$ of the functors E_i and F_i . For $n \geq 1$, set

$$\begin{aligned} E_i^{(n)} &:= \text{pr} \circ (\theta_i^*)^{(n)} \circ \text{infl} : \text{Mod}(R_{\alpha+n\alpha_i}^\Lambda) \rightarrow \text{Mod}(R_\alpha^\Lambda), \\ F_i^{(n)} &:= \text{pr} \circ \theta_i^{(n)} \circ \text{infl} \langle n^2 - n(\Lambda - \alpha, \alpha_i) \rangle : \text{Mod}(R_\alpha^\Lambda) \rightarrow \text{Mod}(R_{\alpha+n\alpha_i}^\Lambda), \end{aligned}$$

recalling (2.24)–(2.25). Again we use the same notation $E_i^{(n)}$ and $F_i^{(n)}$ for the direct sums of these functors over all $\alpha \in Q_+$.

Lemma 4.8. *There are isomorphisms $E_i^n \cong [n]! \cdot E_i^{(n)}$ and $F_i^n \cong [n!] \cdot F_i^{(n)}$. Hence $E_i^{(n)}$ and $F_i^{(n)}$ are exact, and they send finite dimensional (resp. finitely generated projective) modules to finite dimensional (resp. finitely generated projective) modules.*

Proof. This follows by Lemma 4.4, (4.19) and (2.26). \square

4.7. Ungraded irreducible representations and branching rules. It is time to recall Grojnowski's classification [G2] of finite dimensional irreducible H_α^Λ -modules in terms of the crystal associated to the highest weight module $V(\Lambda)$, which was inspired by the modular branching rules of [K1]. We will use the explicit realization $(\mathcal{R}\mathcal{P}^\Lambda, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \text{wt})$ of the crystal from §3.7.

For $\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda$, we define an H_α^Λ -module $\underline{D}(\lambda)$ recursively as follows. To start with, let $\underline{D}(\emptyset)$ denote the trivial representation of $H_0^\Lambda \cong F$. Now suppose that we are given $\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda$ for $\text{ht}(\alpha) > 0$. Choose any $i \in I$ such that $\varepsilon_i(\lambda) \neq 0$ and define

$$\underline{D}(\lambda) := \text{soc}(f_i \underline{D}(\tilde{e}_i \lambda)), \quad (4.23)$$

where $\underline{D}(\tilde{e}_i \lambda)$ is the recursively defined $H_{\alpha-\alpha_i}^\Lambda$ -module. It is known that $\underline{D}(\lambda)$ does not depend up to isomorphism on the particular choice of i . Moreover:

Theorem 4.9 (Grojnowski). *The modules $\{\underline{D}(\lambda) \mid \lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda\}$ give a complete set of pairwise non-isomorphic irreducible H_α^Λ -modules. Moreover, the following hold for any $i \in I$ and $\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda$:*

- (1) $e_i \underline{D}(\lambda)$ is non-zero if and only if $\varepsilon_i(\lambda) \neq 0$, in which case $e_i \underline{D}(\lambda)$ has irreducible socle and head both isomorphic to $\underline{D}(\tilde{e}_i \lambda)$.
- (2) $f_i \underline{D}(\lambda)$ is non-zero if and only if $\varphi_i(\lambda) \neq 0$, in which case $f_i \underline{D}(\lambda)$ has irreducible socle and head both isomorphic to $\underline{D}(\tilde{f}_i \lambda)$.
- (3) In the Grothendieck group we have that

$$\begin{aligned} [e_i \underline{D}(\lambda)] &= \varepsilon_i(\lambda) [\underline{D}(\tilde{e}_i \lambda)] + \sum_{\substack{\mu \in \mathcal{R}\mathcal{P}_{\alpha-\alpha_i}^\Lambda \\ \varepsilon_i(\mu) < \varepsilon_i(\lambda) - 1}} u_{\mu, \lambda; i}(1) [\underline{D}(\mu)], \\ [f_i \underline{D}(\lambda)] &= \varphi_i(\lambda) [\underline{D}(\tilde{f}_i \lambda)] + \sum_{\substack{\mu \in \mathcal{R}\mathcal{P}_{\alpha+\alpha_i}^\Lambda \\ \varphi_i(\mu) < \varphi_i(\lambda) - 1}} v_{\mu, \lambda; i}(1) [\underline{D}(\mu)], \end{aligned}$$

for some coefficients $u_{\mu, \lambda; i}(1), v_{\mu, \lambda; i}(1) \in \mathbb{Z}_{\geq 0}$. (The first term on the right hand side of these formulae should be interpreted as zero if $\varepsilon_i(\lambda) = 0$ (resp. $\varphi_i(\lambda) = 0$).

- (4) There are algebra isomorphisms

$$\begin{aligned} f &: F[x]/(x^{\varepsilon_i(\lambda)}) \xrightarrow{\sim} \underline{\text{End}}_{H_{\alpha-\alpha_i}^\Lambda}(e_i \underline{D}(\lambda)), \\ g &: F[x]/(x^{\varphi_i(\lambda)}) \xrightarrow{\sim} \underline{\text{End}}_{H_{\alpha+\alpha_i}^\Lambda}(f_i \underline{D}(\lambda)). \end{aligned}$$

Proof. See [G2] or [K3] which gives an exposition of Grojnowski's methods in the degenerate case. More precisely, the first statement is [K3, Theorem 10.3.4], then [K3, Theorem 8.3.2] gives (1) and (2), and [K3, Theorems 5.5.1 and 8.5.9] gives (3) and (4). \square

Remark 4.10. The results of Theorem 4.9(1)–(4) are extended to the divided powers $e_i^{(n)}$ and $f_i^{(n)}$ of the functors e_i and f_i in [CR, Proposition 5.20].

4.8. Graded irreducible representations and branching rules. Now we lift the parametrization of irreducible modules from H_α^Λ to R_α^Λ using Theorem 4.1.

Theorem 4.11. *For each $\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda$, there exists an irreducible graded R_α^Λ -module $D(\lambda)$ such that*

- (1) $D(\lambda)^\circledast \cong D(\lambda)$;
- (2) $\underline{D}(\lambda) \cong \underline{D}(\lambda)$ as an H_α^Λ -module.

The module $D(\lambda)$ is determined uniquely up to isomorphism by these two conditions. Moreover, the modules $\{D(\lambda)\langle m \rangle \mid \lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda, m \in \mathbb{Z}\}$ give a complete set of pairwise non-isomorphic irreducible graded R_α^Λ -modules.

Proof. By Lemma 2.3 and Theorem 4.1, there exists an irreducible graded R_α^Λ -module $D(\lambda)$ satisfying (2), but this is only unique up to isomorphism and grading shift. The fact that $D(\lambda)$ can be chosen so that it also satisfies (1) is explained at the end of [KL1, §3.2]. This pins down the choice of grading shift and makes $D(\lambda)$ unique up to isomorphism. The final statement follows from Theorem 4.9 and Lemma 2.2. \square

The following theorem lifts the remaining parts of Theorem 4.9 to the graded setting.

Theorem 4.12. *For any $\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda$ and $i \in I$, we have:*

- (1) $E_i D(\lambda)$ is non-zero if and only if $\varepsilon_i(\lambda) \neq 0$, in which case $E_i D(\lambda)$ has irreducible socle isomorphic to $D(\tilde{e}_i \lambda)\langle \varepsilon_i(\lambda) - 1 \rangle$ and head isomorphic to $D(\tilde{e}_i \lambda)\langle 1 - \varepsilon_i(\lambda) \rangle$.
- (2) $F_i D(\lambda)$ is non-zero if and only if $\varphi_i(\lambda) \neq 0$, in which case $F_i D(\lambda)$ has irreducible socle isomorphic to $D(\tilde{f}_i \lambda)\langle \varphi_i(\lambda) - 1 \rangle$ and head isomorphic to $D(\tilde{f}_i \lambda)\langle 1 - \varphi_i(\lambda) \rangle$.
- (3) In the Grothendieck group we have that

$$[E_i D(\lambda)] = [\varepsilon_i(\lambda)] [D(\tilde{e}_i \lambda)] + \sum_{\substack{\mu \in \mathcal{R}\mathcal{P}_{\alpha - \alpha_i}^\Lambda \\ \varepsilon_i(\mu) < \varepsilon_i(\lambda) - 1}} u_{\mu, \lambda; i}(q) [D(\mu)],$$

$$[F_i D(\lambda)] = [\varphi_i(\lambda)] [D(\tilde{f}_i \lambda)] + \sum_{\substack{\mu \in \mathcal{R}\mathcal{P}_{\alpha + \alpha_i}^\Lambda \\ \varphi_i(\mu) < \varphi_i(\lambda) - 1}} v_{\mu, \lambda; i}(q) [D(\mu)],$$

for some $u_{\mu, \lambda; i}(q), v_{\mu, \lambda; i}(q) \in \mathbb{Z}[q, q^{-1}]$ with non-negative coefficients. (The first term on the right hand side of these formulae should be interpreted as zero if $\varepsilon_i(\lambda) = 0$ (resp. $\varphi_i(\lambda) = 0$).

- (4) Viewing $F[x]$ as a graded algebra by putting x in degree 2, there are graded algebra isomorphisms

$$f : F[x]/(x^{\varepsilon_i(\lambda)}) \xrightarrow{\sim} \text{END}_{R_{\alpha - \alpha_i}^\Lambda}(E_i D(\lambda)),$$

$$g : F[x]/(x^{\varphi_i(\lambda)}) \xrightarrow{\sim} \text{END}_{R_{\alpha + \alpha_i}^\Lambda}(F_i D(\lambda)).$$

Proof. We first consider (4). Let $d := \text{ht}(\alpha)$. By Theorem 4.9 and (2.15), we have an isomorphism $f : F[x]/(x^{\varepsilon_i(\lambda)}) \xrightarrow{\sim} \text{END}_{R_{\alpha - \alpha_i}^\Lambda}(E_i D(\lambda))$, but we do not know yet that this is an isomorphism of graded algebras. For this, we go back into the proof of Theorem 4.9(4) to find that the map f sends x to the endomorphism of $E_i D(\lambda) = e_{\alpha - \alpha_i, \alpha_i} D(\lambda)$ defined by multiplication by y_d (which centralizes elements in the image of $\iota_{\alpha - \alpha_i, \alpha_i}$). Since $\deg(y_d) = 2$ this shows that f is indeed an isomorphism of graded algebras. Similarly the isomorphism $g : F[x]/(x^{\varphi_i(\lambda)}) \xrightarrow{\sim} \text{END}_{R_{\alpha + \alpha_i}^\Lambda}(F_i D(\lambda))$ from Theorem 4.9(4) maps x to the endomorphism of $F_i D(\lambda)$ defined by multiplication by the central

element $y_1 + \cdots + y_{d+1} \in R_{\alpha+\alpha_i}^\Lambda$. So this is an isomorphism of graded algebras too. This completes the proof of (4).

