Semisimple restrictions from $GL(n)$ to $GL(n-1)$[1]

Jonathan Brundan
Department of Mathematics
University of Oregon
Eugene, Oregon
U.S.A.
E-mail: `brundan@darkwing.uoregon.edu`

Alexander Kleshchev
Department of Mathematics
University of Oregon
Eugene, Oregon
U.S.A.
E-mail: `klesh@math.uoregon.edu`

Irina Suprunenko
Institute of Mathematics
Academy of Sciences of Belarus
Minsk, 220072
Belarus
E-mail: `imanb%imanb.belpak.minsk.by@demos.su`

Abstract

We obtain a criterion for the restriction of an irreducible rational $GL(n)$-module to the naturally embedded subgroup $GL(n-1)$ to be semisimple, over an arbitrary algebraically closed field. In that case, we describe the composition factors of the restriction explicitly. As an application, we classify the completely splittable representations of general linear groups and give an exact character formula for these modules.

Introduction

In this paper, we study rational representations of the algebraic group $GL(n) = GL(n, \mathbb{F})$ defined over an algebraically closed field $\mathbb{F}$ of characteristic $p \geq 0$. Our main result gives a combinatorial criterion for the restriction of an irreducible $GL(n)$-module to the naturally embedded subgroup $GL(n-1)$ to be semisimple. In that case, we describe the composition factors of the restriction explicitly. This extends earlier work of Kleshchev [K3] and Brundan [B1] where analogous results for symmetric groups and Hecke algebras were proved.

As an application – in section 6 – we will classify all completely splittable representations of general linear groups, extending [K4]. By definition, a ***completely splittable representation*** is an irreducible module which is semisimple on restriction to *every* Levi subgroup of $GL(n)$. For example, all irreducible $GL(n)$-modules of high weight a multiple of a fundamental dominant weight are completely splittable, by Theorem 6.2. Our results allow the dimensions of all weight spaces of a completely splittable representation to be determined exactly. Character formulae for these representations have recently also been obtained (independently) by Mathieu and Papadopoulo [MP], using completely different methods.

We now describe the main result of the paper in detail. Recall that rational irreducible $\mathbb{F}GL(n)$-modules are parametrised by dominant weights, which can be identified with n-tuples of integers $\lambda = (\lambda_1, \dots, \lambda_n)$ where $\lambda_1 \geq \cdots \geq \lambda_n$ and λ_n is arbitrary. Let $L_n(\lambda)$ denote the corresponding irreducible $GL(n)$-module.

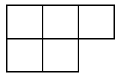
[1]1991 *subject classification.* Primary 20G05.

The dominant weight λ is ***p-restricted*** if $p = 0$, or $p \neq 0$ and $\lambda_i - \lambda_{i+1} < p$ for all $1 \leq i < n$. If $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ are dominant weights, we denote by $\lambda + \mu$ the weight whose ith part is $\lambda_i + \mu_i$ for $i = 1, 2, \dots, n$, and for a positive integer m we write $m\lambda$ for the weight whose ith part is $m\lambda_i$. Given this notation, we may write an arbitrary dominant weight λ (non-uniquely) as $\lambda = \lambda(0) + p\lambda(1) + \cdots + p^d\lambda(d)$ for some $d \geq 0$, with each $\lambda(i)$ p-restricted. We call any such expansion a ***p-adic expansion*** of λ.

Fix now a dominant weight $\lambda = (\lambda_1, \dots, \lambda_n)$. If $\lambda_n \geq 0$, λ should be regarded as a ***partition***, with corresponding ***diagram*** $[\lambda]$. This is the set

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid \lambda_i \neq 0, 1 \leq j \leq \lambda_i\}$$

which we always identify with an array of boxes in the plane. For example, if $\lambda = (3, 2)$, then $[\lambda]$ is



We say i is a ***removable row*** (for λ) if $1 \leq i < n$ and $\lambda_i \neq \lambda_{i+1}$. Notice that if $\lambda_n \geq 0$ and i is a removable row, then the node (i, λ_i) at the end of the ith row of $[\lambda]$ can be removed to leave the diagram of a proper partition – that is, removable rows contain ***removable nodes***.

Given $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ (not necessarily $\mathbb{N} \times \mathbb{N}$), define the corresponding ***p-residue*** $\mathrm{res}_p(i, j)$ to be $i - j$ regarded as an element of the ring $\mathbb{Z}/p\mathbb{Z}$. In the above example, the 3-residues of $[\lambda]$ are:

0	2	1
1	0	

For $1 \leq i \leq j < n$, define $B^\lambda(i, j) := j - i + \lambda_i - \lambda_{j+1} \in \mathbb{Z}/p\mathbb{Z}$. We shall use the fact that this is just the difference between the p-residues of $(j + 1, \lambda_{j+1} + 1)$ and (i, λ_i). Given i, j with $1 \leq i < j < n$, a ***B-chain from** i **to** j* is a chain $i = i_0 < \cdots < i_r = j$ such that $B^\lambda(i_{s-1}, i_s) = 0$ (in $\mathbb{Z}/p\mathbb{Z}$) for all $1 \leq s \leq r$. Finally, let $R_i(\lambda) := \{\mathrm{res}_p(i, j) \mid \lambda_{i+1} < j \leq \lambda_i\}$, $i = 1, 2, \dots, n - 1$.

We can now give the two main combinatorial definitions of the paper. If λ is a p-restricted dominant weight, we say that λ is a ***generalized Jantzen-Seitz weight*** (GJS for short) if whenever there are $1 \leq i < j < n$ with $R_i(\lambda) \cap R_j(\lambda) \neq \varnothing$, then there is a B-chain from i to j. More generally, if λ is an arbitrary weight, we say that λ is a generalized Jantzen-Seitz weight if λ is dominant and each $\lambda(i)$ in a p-adic expansion of λ is GJS.

Next, suppose that $\lambda = (\lambda_1, \dots, \lambda_n)$ is a p-restricted GJS weight. The weight $\mu = (\mu_1, \dots, \mu_{n-1})$ is ***allowable*** (for λ) provided

(1) $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for all $1 \leq i < n$;

(2) if $\mu_i \neq \lambda_i$ and there is a B-chain from i to j for some $1 \leq i < j < n$, then $\mu_j = \lambda_{j+1}$.

The definition ensures that such weights are dominant. Moreover, they are p-restricted: indeed, since λ is p-restricted and μ is allowable for λ, $\mu_i - \mu_{i+1} \geq p$ implies that $R_i(\lambda)$ meets $R_{i+1}(\lambda)$. Hence, as λ is GJS, $B^\lambda(i, i + 1) = 0$ modulo p. For p-restricted λ, $B^\lambda(i, i+1) = \lambda_i - \lambda_{i+2} + 1$ is at most $2p - 1$, so in fact $\lambda_i - \lambda_{i+2} + 1 = p$, which implies that $\mu_i - \mu_{i+1} \leq \lambda_i - \lambda_{i+2} = p - 1$.

More generally, if λ is an arbitrary GJS weight with p-adic expansion $\lambda = \lambda(0) + p\lambda(1) + \cdots + p^d\lambda(d)$, we say that the weight $\mu = (\mu_1, \ldots, \mu_{n-1})$ is allowable (for λ) if $\mu = \mu(0) + p\mu(1) + \cdots + p^d\mu(d)$ for weights $\mu(0), \ldots, \mu(d)$ such that $\mu(i)$ is allowable for $\lambda(i)$ for $i = 0, 1, \ldots, d$.

Note that these two definitions are independent of the choice of p-adic expansion of λ, in the case that λ is not p-restricted. We shall shortly give some examples to illustrate the definitions, but first, we state the main result of the paper.

Main Theorem. *Let λ be a dominant weight. Then, $\operatorname{res}^{GL(n)}_{GL(n-1)} L_n(\lambda)$ is semisimple if and only if λ is GJS. In that case,*

$$\operatorname{res}^{GL(n)}_{GL(n-1)} L_n(\lambda) \cong \bigoplus_{\mu \textit{ allowable}} L_{n-1}(\mu).$$

Examples. The following are examples of p-restricted GJS weights if $p = 3$ and $n \geq 7$:

$$\text{Example one: } \begin{array}{|c|c|c|} \hline 0 & 2 & 1 \\ \hline 1 & 0 \\ \cline{1-2} 2 & 1 \\ \cline{1-2} 0 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} 2 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array} \qquad \text{Example two: } \begin{array}{|c|c|c|c|c|c|} \hline 0 & 2 & 1 & 0 & 2 & 1 \\ \hline 1 & 0 & 2 & 1 \\ \cline{1-4} 2 & 1 & 0 & 2 \\ \cline{1-4} 0 & 2 & 1 & 0 \\ \cline{1-4} 1 & 0 & 2 \\ \cline{1-3} 2 & 1 \\ \cline{1-2} 0 \\ \cline{1-1} \end{array}$$

In example one, the following are the allowable weights if $n > 7$: $(3, 2^3, 1^3)$, $(3, 2^2, 1^4)$, $(3, 2^3, 1^2)$, $(3, 2^2, 1^3)$, $(2^4, 1^2)$ and $(2^3, 1^3)$. In the case $n = 7$, omit the first two weights from this list (as they contain more than $(n-1)$ non-zero parts). In example two, allowable weights are obtained by removing nodes 'from the bottom up' – the possibilities for $n > 7$ are $(6, 4^3, 3, 2, 1)$,$(6, 4^3, 3, 2)$,$(6, 4^3, 3, 1)$,$(6, 4^3, 2, 1)$,$(6, 4^2, 3, 2, 1)$,$(5, 4^2, 3, 2, 1)$ and $(4^3, 3, 2, 1)$. Again, if $n = 7$, omit the first weight from this list.

