Unipotent Brauer character values of $GL(n, \mathbb{F}_q)$ and the forgotten basis of the Hall algebra *

Jonathan Brundan†‡

Abstract

We give a formula for the values of irreducible unipotent $p$-modular Brauer characters of $GL(n, \mathbb{F}_q)$ at unipotent elements, where $p$ is a prime not dividing $q$, in terms of (unknown!) weight multiplicities of quantum $GL_n$ and certain generic polynomials $S_{\lambda, \mu}(q)$. These polynomials arise as entries of the transition matrix between the renormalized Hall-Littlewood symmetric functions and the forgotten symmetric functions. We also provide an alternative combinatorial algorithm working in the Hall algebra for computing $S_{\lambda, \mu}(q)$.

1 Introduction

In the character theory of the finite general linear group $G_n = GL(n, \mathbb{F}_q)$, the Gelfand-Graev character $\Gamma_n$ plays a fundamental role. By definition [5], $\Gamma_n$ is the character obtained by inducing a “general position” linear character from a maximal unipotent subgroup. It has support in the set of unipotent elements of $G_n$ and for a unipotent element $u$ of type $\lambda$ (i.e. the block sizes of the Jordan normal form of $u$ are the parts of the partition $\lambda$) Kawanaka [7, §3.2.24] has shown that

$$\Gamma_n(u) = (-1)^n(1 - q)(1 - q^2)\cdots(1 - q^{h(\lambda)}),$$

(1.1)

where $h(\lambda)$ is the number of non-zero parts of $\lambda$. The starting point for this article is the problem of calculating the operator determined by Harish-Chandra multiplication by $\Gamma_n$.

We have restricted our attention throughout to character values at unipotent elements, when it is convenient to work in terms of the Hall algebra, that is [13, §10.1], the vector space $g = \bigoplus_{n \geq 0} g_n$, where $g_n$ denotes the set of unipotent-supported $\mathbb{C}$-valued class functions on $G_n$, with multiplication coming from the Harish-Chandra induction operator. For a partition $\lambda$ of $n$, let $\pi_\lambda \in g_n$ denote the class function which is 1 on unipotent elements of type $\lambda$ and zero on all other conjugacy classes of $G_n$. Then, $\{\pi_\lambda\}$ is a basis for the Hall algebra labelled by all partitions. Let $\gamma_n : g \to g$ be the linear operator determined by multiplication in $g$ by $\Gamma_n$. We describe in §2 an explicit recursive algorithm, involving the combinatorics of addable and removable nodes, for calculating the effect of $\gamma_n$ on the basis $\{\pi_\lambda\}$. As an illustration of the algorithm, we rederive Kawanaka’s formula (1.1) in Example 2.12.

Now recall from [13] that $g$ is isomorphic to the algebra $\Lambda_\mathbb{C}$ of symmetric functions over $\mathbb{C}$, the isomorphism sending the basis element $\pi_\lambda$ of $g$ to the Hall-Littlewood symmetric function $\tilde{P}_\lambda \in \Lambda_\mathbb{C}$ (renormalized as in [9, §II.3, ex.2]). Consider instead the

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element $\vartheta_\lambda \in \mathfrak{g}$ which maps under this isomorphism to the forgotten symmetric function $f_\lambda \in \Lambda_C$ (see [9, §I.2]). Introduce the renormalized Gelfand-Graev operator $\hat{\gamma}_n = \delta \circ \gamma_n$, where $\delta : \mathfrak{g} \rightarrow \mathfrak{g}$ is the linear map with $\delta(\pi_\lambda) = \frac{1}{q^{\pi_\lambda(n-1)}}\pi_\lambda$ for all partitions $\lambda$. We show in Theorem 3.5 that

$$\vartheta_\lambda = \sum_{(n_1, \ldots, n_h)} \hat{\gamma}_{n_1} \circ \hat{\gamma}_{n_2} \circ \cdots \circ \hat{\gamma}_{n_h}(\pi(0)), \quad (1.2)$$

summing over all $(n_1, \ldots, n_h)$ obtained by reordering the non-zero parts $\lambda_1, \ldots, \lambda_h$ of $\lambda$ in all possible ways. Thus, we obtain a direct combinatorial construction of the ‘forgotten basis’ $\{\vartheta_\lambda\}$ of the Hall algebra.

Let $K = (K_{\lambda,\mu})$ denote the matrix of Kostka numbers [9, I, (6.4)], $\tilde{K} = (\tilde{K}_{\mu,\nu}(q))$ denote the matrix of Kostka-Foulkes polynomials (renormalized as in [9, III, (7.11)]) and $J = (J_{\lambda,\mu})$ denote the matrix with $J_{\lambda,\mu} = 0$ unless $\mu = \lambda'$ when it is 1, where $\lambda'$ is the conjugate partition to $\lambda$. Consulting [9, §I.6, §III.6], the transition matrix between the bases $\{\pi_\lambda\}$ and $\{\vartheta_\lambda\}$, i.e. the matrix $S = (S_{\lambda,\mu}(q))$ of coefficients such that

$$\vartheta_\lambda = \sum_{\mu} S_{\lambda,\mu}(q)\pi_\mu, \quad (1.3)$$

is then given by the formula $S = K^{-1}J\tilde{K}$; in particular, this implies that $S_{\lambda,\mu}(q)$ is a polynomial in $q$ with integer coefficients. Our alternative approach to computing $\vartheta_\lambda$ using (1.2) allows explicit computation of the polynomials $S_{\lambda,\mu}(q)$ in some extra cases (e.g. when $\mu = (1^n)$) not easily deduced from the matrix product $K^{-1}J\tilde{K}$.