Now consider (1). We know already that $E_i D(\lambda)$ is non-zero if and only if $\varepsilon_i(\lambda) \neq 0$ by Theorem 4.9(1), and moreover, assuming this is the case, the head (resp. socle) of $E_i D(\lambda)$ must be isomorphic to $D(\tilde{e}_i \lambda) \langle m \rangle$ (resp. $D(\tilde{e}_i \lambda) \langle n \rangle$) for some $m, n \in \mathbb{Z}$. Applying (4), we define a filtration

$$\{0\} = M_{\varepsilon_i(\lambda)} \subset \cdots \subset M_1 \subset M_0 = E_i D(\lambda)$$

by setting $M_k := \text{im} f(x)^k$. By the simplicity of the head of $E_i D(\lambda)$, each section M_{k-1}/M_k for $k = 1, \dots, \varepsilon_i(\lambda)$ must have irreducible head isomorphic to $D(\tilde{e}_i \lambda) \langle m + 2k - 2 \rangle$. Since $[e_i \underline{D}(\lambda) : \underline{D}(\tilde{e}_i \lambda)] = \varepsilon_i(\lambda)$ by Theorem 4.9(3), there can be no other composition factors in M_{k-1}/M_k that are isomorphic to $\underline{D}(\tilde{e}_i \lambda)$ on forgetting the grading. This argument shows that

$$[E_i D(\lambda) : D(\tilde{e}_i \lambda)] = q^{m+\varepsilon_i(\lambda)-1} [\varepsilon_i(\lambda)].$$

Moreover, the submodule $M_{\varepsilon_i(\lambda)-1}$ at the bottom of our filtration has irreducible socle isomorphic to $D(\tilde{e}_i \lambda) \langle n \rangle$, so in fact $n = m + 2\varepsilon_i(\lambda) - 2$ and $M_{\varepsilon_i(\lambda)-1}$ is irreducible. In view of the first part of Theorem 4.9(3), to complete the proof of (1) and the first part of (3), it remains to show that $m = 1 - \varepsilon_i(\lambda)$. But $D(\lambda)$ is self-dual and E_i commutes with duality by Lemma 4.5, hence $E_i D(\lambda)$ is self-dual too. So the q -multiplicity $q^{m+\varepsilon_i(\lambda)-1} [\varepsilon_i(\lambda)]$ computed above must be bar-invariant. This implies that $m = 1 - \varepsilon_i(\lambda)$ as required.

Finally consider (2) and the second part of (3). Assume $\varphi_i(\lambda) \neq 0$. Entirely similar argument to the previous paragraph shows that $F_i D(\lambda)$ has irreducible head $D(\tilde{f}_i \lambda) \langle m \rangle$, socle $D(\tilde{f}_i \lambda) \langle m + 2\varphi_i(\lambda) - 2 \rangle$, and

$$[F_i D(\lambda) : D(\tilde{f}_i \lambda)]_q = q^{m+1-\varphi_i(\lambda)} [\varphi_i(\lambda)],$$

for some $m \in \mathbb{Z}$. To complete the proof of the theorem, we need to show that $m = 1 - \varphi_i(\lambda)$. This time we do not know that F_i commutes with duality, so we must give a different argument than before. By (1), we have that $E_i D(\tilde{f}_i \lambda) \langle -\varepsilon(\lambda) \rangle$ has irreducible socle isomorphic to $D(\lambda)$. Hence using Lemma 4.3, we get that

$$\begin{aligned} \text{HOM}_{R_\alpha^\Lambda}(D(\lambda), D(\lambda)) &\cong \text{HOM}_{R_\alpha^\Lambda}(D(\lambda), E_i D(\tilde{f}_i \lambda) \langle -\varepsilon_i(\lambda) \rangle) \\ &\cong \text{HOM}_{R_{\alpha+\alpha_i}^\Lambda}(F_i D(\lambda) \langle d_i(\lambda) - 1 \rangle, D(\tilde{f}_i \lambda) \langle -\varepsilon_i(\lambda) \rangle) \\ &\cong \text{HOM}_{R_{\alpha+\alpha_i}^\Lambda}(D(\tilde{f}_i \lambda) \langle m + d_i(\lambda) - 1 \rangle, D(\tilde{f}_i \lambda) \langle -\varepsilon_i(\lambda) \rangle). \end{aligned}$$

We deduce by Schur's lemma that $m + d_i(\lambda) - 1 = -\varepsilon_i(\lambda)$. An application of (3.26) completes the proof. \square

Remark 4.13. It is also known that the polynomials $u_{\mu, \lambda; i}(q)$ and $v_{\mu, \lambda; i}(q)$ in Theorem 4.12(3) are bar-invariant. This follows because $D(\lambda)^\otimes \cong D(\lambda)$ and the linear endomorphisms of the Grothendieck group induced by the functors E_i and F_i commute with \otimes ; the latter statement is a consequence of Theorem 4.18 below and the bar-invariance of $E_i, F_i \in U_q(\mathfrak{g})$.

4.9. Mixed relations. The next goal is to check that the “mixed relation” from (3.5) holds on the Grothendieck group. We do this by appealing to some general results of Chuang and Rouquier from [CR] and Rouquier from [R2]. We remark that in a previous version of this article, we established the required relations in Corollary 4.15 (but *not* the stronger Theorem 4.14) in a more elementary way, using instead Theorem 4.12 and a graded version of an argument of Grojnowski; see e.g. [K3, Lemma 9.4.3].

To start with, following the formalism of [R2] (but inevitably using somewhat different notation, in particular, we have switched the roles of E_i and F_i), we need to define endomorphisms of functors

$$y : F_i \rightarrow F_i, \quad \psi : F_i F_j \rightarrow F_j F_i \quad (4.24)$$

for all $i, j \in I$. It suffices by additivity to define natural homomorphisms $y_M : F_i M \rightarrow F_i M$ and $\psi_M : F_i F_j(M) \rightarrow F_j F_i(M)$ for each $M \in \text{Rep}(R_\alpha^\Lambda)$ and $\alpha \in Q_+$ of height d . For this, y_M is the homogeneous endomorphism of degree 2 defined by right multiplication by $y_{d+1} \in R_{\alpha+\alpha_i}^\Lambda$, and ψ_M is the homogeneous endomorphism of degree $-a_{i,j}$ defined by right multiplication by $\psi_{d+1} \in R_{\alpha+\alpha_i+\alpha_j}^\Lambda$. See [CR, §7.2] for similar definitions in the ungraded setting.

Now, given $i \in I$, let us denote the unit and the counit arising from the adjunction from Lemma 4.3 by $\eta : \text{Id} \rightarrow E_i F_i$ and $\epsilon : F_i E_i \rightarrow \text{Id}$, respectively. On modules from $\text{Rep}(R_\alpha^\Lambda)$, these natural transformations define maps that are homogeneous of degrees $1 - (\Lambda - \alpha, \alpha_i)$ and $1 + (\Lambda - \alpha, \alpha_i)$, respectively. Note also that the natural transformations

$$\mathbf{1}(y^n) \circ \eta : \text{Id} \rightarrow E_i F_i, \quad \epsilon \circ (y^n) \mathbf{1} : F_i E_i \rightarrow \text{Id}$$

define homogeneous maps of degrees $2n + 1 - (\Lambda - \alpha, \alpha_i)$ and $2n + 1 + (\Lambda - \alpha, \alpha_i)$, respectively, on modules from $\text{Rep}(R_\alpha^\Lambda)$. Finally given also $j \in I$, define

$$\sigma : F_j E_i \xrightarrow{\eta \mathbf{1} \mathbf{1}} E_i F_i F_j E_i \xrightarrow{\mathbf{1} \psi \mathbf{1}} E_i F_j F_i E_i \xrightarrow{\mathbf{1} \mathbf{1} \epsilon} E_i F_j,$$

which yields a map that is homogeneous of degree zero on every module; cf. [R2, §4.1.3].

Theorem 4.14 ([CR, R2]). *Suppose that $\alpha \in Q_+$, $i, j \in I$, and set $a := (\Lambda - \alpha, \alpha_i)$. Let $M \in \text{Rep}(R_\alpha^\Lambda)$.*

- (1) *If $i = j$ and $a \geq 0$ the natural transformation $\sigma + \sum_{n=0}^{a-1} \mathbf{1}(y^n) \circ \eta$ defines an isomorphism of graded modules*

$$F_j E_i(M) \oplus \bigoplus_{n=0}^{a-1} M \langle 2n + 1 - a \rangle \xrightarrow{\sim} E_i F_j(M).$$

- (2) *If $i = j$ and $a \leq 0$ the natural transformation $\sigma + \sum_{n=0}^{-a-1} \epsilon \circ (y^n) \mathbf{1}$ defines an isomorphism of graded modules*

$$F_j E_i(M) \xrightarrow{\sim} E_i F_j(M) \oplus \bigoplus_{n=0}^{-a-1} M \langle -2n - 1 - a \rangle.$$

- (3) *If $i \neq j$ the natural transformation σ defines an isomorphism of graded modules*

$$F_j E_i(M) \xrightarrow{\sim} E_i F_j(M).$$

Proof. Since the maps in all cases are homogeneous of degree zero, it suffices to prove the theorem on forgetting the gradings everywhere. For (1) and (2), using Theorem 4.1, we reduce to establishing the analogous isomorphism for H_α^Λ , which is proved in [CR, Theorem 5.27] (using also [CR, §7.2] which verifies that the axioms of \mathfrak{sl}_2 -categorification are satisfied in that setting). Once (1) and (2) have been established, (3) follows by [R2, §5.3.5]. \square

Corollary 4.15. *For all $i, j \in I$ and $\alpha \in Q_+$, we have that*

$$[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

as endomorphisms of $[\text{Rep}(R_\alpha^\Lambda)]$.

4.10. The graded categorification theorem. Like in (2.27), let us abbreviate

$$[\text{Proj}(R^\Lambda)] := \bigoplus_{\alpha \in Q_+} [\text{Proj}(R_\alpha^\Lambda)], \quad [\text{Rep}(R^\Lambda)] := \bigoplus_{\alpha \in Q_+} [\text{Rep}(R_\alpha^\Lambda)]. \quad (4.25)$$

The exact functors $E_i^{(n)}, F_i^{(n)}$ and K_i induce \mathcal{A} -linear endomorphisms of the Grothendieck groups $[\text{Proj}(R^\Lambda)]$ and $[\text{Rep}(R^\Lambda)]$. In view of Theorem 4.11, $[\text{Rep}(R^\Lambda)]$ is a free \mathcal{A} -module on basis $\{[D(\lambda)] \mid \lambda \in \mathcal{RP}^\Lambda\}$. Also let $Y(\lambda)$ denote the projective cover of $D(\lambda)$ in $\text{Rep}(R_\alpha^\Lambda)$, for each $\lambda \in \mathcal{RP}^\Lambda$. Thus there is a degree-preserving surjection

$$Y(\lambda) \twoheadrightarrow D(\lambda).$$

The classes $\{[Y(\lambda)] \mid \lambda \in \mathcal{RP}^\Lambda\}$ give a basis for $[\text{Proj}(R^\Lambda)]$ as a free \mathcal{A} -module. Note by Corollary 2.6 and (4.22) that

$$Y(\lambda)^\# \cong Y(\lambda). \quad (4.26)$$

Comparing the relations of $U_q(\mathfrak{g})$ and \mathbf{f} , it follows easily that there is an algebra anti-homomorphism

$$\mathbf{f} \rightarrow U_q(\mathfrak{g}), \quad x \mapsto x^b \quad (4.27)$$

such that $\theta_i^b := q^{-1}F_iK_i$ (which is the same thing as $\tau^{-1}(E_i)$ according to (3.9)).