All the results of the paper are expected to generalize (without significant alterations to the proofs) to quantum $GL(n)$ – by which we mean the quantum algebra obtained by base change from Lusztig's integral form for the Drinfeld-Jimbo quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ [L]. The results in the quantum case should be valid over an arbitrary field $\mathbb{F}$, and at an arbitrary root of unity $v \in \mathbb{F}^\times$. To obtain the correct statements of the results for quantum $GL(n)$ – in the p-restricted case only – replace the integer p in all the above definitions with the integer ℓ, which by definition is the smallest positive integer such that $v^{-\ell+1} + v^{-\ell+3} + \cdots + v^{\ell-1} = 0$ in $\mathbb{F}$, or 0 if no such integer exists. More complicated modifications are needed in the non-p-restricted case, using the quantum version of Steinberg's tensor product theorem. The necessary technical theory to generalize the proofs here to the quantum case can be found in [B1] (see also Remark 2.7).

The paper is organised as follows. In section 1, we will give some equivalent statements of the above definitions and results. In particular, we restate results in terms of dominant weights for the algebraic group $SL(n, \mathbb{F})$, since this may be more familiar to some readers. In section 2, we set up notation and recall well known basis theorems for standard and costandard modules for $GL(n)$. In section 3, we use these basis theorems to obtain a useful theoretical criterion for $\operatorname{res}^{GL(n)}_{GL(n-1)} L_n(\lambda)$ to be semisimple.

The central tool in the proof of the Main Theorem is introduced in section 4, where we define Kleshchev's lowering operators and prove some technical properties. We then prove the Main Theorem in section 5. Finally, in section 6, we give the application to classifying completely splittable representations. There is also an appendix containing a short proof of the standard basis theorem and straightening rule.

Acknowledgements. The first author would like to thank G. Seitz for making this collaboration possible and the University of Oregon for its hospitality whilst this work was completed. We also thank J. C. Jantzen for pointing out [J2], Satz II.6. The second author was partially supported by the NSF. The third author was partially supported by the Basic Research Foundation of Belarus.

1 Some equivalent statements of the main results

We wish first to reformulate the definition of GJS. Recall the two examples of GJS weights given in the introduction. The first of these examples is 'trivially' a GJS weight as in fact $R_i(\lambda) \cap R_j(\lambda) = \varnothing$ for all $1 \le i < j < n$. The second example has the (stronger) property that there is a B-chain between *all* pairs of removable rows $1 \le i < j < n$. Weights with this latter property are the usual ***Jantzen-Seitz weights*** of [JS]. We want to show that these are essentially the *only* two ways p-restricted generalized Jantzen-Seitz weights can occur.

Given removable rows $1 \le i < j < n$, a B-chain $i = i_0 < \cdots < i_r = j$ is ***proper*** if every i_s is a removable row. Note that if $i < k < j$ and k is not a removable row then $B^\lambda(i,k) + B^\lambda(k,j) = B^\lambda(i,j)$. So if a B-chain from i to j exists, then a proper B-chain from i to j exists.

1.1. **Lemma.** *Let λ be GJS. Suppose $i_0 < i_1 < \cdots < i_r$ and $j_0 < j_1 < \cdots < j_s$ are two proper B-chains and that either $i_0 = j_0$ or $i_r = j_s$. Then there is a proper B-chain $\min(i_0, j_0) = k_0 < k_1 < \cdots < k_t = \max(i_r, j_s)$ such that $\{i_0, \ldots, i_r, j_0, \ldots, j_s\} \subseteq \{k_0, \ldots, k_t\}$.*

Proof. We prove this in the case that $i_0 = j_0$, leaving the similar case $i_r = j_s$ to the reader. Suppose first that $i_1 < j_1$. Then, $B^\lambda(i_0, i_1) = 0 = B^\lambda(j_0, j_1)$. Hence, as $i_0 = j_0$, the nodes $(i_1, \lambda_{i_1+1} + 1)$ and $(j_1, \lambda_{j_1+1} + 1)$ have the same residue. These residues are elements of $R_{i_1}(\lambda)$ and $R_{j_1}(\lambda)$ by definition, so $R_{i_1}(\lambda)$ meets $R_{j_1}(\lambda)$. So there is a proper B-chain from i_1 to j_1 by definition of GJS. So, we may refine the chain $j_0 < j_1 < \cdots < j_s$ to assume that $j_1 = i_1$. Similarly, if $j_1 < i_1$, we may refine the chain $i_0 < i_1 < \cdots < i_r$ to assume that $i_1 = j_1$. Now, the proof is easily completed by induction. $\square$

The lemma implies in particular that if $i < j$ are removable rows and there is a B-chain from i to j for a GJS weight λ, then there is a unique maximal proper B-chain from i to j.

1.2. **Lemma.** *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a p-restricted dominant weight. Then, λ is GJS if and only if one of the following two conditions holds:*

(JS1) $R_i(\lambda) \cap R_j(\lambda) = \varnothing$ for all $1 \le i < j < n$;

(JS2) whenever $1 \le i < j < n$ are consecutive *removable rows, then $B^\lambda(i,j) = 0$.*

Proof. ($\Leftarrow$) is obvious.

($\Rightarrow$) Let λ be GJS. Suppose that (JS1) is false, so that we can find $1 \le i < j < n$ with $R_i(\lambda) \cap R_j(\lambda) \neq \varnothing$. Let $i = i_0 < i_1 < \cdots < i_r = j$ be a maximal proper B-chain from i to j.

We first claim that $R_{i_0}(\lambda) \cup R_{i_1}(\lambda) \cup \cdots \cup R_{i_r}(\lambda) = \mathbb{Z}/p\mathbb{Z}$. Let $a_s = \mathrm{res}_p(i_s, \lambda_{i_s})$ for all $0 \le s \le r$. Take any $a \in R_i(\lambda) \cap R_j(\lambda)$. Then, $R_{i_0}(\lambda)$ contains $a, a-1, \dots, a_0$, $R_{i_1}(\lambda)$ equals $a_0 - 1, a_0 - 2, \dots, a_1$, $R_{i_2}(\lambda)$ equals $a_1 - 1, a_1 - 2, \dots, a_2$ and so on. Finally, $R_{i_r}(\lambda)$ contains $a_{r-1} - 1, \dots, a$. This proves the claim.

Next, we claim that in fact $\{i_0, i_1, \dots, i_r\}$ are all of the removable rows between i and j. Well, suppose not. Then, there is some $i < k < j$ with k removable and equal to no i_s. By the previous claim, $R_k(\lambda)$ meets some $R_{i_s}(\lambda)$ for some $0 \le s \le r$. If $i_s < k$, then there is a B-chain from i_s to k and from i_s to j, so that the B-chain from i_s to j can be refined to a strictly longer proper B-chain from i_s to j by Lemma 1.1, contradicting maximality of the original B-chain from i to j. So, $i_s > k$ and there is a B-chain from k to i_s. This time, the B-chain from i to i_s can be refined to a strictly longer proper B-chain by Lemma 1.1, giving the required contradiction in this case.