To explain our interest in this, let $\chi_\lambda$ denote the irreducible unipotent character of $G_n$ labelled by the partition $\lambda$, as constructed originally in [12], and let $\sigma_\lambda \in \mathfrak{g}$ denote its projection to unipotent-supported class functions. So, $\sigma_\lambda$ is the element of $\mathfrak{g}$ mapping to the Schur function $s_\lambda$ under the isomorphism $\mathfrak{g} \rightarrow \Lambda_C$ (see [13]). Since $\sigma_{\lambda'} = \sum_{\mu} K_{\lambda,\mu}\sigma_\mu$ [9, §I.6], we deduce that the value of $\chi_{\lambda'}$ at a unipotent element $u$ of type $\nu$ can be expressed in terms of the Kostka numbers $K_{\lambda,\mu}$ and the polynomials $S_{\mu,\nu}(q)$ as

$$\chi_{\lambda'}(u) = \sum_{\mu} K_{\lambda,\mu}S_{\mu,\nu}(q). \quad (1.4)$$

This is a rather clumsy way of expressing the unipotent character values in the ordinary case, but this point of view turns out to be well-suited to describing the irreducible unipotent Brauer characters.

So now suppose that $p$ is a prime not dividing $q$, $k$ is a field of characteristic $p$ and let the multiplicative order of $q$ modulo $p$ be $\ell$. In [6], James constructed for each partition $\lambda$ of $n$ an absolutely irreducible, unipotent $kG_n$-module $D_\lambda$ (denoted $L(1,\lambda)$ in [1]), and showed that the set of all $D_\lambda$ gives the complete set of non-isomorphic irreducible modules that arise as constituents of the permutation representation of $kG_n$ on cosets of a Borel subgroup. Let $\chi_\lambda^p$ denote the Brauer character of the module $D_\lambda$, and $\sigma_\lambda^p \in \mathfrak{g}$ denote the projection of $\chi_\lambda^p$ to unipotent-supported class functions. Then, as a direct consequence of the results of Dipper and James [3], we show in Theorem 4.6
that \( \sigma^p_\lambda = \sum_\mu K^p_\lambda,\mu \varphi_{\mu} \) where \( K^p_\lambda,\mu \) denotes the weight multiplicity of the \( \mu \)-weight space in the irreducible high-weight module of high-weight \( \lambda \) for quantum \( GL_n \), at an \( \ell \)th root of unity over a field of characteristic \( p \). In other words, for a unipotent element \( u \) of type \( \nu \), we have the modular analogue of (1.4):

\[
\chi^p_\lambda(u) = \sum_\mu K^p_\lambda,\mu S_{\mu,\nu}(q)
\]

This formula reduces the problem of calculating the values of the irreducible unipotent Brauer characters at unipotent elements to knowing the modular Kostka numbers \( K^p_\lambda,\mu \) and the polynomials \( S_{\mu,\nu}(q) \).

Most importantly, taking \( \nu = (1^n) \) in (1.5), we obtain the degree formula:

\[
\chi^p_\lambda(1) = \sum_\mu K^p_\lambda,\mu S_{\mu,(1^n)}(q)
\]

where, as a consequence of (1.2) (see Example 3.7),

\[
S_{\mu,(1^n)}(q) = \sum_{(n_1, \ldots, n_h)} \left[ \prod_{i=1}^{n} (q^i - 1) \right] / \left[ \prod_{i=1}^{h} (q^{n_1 + \cdots + n_i} - 1) \right]
\]

summing over all \( (n_1, \ldots, n_h) \) obtained by reordering the non-zero parts \( \mu_1, \ldots, \mu_h \) of \( \mu \) in all possible ways. This formula was first proved in [1, §5.5], as a consequence of a result which can be regarded as the modular analogue of Zelevinsky’s branching rule [13, §13.5] involving the affine general linear group. The proof presented here is independent of [1] (excepting some self-contained results from [1, §5.1]), appealing instead directly to the original characteristic 0 branching rule of Zelevinsky, together with the work of Dipper and James on decomposition matrices. We remark that since all of the integers \( S_{\mu,(1^n)}(q) \) are positive, the formula (1.6) can be used to give quite powerful lower bounds for the degrees of the irreducible Brauer characters, by exploiting a \( q \)-analogue of the Premet-Suprunenko bound for the \( K^p_\lambda,\mu \). The details can be found in [2].