Proposition 4.16. *There is a unique \mathcal{A} -module isomorphism δ making the following diagram commute*

$$\begin{array}{ccc} \mathcal{A}\mathbf{f} & \xrightarrow[\gamma]{\sim} & [\text{Proj}(R)] \\ \beta \downarrow & & \downarrow \text{pr} \\ V(\Lambda)_{\mathcal{A}} & \xrightarrow[\delta]{\sim} & [\text{Proj}(R^\Lambda)], \end{array}$$

where β denotes the surjection $x \mapsto x^b v_\Lambda$, γ is the isomorphism from Theorem 2.7, and pr is the \mathcal{A} -linear map induced by the additive functor pr from (4.20). Moreover, δ intertwines the left action of $F_i^{(n)} \in U_q(\mathfrak{g})_{\mathcal{A}}$ on $V(\Lambda)_{\mathcal{A}}$ with the endomorphism of $[\text{Proj}(R^\Lambda)]$ induced by the divided power functor $F_i^{(n)}$, for every $i \in I$ and $n \geq 1$. It obviously intertwines the K_i 's too.

Proof. We first show there exists an \mathcal{A} -module homomorphism δ making the diagram commute. Remembering that the map β involves a twist by the anti-homomorphism $x \mapsto x^b$, the well-known description of $V(\Lambda)$ by generators and relations implies that $\ker \beta$ is generated as a right ideal by the elements $\{\theta_i^{((\Lambda, \alpha_i)+1)} \mid i \in I\}$. Therefore it suffices to check that $\ker \text{pr}$ is a right ideal of the algebra $[\text{Proj}(R)]$ and that $\text{pr}(\gamma(\theta_i^{((\Lambda, \alpha_i)+1)})) = 0$ for each i . To show that $\ker \text{pr}$ is a right ideal, take $P \in \text{Proj}(R_\alpha)$ and $Q \in \text{Proj}(R_\beta)$ such that $\text{pr} P = \{0\}$. We need to show that $\text{pr} \text{Ind}_{\alpha, \beta}^{\alpha+\beta}(P \boxtimes Q) = \{0\}$, or equivalently, that $\text{HOM}_{R_{\alpha+\beta}}(\text{Ind}_{\alpha, \beta}^{\alpha+\beta}(P \boxtimes Q), L) = \{0\}$ for every $R_{\alpha+\beta}^\Lambda$ -module L . This follows from the isomorphism

$$\text{HOM}_{R_{\alpha+\beta}}(\text{Ind}_{\alpha, \beta}^{\alpha+\beta}(P \boxtimes Q), L) \cong \text{HOM}_{R_{\alpha, \beta}}(P \boxtimes Q, \text{Res}_{\alpha, \beta}^{\alpha+\beta} L).$$

To show that $\text{pr}(\gamma(\theta_i^{((\Lambda, \alpha_i)+1)})) = 0$, we show equivalently that

$$\text{pr} \theta_i^{(\Lambda, \alpha_i)+1} \text{infl}(D(\emptyset)) = \{0\}.$$

In view of Lemma 4.4, this follows if we can show that $F_i^{(\Lambda, \alpha_i)+1} D(\emptyset) = \{0\}$. This holds thanks to Theorem 4.12(3) since $\varphi_i(\emptyset) = (\Lambda, \alpha_i)$.

The map pr is obviously surjective, and γ is an isomorphism, hence we get that δ is surjective. It sends the $(\Lambda - \alpha)$ -weight space of $V(\Lambda)_{\mathcal{A}}$ onto $[\text{Proj}(R_\alpha^\Lambda)]$. It is an isomorphism because $[\text{Proj}(R_\alpha^\Lambda)]$ is a free \mathcal{A} -module of rank $|\mathcal{R} \mathcal{P}_\alpha^\Lambda|$ thanks to Theorem 4.11, which is the same as the rank of the $(\Lambda - \alpha)$ -weight space of $V(\Lambda)_{\mathcal{A}}$.

The fact that δ intertwines the $F_i^{(n)}$'s follows from the definitions. \square

Recall the Cartan pairing $\langle \cdot, \cdot \rangle : [\text{Proj}(R^\Lambda)] \times [\text{Rep}(R^\Lambda)] \rightarrow \mathcal{A}$ from §2.4 and the Shapovalov pairing $\langle \cdot, \cdot \rangle : V(\Lambda)_{\mathcal{A}} \times V(\Lambda)_{\mathcal{A}}^* \rightarrow \mathcal{A}$ from §3.3. Let

$$\varepsilon : [\text{Rep}(R^\Lambda)] \xrightarrow{\sim} V(\Lambda)_{\mathcal{A}}^* \tag{4.28}$$

be the dual map to the isomorphism δ of Proposition 4.16 with respect to these pairings. Thus, ε is the \mathcal{A} -module isomorphism defined by the equation

$$\langle \delta(x), y \rangle = \langle x, \varepsilon(y) \rangle \tag{4.29}$$

for all $x \in V(\Lambda)_{\mathcal{A}}^*$ and $y \in [\text{Rep}(R^\Lambda)]$.

Also let

$$\circledast : [\text{Rep}(R^\Lambda)] \rightarrow [\text{Rep}(R^\Lambda)] \tag{4.30}$$

be the anti-linear involution induced by the duality \circledast .

Proposition 4.17. *The isomorphism ε from (4.28) intertwines the endomorphism of $[\text{Rep}(R^\Lambda)]$ induced by the divided power functor $E_i^{(n)}$ with the left action of $E_i^{(n)} \in U_q(\mathfrak{g})_{\mathcal{A}}$ on $V(\Lambda)_{\mathcal{A}}^*$, for every $i \in I$ and $n \geq 1$. It obviously intertwines the K_i 's too. Finally, it intertwines the anti-linear involution \circledast with the bar-involution on $V(\Lambda)_{\mathcal{A}}^*$.*

Proof. For the first statement, it suffices by Lemma 4.8 to show that $E_i \circ \varepsilon = \varepsilon \circ E_i$. This is immediate from Proposition 4.16 and the defining property (1) of the Shapovalov form from §3.3, because the functor E_i is right adjoint to $F_i K_i \langle -1 \rangle$ by Lemma 4.3, while $\tau^{-1}(E_i) = q^{-1} F_i K_i$ according to (3.9).

For the last statement, we show that $\varepsilon(v^{\otimes}) - \overline{\varepsilon(v)} = 0$ for each $v \in [\text{Rep}(R_\alpha^\Lambda)]$ by induction on height. This is clear in the case $\alpha = 0$. Now take $\alpha \in Q_+$ with $\text{ht}(\alpha) > 0$. Since E_i commutes with ε , with the bar-involution, and with the duality \otimes by Lemma 4.5, we get from the induction hypothesis that $E_i(\varepsilon(v^{\otimes}) - \overline{\varepsilon(v)}) = 0$ for every $i \in I$. Hence, $\varepsilon(v^{\otimes}) - \overline{\varepsilon(v)}$ is a highest weight vector of weight different from Λ , so it is zero. \square

Now we can prove the following fundamental theorem which makes precise a sense in which $\text{Proj}(R^\Lambda)$ categorifies the $U_q(\mathfrak{g})_{\mathcal{A}}$ -module $V(\Lambda)_{\mathcal{A}}$ and $\text{Rep}(R^\Lambda)$ categorifies $V(\Lambda)_{\mathcal{A}}^*$.

Theorem 4.18. *The following diagram commutes:*

$$\begin{array}{ccc} V(\Lambda)_{\mathcal{A}} & \xrightarrow[\delta]{\sim} & [\text{Proj}(R^\Lambda)] \\ a \downarrow & & \downarrow b \\ V(\Lambda)_{\mathcal{A}}^* & \xleftarrow[\varepsilon]{\sim} & [\text{Rep}(R^\Lambda)], \end{array}$$

where $a : V(\Lambda)_{\mathcal{A}} \hookrightarrow V(\Lambda)_{\mathcal{A}}^*$ is the canonical inclusion, and $b : [\text{Proj}(R^\Lambda)] \rightarrow [\text{Rep}(R^\Lambda)]$ is the \mathcal{A} -linear map induced by the natural inclusion of $\text{Proj}(R_\alpha^\Lambda)$ into $\text{Rep}(R_\alpha^\Lambda)$ for each $\alpha \in Q_+$. Hence:

- (1) b is injective and becomes an isomorphism over $\mathbb{Q}(q)$;
- (2) both maps δ and ε commute with the actions of $E_i^{(n)}, F_i^{(n)}$ and K_i ;
- (3) both maps δ and ε intertwine the involution \otimes coming from duality with the bar-involution;
- (4) the isomorphism δ identifies the Shapovalov form on $V(\Lambda)_{\mathcal{A}}$ with Cartan form on $[\text{Proj}(R^\Lambda)]$.

Proof. Everything in sight is a free \mathcal{A} -module, so it does no harm to extend scalars from \mathcal{A} to $\mathbb{Q}(q)$. Denote the resulting $\mathbb{Q}(q)$ -linear maps by $\hat{a}, \hat{b}, \hat{\delta}$ and $\hat{\varepsilon}$. Actually, we may as well identify $\mathbb{Q}(q) \otimes_{\mathcal{A}} V(\Lambda)_{\mathcal{A}}$ and $\mathbb{Q}(q) \otimes_{\mathcal{A}} V(\Lambda)_{\mathcal{A}}^*$ both with $V(\Lambda)$, so that \hat{a} is just the identity map, and then we need to show the following diagram commutes:

$$\begin{array}{ccc} V(\Lambda) & \xrightarrow[\hat{\delta}]{\sim} & \mathbb{Q}(q) \otimes_{\mathcal{A}} [\text{Proj}(R^\Lambda)] \\ \parallel & & \downarrow \hat{b} \\ V(\Lambda) & \xleftarrow[\hat{\varepsilon}]{\sim} & \mathbb{Q}(q) \otimes_{\mathcal{A}} [\text{Rep}(R^\Lambda)]. \end{array}$$

Note also that \hat{b} obviously commutes with $E_i^{(n)}, F_i^{(n)}, K_i$ and \otimes , $\hat{\delta}$ commutes with $F_i^{(n)}$ and K_i by Proposition 4.16, and $\hat{\varepsilon}$ commutes with $E_i^{(n)}, K_i$ and \otimes by Proposition 4.17. Hence (1), (2) and (3) all follow easily from the commutativity of this diagram. Also once the commutativity is established, (4) follows immediately by (4.29).