Hence, $B^\lambda(a,b) = 0$ for all consecutive removable rows $i \le a < b \le j$. Now, let i^-, j^+ be the smallest and largest removable rows respectively, so $i^- \le i$ and $j^+ \ge j$. By the first claim, $R_{i^-}(\lambda)$ meets $R_{i^+}(\lambda)$ for some $i \le i^+ \le j$. If $i^- = i$, we take $i^+ = j$ so that $i^- < i^+$ always. Now repeating the previous argument shows that $B^\lambda(a,b) = 0$ for all consecutive removable rows $i^- \le a < b \le i^+$. Similarly, there is $i \le j^- \le j$ with $j^- < j^+$ such that $B^\lambda(a,b) = 0$ for all consecutive removable rows $j^- \le a < b < j^+$. Since the intervals $\{i^-, \dots, i^+\}, \{i, \dots, j\}, \{j^-, \dots, j^+\}$ overlap, this shows that (JS2) holds, completing the proof. □

As a corollary of the Main Theorem and Lemma 1.2, we now describe the semisimple restrictions from $SL(n)$ to $SL(n-1)$ (embedded as in the GL case). Let $\ell = n-1$, and $\omega_1, \dots, \omega_\ell$ be the fundamental dominant weights for the root system A_ℓ (as in [Bou]). If $\Lambda = b_1\omega_1 + \cdots + b_{l-1}\omega_{l-1} + b_\ell\omega_\ell$ is a weight for A_ℓ we define $\mathrm{res}(\Lambda)$ to be $b_1\omega_1 + \cdots + b_{\ell-1}\omega_{\ell-1}$ considered as a weight for $A_{\ell-1}$.

A dominant weight $\Lambda = b_1\omega_1 + \cdots + b_\ell\omega_\ell$ is called p-restricted if $p = 0$ or $b_1, \dots, b_\ell < p$. Every dominant weight can be represented in the form

$$\Lambda = a_1\omega_{i_1} + \cdots + a_k\omega_{i_k}, \quad \text{with } i_1 < \cdots < i_k \text{ and } a_1, \dots, a_k > 0. \tag{1}$$

By convention, we represent the weight $\Lambda = 0$ in (1) by taking $k = 0$. For such Λ and $1 \le r \le s \le k$ define

$$B^\Lambda_{r,s} := i_s - i_r + \sum_{v=r}^{s} a_v \in \mathbb{Z}/p\mathbb{Z}.$$

We say that Λ satisfies (JS1′) if, for every $1 \le r < s \le k$, one has $B^\Lambda_{r,s} - v \neq 0$ (in $\mathbb{Z}/p\mathbb{Z}$) for all $0 < v < a_r + a_s$. We say that Λ satisfies (JS2′) if $B^\Lambda_{r,r+1} = 0$ for all $1 \le r < k$. Finally, we say that a p-restricted dominant weight Λ is GJS if Λ satisfies (JS1′) or (JS2′). The zero weight is GJS by convention.

Let $\alpha_1, \dots, \alpha_\ell$ be simple roots in A_ℓ corresponding to $\omega_1, \dots, \omega_\ell$, respectively. Denote the root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_\ell$ by $\alpha(i, \ell)$. Fix Λ written in the form (1). An A_ℓ-weight M is called Λ-allowable if $M = \Lambda - n_1\alpha(i_1, \ell) - \cdots - n_k\alpha(i_k, \ell)$ for some integers n_i such that

(1) $0 \le n_i \le a_i,\ i = 1, 2, \dots, k$;

(2) if $n_i \neq 0$ and $B^{\Lambda}_{i,j} = 0$ for some $1 \leq i < j \leq k$ then $n_j = a_j$.

Recall Steinberg's tensor product theorem [St], Theorem 41. Let $\Lambda = \Lambda_0 + p\Lambda_1 + \dots + p^d\Lambda_d$ be the (coordinate-wise) p-adic expansion. Denote the irreducible $SL(n)$-module with high weight Λ by $L(\Lambda)$. Steinberg's theorem implies that for $p > 0$, $L(\Lambda)$ can be represented as the tensor product $L(\Lambda) = L(\Lambda_0) \otimes L(\Lambda_1)^{[1]} \otimes \dots \otimes L(\Lambda_d)^{[d]}$ where ${}^{[j]}$ means the j-th Frobenius twist.

1.3. **Theorem (*SL*-version of Main Theorem)**. *Let Λ be an arbitrary dominant weight with p-adic expansion $\Lambda = \Lambda_0 + p\Lambda_1 + \dots + p^d\Lambda_d$, where $d = 0$ if $p = 0$. The restriction $\mathrm{res}^{SL(n)}_{SL(n-1)} L(\Lambda)$ is semisimple if and only if every Λ_j is GJS. In that case*

$$\mathrm{res}^{SL(n)}_{SL(n-1)} L(\Lambda) = \bigoplus L(\mathrm{res}(M_0)) \otimes L(\mathrm{res}(M_1))^{[1]} \otimes \dots \otimes L(\mathrm{res}(M_d))^{[d]}$$

where the sum is over all tuples $(M_0, M_1, \dots, M_d)$ such that M_j is Λ_j-allowable, $j = 0, 1, \dots, d$.

2 The standard and costandard basis theorems

To prove the Main Theorem, we will work with the hyperalgebra $U(n)$ corresponding to the algebraic group $GL(n)$, which we may do as the category of rational $GL(n)$-modules is equivalent to the full subcategory of the category of $U(n)$-modules consisting of all integrable $U(n)$-modules (see [J1], II.1.20 for $SL(n)$ and [J1], I.7.13 for tori). We begin by recalling the definition of this hyperalgebra, and then introduce notation for the standard and costandard bases of standard and costandard modules.

2.1 **The hyperalgebra**. The hyperalgebra $U(n) = U(n, \mathbb{F})$ corresponding to the algebraic group $GL(n) = GL(n, \mathbb{F})$, can be defined by base change from a Kostant $\mathbb{Z}$-form $U(n, \mathbb{Z})$ for the universal enveloping algebra $U(n, \mathbb{C})$ of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$, as in [CL]. For $1 \leq i, j \leq n$, let $X_{i,j}$ denote the element of $\mathfrak{gl}(n, \mathbb{C})$ corresponding to the $n \times n$ matrix with a 1 in the ij-entry and zeros elsewhere. Let

$$X^{(r)}_{i,j} := \frac{(X_{i,j})^r}{r!}, \qquad \binom{X_{i,j}}{r} := \frac{X_{i,j}(X_{i,j} - 1) \dots (X_{i,j} - r + 1)}{r!}$$

in $U(n, \mathbb{C})$. Then, $U(n, \mathbb{Z})$ is the $\mathbb{Z}$-subalgebra of $U(n, \mathbb{C})$ generated by

$$\left\{ 1, X^{(r)}_{i,j}, \binom{X_{i,i}}{r} \;\middle|\; 1 \leq i, j \leq n, i \neq j, r \geq 1 \right\}$$

and the ***hyperalgebra*** $U(n)$ over $\mathbb{F}$ is $U(n, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}$. For $1 \leq i < j \leq n$, we denote the image of $X^{(r)}_{i,j}$ and $X^{(r)}_{j,i}$ in $U(n)$ by $E^{(r)}_{i,j}$ and $F^{(r)}_{i,j}$ respectively, and the image of $\binom{X_{i,i}}{r}$ by $\binom{H_i}{r}$. We use $E^{(r)}_i$ and $F^{(r)}_i$ as shorthands for $E^{(r)}_{i,i+1}$ and $F^{(r)}_{i,i+1}$.

Let $U^+(n)$ and $U^-(n)$ respectively be the subalgebras of $U(n)$ generated by the elements $\left\{1, E^{(r)}_i \mid 1 \leq i < n, r \geq 1\right\}$ and $\left\{1, F^{(r)}_i \mid 1 \leq i < n, r \geq 1\right\}$. Let $U^0(n)$ denote the 'diagonal' subalgebra generated by $\left\{1, \binom{H_i}{r} \;\middle|\; 1 \leq i \leq n, r \geq 1\right\}$. Recall that $U^-(n)$ has a PBW basis which we always order as follows:

$$F^{(N_{1,2})}_{1,2} F^{(N_{1,3})}_{1,3} F^{(N_{2,3})}_{2,3} \dots F^{(N_{1,n})}_{1,n} \dots F^{(N_{n-1,n})}_{n-1,n}$$

as N runs over all upper triangular matrices with entries in $\mathbb{Z}_{\geq 0}$ and zeros on the diagonal.

We identify the n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with the ***weight*** $\lambda : U^0(n) \to \mathbb{F}$ which maps $\binom{H_i}{r}$ to $\binom{\lambda_i}{r}$ for all $1 \leq i \leq n, r \geq 1$. Let ε_i denote the $U(n)$-weight $(0, \dots, 0, 1, 0, \dots, 0)$ (where the 1 is in the ith position). As usual, there is a ***dominance order*** defined on the set $X(n) := \mathbb{Z}^n$ of all weights: for $\mu, \lambda \in X(n)$, $\mu \leq \lambda$ if $\lambda - \mu$ can be written as $\sum_{i=1}^{n-1} a_i(\varepsilon_i - \varepsilon_{i+1})$ with each a_i a non-negative integer. The weight $\lambda \in X(n)$ is ***dominant*** if $\lambda_1 \geq \cdots \geq \lambda_n$ with $\lambda_n \in \mathbb{Z}$ arbitrary. Let $X^+(n)$ denote the set of all dominant weights.