To conclude this introduction, we list in the table below the polynomials \( S_{\lambda,\mu}(q) \) for \( n \leq 4 \):

\[
\begin{array}{cccccccc}
1 & | & 1 & | & 2 & | & 3 & | & 4 \\
1 & | & 2 & | & 1^2 & | & 3 & | & 21^2 \\
1 & | & 1 & | & 1 & | & 1 & | & 1^3 \\
\end{array}
\begin{array}{cccccccc}
4 & | & -1 & | & q - 1 & | & (1 - q)(q^2 - 1)(q - 1) \\
31 & | & 2 & | & (1 - q)(q + 2) & | & (q^2 - 1)(q - 2)(q - 1)(q^3 + q^2 - 2) \\
2^2 & | & 1 & | & 1 - q & | & (q^3 - 1)(q - 1) \\
21^2 & | & -3 & | & q - 3 & | & q^2 + q - 3 & | & q^3 + q^2 + q - 3 \\
1^4 & | & 1 & | & 1 & | & 1 & | & 1 \\
\end{array}
\]

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2 An algorithm for computing \( \gamma_n \)

We will write \( \lambda \vdash n \) to indicate that \( \lambda \) is a partition of \( n \), that is, a sequence \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \) of non-negative integers summing to \( n \). Given \( \lambda \vdash n \), we denote its Young diagram by \([\lambda]\); this is the set of nodes

\[ [\lambda] = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq j \leq \lambda_i\}. \]

By an addable node (for \( \lambda \)), we mean a node \( A \in \mathbb{N} \times \mathbb{N} \) such that \([\lambda] \cup \{A\}\) is the diagram of a partition; we denote the new partition obtained by adding the node \( A \) to \( \lambda \) by \( \lambda \cup A \).

By a removable node (for \( \lambda \)) we mean a node \( B \in [\lambda] \) such that \([\lambda] \setminus \{B\}\) is the diagram of a partition; we denote the new partition obtained by removing \( B \) from \( \lambda \) by \( \lambda \setminus B \).

The depth \( d(B) \) of the node \( B = (i, j) \in \mathbb{N} \times \mathbb{N} \) is the row number \( i \). If \( B \) is removable for \( \lambda \), it will also be convenient to define \( e(B) \) (depending also on \( \lambda \)!) to be the depth of the next removable node above \( B \) in the partition \( \lambda \), or 0 if no such node exists.

For example consider the partition \( \lambda = (4, 4, 2, 1) \), and let \( A, B, C \) be the removable nodes in order of increasing depth:

```
    A
   /\ /
  /  B
  \  /  \\
   \ C
```

Then, \( e(A) = 0 \), \( e(B) = d(A) = 2 \), \( e(C) = d(B) = 3 \), \( d(C) = 4 \).

Now fix a prime power \( q \) and let \( G_n \) denote the finite general linear group \( GL(n, \mathbb{F}_q) \) as in the introduction. Let \( V_n = \mathbb{F}_q^n \) denote the natural \( n \)-dimensional left \( G_n \)-module, with standard basis \( v_1, \ldots, v_n \). Let \( H_n \) denote the affine general linear group \( AGL(n, \mathbb{F}_q) \). This is the semidirect product \( H_n = V_n \rtimes G_n \) of \( G_n \) acting on the elementary Abelian group \( V_n \). We always work with the standard embedding \( H_n \hookrightarrow G_{n+1} \) that identifies \( H_n \) with the subgroup of \( G_{n+1} \) consisting of all matrices of the form:

\[
\begin{bmatrix}
* & * & \cdots & 1 \\
0 & \cdots & 0 & 1
\end{bmatrix}
\]

Thus, we have a chain of subgroups \( 1 = H_0 \subset G_1 \subset H_1 \subset G_2 \subset H_2 \subset \ldots \) (by convention, we allow the notations \( G_0 \), \( H_0 \) and \( V_0 \), all of which denote groups with one element.)

For \( \lambda \vdash n \), let \( u_\lambda \in G_n \subset H_n \) denote the upper uni-triangular matrix consisting of Jordan blocks of sizes \( \lambda_1, \lambda_2, \ldots \) down the diagonal. As is well-known, \( \{u_\lambda \mid \lambda \vdash n\} \) is a set of representatives of the unipotent conjugacy classes in \( G_n \). We wish to describe instead the unipotent classes in \( H_n \). These were determined in [10, §1], but the notation here will be somewhat different. For \( \lambda \vdash n \) and an addable node \( A \) for \( \lambda \), define the upper uni-triangular \((n+1) \times (n+1)\) matrix \( u_{\lambda,A} \in H_n \subset G_{n+1} \) by

\[
u_{\lambda,A} = \begin{cases}
u_{\lambda} & \text{if } A \text{ is the deepest addable node,} \\
\nu_{\lambda_1+\cdots+\lambda_{d(A)}} \nu_{\lambda} & \text{otherwise.}
\end{cases}
\]
If instead $\lambda \vdash (n+1)$ and $B$ is removable for $\lambda$ (hence addable for $\lambda \setminus B$) define $u_{\lambda,B}$ to be a shorthand for $u_{\lambda \setminus B,B} \in H_n$. To aid translation between our notation and that of [10], we note that $u_{\lambda,B}$ is conjugate to the element denoted $c_{n+1}(1^{(k)},\mu)$ there, where $k = \lambda[d(B)]$ and $\mu$ is the partition obtained from $\lambda$ by removing the $d(B)$th row. Then:

2.1. Lemma. (i) The set

$$\{u_{\lambda,A} \mid \lambda \vdash n, A \text{ addable for } \lambda\} = \{u_{\lambda,B} \mid \lambda \vdash (n+1), B \text{ removable for } \lambda\}$$

is a set of representatives of the unipotent conjugacy classes of $H_n$.