To prove that the above diagram commutes, we show by induction that it commutes on restriction to the $(\Lambda - \alpha)$ -weight spaces for each $\alpha \in Q_+$. The diagram obviously commutes on restriction to the highest weight space, so assume now that $\alpha > 0$ and that we have already established the commutativity

on restriction to the $(\Lambda - \beta)$ -weight spaces for all $0 \leq \beta < \alpha$. It suffices to show that $\hat{\varepsilon}\hat{b}\hat{\delta}(F_j w) = F_j w$ for all $j \in I$ and w in the $(\Lambda - \alpha + \alpha_j)$ -weight space of $V(\Lambda)$. This follows if we can check that

$$\langle F_i v, \hat{\varepsilon}\hat{b}\hat{\delta}(F_j w) \rangle = \langle F_i v, F_j w \rangle \quad (4.31)$$

for all $i \in I$ and v in the $(\Lambda - \alpha + \alpha_i)$ -weight space of $V(\Lambda)$. For this we compute using the defining property of the Shapovalov form, Propositions 4.16 and 4.17, and Corollary 4.15:

$$\begin{aligned} \langle F_i v, \hat{\varepsilon}\hat{b}\hat{\delta}(F_j w) \rangle &= \langle v, q^{-1} K_i E_i \hat{\varepsilon}\hat{b}\hat{\delta}(F_j w) \rangle = \langle v, q^{-1} K_i \hat{\varepsilon} E_i F_j \hat{b}\hat{\delta}(w) \rangle \\ &= \left\langle v, q^{-1} K_i \hat{\varepsilon} \left(F_j E_i + \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}} \right) \hat{b}\hat{\delta}(w) \right\rangle. \end{aligned}$$

By the inductive hypothesis, we know already that our diagram commutes on the $(\Lambda - \alpha + \alpha_i)$ - and $(\Lambda - \alpha + \alpha_i + \alpha_j)$ -weight spaces, hence Proposition 4.16 allows us to commute the $\hat{\varepsilon}$ and the F_j past each other, to get that

$$\begin{aligned} \langle F_i v, \hat{\varepsilon}\hat{b}\hat{\delta}(F_j w) \rangle &= \left\langle v, q^{-1} K_i \left(F_j E_i + \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}} \right) \hat{\varepsilon}\hat{b}\hat{\delta}(w) \right\rangle \\ &= \langle v, q^{-1} K_i E_i F_j \hat{\varepsilon}\hat{b}\hat{\delta}(w) \rangle = \langle F_i v, F_j \hat{\varepsilon}\hat{b}\hat{\delta}(w) \rangle. \end{aligned}$$

Finally by the inductive hypothesis, we know already that $\hat{\varepsilon}\hat{b}\hat{\delta}(w) = w$, so this completes the proof of (4.31). \square

Remark 4.19. In view of Theorem 4.14, Theorem 4.18 can also be formulated as an example of a 2-representation of the 2-Kac-Moody algebra $\mathfrak{A}(\mathfrak{g})$ in the sense of Rouquier [R2]. The required data as specified in [R2, Definition 5.1.1] comes from the categories $\text{Rep}(R_\alpha^\Lambda)$ for all $\alpha \in Q_+$, together with the functors F_i and E_i from (4.13)–(4.14), the adjunction from Lemma 4.3, and the endomorphisms (4.24).

4.11. A graded dimension formula. As the first application of Theorem 4.18, we can derive a combinatorial formula for the graded dimension of R_α^Λ .

Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \mathcal{P}^\Lambda$ be an l -multipartition, and set $d := |\lambda|$. A *standard λ -tableau* $\mathbf{T} = (\mathbf{T}^{(1)}, \dots, \mathbf{T}^{(l)})$ is obtained from the diagram of λ by inserting the integers $1, \dots, d$ into the nodes, allowing no repeats, so that the entries in each individual $\mathbf{T}^{(m)}$ are strictly increasing along rows from left to right and down columns from top to bottom. The set of all standard λ -tableaux will be denoted by $\mathcal{T}(\lambda)$.

To each $\mathbf{T} \in \mathcal{T}(\lambda)$ we associate its *residue sequence*

$$\mathbf{i}^{\mathbf{T}} = (i_1, \dots, i_d) \in I^d, \quad (4.32)$$

where $i_r \in I$ is the residue of the node occupied by r in \mathbf{T} ($1 \leq r \leq d$) in the sense of (3.13) (reduced modulo e). Recalling (3.20), define the *degree* of \mathbf{T} inductively from

$$\deg(\mathbf{T}) := \begin{cases} \deg(\mathbf{T}_{\leq (d-1)}) + d_A(\lambda) & \text{if } d > 0, \\ 0 & \text{if } d = 0, \end{cases} \quad (4.33)$$

where for $d > 0$ we let A denote the node of \mathbf{T} containing entry d , and $\mathbf{T}_{\leq(d-1)}$ denotes the tableau obtained from \mathbf{T} by removing this entry.

Theorem 4.20. *For $\alpha \in Q_+$ and $\mathbf{i}, \mathbf{j} \in I^\alpha$, we have that*

$$\mathrm{qdim} e(\mathbf{i})R_\alpha^\Lambda e(\mathbf{j}) = \sum_{\substack{\lambda \in \mathcal{P}^\Lambda \\ \mathbf{s}, \mathbf{T} \in \mathcal{T}(\lambda) \\ \mathbf{i}^\mathbf{s} = \mathbf{i}, \mathbf{i}^\mathbf{T} = \mathbf{j}}} q^{2\mathrm{def}(\alpha) - \mathrm{deg}(\mathbf{S}) - \mathrm{deg}(\mathbf{T})} = \sum_{\substack{\lambda \in \mathcal{P}^\Lambda \\ \mathbf{s}, \mathbf{T} \in \mathcal{T}(\lambda) \\ \mathbf{i}^\mathbf{s} = \mathbf{i}, \mathbf{i}^\mathbf{T} = \mathbf{j}}} q^{\mathrm{deg}(\mathbf{S}) + \mathrm{deg}(\mathbf{T})}.$$

Proof. Given $\mathbf{i} \in I^\alpha$, let $F_{\mathbf{i}} := F_{i_d} \cdots F_{i_1}$ for short. The definition (4.14) implies easily that $R_\alpha^\Lambda e(\mathbf{i}) \cong F_{\mathbf{i}} R_0^\Lambda \langle \mathrm{def}(\alpha) \rangle$ as graded left R_α^Λ -modules. So

$$\begin{aligned} \mathrm{qdim} e(\mathbf{i})R_\alpha^\Lambda e(\mathbf{j}) &= \mathrm{qdim} \mathrm{HOM}_{R_\alpha^\Lambda}(R_\alpha^\Lambda e(\mathbf{i}), R_\alpha^\Lambda e(\mathbf{j})) \\ &= \mathrm{qdim} \mathrm{HOM}_{R_\alpha^\Lambda}(F_{\mathbf{i}} R_0^\Lambda, F_{\mathbf{j}} R_0^\Lambda) = \langle F_{\mathbf{i}} R_0^\Lambda, F_{\mathbf{j}} R_0^\Lambda \rangle. \end{aligned}$$

Invoking Theorem 4.18 (especially part (4)), we deduce that

$$\mathrm{qdim} e(\mathbf{i})R_\alpha^\Lambda e(\mathbf{j}) = \langle F_{\mathbf{i}} v_\Lambda, F_{\mathbf{j}} v_\Lambda \rangle.$$

We now proceed to compute this by working in terms of the monomial basis of the Fock space $F(\Lambda)$ from §3.6. In particular, we will exploit the sesquilinear form $\langle \cdot, \cdot \rangle$ on $F(\Lambda)$ from (3.39).

By considerations involving (3.57), we that

$$F_{\mathbf{j}} v_\Lambda = \sum_{\mu \in \mathcal{P}^\Lambda, \mathbf{T} \in \mathcal{T}(\mu), \mathbf{i}^\mathbf{T} = \mathbf{j}} q^{-\mathrm{codeg}(\mathbf{T})} M_\mu,$$

where $\mathrm{codeg}(\mathbf{T})$ is defined inductively by

$$\mathrm{codeg}(\mathbf{T}) := \begin{cases} \mathrm{codeg}(\mathbf{T}_{\leq(d-1)}) + d^A(\lambda_A) & \text{if } d > 0, \\ 0 & \text{if } d = 0, \end{cases} \quad (4.34)$$

adopting the same notations as in (4.33). By [BKW, Lemma 3.12], we have that $-\mathrm{codeg}(\mathbf{T}) = \mathrm{deg}(\mathbf{T}) - \mathrm{def}(\alpha)$. Also $F_{\mathbf{j}} v_\Lambda$ is bar-invariant. Putting these things together and simplifying, we get that

$$\mathrm{qdim} e(\mathbf{i})R_\alpha^\Lambda e(\mathbf{j}) = \sum_{\substack{\lambda \in \mathcal{P}^\Lambda, \mathbf{s} \in \mathcal{T}(\lambda), \mathbf{i}^\mathbf{s} = \mathbf{i} \\ \mu \in \mathcal{P}^\Lambda, \mathbf{T} \in \mathcal{T}(\mu), \mathbf{i}^\mathbf{T} = \mathbf{j}}} q^{\mathrm{def}(\alpha) - \mathrm{deg}(\mathbf{S})} q^{\mathrm{def}(\alpha) - \mathrm{deg}(\mathbf{T})} \langle M_\lambda, \overline{M_\mu} \rangle.$$

In view of (3.39) this gives the first expression for $\mathrm{qdim} e(\mathbf{i})R_\alpha^\Lambda e(\mathbf{j})$ from the statement of the theorem.

Finally, we note by Lemma 3.1(3) that

$$\mathrm{qdim} e(\mathbf{i})R_\alpha^\Lambda e(\mathbf{j}) = \langle F_{\mathbf{i}} v_\Lambda, F_{\mathbf{j}} v_\Lambda \rangle = q^{2\mathrm{def}(\alpha)} \overline{\langle F_{\mathbf{j}} v_\Lambda, F_{\mathbf{i}} v_\Lambda \rangle}.$$

Then we compute the right hand side of this by similar substitutions to the previous paragraph. This gives the second expression from the statement of the theorem. \square

4.12. Extremal sequences. For later use, we recall here an elementary but useful observation from [BK1] which generalizes almost at once to the present graded setting. Given $\mathbf{i} = (i_1, \dots, i_d) \in I^d$ we can gather consecutive equal entries together to write it in the form

$$\mathbf{i} = (j_1^{m_1} \dots j_n^{m_n}) \quad (4.35)$$

where $j_r \neq j_{r+1}$ for all $1 \leq r < n$. For example $(2, 2, 2, 1, 1, 2) = (2^3 1^2 2)$.

Now take $\alpha \in Q_+$ with $\text{ht}(\alpha) = d$. Given a non-zero $M \in \text{Rep}(R_\alpha^\Lambda)$ and $\mathbf{i} \in I$, we let

$$\varepsilon_{\mathbf{i}}(M) := \max\{k \geq 0 \mid E_{\mathbf{i}}^k(M) \neq \{0\}\}. \quad (4.36)$$

For example, $\varepsilon_{\mathbf{i}}(D(\lambda)) = \varepsilon_{\mathbf{i}}(\lambda)$ by Theorem 4.12(3). We say that a sequence \mathbf{i} of the form (4.35) is an *extremal sequence* for M if $m_r = \varepsilon_{j_r}(E_{j_r}^{m_{r+1}} \dots E_{j_n}^{m_n} M)$ for all $r = n, n-1, \dots, 1$. Informally speaking this means that among all $\mathbf{i} \in I^\alpha$ such that $e(\mathbf{i})M \neq \{0\}$, we first choose those with the longest j_n -string at the end, then among these we choose the ones with the longest j_{n-1} -string preceding the j_n -string at the end, and so on. It is obvious that if \mathbf{i} is an extremal sequence for M , then $e(\mathbf{i})M \neq \{0\}$.

Lemma 4.21 ([BK1, Corollary 2.17]). *If $\mathbf{i} = (i_1, \dots, i_d)$ is an extremal sequence for $M \in \text{Rep}(R_\alpha^\Lambda)$ of the form (4.35), then $\lambda := f_{i_d} \dots f_{i_1} \emptyset$ is a well-defined element of $\mathcal{RP}_\alpha^\Lambda$, and*

$$[M : D(\lambda)]_q = (\text{qdim } e(\mathbf{i})M) / ([m_1]! \dots [m_n]!).$$

5. GRADED SPECHT MODULES AND DECOMPOSITION NUMBERS

We continue with notation as in the previous section, so F is any algebraically closed field, $\xi \in F^\times$ is of “quantum characteristic” e as defined at the start of the previous section, and Λ is fixed according to (3.1). To this data and every $\alpha \in Q_+$ we have associated a block H_α^Λ of a cyclotomic Hecke algebra with parameter $\xi \in F^\times$ (degenerate if $\xi = 1$), which is isomorphic to the algebra R_α^Λ according to Theorem 4.1.