For $\lambda \in X^+(n)$, let $L_n(\lambda), \Delta_n(\lambda)$ and $\nabla_n(\lambda)$ denote the irreducible, standard (or Weyl) and costandard (or co-Weyl) $U(n)$-modules of high weight λ respectively. By definition, a ***high weight vector*** in a $U(n)$-module is a weight vector annihilated by $E_i^{(r)}$ for all $1 \leq i < n, r \geq 1$. So the standard module $\Delta_n(\lambda)$ is generated by a high weight vector e_λ (unique up to scalars), and $U^0(n)$ acts on e_λ by the weight λ. Recall that $L_n(\lambda) = \Delta_n(\lambda)/\operatorname{rad}\Delta_n(\lambda)$ where $\operatorname{rad}\Delta_n(\lambda)$ denotes the unique maximal proper submodule, and let f_λ be the image of e_λ in this quotient. Also, $L_n(\lambda)$ is the simple socle of $\nabla_n(\lambda)$.

Let δ denote the n-tuple $(1, 1, \dots, 1)$ which is the weight of the one dimensional determinant module det. Then, for any $c \in \mathbb{Z}$, $\Delta_n(\lambda) \otimes \det^c \cong \Delta_n(\lambda + c\delta)$, and similarly for $L_n(\lambda)$. Using this observation, it is easy to reduce all the results in the introduction to the case that $\lambda_n \geq 0$, when we may identify λ with a partition. We will do this from now on in the paper, and let $\Lambda^+(n)$ denote the set of all partitions $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. For convenience, let $\Lambda(n) \subset X(n)$ denote the set of all n-tuples of non-negative integers, so $\Lambda^+(n) = \Lambda(n) \cap X^+(n)$. Elements of $\Lambda(n)$ are ***compositions***.

Let $U(n-1), U^-(n-1), U^0(n-1), U^+(n-1)$ denote the naturally embedded subalgebras of $U(n), U^-(n), U^0(n), U^+(n)$ respectively corresponding to the subgroup $GL(n-1) < GL(n)$ (embedded into the top left hand corner of the matrices). We will talk about $U(n-1)$-weights and $U(n-1)$-high weight vectors when it is necessary to distinguish between these notions and the corresponding notions for $U(n)$.

For $\lambda \in X(n)$ and μ in either $X(n-1)$ or $X(n)$, and $1 \leq i \leq j < n$, define

$$\begin{aligned}
C^\lambda(i,j) &:= j - i + \lambda_i - \lambda_j, \\
B^\lambda(i,j) &:= j - i + \lambda_i - \lambda_{j+1}, \\
B^{\mu,\lambda}(i,j) &:= j - i + \mu_i - \lambda_{j+1},
\end{aligned}$$

all regarded as elements of $\mathbb{Z}/p\mathbb{Z}$. Next, given $1 \leq i < j \leq n$, define

$$\begin{aligned}
\mathscr{C}^\lambda(i,j) &:= \{a \mid i < a < j,\ C^\lambda(i,a) = 0\}, \\
\mathscr{B}^\lambda(i,j) &:= \{a \mid i \leq a < j,\ B^\lambda(i,a) = 0\}, \\
\mathscr{B}^{\mu,\lambda}(i,j) &:= \{a \mid i \leq a < j,\ B^{\mu,\lambda}(i,a) = 0\}.
\end{aligned}$$

Observe that $B^\lambda(i,j) = B^{\lambda,\lambda}(i,j)$ and $\mathscr{B}^\lambda(i,j) = \mathscr{B}^{\lambda,\lambda}(i,j)$.

2.2 **Tableaux**. Suppose that $\lambda \in \Lambda(n)$ is a composition, where $\lambda = (\lambda_1, \dots, \lambda_n)$. Recall the definition of the diagram $[\lambda]$ of λ from the introduction. A λ-***tableau*** is a function $t : [\lambda] \to \{1, \dots, n\}$, which we regard just as the diagram $[\lambda]$ with boxes filled with integers in $\{1, \dots, n\}$. A ***tableau*** t is simply a λ-tableau for some $\lambda \in \Lambda(n)$, and in which case λ is the ***shape*** of t. If 1 appears μ_1 times, 2 appears μ_2 times, $\dots$, n appears μ_n times in t then we say that the ***weight*** of t is $(\mu_1, \mu_2, \dots, \mu_n)$.

A tableau is ***row standard*** if the entries increase weakly along the rows. If $\lambda \in \Lambda^+(n)$ is a partition, we say a λ-tableau is ***standard*** if the entries increase weakly along rows and strictly down columns. The λ-tableau with every entry on the ith row equal to i is denoted 1_λ. It is the unique standard λ-tableau of weight λ.

Given a row standard tableau t and $1 \leq m \leq n$, let $t[m]$ denote the tableau obtained by removing all nodes with entry greater than m. As t is row standard, $t[m]$ is well-defined and has the shape of some composition. Note also that t equals $t[n]$ as the entries in t are at most n. If in addition every row on the ith row is at least i for all i, then $t[m]$ has at most m non-empty rows, and we can define $\mathrm{shape}(t[m])$ to be the shape of the tableau $t[m]$ regarded as an element of $\Lambda(m)$.

We can now define a partial order on the set of all row standard tableaux, called the ***dominance order*** on tableaux, as follows. Given λ-tableaux s and t, write $s \leq t$ if either $s = t$, or there is some $1 \leq m \leq n$ such that $s[i]$ has the same shape as $t[i]$ for $i = m+1, \dots, n$ but the shape of $s[m]$ is strictly lower than the shape of $t[m]$ in the dominance order on $\Lambda(n)$. For example,

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&2\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 2&3\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&4\\\cline{1-2}\end{array}.$$

Say that μ ***belongs to*** λ, written $\mu \longleftarrow \lambda$, if $\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda^+(m)$ and $\mu = (\mu_1, \dots, \mu_{m-1}) \in \Lambda^+(m-1)$ for some $m > 1$, satisfying $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for all $1 \leq i < m$. So, $[\mu]$ is the diagram of a partition, obtained from $[\lambda]$ by removing nodes from the bottoms of columns. Given $\mu \longleftarrow \lambda \in \Lambda^+(n)$, we define the ***level*** of μ to be the integer $\sum_{i=1}^{n-1}(\lambda_i - \mu_i)$, and let $1_{\mu,\lambda}$ denote the standard λ-tableau with (i,j)-entry equal to i if $(i,j) \in [\mu]$ or n otherwise. For example, if $\mu = (2,1,0)$ and $\lambda = (3,2,0,0)$ (so $n = 4$) then

$$1_{\mu,\lambda} = \begin{array}{|c|c|c|}\hline 1&1&4\\\hline 2&4\\\cline{1-2}\end{array}.$$

2.3 **The standard basis theorem**. Fix now a partition $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+(n)$. Given a row standard λ-tableau t such that every entry on row i of t is at least i, let

$$F_t := F_{1,2}^{(N_{1,2})} F_{1,3}^{(N_{1,3})} F_{2,3}^{(N_{2,3})} \dots F_{1,n}^{(N_{1,n})} \dots F_{n-1,n}^{(N_{n-1,n})}$$

where $N_{i,j}$ equals the number of j's on row i of t. This is just a monomial in the PBW basis for $U^-(n)$. For example, F_{1_λ} is the identity in $U^-(n)$. Given $\mu \longleftarrow \lambda$, let $F_{\mu,\lambda}$ denote $F_{1_{\mu,\lambda}}$; explicitly, $F_{\mu,\lambda} = F_{1,n}^{(\lambda_1-\mu_1)} \dots F_{n-1,n}^{(\lambda_{n-1}-\mu_{n-1})}$.

The ***standard basis theorem*** [CL], 3.5 says that

$$\{F_t e_\lambda \mid t \text{ a standard } \lambda\text{-tableau}\}$$

is a basis for $\Delta_n(\lambda)$. We will also need the straightening rule proved in Theorem A.4 in the appendix (also note the remarks after the proof of Theorem A.4): if t is any row standard but *not* standard λ-tableau such that every entry on row i is at least i, then $F_t e_\lambda$ can be written as a linear combination of standard basis elements $F_s e_\lambda$, s standard, where $s > t$ in the dominance order on tableaux. Moreover, as the weight of the vector $F_s e_\lambda$ is precisely the weight of s as defined in (2.2), the standard tableaux s occurring in this expansion of $F_t e_\lambda$ have the same weight as t.