(ii) For $\lambda \vdash (n+1)$ and a removable node $B$, $|C_{G_{n+1}}(u_\lambda)|/|C_{H_n}(u_{\lambda,B})| = q^{d(B)} - q^{e(B)}$.

Proof. Part (i) is a special case of [10, 1.3(i)], where all conjugacy classes of the group $H_n$ are described. For (ii), combine the formula for $|C_{G_{n+1}}(u_\lambda)|$ from [12, 2.2] with [10, 1.3(ii)], or calculate directly.  \[ \square \]

For any group $G$, we write $C(G)$ for the set of $C$-valued class functions on $G$. Let $g = \bigoplus_{n \geq 0} g_n \subset \bigoplus_{n \geq 0} C(G_n)$ denote the Hall algebra as in the introduction. We recall that $g$ is a graded Hopf algebra in the sense of [13], with multiplication $\circ : g \otimes g \to g$ arising from Harish-Chandra induction and comultiplication $\Delta : g \to g \otimes g$ arising from Harish-Chandra restriction, see [13, §10.1] for fuller details. Also defined in the introduction, $g$ has the natural ‘characteristic function’ basis $\{\pi_\lambda\}$ labelled by all partitions.

By analogy, we introduce an extended version of the Hall algebra corresponding to the affine general linear group. This is the vector space $\mathfrak{h} = \bigoplus_{n \geq 0} \mathfrak{h}_n$ where $\mathfrak{h}_n$ is the subspace of $C(H_n)$ consisting of all class functions with support in the set of unipotent elements of $H_n$, with algebra structure to be explained below. To describe a basis for $\mathfrak{h}$, given $\lambda \vdash n$ and an addable node $A$, define $\pi_{\lambda,A} \in C(H_n)$ to be the class function which takes value 1 on $u_{\lambda,A}$ and is zero on all other conjugacy classes of $H_n$. Given $\lambda \vdash (n+1)$ and a removable $B$, set $\pi_{\lambda,B} = \pi_{\lambda\setminus B,B}$. Then, in view of Lemma 2.1(i), $\{\pi_{\lambda,A} \mid \lambda \text{ a partition, } A \text{ addable for } \lambda\} = \{\pi_{\lambda,B} \mid \lambda \text{ a partition, } B \text{ removable for } \lambda\}$ is a basis for $\mathfrak{h}$.

Now we introduce various operators as in [13, §13.1] (but take notation instead from [1, §5.1]). First, for $n \geq 0$, we have the inflation operator

$$e^n_0 : C(G_n) \to C(H_n)$$

defined by $(e^n_0 \chi)(vg) = \chi(g)$ for $\chi \in C(G_n), v \in V_n, g \in G_n$. Next, fix a non-trivial additive character $\chi : \mathbb{F}_q \to \mathbb{C}^\times$ and let $\chi_n : V_n \to \mathbb{C}^\times$ be the character defined by $\chi_n(\sum_{i=1}^n c_i v_i) = \chi(c_n)$. The group $G_n$ acts naturally on the characters $C(V_n)$ and one easily checks that the subgroup $H_{n-1} < G_n$ centralizes $\chi_n$. In view of this, it makes sense to define for each $n \geq 1$ the operator

$$e^n_i : C(H_{n-1}) \to C(H_n),$$

namely, the composite of inflation from $H_{n-1}$ to $V_n H_{n-1}$ with the action of $V_n$ being via the character $\chi_n$, followed by ordinary induction from $V_n H_{n-1}$ to $H_n$. Finally, for $n \geq 1$ and $1 \leq i \leq n$, we have the operator

$$e^n_i : C(G_{n-i}) \to C(H_n)$$
defined inductively by $e^n_0 = e^n_+ \circ e^{n-1}_-$. The significance of these operators is due to the following lemma [13, §13.2]:

2.2. Lemma. The operator $e^n_0 \oplus e^n_1 \oplus \cdots \oplus e^n_n : C(G_n) \oplus C(G_{n-1}) \oplus \cdots \oplus C(G_0) \to C(H_n)$ is an isometry.