5.1. Input from geometric representation theory. Let us specialize the setup of §3.3 at $q = 1$, setting

$$V(\Lambda)_{\mathbb{Z}} := \mathbb{Z} \otimes_{\mathcal{A}} V(\Lambda)_{\mathcal{A}}, \quad V(\Lambda)_{\mathbb{Z}}^* := \mathbb{Z} \otimes_{\mathcal{A}} V(\Lambda)_{\mathcal{A}}^*, \quad (5.1)$$

where we view \mathbb{Z} as an \mathcal{A} -module so that q acts as 1. Recalling that $\mathfrak{g} = \widehat{\mathfrak{sl}}_e(\mathbb{C})$ if $e > 0$ or $\mathfrak{sl}_\infty(\mathbb{C})$ if $e = 0$, let $U(\mathfrak{g})_{\mathbb{Z}}$ denote the Kostant \mathbb{Z} -form for the universal enveloping algebra of \mathfrak{g} , generated by the usual divided powers $e_i^{(n)}$ and $f_i^{(n)}$ in its Chevalley generators. This acts on $V(\Lambda)_{\mathbb{Z}}$ and $V(\Lambda)_{\mathbb{Z}}^*$ so that $e_i^{(n)}$ and $f_i^{(n)}$ act as $1 \otimes E_i^{(n)}$ and $1 \otimes F_i^{(n)}$, respectively. In other words, $V(\Lambda)_{\mathbb{Z}}$ is the standard \mathbb{Z} -form for the irreducible highest weight module for \mathfrak{g} of highest weight Λ , and $V(\Lambda)_{\mathbb{Z}}^*$ is the dual lattice under the usual contravariant form (\cdot, \cdot) (which at $q = 1$ coincides with the Shapovalov form).

Paralleling (4.25) in the ungraded setting, we set

$$[\text{Proj}(H^\Lambda)] := \bigoplus_{\alpha \in Q_+} [\text{Proj}(H_\alpha^\Lambda)], \quad [\text{Rep}(H^\Lambda)] := \bigoplus_{\alpha \in Q_+} [\text{Rep}(H_\alpha^\Lambda)]. \quad (5.2)$$

The exact functors e_i and f_i from (4.11)–(4.12) induce \mathbb{Z} -linear endomorphisms of these spaces. Also we have the Cartan pairing

$$(\cdot, \cdot) : [\mathbf{Proj}(H^\Lambda)] \times [\mathbf{Rep}(H^\Lambda)] \rightarrow \mathbb{Z}, \quad ([P], [M]) := \dim \mathbf{Hom}_{H^\Lambda}(P, M)$$

for $\alpha \in Q_+$, $P \in \mathbf{Proj}(H^\alpha)$ and $M \in \mathbf{Rep}(H^\alpha)$.

If we forget the grading in Theorem 4.18, we deduce that there is a commuting square

$$\begin{array}{ccc} V(\Lambda)_{\mathbb{Z}} & \xrightarrow[\underline{\delta}]{\sim} & [\mathbf{Proj}(H^\Lambda)] \\ \underline{a} \downarrow & & \downarrow \underline{b} \\ V(\Lambda)_{\mathbb{Z}}^* & \xleftarrow[\underline{\varepsilon}]{\sim} & [\mathbf{Rep}(H^\Lambda)], \end{array} \quad (5.3)$$

where $\underline{a} : V(\Lambda)_{\mathbb{Z}} \hookrightarrow V(\Lambda)_{\mathbb{Z}}^*$ is the canonical inclusion, $\underline{b} : [\mathbf{Proj}(H^\Lambda)] \hookrightarrow [\mathbf{Rep}(H^\Lambda)]$ is induced by the natural inclusion of categories, $\underline{\delta}$ is the unique \mathbb{Z} -module isomorphism that sends the highest weight vector $v_\Lambda \in V(\Lambda)_{\mathbb{Z}}$ to the isomorphism class of the trivial H_0^Λ -module and commutes with the f_i 's, and finally $\underline{\varepsilon}$ is the dual map to $\underline{\delta}$ with respect to the pairings (\cdot, \cdot) .

The following is a deep result underlying almost all subsequent work in this paper. It was proved by Ariki [A1] as a consequence of the geometric representation theory of quantum algebras and affine Hecke algebras developed by Kazhdan, Lusztig and Ginzburg, as the key step in his proof of the (generalized) Lascoux-Leclerc-Thibon conjecture. For an exposition of the proof and a fuller historical account, we refer to [A3, Theorem 12.5]. We cite also our recent work [BK5] which gives a quite different proof in the degenerate case $\xi = 1$ based on Schur-Weyl duality for higher levels and the Kazhdan-Lusztig conjecture in finite type A .

Theorem 5.1 ([A1, Theorem 4.4]). *Assume $\text{char } F = 0$. The isomorphism $\underline{\delta}$ from (5.3) maps the canonical basis of $V(\Lambda)_{\mathbb{Z}}$ to the basis of $[\mathbf{Proj}(H^\Lambda)]$ arising from projective indecomposable modules.*

Corollary 5.2. *Assume $\text{char } F = 0$. The isomorphism $\underline{\varepsilon}$ from (5.3) maps the basis of $[\mathbf{Rep}(H^\Lambda)]$ arising from irreducible modules to the dual-canonical basis of $V(\Lambda)_{\mathbb{Z}}^*$.*

Proof. This follows from the definition of the map $\underline{\varepsilon}$, since the basis arising from the irreducible modules is dual to the basis arising from the projective indecomposable modules under the Cartan pairing, and the dual-canonical basis is dual to the canonical basis under the contravariant form. \square

We have not yet incorporated any particular parametrization for the bases mentioned in either Theorem 5.1 or Corollary 5.2. This is addressed in detail in Ariki's work via the theory of Specht modules, as we will explain in the next subsections. Even before we introduce Specht modules into the picture, we can show that the bijection between the isomorphism classes of irreducible modules and the dual-canonical basis from Corollary 5.2 is consistent with the parametrizations of these two sets by restricted multipartitions from Theorem 4.9 and §3.8, respectively.

To do this, recall the quasi-canonical basis $\{Y_\lambda\}$ from §3.10 and the dual-canonical basis $\{D_\lambda\}$ from §3.8, respectively, both of which are parametrized by the set \mathcal{RP}^Λ of restricted multipartitions. Denote their specializations at $q = 1$ by

$$\underline{Y}_\lambda := 1 \otimes Y_\lambda \in V(\Lambda)_\mathbb{Z}, \quad \underline{D}_\lambda := 1 \otimes D_\lambda \in V(\Lambda)_\mathbb{Z}^*, \quad (5.4)$$

for $\lambda \in \mathcal{RP}^\Lambda$. By Lemma 3.12, $\{\underline{Y}_\lambda \mid \lambda \in \mathcal{RP}^\Lambda\}$ and $\{\underline{D}_\lambda \mid \lambda \in \mathcal{RP}^\Lambda\}$ are the canonical and dual-canonical bases of $V(\Lambda)_\mathbb{Z}$ and $V(\Lambda)_\mathbb{Z}^*$, respectively.

Recall also from Theorem 4.9 that the irreducible H_α^Λ -modules are denoted $\{\underline{D}(\lambda) \mid \lambda \in \mathcal{RP}_\alpha^\Lambda\}$; they are defined recursively in terms of the crystal graph by (4.23). For $\lambda \in \mathcal{RP}_\alpha^\Lambda$, let $\underline{Y}(\lambda)$ denote the projective cover of $\underline{D}(\lambda)$, so that

$$\underline{Y}(\lambda) \cong \underline{Y}(\lambda). \quad (5.5)$$

Now we reformulate Theorem 5.1 and Corollary 5.2 incorporating these explicit parametrizations as follows:

Theorem 5.3. *Assume $\text{char } F = 0$. For each $\lambda \in \mathcal{RP}_\alpha^\Lambda$, we have that $\underline{\varepsilon}(\underline{Y}_\lambda) = [\underline{Y}(\lambda)]$ and $\underline{\varepsilon}([\underline{D}(\lambda)]) = \underline{D}_\lambda$.*

Proof. Reversing the argument with duality from the proof of Corollary 5.2, it suffices to prove the second statement. For that, we know already from Corollary 5.2 that there is some bijection $\sigma : \mathcal{RP}_\alpha^\Lambda \rightarrow \mathcal{RP}_\alpha^\Lambda$ such that

$$\underline{\varepsilon}([\underline{D}(\lambda)]) = \underline{D}_{\sigma(\lambda)}.$$

We need to show that σ is the identity map. For this we repeat an easy argument from the proof of [BK2, Theorem 4.4], as follows.

Proceed by induction on $\text{ht}(\alpha)$, the statement being trivial for $\text{ht}(\alpha) = 0$. For the induction step, take $\alpha > 0$ and $\lambda \in \mathcal{RP}_\alpha^\Lambda$. Write $\lambda = \tilde{f}_i \mu$ for some $i \in I$ and $\mu \in \mathcal{RP}_{\alpha - \alpha_i}^\Lambda$. By induction, we know that $\sigma(\mu) = \mu$. By Proposition 3.5 specialized at $q = 1$, we know that $f_i^{\varphi_i(\mu)} \underline{D}_\mu \neq 0$, and $f_i \underline{D}_\mu = \varphi_i(\mu) \underline{D}_\lambda + (*)$ where $(*)$ is a linear combination of \underline{D}_ν 's such that $f_i^{\varphi_i(\mu)-1} \underline{D}_\nu = 0$. Since $f_i^{\varphi_i(\mu)} \underline{D}_\mu \neq 0$ we have that $f_i^{\varphi_i(\mu)-1} \underline{D}_\lambda \neq 0$.

Applying the commutativity of (5.3) we deduce that $f_i \underline{D}(\mu)$ has a unique (up to isomorphism) composition factor D such that $f_i^{\varphi_i(\mu)-1} [D] \neq 0$, and $\underline{\varepsilon}([D]) = \underline{D}_\lambda$. On the other hand, by Theorem 4.9, $f_i \underline{D}(\mu)$ has a composition factor isomorphic to $\underline{D}(\lambda)$ and $f_i^{\varphi_i(\mu)-1} [\underline{D}(\lambda)] \neq 0$. Hence $\underline{\varepsilon}([\underline{D}(\lambda)]) = \underline{D}_\lambda$, i.e. $\sigma(\lambda) = \lambda$. \square

5.2. Graded Specht modules. The cyclotomic Hecke algebra H_d^Λ is a cellular algebra in the sense of [GL] with weight poset $\{\lambda \in \mathcal{P}^\Lambda \mid |\lambda| = d\}$ partially ordered by \trianglelefteq . For the explicit construction of the underlying cell datum, we refer the reader to [DJM]; see also [AMR, §6] for the appropriate modifications in the degenerate case. The associated cell modules are the so-called *Specht modules* $\underline{S}(\lambda)$ for each $\lambda \in \mathcal{P}^\Lambda$ with $|\lambda| = d$. If $\text{cont}(\lambda) = \alpha$ in the sense of (3.14) then $\underline{S}(\lambda)$ belongs to the block parametrized by α in the block decomposition (4.7); this follows from the character formula (5.7) below. Hence, invoking the following standard lemma (taking $e := e_\alpha$), we can project the cellular structure

on H_d^Λ to the block H_α^Λ , to get also that H_α^Λ is a cellular algebra with weight poset $(\mathcal{P}_\alpha^\Lambda, \trianglelefteq)$ and cell modules $\{\underline{S}(\lambda) \mid \lambda \in \mathcal{P}_\alpha^\Lambda\}$.