2.4. **Lemma.** *Given $\mu \longleftarrow \lambda$ and an element $S \in U^-(n)$, the $F_{\mu,\lambda}e_\lambda$-coefficient of Se_λ when expanded in terms of the standard basis for $\Delta_n(\lambda)$ is equal to the $F_{\mu,\lambda}$-coefficient of S when written in terms of the PBW basis for $U^-(n)$.*

Proof. Take a monomial $X = F_{1,2}^{(N_{1,2})} F_{1,3}^{(N_{1,3})} F_{2,3}^{(N_{2,3})} \dots F_{1,n}^{(N_{1,n})} \dots F_{n-1,n}^{(N_{n-1,n})}$ in the PBW basis for $U^-(n)$. Let $M_i = N_{i,i+1} + N_{i,i+2} + \dots + N_{i,n}$. Suppose first that X cannot be written in the form F_t for some row standard λ-tableau t with every entry on row i at least i. This means that for some $1 \le i < n$, $M_i > \lambda_i$. Now note that X can be rearranged to equal

$$X = F_{1,2}^{(N_{1,2})} F_{1,3}^{(N_{1,3})} \dots F_{1,n}^{(N_{1,n})} \dots F_{i,i+1}^{(N_{i,i+1})} \dots F_{i,n}^{(N_{i,n})} \dots F_{n-1,n}^{(N_{n-1,n})}.$$

All weights of $\Delta_n(\lambda)$ lie in $\Lambda(n)$. So, the assumption that $M_i > \lambda_i$ means that $F_{i,i+1}^{(N_{i,i+1})} \dots F_{i,n}^{(N_{i,n})} \dots F_{n-1,n}^{(N_{n-1,n})} e_\lambda$ is zero as its weight is not in $\Lambda(n)$. Hence, $Xe_\lambda = 0$.

So, we may assume that $X = F_t$ for some row standard λ-tableau t with every entry on row i at least i. In that case, if t is standard, it contributes to the $F_{\mu,\lambda}e_\lambda$ coefficient of Se_λ precisely when $t = 1_{\mu,\lambda}$. If t is not standard, then by the straightening rule, $F_t e_\lambda$ expands as a sum of standard basis elements $F_s e_\lambda$ for $s > t$ and s of the same weight as t. But the tableau $1_{\mu,\lambda}$ is minimal amongst row standard λ-tableaux of the same weight as $1_{\mu,\lambda}$ for which every entry on row i is at least i, in the dominance order on tableaux. So $s \neq 1_{\mu,\lambda}$ for each s. Hence, no term $F_s e_\lambda$ in the expansion contributes to the $F_{\mu,\lambda}e_\lambda$-coefficient of Se_λ. □

2.5 **The costandard basis theorem**. The basic reference here is [G]. Let $A(n) = \mathbb{F}[c_{i,j} \mid 1 \le i, j \le n]$, a free polynomial ring. Regarding elements of $A(n)$ as functions $GL(n) \to \mathbb{F}$, $A(n)$ is a (left) rational $GL(n)$-module with action

$$(g.f)(g') = f(g'g)$$

for all $g, g' \in GL(n), f \in A(n)$. Hence, $A(n)$ is a $U(n)$-module.

Suppose first that λ is a partition with diagram consisting of a single column of length r. Let t be a λ-tableau with entries $t_1, \dots, t_r$ reading down this column. Define $f_t \in A(n)$ to be the ***bideterminant***

$$\det \begin{pmatrix} c_{1,t_1} & \cdots & c_{1,t_r} \\ \vdots & \ddots & \vdots \\ c_{r,t_1} & \cdots & c_{r,t_r} \end{pmatrix}.$$

More generally, given any $\lambda \in \Lambda^+(n)$ and a λ-tableau t with columns $t^{(1)}, \dots, t^{(s)}$, define $f_t \in A(n)$ to be the product $f_t := f_{t^{(1)}} \dots f_{t^{(s)}}$. By [G], 4.4, $\nabla_n(\lambda)$ can be defined as the submodule of $A(n)$ spanned by the f_t for all λ-tableaux t. [Note that our f_t corresponds to Green's bideterminant $(T_l : T_i)$ where l and i are the row sequences of the λ-tableaux 1_λ and t, respectively.] The following elementary facts about the f_t are known [G]. (One can use the map φ defined in [G], 4.4 to prove (1) and (2). For (3) see [G], 4.6b.)

(1) $F_{i,j}^{(r)}.f_t = \sum_s f_s$ and $E_{j,i}^{(r)}.f_t = \sum_s f_s$, where the sums are over all λ-tableaux s obtained from t by replacing r entries equal to i by j's in all possible ways.

(2) $\binom{H_i}{r}.f_t = \binom{N_i}{r} f_t$ where N_i is the number of entries equal to i in t.

(3) $f_t = 0$ if t contains repeated entries in some column.

By [G], 4.5, we have the ***costandard basis theorem***:

$$\{f_t \,|\, t \text{ a standard } \lambda\text{-tableau}\}$$

is a basis for $\nabla_n(\lambda)$. By (1) and (3), f_{1_λ} is a $U(n)$-high weight vector, of weight λ by (2), so we may choose the high weight vector f_λ defined in (2.1) to be f_{1_λ}. Given $\mu \longleftarrow \lambda$, let $f_{\mu,\lambda}$ denote $f_{1_{\mu,\lambda}}$.

2.6. **Lemma.** *Let $\mu \longleftarrow \lambda$.*

(i) *The vector $f_{\mu,\lambda}$ is a $U(n-1)$-high weight vector in $\nabla_n(\lambda)$ of $U(n-1)$-weight μ.*

(ii) *For $i = 1, \dots, n-1$, let $a_i = \sum_{s=1}^{i} (\lambda_s - \mu_s)$. Then, $E_1^{(a_1)} \dots E_{n-1}^{(a_{n-1})} f_{\mu,\lambda} = f_\lambda$.*

Proof. Use (1)–(3) and the definition of the tableau $1_{\mu,\lambda}$. □

2.7. **Remark.** Part (ii) of this lemma will allow us to raise $U(n-1)$-high weight vectors in $L_n(\lambda)$ to f_λ. In the quantum case – where we do not know of an analogue of the costandard basis theorem in the literature – one would need to argue more carefully here, using induction together with the explicit construction of $f_{\mu,\lambda}$ given below in Theorem 5.5.

3 Branching rules

We now review some of the results on branching rules proved in [K1, K3, B1, B2], and prove some other important preliminary results. Throughout the section, $\lambda \in \Lambda^+(n)$ denotes a fixed partition with $\lambda = (\lambda_1, \dots, \lambda_n)$.

The following result is proved in [B1], Theorem 3.19, as a consequence of the standard basis theorem. It is also proved in Proposition A.2 in the appendix.

3.1. **Proposition.** *Let $\mu(1), \mu(2), \dots, \mu(N)$ be all partitions $\mu \longleftarrow \lambda$ ordered so that $\mu(i) > \mu(j)$ in the usual dominance order implies $i < j$. Define Δ^i to be the $U(n-1)$-submodule of $\Delta = \Delta_n(\lambda)$ generated by $\{F_{\mu(1),\lambda} e_\lambda, \dots, F_{\mu(i),\lambda} e_\lambda\}$. Then*

$$(0) = \Delta^0 \subset \Delta^1 \subset \cdots \subset \Delta^N = \Delta_n(\lambda) \tag{2}$$

is a $U(n-1)$-stable filtration, with $\Delta^i/\Delta^{i-1} \cong \Delta_{n-1}(\mu(i))$ for $i = 1, 2, \dots, N$. Moreover, the image of $F_{\mu(i),\lambda} e_\lambda$ in Δ^i/Δ^{i-1} is a $U(n-1)$-high weight vector.

The result shows that $\Delta_n(\lambda)$ has a ***Weyl filtration*** as a $U(n-1)$-module, with factors $\{\Delta_{n-1}(\mu) \,|\, \mu \longleftarrow \lambda\}$ each occurring with multiplicity one. This well known fact seems to have been noticed first by Jantzen in the proof of [J2], Satz II.6.

3.2. **Corollary.** *For $\mu \in X^+(n-1)$,*

$$\dim \mathrm{Hom}_{U(n-1)}(\Delta_{n-1}(\mu), \nabla_n(\lambda)) = \begin{cases} 1 & \text{if } \mu \longleftarrow \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mu \longleftarrow \lambda$, the $U(n-1)$-high weight vector in the image of $\Delta_{n-1}(\mu)$ under any such non-zero homomorphism is a scalar multiple of $f_{\mu,\lambda}$.