We also have the usual restriction and induction operators

$$\text{res}^G_{H_{n-1}} : C(G_n) \to C(H_{n-1}), \quad \text{ind}^G_{H_{n-1}} : C(H_{n-1}) \to C(G_n).$$

One checks that all of the operators $e^n_0, e^n_+, \text{res}^G_{H_{n-1}}$ and $\text{ind}^G_{H_{n-1}}$ send class functions with unipotent support to class functions with unipotent support. So, we can define the following operators between $g$ and $h$, by restricting the operators listed to unipotent-supported class functions:

$$e_+ : h \to h, \quad e_+ = \bigoplus_{n \geq 1} e^n_+; \quad (2.3)$$

$$e_i : g \to h, \quad e_i = \bigoplus_{n \geq i} e^n_i = (e_+)^i \circ e_0; \quad (2.4)$$

$$\text{ind} : h \to g, \quad \text{ind} = \bigoplus_{n \geq 1} \text{ind}^G_{H_{n-1}}; \quad (2.5)$$

$$\text{res} : g \to h, \quad \text{res} = \bigoplus_{n \geq 0} \text{res}^G_{H_{n-1}} \quad (2.6)$$

where for the last definition, $\text{res}^G_{H_{n-1}}$ should be interpreted as the zero map.

Now we indicate briefly how to make $h$ into a graded Hopf algebra. In view of Lemma 2.2, there are unique linear maps $\cdot : h \otimes h \to h$ and $\Delta : h \to h \otimes h$ such that

$$(e_i \chi) \cdot (e_j \tau) = e_{i+j} \cdot (\chi \cdot \tau), \quad (2.7)$$

$$\Delta(e_k \psi) = \sum_{i+j=k} (e_i \otimes e_j) \Delta(\psi), \quad (2.8)$$

for all $i, j, k \geq 0$ and $\chi, \tau, \psi \in g$. One can check, using Lemma 2.2 and the fact that $g$ is a graded Hopf algebra, that these operations endow $h$ with the structure of a graded Hopf algebra (the unit element is $e_0 \pi(0)$, and the counit is the map $e_i \chi \mapsto \delta_{i,0} \varepsilon(\chi)$ where $\varepsilon : g \to \mathbb{C}$ is the counit of $g$). Moreover, the map $e_0 : g \to h$ is a Hopf algebra embedding. Unlike for the operations of $g$, we do not know of a natural representation theoretic interpretation for these operations on $h$ except in special cases, see [1, §5.2].

The effect of the operators (2.3)–(2.6) on our characteristic function bases is described explicitly by the following lemma:

2.9. Lemma. Let $\lambda \vdash n$ and label the addable nodes (resp. removable nodes) of $\lambda$ as $A_1, A_2, \ldots, A_s$ (resp. $B_1, B_2, \ldots, B_{s-1}$) in order of increasing depth. Also let $B = B_r$ be some fixed removable node. Then,
Proof. (i) For \( \mu \vdash n \) and \( A \) addable, we have by definition that \((e_0 \pi_\lambda)(u_\mu, A) = \pi_\lambda(u_\mu) = \delta_{\lambda, \mu}\). Hence, \( e_0 \pi_\lambda = \sum_A \pi_\lambda, A \), summing over all addable nodes \( A \) for \( \lambda \).

(ii) This is a special case of [10, 2.4] translated into our notation.

(iii) For \( \mu \vdash n \) and \( B \) removable, \((\text{res} \pi_\lambda)(u_\mu, B)\) is zero unless \( u_\mu, B \) is conjugate in \( G_n \) to \( u_\lambda \), when it is one. So the result follows on observing that \( u_\mu, B \) is conjugate in \( G_n \) to \( u_\lambda \) if and only if \( \mu = \lambda \).

(iv) We can write \( \text{ind} \pi_\lambda, B = \sum_{\mu \vdash n} c_\mu \pi_\mu \). To calculate the coefficient \( c_\mu \) for fixed \( \mu \vdash n \), we use (iii), Lemma 2.1(ii) and Frobenius reciprocity.

(v) This follows at once from (iii) and (iv) since \( \sum_B (q^{d(B)} - q^{e(B)}) \pi_\lambda \).

Lemma 2.9(i),(ii) give explicit formulae for computing the operator \( e_n = (e_0 + e_0^n) \circ e_0 \).

The connection between \( e_n \) and the Gelfand-Graev operator \( \gamma_n \) defined in the introduction comes from the following result:

2.10. Theorem. For \( n \geq 1, \gamma_n = \text{ind} \circ e_{n-1} \).

Proof. In [1, Theorem 5.1e], we showed directly from the definitions that for any \( \chi \in C(G_m) \) and any \( n \geq 1 \), the class function \( \chi, \Gamma_n \in C(G_{m+n}) \) obtained by Harish-Chandra induction from \( (\chi, \Gamma_n) \in C(G_m) \times C(G_n) \) is equal to \( \text{res}_{G_{m+n}}(e^m_{n} \chi) \). Moreover, by [1, Lemma 5.1e(iii)], we have that \( \text{res}_{G_{m+n}}(e^m_{n} \chi) = \text{ind}_{H_{m+n-1}}(e^m_{n-1} \chi) \). Hence, \( \chi, \Gamma_n = \text{ind}_{H_{m+n-1}}(e^m_{n-1} \chi) \).

The theorem is just a restatement of this formula at the level of unipotent-supported class functions.