Lemma 5.4. *Let A be a cellular algebra with cell datum $(I, M, C, *)$ and associated cell modules $\{V(\lambda) \mid \lambda \in I\}$. Let $e \in A$ be a central idempotent. Then eAe is a cellular algebra with cell datum $(\bar{I}, \bar{M}, \bar{C}, \bar{*})$ and associated cell modules $\{V(\lambda) \mid \lambda \in \bar{I}\}$ where:*

- (1) $\bar{I} = \{\lambda \in I \mid eV(\lambda) = V(\lambda)\}$;
- (2) $\bar{M}(\lambda) = M(\lambda)$ for each $\lambda \in \bar{I}$;
- (3) $\bar{C}_{s,t}^\lambda = eC_{s,t}^\lambda e$ for each $\lambda \in \bar{I}$ and $s, t \in \bar{M}(\lambda)$;
- (4) $\bar{*}$ is the restriction of $*$ (which necessarily leaves eAe invariant).

In [BKW], we constructed a canonical graded lift of the Specht module $\underline{S}(\lambda)$, i.e. we gave an explicit construction of a graded R_α^Λ -module $S(\lambda)$ such that

$$S(\lambda) \cong \underline{S}(\lambda) \quad (5.6)$$

as H_α^Λ -modules. We refer to $S(\lambda)$ as a *graded Specht module*. Rather than repeat the definition here, we just note that the construction produces an explicit homogeneous basis $\{v_{\mathbf{T}} \mid \mathbf{T} \in \mathcal{T}(\lambda)\}$ for $S(\lambda)$, in which the vector $v_{\mathbf{T}}$ belongs to $e(\mathbf{i}^{\mathbf{T}})S(\lambda)$ and is of degree $\deg(\mathbf{T})$, notation as in (4.32)–(4.33). In particular, this means that the q -character of $S(\lambda)$ (by which we mean the q -character of its inflation to R_α in the sense of (2.28)) is given by

$$\mathrm{ch}_q S(\lambda) = \sum_{\mathbf{T} \in \mathcal{T}(\lambda)} q^{\deg(\mathbf{T})} \mathbf{i}^{\mathbf{T}}. \quad (5.7)$$

We also derived the following branching rule for graded Specht modules:

Proposition 5.5 ([BKW]). *Let $\lambda \in \mathcal{P}_\alpha^\Lambda$, $i \in I$, and A_1, \dots, A_c be all the removable i -nodes of λ in order from bottom to top. Then $E_i S(\lambda)$ has a filtration*

$$\{0\} = V_0 \subset V_1 \subset \dots \subset V_c = E_i S(\lambda)$$

as a graded $R_{\alpha-\alpha_i}^\Lambda$ -module such that $V_m/V_{m-1} \cong S(\lambda_{A_m})\langle d_{A_m}(\lambda) \rangle$ for all $1 \leq m \leq c$.

Proof. This follows from [BKW, Theorem 4.11] on projecting to $R_{\alpha-\alpha_i}^\Lambda$. \square

Using this we can identify the image of $[S(\lambda)] \in \mathrm{Rep}(R^\Lambda)$ under the isomorphism $\varepsilon : [\mathrm{Rep}(R^\Lambda)] \rightarrow V(\Lambda)_{\mathcal{A}}^*$ from Theorem 4.18. Of course it is the standard monomial S_λ from (3.29):

Theorem 5.6. *For each $\lambda \in \mathcal{P}_\alpha^\Lambda$, we have that $\varepsilon([S(\lambda)]) = S_\lambda$.*

Proof. We proceed by induction on $\mathrm{ht}(\alpha)$. The result is trivial in the case $\mathrm{ht}(\alpha) = 0$, so suppose that $\mathrm{ht}(\alpha) > 0$. We must show that $\varepsilon([S(\lambda)]) - S_\lambda = 0$ in $V(\Lambda)$. Since $V(\Lambda)$ is an irreducible highest weight module, this follows if we can check that $E_i(\varepsilon([S(\lambda)]) - S_\lambda) = 0$ for every $i \in I$. For this, we have by Theorem 4.18(2), Proposition 5.5 and the induction hypothesis that

$$E_i \varepsilon([S(\lambda)]) = \varepsilon E_i([S(\lambda)]) = \sum_A q^{d_A(\lambda)} \varepsilon([S(\lambda_A)]) = \sum_A q^{d_A(\lambda)} S_{\lambda_A}.$$

By (3.30) this is equal to $E_i S_\lambda$. \square

Corollary 5.7. *The classes $\{[S(\lambda)] \mid \lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda\}$ give a basis for $\text{Rep}(R_\alpha^\Lambda)$ as a free \mathcal{A} -module.*

Proof. This follows from Theorems 5.6 and 3.9(1). \square

Corollary 5.8. *For $\lambda \in \mathcal{P}^\Lambda$ and $i \in I$, the following hold in $[\text{Rep}(R^\Lambda)]$:*

$$E_i[S(\lambda)] = \sum_A q^{d_A(\lambda)} [S(\lambda_A)], \quad F_i[S(\lambda)] = \sum_B q^{-d^B(\lambda)} [S(\lambda^B)],$$

where the first sum is over all removable i -nodes A for λ , and the second sum is over all addable i -nodes B for λ .

Proof. This follows from (3.30), Theorem 4.18(2) and Theorem 5.6. \square

5.3. Ungraded decomposition numbers in characteristic zero. Combining Theorem 5.6 with Theorem 5.3 (which we recall was a reformulation of the geometric Theorem 5.1), we recover the following result which computes decomposition numbers of Specht modules in characteristic zero. These decomposition numbers were computed originally by Ariki in [A1] in his proof of the Lascoux-Leclerc-Thibon conjecture from [LLT] (generalized to higher levels).

Theorem 5.9 (Ariki). *Assume that $\text{char } F = 0$. For any $\mu \in \mathcal{P}_\alpha^\Lambda$ we have that*

$$[\underline{S}(\mu)] = \sum_{\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda} d_{\lambda,\mu}(1) [\underline{D}(\lambda)]$$

in the Grothendieck group $[\underline{\text{Rep}}(H_\alpha^\Lambda)]$, where $d_{\lambda,\mu}(1)$ denotes the polynomial from (3.33) evaluated at $q = 1$. In other words, for $\mu \in \mathcal{P}_\alpha^\Lambda$ and $\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda$, we have that

$$[\underline{S}(\mu) : \underline{D}(\lambda)] = d_{\lambda,\mu}(1),$$

Proof. Let $\underline{S}_\mu := 1 \otimes S_\lambda \in V(\Lambda)_{\mathbb{Z}}^*$ denote the standard monomial from (3.29) specialized at $q = 1$. By (3.33) at $q = 1$, we have that

$$\underline{S}_\mu = \sum_{\lambda \in \mathcal{R}\mathcal{P}_\alpha^\Lambda} d_{\lambda,\mu}(1) \underline{D}_\lambda.$$

Now apply $\underline{\varepsilon}^{-1}$ and use Theorems 5.6 and 5.3. \square

In [A1], Ariki formulated his results in different terms, involving lifting projectives from H_d^Λ to the semisimple Hecke algebras whose irreducible representations were classified in [AK]. The Specht modules in our formulation of the above theorem were not introduced in full generality until [DJM] (after the time of [A1]). They are canonical “modular reductions” of the irreducible representations of the aforementioned semisimple Hecke algebras. Invoking a form of Brauer reciprocity, Ariki’s results can be reformulated equivalently in terms of decomposition numbers of Specht modules, as we have done above.

Putting this technical difference aside, Theorem 5.9 is still not strictly the same as Ariki’s original theorem from [A1], since we are using the parametrization of irreducible modules coming from the crystal graph, whereas Ariki was implicitly using a parametrization coming from the triangularity properties of the decomposition matrices of Specht modules. We discuss this subtle labelling issue in the next subsection.

5.4. Another classification of irreducible representations. The general theory of cellular algebras leads to alternative way to classify the irreducible H_α^Λ -modules, which was worked out originally by Ariki in [A2]. In the following theorem, we reprove the main points of this alternative classification, keeping track of gradings as we go.

Theorem 5.10. *For $\lambda \in \mathcal{RP}_\alpha^\Lambda$, the graded Specht module $S(\lambda)$ has irreducible head denoted $\dot{D}(\lambda)$. Moreover:*

- (1) *The modules $\{\dot{D}(\lambda)\langle m \rangle \mid \lambda \in \mathcal{RP}_\alpha^\Lambda, m \in \mathbb{Z}\}$ give a complete set of pairwise non-isomorphic irreducible graded R_α^Λ -modules.*
- (2) *For $\lambda \in \mathcal{P}_\alpha^\Lambda$, we have in $[\text{Rep}(R_\alpha^\Lambda)]$ that*

$$[S(\lambda)] = \begin{cases} [\dot{D}(\lambda)] + (*) & \text{if } \lambda \text{ is restricted,} \\ (*) & \text{otherwise,} \end{cases}$$

where $(*)$ denotes a $\mathbb{Z}[q, q^{-1}]$ -linear combination of $[\dot{D}(\mu)]$'s for $\mu \triangleleft \lambda$.

- (3) *We have that $\dot{D}(\lambda)^\circledast \cong \dot{D}(\lambda)$ for each $\lambda \in \mathcal{RP}_\alpha^\Lambda$.*

Proof. Recall from Hypothesis 3.7(2) (which was verified in §3.9 in the case $e = 0$ and §3.12 in the case $e > 0$) that \leq_{lex} is a total order on \mathcal{P}^Λ refining the partial order \leq . So, applying ε^{-1} to Theorem 3.9(1)–(3) and using Theorems 5.6 and 4.18(3), we deduce:

- (a) The classes $\{[S(\lambda)] \mid \lambda \in \mathcal{RP}_\alpha^\Lambda\}$ are linearly independent.
- (b) For $\lambda \in \mathcal{P}_\alpha^\Lambda \setminus \mathcal{RP}_\alpha^\Lambda$, we can express $[S(\lambda)]$ as a $q\mathbb{Z}[q]$ -linear combination of $[S(\mu)]$'s for $\mu \in \mathcal{RP}_\alpha^\Lambda$ with $\mu <_{\text{lex}} \lambda$.
- (c) For $\lambda \in \mathcal{RP}_\alpha^\Lambda$, $[S(\lambda)] - [S(\lambda)^\circledast]$ is a $\mathbb{Z}[q, q^{-1}]$ -linear combination of $[S(\mu)]$'s for $\mu \in \mathcal{RP}_\alpha^\Lambda$ with $\mu <_{\text{lex}} \lambda$.