Proof. The first statement follows from the proposition by contravariant duality and [J1], II.4.16(a). For the second, use Lemma 2.6(i). □

Following [B1], we say that μ is ***normal*** for λ, written $\mu \overset{\text{norm}}{\longleftarrow} \lambda$, if $\mu \in \Lambda^+(n-1)$ and

$$\dim \mathrm{Hom}_{U(n-1)}(\Delta_{n-1}(\mu), L_n(\lambda)) \neq 0.$$

We say that μ is ***good*** for λ, written $\mu \overset{\text{good}}{\longleftarrow} \lambda$, if $\mu \in \Lambda^+(n-1)$ and

$$\dim \mathrm{Hom}_{U(n-1)}(L_{n-1}(\mu), L_n(\lambda)) \neq 0.$$

Since $L_{n-1}(\lambda)$ is a quotient of $\Delta_{n-1}(\mu)$, we know that $\mu \overset{\text{good}}{\longleftarrow} \lambda$ implies $\mu \overset{\text{norm}}{\longleftarrow} \lambda$. Moreover, since $L_n(\lambda) \subseteq \nabla_n(\lambda)$, we may use Corollary 3.2 to conclude that $\mu \overset{\text{norm}}{\longleftarrow} \lambda$ implies $\mu \longleftarrow \lambda$.

We will use the following criteria for normality; the first follows from Corollary 3.2, the second is proved in [B2], Theorem 5.2.

3.3. **Criterion.** *Fix $\mu \longleftarrow \lambda$. Then, $\mu \overset{\text{norm}}{\longleftarrow} \lambda$ if and only if $f_{\mu,\lambda} \in L_n(\lambda)$.*

3.4. **Criterion.** *Fix $\mu \longleftarrow \lambda$. Then, $\mu \overset{\text{norm}}{\longleftarrow} \lambda$ if and only if every vector $e \in \operatorname{rad} \Delta_n(\lambda)$ has zero $F_{\mu,\lambda}e_\lambda$-coefficient when written in terms of the standard basis for $\Delta_n(\lambda)$.*

By Criterion 3.4 and Lemma 2.4, we obtain:

3.5. **Lemma.** *If $\mu \longleftarrow \lambda$ and there exists $S \in U^-(n)$ such that*

(i) *S has non-zero $F_{\mu,\lambda}$-coefficient when written in terms of the PBW basis for $U^-(n)$,*

(ii) *$Sf_\lambda = 0$ (in $L_n(\lambda)$)*

then μ is not normal for λ.

In this paper, we are concerned with the property that $\mathrm{res}^{U(n)}_{U(n-1)} L_n(\lambda)$ is semisimple. Our aim now is to show that this holds if and only if every $\mu \overset{\text{norm}}{\longleftarrow} \lambda$ is in fact good. As in Proposition 3.1, let $\mu(1), \mu(2), \ldots, \mu(N)$ be all partitions $\mu \longleftarrow \lambda$ ordered so that $\mu(i) > \mu(j)$ in the usual dominance order implies $i < j$. Let

$$(0) = \Delta^0 \subset \Delta^1 \subset \cdots \subset \Delta^N = \Delta_n(\lambda)$$

be the filtration of (2). Denote the image of Δ^i under the natural map $\Delta \to L = \Delta / \operatorname{rad} \Delta$ by L^i. We get a filtration

$$(0) = L^0 \subseteq L^1 \subseteq \cdots \subseteq L^N = L.$$

With this notation, we obtain a third criterion for normality:

3.6. **Lemma.** *Given $1 \le i \le N$ as above, $\mu(i) \overset{\text{norm}}{\longleftarrow} \lambda$ if and only if $L^i \neq L^{i-1}$.*

Proof. First, consider the weights of Δ/Δ^{i-1}. In $\Delta^i/\Delta^{i-1} \cong \Delta_{n-1}(\mu(i))$, the weight $\mu(i)$ occurs with multiplicity one, all other weights being lower in the dominance order. The remaining weights of Δ/Δ^{i-1} are of the form $\{\gamma | \text{there exists } j > i \text{ with } \gamma \le \mu(j)\}$. But $\mu(j) \not\geq \mu(i)$ if $j > i$, so $\mu(i)$ only occurs with multiplicity one in Δ/Δ^{i-1}, and moreover, other weights of Δ/Δ^{i-1} do not dominate $\mu(i)$. Hence the span of all weight spaces in Δ/Δ^{i-1} different from $\mu(i)$ is stable under the action of $U^-(n-1)U^0(n-1)$. So there exists a non-zero $U^-(n-1)U^0(n-1)$-homomorphism $\Delta/\Delta^{i-1} \to \mathbb{F}_{\mu(i)}$. By Frobenius reciprocity for algebraic groups [J1], I.3.4, we obtain a non-zero $U(n-1)$-homomorphism

$$\theta : \Delta/\Delta^{i-1} \to \nabla_{n-1}(\mu(i)).$$

We claim that θ is non-zero on Δ^i/Δ^{i-1}. Indeed, $\operatorname{soc} \nabla_{n-1}(\mu(i)) = L_{n-1}(\mu(i))$, hence $\operatorname{im} \theta \supseteq L_{n-1}(\mu(i))$. So the $\mu(i)$-weight space of $\operatorname{im}\theta$ is non-zero. Now we can use the fact that the $\mu(i)$-weight space of Δ/Δ^{i-1} is contained in Δ^i/Δ^{i-1}.

Extend θ to the map $\hat{\theta} : \Delta \to \nabla_{n-1}(\mu(i))$, zero on Δ^{i-1}, and non-zero on Δ^i.

Suppose that $\mu(i) \overset{\text{norm}}{\longleftarrow} \lambda$. By contravariant duality, this is equivalent to the fact that there exists a non-zero $U(n-1)$-homomorphism $\varphi : L \to \nabla_{n-1}(\mu(i))$. This homomorphism can be extended to a (non-zero) homomorphism $\hat{\varphi} : \Delta \to \nabla_{n-1}(\mu(i))$. Since $\dim \operatorname{Hom}_{U(n-1)}(\Delta, \nabla_{n-1}(\mu(i))) = 1$ (which follows from Corollary 3.2), we conclude that $\hat{\varphi}$ is proportional to $\hat{\theta}$. This implies $\operatorname{rad}\Delta \subseteq \ker\hat{\theta}$, i.e. $\hat{\theta}$ factors through $L = \Delta/\operatorname{rad}\Delta$. Now, $\hat{\theta}$ is zero on Δ^{i-1} and non-zero on Δ^i, so it factors to give a map which is zero on L^{i-1} and non-zero on L^i. Hence $L^i \neq L^{i-1}$.

Conversely, suppose $L^i \neq L^{i-1}$. Then $F_{\mu(i),\lambda} e_\lambda + L^{i-1}$ is a non-zero high weight vector of L/L^{i-1}. So there is a non-zero $U^-(n-1)U^0(n-1)$-homomorphism $L \to \mathbb{F}_{\mu(i)}$. By Frobenius reciprocity again, this gives a non-zero $U(n-1)$-homomorphism $L \to \nabla_{n-1}(\mu(i))$, and consequently by contravariant duality we obtain a non-zero homomorphism $\Delta_{n-1}(\mu(i)) \to L$. □

3.7. **Corollary.** *$L_n(\lambda)$ is generated as a $U^-(n-1)$-module by $\{F_{\mu,\lambda} f_\lambda \mid \mu \overset{\text{norm}}{\longleftarrow} \lambda\}$.*

Proof. Note that L^i/L^{i-1} is a homomorphic image of Δ^i/Δ^{i-1}, and hence is a module generated by a high weight vector $F_{\mu(i),\lambda} f_\lambda + L_{i-1}$. Now apply Lemma 3.6. □

Now we are ready to prove the main result of this section.

3.8. **Proposition.** *Let $\lambda \in \Lambda^+(n)$. The following statements are equivalent:*

(i) $\operatorname{res}^{U(n)}_{U(n-1)} L_n(\lambda)$ *is semisimple;*

(ii) *every $\mu \overset{\text{norm}}{\longleftarrow} \lambda$ is good;*

(iii) *for each $\mu \overset{\text{norm}}{\longleftarrow} \lambda$, $f_{\mu,\lambda}$ is a scalar multiple of*

$$F_{\mu,\lambda} f_\lambda + \sum_{\substack{\nu > \mu, \\ \nu \overset{\text{norm}}{\longleftarrow} \lambda}} S_\nu F_{\nu,\lambda} f_\lambda$$

for certain coefficients $S_\nu \in U^-(n-1)$;

(iv) *for each $\mu \overset{\text{norm}}{\longleftarrow} \lambda$ and every $S \in U^-(n)$ such that $f_{\mu,\lambda}$ is a scalar multiple of $S f_\lambda$, the element S has non-zero $F_{\mu,\lambda}$-coefficient when written in terms of the PBW basis.*

Proof. (i)⇒(ii). If $\mu \overset{\text{norm}}{\longleftarrow} \lambda$ and the restriction is semisimple, then $f_{\mu,\lambda}$ must generate an irreducible $U(n-1)$-module, so that μ is good as required.