2.11. Example. We show how to calculate \( \gamma_2 \pi_{(3,2)} \) using Lemma 2.9 and the theorem. We omit the label \( \pi \) in denoting basis elements, and in the case of the intermediate basis
elements of $\mathfrak{h}$, we mark removable nodes with $\times$.

$$\gamma_2 = \text{ind } o e_+ \left( \begin{array}{c}
\begin{array}{c}
\times \\
\end{array}
\end{array} \right) + \begin{array}{c}
\begin{array}{c}
\times \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\times \\
\end{array}
\end{array} \right)$$

$$= \text{ind } \left( - \begin{array}{c}
\begin{array}{c}
\times \\
\end{array} \right) + (q - 1) \begin{array}{c}
\begin{array}{c}
\times \\
\end{array} \right) + (q - 1) \begin{array}{c}
\begin{array}{c}
\times \\
\end{array} \right) \right)$$

$$\gamma_2 = -(q - 1) \begin{array}{c}
\begin{array}{c}
\times \\
\end{array} \right) + (q - 1)(q^2 - q - 1) \begin{array}{c}
\begin{array}{c}
\times \\
\end{array} \right) \right)$$

$$+ (q - 1)(q^3 - q^2) \begin{array}{c}
\begin{array}{c}
\times \\
\end{array} \right) + (q^2 - 1)(q^3 - q^2) \begin{array}{c}
\begin{array}{c}
\times \\
\end{array} \right) \right)$$

$$- q^2(q^3 - q) \begin{array}{c}
\begin{array}{c}
\times \\
\end{array} \right) + (q^3 - q^2)(q^4 - q^2) \begin{array}{c}
\begin{array}{c}
\times \\
\end{array} \right) \right).$$

2.12. Example. We apply Theorem 2.10 to rederive the explicit formula (1.1) for the Gelfand-Graev character $\Gamma_n$ itself. Of course, by Theorem 2.10, $\Gamma_n = \text{ind } o e_{n-1}(\pi(0))$. We will in fact prove that

$$e_{n-1}(\pi(0)) = (-1)^{n-1} \sum_{\lambda \vdash n} \sum_{B \text{ removable}} (1 - q)(1 - q^2) \ldots (1 - q^{h(\lambda) - 1}) \pi_{\lambda, B} \quad (2.13)$$

Then (1.1) follows easily on applying $\text{ind}$ using Lemma 2.9(iv) and the calculation in the proof of Lemma 2.9(v).

To prove (2.13), use induction on $n$, $n = 1$ being immediate from Lemma 2.9(i). For $n > 1$, fix some $\lambda \vdash n$, label the addable and removable nodes of $\lambda$ as in Lemma 2.9 and take $1 \leq r \leq s$. Thanks to Lemma 2.9(ii), the $\pi_{\lambda, A_r}$-coefficient of $e_n(\pi(0)) = e_+ \circ e_{n-1}(\pi(0))$ only depends on the $\pi_{\lambda, B_i}$-coefficients of $e_{n-1}(\pi(0))$ for $1 \leq i \leq \min(r, s - 1)$. So by the induction hypothesis the $\pi_{\lambda, A_r}$-coefficient of $e_n(\pi(0))$ is the same as the $\pi_{\lambda, A_r}$-coefficient of

$$(-1)^{n-1}(1 - q) \ldots (1 - q^{h(\lambda) - 1}) \sum_{i=1}^{\min(r, s-1)} e_+ \pi_{\lambda, B_i},$$

which using Lemma 2.9(ii) equals

$$(-1)^{n-1}(1 - q) \ldots (1 - q^{h(\lambda) - 1}) \sum_{i=1}^{\min(r, s-1)} (\delta_{r,i}q^{d(B_i)} - q^{e(B_i)}).$$
This simplifies to \((-1)^n(1-q)\ldots(1-q^{h(\lambda)-1})\) if \(r < s\) and \((-1)^n(1-q)\ldots(1-q^{h(\lambda)})\) if \(r = s\), as required to prove the induction step.

3 The forgotten basis

Recall from the introduction that for \(\lambda \vdash n\), \(\chi_\lambda \in C(G_n)\) denotes the irreducible unipotent character parametrized by the partition \(\lambda\), and \(\sigma_\lambda \in \mathfrak{g}\) is its projection to unipotent-supported class functions. Also, \(\{\vartheta_\lambda\}\) denotes the ‘forgotten’ basis of \(\mathfrak{g}\), which can be defined as the unique basis of \(\mathfrak{g}\) such that for each \(n\) and each \(\lambda \vdash n\),

\[
\sigma_\lambda' = \sum_{\mu \vdash n} K_{\lambda,\mu} \vartheta_\mu.
\]  

(3.1)

Given \(\lambda \vdash n\), we write \(\mu \perp \lambda\) if \(\mu \vdash (n-j)\) and \(\lambda_{i+1} \leq \mu_i \leq \lambda_i\) for all \(i = 1, 2, \ldots\). This definition arises in the following well-known inductive formula for the Kostka number \(K_{\lambda,\mu}\), i.e. the number of standard \(\lambda\)-tableaux of weight \(\mu\) [9, §I(6.4)]:

3.2. Lemma. For \(\lambda \vdash n\) and any composition \(\nu \vdash n\), \(K_{\lambda,\nu} = \sum_{\mu \perp \lambda} \chi_{\mu}'\), where \(j\) is the last non-zero part of \(\nu\) and \(\hat{\nu}\) is the composition obtained from \(\nu\) by replacing this last non-zero part by zero.