Now we forget gradings for a moment. Recall that H_α^Λ is a cellular algebra with weight poset $(\mathcal{P}_\alpha^\Lambda, \leq)$, and the Specht modules are its cell modules. We claim that the cell modules $\underline{S}(\lambda)$ for $\lambda \in \mathcal{RP}_\alpha^\Lambda$ have irreducible head denoted $\underline{\dot{D}}(\lambda)$, the modules $\{\underline{\dot{D}}(\lambda) \mid \lambda \in \mathcal{RP}_\alpha^\Lambda\}$ give a complete set of pairwise non-isomorphic irreducible H_α^Λ -modules, and

$$[\underline{S}(\lambda)] = \begin{cases} [\underline{\dot{D}}(\lambda)] + (*) & \text{if } \lambda \text{ is restricted,} \\ (*) & \text{otherwise,} \end{cases}$$

for any $\lambda \in \mathcal{P}_\alpha^\Lambda$, where $(*)$ denotes a linear combination of $[\underline{\dot{D}}(\mu)]$'s for $\mu \triangleleft \lambda$. To prove the claim, recall by the general theory of cellular algebras from [GL] that certain of the cell modules are distinguished, the distinguished cell modules have irreducible heads which give a complete set of non-isomorphic irreducible modules, and finally every composition factor of an arbitrary cell module $\underline{S}(\lambda)$ is isomorphic to the irreducible head of a distinguished cell module $\underline{S}(\mu)$ for $\mu \leq \lambda$. Therefore to prove the claim it suffices to show that the distinguished cell modules are the $\underline{S}(\lambda)$'s indexed by $\lambda \in \mathcal{RP}_\alpha^\Lambda$. Proceed by induction on the total order \leq_{lex} that refines \leq . For the induction step, consider $\underline{S}(\lambda)$ for $\lambda \in \mathcal{P}_\alpha^\Lambda$. If λ is not restricted then $[\underline{S}(\lambda)]$ is a sum of earlier $[\underline{S}(\mu)]$'s by (b), so $\underline{S}(\lambda)$ cannot be distinguished. If λ is restricted then $[\underline{S}(\lambda)]$ is not a sum of earlier $[\underline{S}(\mu)]$'s by (a), so $\underline{S}(\lambda)$ must be distinguished. The claim follows.

Re-introducing the grading using Lemmas 2.1–2.3, it follows from the claim that $S(\lambda)$ has irreducible head $\dot{D}(\lambda)$ for each $\lambda \in \mathcal{RP}_\alpha^\Lambda$ such that $\underline{D}(\lambda) \cong \dot{D}(\lambda)$. Moreover, (1) and (2) hold. It remains to deduce (3). We certainly have that $\dot{D}(\lambda)^\circledast \cong \dot{D}(\lambda)\langle m \rangle$ for some $m \in \mathbb{Z}$. Now (2) gives us that $[S(\lambda)] - [S(\lambda)^\circledast]$ is equal to $(1 - q^m)[\dot{D}(\lambda)] + (*)$ where $(*)$ is a $\mathbb{Z}[q, q^{-1}]$ -linear combination of $[\dot{D}(\mu)]$'s for $\mu \in \mathcal{RP}_\alpha^\Lambda$ with $\mu <_{\text{lex}} \lambda$. On the other hand (c) gives that $[S(\lambda)] - [S(\lambda)^\circledast]$ is a $\mathbb{Z}[q, q^{-1}]$ -linear combination just of the $[\dot{D}(\mu)]$'s. Hence $1 - q^m = 0$, so $m = 0$ as required. \square

Corollary 5.11. *For each $\mu \in \mathcal{RP}_\alpha^\Lambda$, we have that*

$$\varepsilon^{-1}(D_\mu) = \sum_{\lambda \in \mathcal{RP}_\alpha^\Lambda} a_{\lambda, \mu}(q) [\dot{D}(\lambda)]$$

for some unique bar-invariant Laurent polynomials $a_{\lambda, \mu}(q) \in \mathbb{Z}[q, q^{-1}]$ such that $a_{\mu, \mu}(q) = 1$ and $a_{\lambda, \mu}(q) = 0$ unless $\lambda \leq_{\text{lex}} \mu$.

Proof. Expand D_μ in terms of S_ν 's using Theorem 3.9 (recalling Hypothesis 3.7(2)). Then apply ε^{-1} and use Theorem 5.6 to get a linear combination of $[S(\nu)]$'s. Finally replace each $[S(\nu)]$ with $[\dot{D}(\lambda)]$'s using Theorem 5.10(2). This yields an expression of the form $\sum_{\lambda \in \mathcal{RP}_\alpha^\Lambda} a_{\lambda, \mu}(q) [\dot{D}(\lambda)]$ such that $a_{\mu, \mu}(q) = 1$ and $a_{\lambda, \mu}(q) = 0$ unless $\lambda \leq_{\text{lex}} \mu$. As D_μ is bar-invariant, this expression is too, so all the $a_{\lambda, \mu}(q)$'s are bar-invariant thanks to Theorem 5.10(3). \square

Remark 5.12. In Theorem 5.17 below we will show further that $a_{\lambda, \mu}(q) = 0$ unless $\lambda \trianglelefteq \mu$, and that all the coefficients of $a_{\lambda, \mu}(q)$ are non-negative integers.

Now that we have two different parametrizations of irreducible representations, one from Theorem 4.11, the other from Theorem 5.10, we must address the problem of identifying the two labellings; eventually it will emerge that

$$D(\lambda) \cong \dot{D}(\lambda) \tag{5.8}$$

for each λ . In level one, this fact has an elementary proof by-passing the geometric Theorem 5.1; see [K1, B1]. However, in higher levels, this identification turns out to be surprisingly subtle and the only known proofs when $e > 0$ rely ultimately on geometry. The labelling problem for higher levels was solved originally by Ariki in [A4]. The first author was already aware at that time of a slightly different argument to solve the same problem (see [A4, footnote 8]), which we present below. We begin in this subsection by solving the identification problem in characteristic zero; see Theorem 5.18 for the general case.

Theorem 5.13. *Assume that $\text{char } F = 0$. For every $\lambda \in \mathcal{RP}_\alpha^\Lambda$, we have that $D(\lambda) \cong \dot{D}(\lambda)$, where $D(\lambda)$ is as in Theorem 4.11 and $\dot{D}(\lambda)$ is as in Theorem 5.10.*

Proof. In view of Theorems 4.11(1) and 5.10(3), it suffices to prove this in the ungraded setting, i.e. we need to show that $\underline{D}(\lambda) \cong \dot{D}(\lambda)$ for each $\lambda \in \mathcal{RP}_\alpha^\Lambda$. By Theorems 5.10(2) and 5.9, using also Theorem 3.9 and Hypothesis 3.7(2)

for the triangularity in the second case, we have that

$$\begin{aligned} [\underline{S}(\lambda)] &= [\underline{\dot{D}}(\lambda)] + (\text{a } \mathbb{Z}\text{-linear combination of } \underline{\dot{D}}(\mu)\text{'s for } \mu <_{\text{lex}} \lambda), \\ [\underline{S}(\lambda)] &= [\underline{D}(\lambda)] + (\text{a } \mathbb{Z}\text{-linear combination of } \underline{D}(\mu)\text{'s for } \mu <_{\text{lex}} \lambda), \end{aligned}$$

for each $\lambda \in \mathcal{RP}_\alpha^\Lambda$. By induction on the lexicographic ordering, we deduce from this that $[\underline{\dot{D}}(\lambda)] = [\underline{D}(\lambda)]$, and the corollary follows. \square

5.5. Graded decomposition numbers in characteristic zero. We can now prove the graded versions of Theorems 5.3 and 5.9. These statements should be viewed as a graded version of the Lascoux-Leclerc-Thibon conjecture (generalized to higher levels).

Theorem 5.14. *Assume that $\text{char } F = 0$. For each $\lambda \in \mathcal{RP}_\alpha^\Lambda$, we have that $\delta(Y_\lambda) = [Y(\lambda)]$ and $\varepsilon([D(\lambda)]) = D_\lambda$, where δ and ε are the maps from Theorem 4.18.*

Proof. It suffices to prove the second statement, since the first follows from it by dualizing as in the proof of Theorem 5.3. By Corollary 5.11 and Theorem 5.13, we have that

$$\varepsilon^{-1}(D_\lambda) = [D(\lambda)] + \sum_{\mu \in \mathcal{RP}_\alpha^\Lambda, \mu <_{\text{lex}} \lambda} a_{\mu,\lambda}(q) [D(\mu)] \quad (5.9)$$

for some bar-invariant Laurent polynomials $a_{\mu,\lambda}(q) \in \mathbb{Z}[q, q^{-1}]$. Moreover we know from Theorem 5.3 that $a_{\mu,\lambda}(1) = 0$.

Now we proceed to show that by induction on the lexicographic ordering that $\varepsilon([D(\lambda)]) = D_\lambda$ for all $\lambda \in \mathcal{RP}_\alpha^\Lambda$. When λ is minimal, this is immediate from (5.9). In general, we have that by (3.33), (5.9), Theorems 5.6 and 3.9, and the induction hypothesis that

$$\begin{aligned} [S(\lambda)] &= \varepsilon^{-1}(S_\lambda) = \varepsilon^{-1} \left(D_\lambda + \sum_{\mu \in \mathcal{RP}_\alpha^\Lambda, \mu <_{\text{lex}} \lambda} d_{\mu,\lambda}(q) D_\mu \right) \\ &= [D(\lambda)] + \sum_{\mu \in \mathcal{RP}_\alpha^\Lambda, \mu <_{\text{lex}} \lambda} (d_{\mu,\lambda}(q) + a_{\mu,\lambda}(q)) [D(\mu)] \end{aligned}$$

for every $\lambda \in \mathcal{RP}_\alpha^\Lambda$. Now consider the coefficient $d_{\mu,\lambda}(q) + a_{\mu,\lambda}(q)$ in this expression for any $\mu \in \mathcal{RP}_\alpha^\Lambda$ with $\mu <_{\text{lex}} \lambda$. As this is the decomposition of a module in the Grothendieck group, all coefficients of $d_{\mu,\lambda}(q) + a_{\mu,\lambda}(q)$ are non-negative integers. As $d_{\mu,\lambda}(q) \in q\mathbb{Z}[q]$, we deduce that the $q^0, q^{-1}, q^{-2}, \dots$ coefficients of $a_{\mu,\lambda}(q)$ are non-negative integers. As $a_{\mu,\lambda}(q)$ is bar-invariant it follows that all its coefficients are non-negative. Finally as $a_{\mu,\lambda}(1) = 0$ we get that $a_{\mu,\lambda}(q) = 0$ too. This holds for all μ , so (5.9) implies that $\varepsilon^{-1}(D_\lambda) = [D(\lambda)]$, as required. \square

Corollary 5.15. *Assume that $\text{char } F = 0$. For $\mu \in \mathcal{P}_\alpha^\Lambda$, we have that*

$$[S(\mu)] = \sum_{\lambda \in \mathcal{RP}_\alpha^\Lambda} d_{\lambda,\mu}(q) [D(\lambda)].$$

In other words, for $\mu \in \mathcal{P}_\alpha^\Lambda$ and $\lambda \in \mathcal{RP}_\alpha^\Lambda$, we have that

$$[S(\mu) : D(\lambda)]_q = d_{\lambda, \mu}(q).$$

Moreover, for all such λ, μ , we have that $d_{\lambda, \mu}(q) = 0$ unless $\lambda \trianglelefteq \mu$.

Proof. The first two statements follow from Theorems 5.14 and 5.6, combined with the definition (3.33). The final statement follows from Theorems 5.13 and 5.10(2). \square

5.6. Graded adjustment matrices. In this subsection we complete the proof of (5.8) for fields F of positive characteristic. Recall we have already established this in the case $\text{char } F = 0$ in Theorem 5.13. We will deduce the result in general from the characteristic zero case by a base change argument.