Assume (ii) holds. For every $1 \leq i \leq N$, put

$$M_i := \{\mu(j) \mid 1 \leq j \leq i, \mu(j) \overset{\text{norm}}{\longleftarrow} \lambda\}.$$

Also put $M_0 = \varnothing$.

We prove by induction on $i = 0, 1, \dots, N$ that L^i is semisimple with composition factors $\{L_{n-1}(\mu) \mid \mu \in M_i\}$. Since $L^0 = (0)$, the induction base is clear. Let $i > 0$. If $\mu(i)$ is not normal then $L^i = L^{i-1}$ by virtue of Lemma 3.6, and we are done. Assume that $\mu(i)$ is normal, hence good by assumption. Then $f_{\mu(i),\lambda} \in L$ by Criterion 3.3, and $f_{\mu(i),\lambda}$ generates a copy of $L_{n-1}(\mu(i))$. By Corollary 3.7, we may write

$$f_{\mu(i),\lambda} = \sum_{\nu \overset{\text{norm}}{\longleftarrow} \lambda} S_\nu F_{\nu,\lambda} f_\lambda$$

for certain $S_\nu \in U^-(n-1)$. By considering weights, we can assume that $S_\nu = 0$ unless $\mu(i) \leq \nu$ and that $S_{\mu(i)} \in \mathbb{F}$. So,

$$f_{\mu(i),\lambda} - a_i F_{\mu(i),\lambda} f_\lambda = \sum_{\substack{\nu > \mu(i), \\ \nu \overset{\text{norm}}{\longleftarrow} \lambda}} S_\nu F_{\nu,\lambda} f_\lambda \tag{3}$$

for a scalar $a_i = S_{\mu(i)} \in \mathbb{F}$. The right hand side of this expression lies in L^{i-1} which, by the inductive hypothesis, is semisimple with composition factors $L_{n-1}(\nu)$ for $\nu \in M_{i-1}$. Since $f_{\mu(i),\lambda}$ generates a copy of $L_{n-1}(\mu(i))$, we have $f_{\mu(i),\lambda} \notin L^{i-1}$. Hence, $a_i \neq 0$. Moreover, by definition, L^i is generated by L^{i-1} and $F_{\mu(i),\lambda} f_\lambda$, hence L^i is also generated by L^{i-1} and $f_{\mu(i),\lambda}$, which implies that L^i is semisimple with composition factors $L_{n-1}(\mu)$ where $\mu \in M_{i-1} \cup \{\mu(i)\} = M_i$.

Now (ii) ⇒ (i) follows immediately by taking $i = N$. Also (ii) ⇒ (iii) follows from (3) since we proved that $a_i \neq 0$.

(iv)⇒(iii). This is obvious using Corollary 3.7, as in the proof of (3).

(iv)⇒(ii). Suppose (iv) holds but there is some $\mu \overset{\text{norm}}{\longleftarrow} \lambda$ that is not good. Then, for some $\nu < \mu$ with $\nu \overset{\text{norm}}{\longleftarrow} \lambda$, $f_{\nu,\lambda} = T f_{\mu,\lambda}$ for some $T \in U^-(n-1)$. By (iii), we may write

$$f_{\nu,\lambda} = \left(\sum_{\substack{\gamma \geq \mu, \\ \gamma \overset{\text{norm}}{\longleftarrow} \lambda}} T S_\gamma F_{\gamma,\lambda} \right) f_\lambda.$$

Each TS_γ lies in $U^-(n-1)$, and $F_{\gamma,\lambda} \neq F_{\nu,\lambda}$ as $\gamma \geq \mu > \nu$. So, by the choice of ordering of the PBW basis, the element in the brackets on the right hand side of this expression has zero $F_{\nu,\lambda}$-coefficient when written in terms of the PBW basis. This contradicts (iv).

(iii)⇒(iv). Suppose that (iii) holds but that we can find $\mu \overset{\text{norm}}{\longleftarrow} \lambda$ and $S \in U^-(n)$ with zero $F_{\mu,\lambda}$-coefficient such that $f_{\mu,\lambda} = S f_\lambda$. By (iii), we can find an element

$$S' = a_\mu F_{\mu,\lambda} + \sum_{\substack{\nu > \mu, \\ \nu \overset{\text{norm}}{\longleftarrow} \lambda}} S_\nu F_{\nu,\lambda}$$

for coefficients $a_\mu \in \mathbb{F}^\times, S_\nu \in U^-(n-1)$ such that $S' f_\lambda = f_{\mu,\lambda} = S f_\lambda$. Hence, $(S' - S) f_\lambda = 0$. But the $F_{\mu,\lambda}$-coefficient of $(S' - S)$ is a_μ which is non-zero, so this contradicts normality of μ by Lemma 3.5. $\square$

4 Lowering operators

We next review the definition of Kleshchev's lowering operators, following the reformulation described in [B1]. We then prove two technical properties of the lowering operators that are central to the proof of the Main Theorem. Throughout the section, $\lambda = (\lambda_1, \dots, \lambda_n)$ denotes a fixed element of $\Lambda^+(n)$.

We use the non-standard notation $(i..j)$ to denote the open interval $\{i+1, \dots, j-1\}$ of $\mathbb{N}$. Given $1 \leq i < j \leq n$ and $B \subseteq (i..j)$, let $F_{i,j}^B$ denote $F_{i,b_1} F_{b_1,b_2} \dots F_{b_r,j}$ if $B = \{b_1 < \dots < b_r\}$ (we interpret $F_{i,j}^\varnothing$ as $F_{i,j}$). Let μ be an element of either $\Lambda^+(n-1)$ or $\Lambda^+(n)$, and take $A \subseteq (i..j)$. The ***lowering operator*** $S_{i,j}^\mu(A)$ can be defined as

$$S_{i,j}^\mu(A) := \sum_{B \subseteq (i..j)} H_{i,j}^\mu(A,B) F_{i,j}^B$$

for certain scalars $H_{i,j}^\mu(A,B)$. These scalars are defined by evaluating a certain *polynomial* $\mathcal{H}_{i,j}(A,B) \in \mathbb{Z}[x_i, \dots, x_{j-1}]$ (defined in the next paragraph) at $x_k := \mathrm{res}_p(k, \mu_k)$ for all k. In particular, note that $(x_k - x_i)$ evaluates to $C^\mu(i,k)$ (see (2.1)).

The polynomial $\mathcal{H}_{i,j}(A,B)$, for arbitrary subsets A, B of $(i..j)$, is defined to be the following rational expression in $\mathbb{Q}(x_i, \dots, x_{j-1})$:

$$\mathcal{H}_{i,j}(A,B) := \sum_{D \subseteq B \setminus A} (-1)^{|D|} \frac{\prod_{t \in A} (x_t - x_{D_i(t)})}{\prod_{t \in B} (x_t - x_{D_i(t)})}.$$

Here, given any set $D \subseteq (i..j)$ and any $t \in (i..j)$, $D_i(t)$ denotes the greatest element of D that is strictly smaller than t, or i if no such element exists. It is not immediate that this rational expression is a polynomial, but this is proved in [B1], Lemma 4.6. The first lemma follows easily from these definitions.

4.1. **Lemma.** *If $B \subseteq A \subseteq (i..j)$, then $H_{i,j}^\mu(A,B) = \prod_{a \in A \setminus B} C^\mu(i,a)$.*

We now prove two technical facts about the operators $S_{i,j}^\mu(A)$. The first of these follows easily from the commutator relations proved in [B1] or [K3]. This will be used to construct $U(n-1)$-high weight vectors.

4.2. **Proposition (First technical fact).** *Let $\lambda \in \Lambda^+(n)$ be p-restricted and $\mu \xleftarrow{\text{norm}} \lambda$, so that $f_{\mu,\lambda} \in L_n(\lambda)$ (see Criterion 3.3). Suppose $A \subseteq (i..n)$ satisfies*

$$S_{j,n}^\mu(A \cap (j..n)) f_{\mu,\lambda} = 0$$

for all $i < j < n$ with $j \notin A$. Then $S_{i,n}^\mu(A) f_{\mu,\lambda}$ is a $U(n-1)$-high weight vector (possibly zero) in $L_n(\lambda)$.