We will need the following special case of Zelevinsky’s branching rule [13, §13.5] (see also [1, Corollary 5.4d(ii)] for its modular analogue):

3.3. Theorem (Zelevinsky). For \(\lambda \vdash n\), \(\text{res}_{H_{n-1}}^{G_n} \chi_\lambda' = \sum_{j \geq 1} \sum_{\mu \perp \lambda} \epsilon_{j-1} \chi_{\mu}'\).

Now define the map \(\delta : \mathfrak{g} \rightarrow \mathfrak{g}\) as in the introduction by setting \(\delta(\pi_\lambda) = \frac{1}{q^{h(\lambda)-1}} \pi_\lambda\) for each partition \(\lambda\) and extending linearly to all of \(\mathfrak{g}\). The significance of \(\delta\) is that by Lemma 2.9(v), \(\delta \circ \text{ind} \circ \text{res}(\pi_\lambda) = \pi_\lambda\) for all \(\lambda\). Set \(\gamma_n = \delta \circ \gamma_n\) and for a partition \(\lambda\), define

\[
\gamma_\lambda = \sum_{(n_1, \ldots, n_h)} \gamma_{n_h} \circ \cdots \circ \gamma_{n_2} \circ \gamma_{n_1}
\]

(3.4)

summing over all compositions \((n_1, \ldots, n_h)\) obtained by reordering the \(h = h(\lambda)\) non-zero parts of \(\lambda\) in all possible ways.

3.5. Theorem. For any \(\lambda \vdash n\), \(\vartheta_\lambda = \gamma_\lambda(\pi_{(0)})\).

Proof. We will show by induction on \(n\) that

\[
\sigma_\lambda' = \sum_{\mu \vdash n} K_{\lambda,\mu} \gamma_\mu(\pi_{(0)}).
\]

(3.6)
The theorem then follows immediately in view of the definition (3.1) of \( \vartheta_\lambda \). Our induction starts trivially with the case \( n = 0 \). So now suppose that \( n > 0 \) and that (3.6) holds for all smaller \( n \). By Theorem 3.3,

\[
\text{res } \sigma_\lambda' = \sum_{j \geq 1} \sum_{\mu \perp \lambda} e_{j-1} \sigma_\mu'.
\]

Applying the operator \( \delta \circ \text{ind} \) to both sides, we deduce that

\[
\sigma_\lambda' = \sum_{j \geq 1} \sum_{\mu \perp \lambda} \sum_{\nu \vdash (n-j)} K_{\mu, \nu} \hat{\gamma}_j \circ \hat{\gamma}_\nu(\pi(0))
\]

(we have applied the induction hypothesis)

\[
= \sum_{j \geq 1} \sum_{\mu \perp \lambda} \sum_{(n_1, \ldots, n_h)} K_{\mu, (n_1, \ldots, n_h)} \hat{\gamma}_j \circ \hat{\gamma}_{n_h} \circ \cdots \circ \hat{\gamma}_{n_2} \circ \hat{\gamma}_{n_1}(\pi(0))
\]

(summing over \( (n_1, \ldots, n_h) \) obtained by reordering the non-zero parts \( \nu \) in all possible ways)

\[
= \sum_{j \geq 1} \sum_{\nu \vdash (n-j)} \sum_{(n_1, \ldots, n_h)} K_{\lambda, (n_1, \ldots, n_h, j)} \hat{\gamma}_j \circ \hat{\gamma}_{n_h} \circ \cdots \circ \hat{\gamma}_{n_2} \circ \hat{\gamma}_{n_1}(\pi(0))
\]

(we have applied Lemma 3.2)

\[
= \sum_{\eta \vdash n} \sum_{(m_1, \ldots, m_k)} K_{\lambda, \eta} \hat{\gamma}_{m_k} \circ \cdots \circ \hat{\gamma}_{m_2} \circ \hat{\gamma}_{m_1}(\pi(0))
\]

(now summing over \( (m_1, \ldots, m_k) \) obtained by reordering the non-zero parts of \( \eta \) in all possible ways)

\[
= \sum_{\eta \vdash n} K_{\lambda, \eta} \hat{\gamma}_{\eta}(\pi(0))
\]

which completes the proof. \( \Box \)

3.7. Example. For \( \chi \in \mathfrak{g}_n \), write \( \deg \chi \) for its value at the identity element of \( G_n \). We wish to derive the formula (1.7) for \( \deg \vartheta_\lambda = S_{\lambda,(1^n)}(q) \) using Theorem 3.5. So, fix \( \lambda \vdash n \). Then, by Theorem 3.5,

\[
\deg \vartheta_\lambda = \sum_{(n_1, \ldots, n_h)} \deg \left[ \hat{\gamma}_{n_h} \circ \cdots \circ \hat{\gamma}_{n_1}(\pi(0)) \right].
\]