So assume now that F is of characteristic $p > 0$, keeping all other notation as at the beginning of section 4. Assume we are given $\alpha \in Q_+$ with $\text{ht}(\alpha) = d$. Let $\hat{\xi} \in \mathbb{C}^\times$ be a primitive e th root of unity (or any non-zero element that is not a root of unity if $e = 0$). As well as the algebra R_α^Λ over F , we consider the corresponding algebra defined from the parameter $\hat{\xi}$ over the ground field \mathbb{C} . To avoid confusion we denote it by \hat{R}_α^Λ , and denote the graded Specht and irreducible modules for \hat{R}_α^Λ by $\hat{S}(\lambda)$ and $\hat{D}(\lambda)$, respectively.

In [BK4, §6], we explained a general procedure to reduce the irreducible \hat{R}_α^Λ -module $\hat{D}(\lambda)$ modulo p to obtain an R_α^Λ -module with the same q -character, for each $\lambda \in \mathcal{RP}_\alpha^\Lambda$. There is some freedom in this procedure related to choosing a lattice in $\hat{D}(\lambda)$. We can make an essentially unique choice as follows. Let v_λ denote the image under some surjection $\hat{S}(\lambda) \twoheadrightarrow \hat{D}(\lambda)$ of the homogeneous basis vector $v_{\mathbb{T}} \in \hat{S}(\lambda)$, where \mathbb{T} is the ‘‘initial’’ standard λ -tableau obtained by writing the numbers $1, 2, \dots, d$ in order along rows starting with the top row. By [BKW, §6.2], $\hat{S}(\lambda)$ is generated as an \hat{R}_α^Λ -module by this vector $v_{\mathbb{T}}$, hence $v_\lambda \in \hat{D}(\lambda)$ is non-zero. Now let

$$J(\lambda) := F \otimes_{\mathbb{Z}} L \tag{5.10}$$

where $L \subset \hat{D}(\lambda)$ denotes the \mathbb{Z} -span of the vectors $\psi_{r_1} \cdots \psi_{r_m} y_1^{n_1} \cdots y_d^{n_d} v_\lambda$ for all $m \geq 0, 1 \leq r_1, \dots, r_m < d$ and $n_1, \dots, n_d \geq 0$. By [BK4, Theorem 6.5], L is a lattice in $\hat{D}(\lambda)$, and $J(\lambda)$ is a well-defined graded R_α^Λ -module with $y_r \in R_\alpha^\Lambda$ acting as $1 \otimes y_r$, ψ_r acting as $1 \otimes \psi_r$, and $e(\mathbf{i})$ acting as $1 \otimes e(\mathbf{i})$.

Lemma 5.16. *For each $\lambda \in \mathcal{RP}_\alpha^\Lambda$, $J(\lambda)$ has the same q -character as $\hat{D}(\lambda)$. Hence, $\varepsilon([J(\lambda)]) = D_\lambda$.*

Proof. The fact that $J(\lambda)$ has the same q -character as $\hat{D}(\lambda)$ is immediate from the construction because, for L as in (5.10), each $e(\mathbf{i})L$ is a graded lattice in $e(\mathbf{i})\hat{D}(\lambda)$. To show that $\varepsilon([J(\lambda)]) = D_\lambda$, it suffices by Theorem 5.14 to show that $\varepsilon([M]) = \varepsilon([N])$ in $V(\Lambda)_{\mathcal{Q}}^*$ whenever we are given $M \in \text{Rep}(R_\alpha^\Lambda)$ and $N \in \text{Rep}(\hat{R}_\alpha^\Lambda)$ with the same q -characters. To see this, use Theorem 2.9, Corollary 5.7 and the observation from (5.7) that graded Specht modules over R_α^Λ and \hat{R}_α^Λ have the same q -characters to reduce to checking the statement in

the special case that $M = S(\lambda)$ and $N = \widehat{S}(\lambda)$ for some $\lambda \in \mathcal{RP}_\alpha^\Lambda$. Then apply Theorem 5.6. \square

Theorem 5.17. *The bar-invariant Laurent polynomials $a_{\lambda,\mu}(q)$ from Corollary 5.11 have the property that $a_{\lambda,\mu}(q) = 0$ unless $\lambda \trianglelefteq \mu$. Moreover, for each $\mu \in \mathcal{RP}_\alpha^\Lambda$, we have that*

$$[J(\mu)] = [\dot{D}(\mu)] + \sum_{\lambda \in \mathcal{RP}_\alpha^\Lambda, \lambda \triangleleft \mu} a_{\lambda,\mu}(q) [\dot{D}(\lambda)].$$

Hence, all coefficients of $a_{\lambda,\mu}(q)$ are non-negative integers, and we have that

$$[S(\mu) : \dot{D}(\lambda)]_q = \sum_{\nu \in \mathcal{RP}_\alpha^\Lambda} a_{\lambda,\nu}(q) d_{\nu,\mu}(q)$$

for any $\lambda \in \mathcal{RP}_\alpha^\Lambda$ and $\mu \in \mathcal{P}_\alpha^\Lambda$.

Proof. The first statement follows by repeating the proof of Corollary 5.11, using the stronger result established in Corollary 5.15 that $d_{\lambda,\mu}(q) = 0$ unless $\lambda \trianglelefteq \mu$ to replace the total order \leq_{lex} by the partial order \trianglelefteq . By Lemma 5.16, we have that $\varepsilon([J(\mu)]) = D_\mu$. Using this, the next statement of the theorem follows from Corollary 5.11. For the final statement, we have that by Theorem 5.6 and (3.33) that

$$[S(\mu)] = \sum_{\nu \in \mathcal{RP}_\alpha^\Lambda} d_{\nu,\mu}(q) \varepsilon^{-1}(D_\nu).$$

Now expand each $\varepsilon^{-1}(D_\nu)$ using the formula from Corollary 5.11. \square

We refer to the matrix $(a_{\lambda,\mu}(q))_{\lambda,\mu \in \mathcal{RP}_\alpha^\Lambda}$ as the *graded adjustment matrix*. For level one and $\xi = 1$, our graded adjustment matrix specializes at $q = 1$ to the adjustment matrix defined originally by James in the modular representation theory of symmetric groups. Curiously we did not yet find an example in which $a_{\lambda,\mu}(q) \notin \mathbb{Z}$; this is related to a question raised by Turner in the introduction of [T]. Now we can complete the identification of the two labellings of irreducible representations in positive characteristic.

Theorem 5.18. *Assume that $\text{char } F > 0$. For every $\lambda \in \mathcal{RP}_\alpha^\Lambda$, we have that $D(\lambda) \cong \dot{D}(\lambda)$, where $D(\lambda)$ is as in Theorem 4.11 and $\dot{D}(\lambda)$ is as in Theorem 5.10.*

Proof. We first claim for any $\lambda \in \mathcal{RP}_\alpha^\Lambda$ that $[J(\lambda) : D(\lambda)]_q = 1$. To see this, let $\mathbf{i} \in I^\alpha$ be an extremal sequence for $J(\lambda)$ in the sense of §4.12. As $J(\lambda)$ has the same q -character as the irreducible $\widehat{R}_\alpha^\Lambda$ -module $\widehat{D}(\lambda)$, \mathbf{i} must also be an extremal sequence for $\widehat{D}(\lambda)$. Now apply Lemma 4.21 twice, once for R_α^Λ and once for $\widehat{R}_\alpha^\Lambda$, to get that $\lambda = \tilde{f}_{i_d} \cdots \tilde{f}_{i_1} \emptyset$ and $[J(\lambda) : D(\lambda)] = 1$.

Using the claim and Theorem 5.17, it is now an easy exercise to show that $[\dot{D}(\mu)] = [D(\mu)]$, proceeding by induction on the dominance ordering. The theorem follows. \square

5.7. The Khovanov-Lauda conjecture in type A. Theorem 5.14 combined with Lemma 3.12 proves for all type A quivers (finite or affine) a conjecture of Khovanov and Lauda formulated in [KL1, §3.4]; see also [BS] for an elementary proof in a very special case. In this subsection we record one consequence which is implicit in [KL1]. Apart from the case $e = 2$, the main result of this subsection is also proved in [VV3] by a more direct method (which includes all other simply-laced types, not just type A).

Let $\mathbf{B} = \dot{\bigcup}_{\alpha \in Q_+} \mathbf{B}_\alpha$ be the canonical basis for $\mathbf{f} = \bigoplus_{\alpha \in Q_+} \mathbf{f}_\alpha$ as in [Lus, §14.4]. Let $U_q(\mathfrak{g})^-$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the F_i 's. There is an isomorphism

$$\mathbf{f} \xrightarrow{\sim} U_q(\mathfrak{g})^-, \quad x \mapsto x^- \quad (5.11)$$

such that $\theta_i^- := F_i$. In view of the results of [Lus, §14.4], \mathbf{B} is the unique weight basis for \mathbf{f} such that the following holds for every $x \in \mathbf{B}$ and every dominant integral weight Λ : the vector $x^- v_\Lambda$ is either zero or it is an element of the canonical basis of $V(\Lambda)$.

We have already observed that every irreducible graded R_α -module can be shifted in degree so that it is self-dual with respect to the duality \otimes ; see [KL1, §3.2]. In view of Corollary 2.6, it follows that every indecomposable projective graded R_α -module P can be shifted in degree so that it is self-dual with respect to the duality $\#$. We say simply that P is a *self-dual projective* if that is the case. So the head of an indecomposable self-dual projective is a self-dual irreducible.

Theorem 5.19. *Assume that $\text{char } F = 0$. For every $\alpha \in Q_+$, the isomorphism $\gamma : \mathbf{f}_\alpha \rightarrow [\text{Proj}(R_\alpha)]$ from Theorem 2.7 maps \mathbf{B}_α to the basis of $[\text{Proj}(R_\alpha)]$ arising from the isomorphism classes of the self-dual indecomposable projective graded R_α -modules.*

Proof. Let $\sigma : \mathbf{f} \rightarrow \mathbf{f}$ be the linear anti-automorphism with $\sigma(\theta_i) = \theta_i$ for all i . It is well known that σ maps the canonical basis of \mathbf{f} to itself. Let P be a self-dual indecomposable projective graded R_α -module. Let $x \in \mathbf{f}$ be its pre-image under γ . To prove that $x \in \mathbf{B}$, we show equivalently that $\sigma(x) \in \mathbf{B}$. By the characterization of \mathbf{B} recalled before the statement of the theorem, this follows if we can show for any Λ that $\sigma(x)^- v_\Lambda$ is either zero or an element of the canonical basis of $V(\Lambda)$. Since P is self-dual, we get from Theorem 2.7(3) that x , hence $\sigma(x)$, is bar-invariant. Therefore it is enough just to show that $\sigma(x)^- v_\Lambda$ is either zero or an element of the canonical basis of $V(\Lambda)$ up to scaling by a power of q . Recalling the map \flat from (4.27), note that $\sigma(x)^- v_\Lambda$ is equal to $x^\flat v_\Lambda$ up to scaling by a power of q . So applying Lemma 3.12, we are reduced to showing that $x^\flat v_\Lambda$ is either zero or an element of the quasi-canonical basis of $V(\Lambda)$ up to scaling by a power of q . Finally, by Proposition 4.16, we have that $x^\flat v_\Lambda = \delta^{-1}(\text{pr } P)$. Clearly $\text{pr } P$ is either zero or a projective indecomposable R_α^Λ -module. Moreover when it is non-zero, Theorem 5.14 gives that $\delta^{-1}(\text{pr } P)$ is an element of the quasi-canonical basis of $V(\Lambda)$ up to scaling by a power of q . This completes the proof. \square

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