(3.8)

We will show that given \( \chi \in \mathfrak{g}_m \),

\[
\deg \hat{\gamma}_n(\chi) = (q^{m+n-1} - 1)(q^{m+n-2} - 1) \cdots (q^{m+1} - 1) \deg \chi;
\]

(3.9)
then the formula (1.7) follows easily from (3.8). Now $\gamma_n$ is Harish-Chandra multiplication by $\Gamma_n$, so
\[
\deg \gamma_n(\chi) = \deg \Gamma_n \deg \chi \cdot \frac{(q^{m+n} - 1) \cdots (q^{m+1} - 1)}{(q^n - 1) \cdots (q - 1)},
\]
the last term being the index in $G_{m+n}$ of the standard parabolic subgroup with Levi factor $G_m \times G_n$. This simplifies using (1.1) to $(q^{m+n} - 1) \cdots (q^{m+1} - 1) \deg \chi$. Finally, to calculate $\deg \hat{\gamma}_n(\chi)$, we need to rescale using $\delta$, which divides this expression by $(q^{m+n} - 1)$.

4 Brauer character values

Finally, we derive the formula (1.5) for the unipotent Brauer character values. So let $p$ be a prime not dividing $q$ and $\chi^p_{\lambda}$ be the irreducible unipotent $p$-modular Brauer character labelled by $\lambda$ as in the introduction. Writing $C^p(G_n)$ for the $C$-valued class functions on $G_n$ with support in the set of $p'$-elements of $G_n$, we view $\chi^p_{\lambda}$ as an element of $C^p(G_n)$. Let $\hat{\chi}_{\lambda}$ denote the projection of the ordinary unipotent character $\chi_{\lambda}$ to $C^p(G_n)$. Then, by [6], we can write
\[
\hat{\chi}_{\lambda} = \sum_{\mu \vdash n} D_{\lambda,\mu}^p \chi^p_{\mu},
\]
and the resulting matrix $D = (D_{\lambda,\mu})$ is the unipotent part of the $p$-modular decomposition matrix of $G_n$. One of the main achievements of the Dipper-James theory from [3] (see e.g. [1, (3.5a)]) relates these decomposition numbers to the decomposition numbers of quantum $GL_n$.

To recall some definitions, let $k$ be a field of characteristic $p$ and $v \in k$ be a square root of the image of $q$ in $k$. Let $U_n$ denote the divided power version of the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ specialized over $k$ at the parameter $v$, as defined originally by Lusztig [8] and Du [4, §2] (who extended Lusztig’s construction from $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$). For each partition $\lambda \vdash n$, there is an associated irreducible polynomial representation of $U_n$ of high-weight $\lambda$, which we denote by $L(\lambda)$. Also let $V(\lambda)$ denote the standard (or Weyl) module of high-weight $\lambda$. Write
\[
\text{ch } V(\lambda) = \sum_{\mu \vdash n} D'_{\lambda,\mu} \text{ch } L(\mu),
\]
so $D' = (D'_{\lambda,\mu})$ is the decomposition matrix for the polynomial representations of quantum $GL_n$ of degree $n$. Then, by [3]:

4.3. Theorem (Dipper and James). $D'_{\lambda,\mu} = D'_{\lambda',\mu'}$.

Let $\sigma^p_{\lambda}$ denote the projection of $\chi^p_{\lambda}$ to unipotent-supported class functions. The $\{\sigma^p_{\lambda}\}$ also give a basis for the Hall algebra $g$. Inverting (4.1) and using (3.1),
\[
\sigma^p_{\lambda} = \sum_{\mu \vdash n} D^{-1}_{\lambda',\mu'} \sigma'_{\mu'} = \sum_{\mu,\nu \vdash n} D^{-1}_{\lambda',\mu'} K_{\mu,\nu} \vartheta_{\nu}
\]
(4.4)

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where $D^{-1} = (D_{\lambda,\mu}^{-1})$ is the inverse of the matrix $D$. On the other hand, writing $K^{p,\ell}_{\lambda,\mu}$ for the multiplicity of the $\mu$-weight space of $L(\lambda)$, and recalling that $K_{\lambda,\mu}$ is the multiplicity of the $\mu$-weight space of $V(\lambda)$, we have by (4.2) that

$$K_{\mu,\nu} = \sum_{\eta \vdash n} D'_{\mu,\eta} K^{p,\ell}_{\eta,\nu}. \quad (4.5)$$

Substituting (4.5) into (4.4) and applying Theorem 4.3, we deduce:

4.6. **Theorem.** $\sigma_p^{\lambda'} = \sum_{\mu \vdash n} K^{p,\ell}_{\lambda,\mu} \vartheta_{\mu}$.

Now (1.5) and (1.6) follow at once. This completes the proof of the formulae stated in the introduction.

**References**


brundan@darkwing.uoregon.edu,
Department of Mathematics, University of Oregon, Eugene, Oregon 97403, U.S.A.