Proof. As λ is p-restricted, it suffices by [K2], Theorem B to show that

$$E_{j-1}S^{\mu}_{i,n}(A)f_{\mu,\lambda} = 0$$

for all $1 < j < n$. Now note that $f_{\mu,\lambda}$ is a weight vector of weight μ. So $S^{\mu}_{i,n}(A)f_{\mu,\lambda} = S_{i,n}(A)f_{\mu,\lambda}$, where $S_{i,n}(A)$ is (the classical analogue of) the operator $S_{i,n}(A)$ defined in [B1] or [B2]. Also, $E_{j-1}f_{\mu,\lambda} = 0$ for $1 < j < n$, so by [B1], Lemma 4.11(i), $E_{j-1}S^{\mu}_{i,n}(A)f_{\mu,\lambda} = 0$ unless $j-1 \in \{i\} \cup A$ and $j \notin A$. In this remaining case, [B1], Lemma 4.11(ii) shows that

$$E_{j-1}S^{\mu}_{i,n}(A)f_{\mu,\lambda} = -S^{\mu}_{i,j-1}(A \cap (i..j-1))S^{\mu}_{j,n}(A \cap (j..n))f_{\mu,\lambda}$$

which is zero by assumption. $\square$

We still need to be able to show that the high weight vectors in Proposition 4.2 are non-zero under suitable circumstances. For this, we need a second technical fact, which is rather harder to prove. First, a preliminary lemma, which is easily verified working in $U(n,\mathbb{Z})$:

4.3. **Lemma.** *Fix $1 \le i < j \le n$ and $a \ge 0$. The following commutator relation holds in $U(n)$:*

$$[E^{(a+1)}_{j-1}, F_{i,j}] = \begin{cases} E^{(a)}_{j-1}F_{i,j-1} & \text{if } i < j-1, \\ E^{(a)}_{j-1}(H_{j-1} - H_j + a) & \text{if } i = j-1. \end{cases}$$

Next, for $1 \le i < j \le n$ and $A \subseteq (i..j)$, we define the polynomial $\mathcal{K}_{i,j}(A) \in \mathbb{Z}[x_i, \dots, x_{j-1}; y_{i+1}, \dots, y_j]$ as in [B1]. By definition,

$$\mathcal{K}_{i,j}(A) := \sum_{B \subseteq (i..j)} \left(\mathcal{H}_{i,j}(A,B) \prod_{t \in B \cup \{i\}} (y_{t+1} - x_t) \right).$$

The key property of $\mathcal{K}_{i,j}(A)$ is as follows:

4.4. **Lemma.** *Let $B, C \subseteq (i..j)$ be any sets such that there is a bijection $\theta : C \to B$ with $\theta(c) \ge c$ for all $c \in C$. Let $A = (i..j) \setminus C$. Then,*

$$\mathcal{K}_{i,j}(A) \equiv \prod_{t \in \{i,\dots,j-1\} \setminus B} (y_{t+1} - x_i) \quad \textit{(modulo } J\textit{)},$$

where J is the ideal of $\mathbb{Z}[x_i, \dots, x_{j-1}; y_{i+1}, \dots, y_j]$ generated by $\{y_{\theta(c)+1} - x_c \mid c \in C\}$.

Proof. This is immediate from [B1], Lemma 4.13(ii). $\square$

Define $K^{\mu,\lambda}_{i,j}(A)$ by evaluating the indeterminates in the polynomial $\mathcal{K}_{i,j}(A)$ at

$$x_k := \mathrm{res}_p(k, \mu_k); \qquad y_{k+1} := \mathrm{res}_p(k+1, \lambda_{k+1} + 1).$$

Note $(x_k - x_i)$ evaluates to $C^{\mu}(i,k)$ and $(y_{k+1} - x_i)$ evaluates to $B^{\mu,\lambda}(i,k)$. So,

$$K^{\mu,\lambda}_{i,j}(A) = \sum_{B \subseteq (i..j)} \left(H^{\mu}_{i,j}(A,B) \prod_{t \in B \cup \{i\}} B^{\mu,\lambda}(t,t) \right).$$

4.5. **Proposition (Second technical fact).** *Let $\mu \overset{\text{norm}}{\longleftarrow} \lambda$, so that $f_{\mu,\lambda} \in L_n(\lambda)$. Let $1 \le i < j \le n$ and $C \subseteq \mathscr{C}^\mu(i,j)$. Set $B = \mathscr{B}^{\mu,\lambda}(i,j)$. Suppose that there exists a bijection $\theta : C \to B$ such that $\theta(c) \ge c$ for all $c \in C$. Let $A = (i..j) \setminus C$. Then, $S^\mu_{i,j}(A) f_{\mu,\lambda} \neq 0$.*

Proof. Suppose μ is in level l (see (2.2)), and let $\bar{\mu} = (\mu_1, \dots, \mu_{n-1}, \lambda_n + l)$ be the $U(n)$-weight of $f_{\mu,\lambda}$. We may write $\bar{\mu} = \lambda - \sum_{s=1}^{n-1} a_s(\varepsilon_i - \varepsilon_{i+1})$ for unique non-negative integers a_s. For notational convenience, we let $X_1 = E_1^{(a_1)} \dots E_{i-1}^{(a_{i-1})}$ and $X_2 = E_j^{(a_j)} \dots E_{n-1}^{(a_{n-1})}$.

Step one. We first show that for any $h < j$,

$$E_{j-1}^{(a_{j-1}+1)} X_2 F_{h,j} f_{\mu,\lambda} = \begin{cases} E_{j-1}^{(a_{j-1})} X_2 F_{h,j-1} f_{\mu,\lambda} & \text{if } h < j-1, \\ B^{\mu,\lambda}(j-1,j-1) E_{j-1}^{(a_{j-1})} X_2 f_{\mu,\lambda} & \text{if } h = j-1. \end{cases}$$

For this, we observe that X_2 commutes with $F_{h,j-1}$ and $F_{h,j}$, and that (by considering weights), $E_{j-1}^{(a_{j-1}+1)} X_2 f_{\mu,\lambda} = 0$. The first case follows immediately from these two observations and the first case of Lemma 4.3. For the second case, where $h = j-1$, Lemma 4.3 implies that $E_{j-1}^{(a_{j-1}+1)} X_2 F_{h,j} f_{\mu,\lambda} = E_{j-1}^{(a_{j-1})} (H_{j-1} - H_j + a_{j-1}) X_2 f_{\mu,\lambda}$. So, we need to evaluate $(H_{j-1} - H_j + a_{j-1})$ on the weight $\lambda - \sum_{t=1}^{j-1} a_t(\varepsilon_t - \varepsilon_{t+1})$; H_j evaluates to $\lambda_j + a_{j-1}$ on this. On the other hand, this weight also equals $\bar{\mu} + \sum_{t=j}^{n-1} a_t(\varepsilon_t - \varepsilon_{t+1})$ and H_{j-1} evaluates to μ_{j-1} on this. Consequently, $(H_{j-1} - H_j + a_{j-1})$ evaluates to $\mu_{j-1} - \lambda_j = B^{\mu,\lambda}(j-1,j-1)$ on $X_2 f_{\mu,\lambda}$, and step one follows.

Step two. We next show that

$$X_1 E_i^{(a_i+1)} \dots E_{j-1}^{(a_{j-1}+1)} X_2 S^\mu_{i,j}(A) f_{\mu,\lambda} = K^{\mu,\lambda}_{i,j}(A) f_\lambda.$$

Note that $E_1^{(a_1)} \dots E_{n-1}^{(a_{n-1})} f_{\mu,\lambda} = f_\lambda$ by Lemma 2.6(ii). So, it suffices to show by definition of $S^\mu_{i,j}(A)$ and $K^{\mu,\lambda}_{i,j}(A)$ that

$$X_1 E_i^{(a_i+1)} \dots E_{j-1}^{(a_{j-1}+1)} X_2 F^D_{i,j} f_{\mu,\lambda} = \left(\prod_{t \in D \cup \{i\}} B^{\mu,\lambda}(t,t) \right) E_1^{(a_1)} \dots E_{n-1}^{(a_{n-1})} f_{\mu,\lambda}$$

for all subsets $D \subseteq (i..j)$. This follows from step one by induction on $(j-i)$.

Step three. We finally show that the right hand side of the expression in step two is non-zero, to prove the lemma. To see this, just note that $B^{\mu,\lambda}(c, \theta(c)) = B^{\mu,\lambda}(i, \theta(c)) - C^\mu(i,c) = 0$ for all $c \in C$, so by Lemma 4.4,

$$K^{\mu,\lambda}_{i,j}(A) = \prod_{t \in \{i,\dots,j-1\} \setminus B} B^{\mu,\lambda}(i,t)$$

which is non-zero by definition of B. $\square$

5 Proof of the Main Theorem

We are now in a position to prove the Main Theorem.

We begin by reducing the proof to the case that λ is p-restricted. So suppose that $p > 0$ and take an arbitrary $\lambda \in \Lambda^+(n)$ and let $\lambda = \lambda(0) + p\lambda(1) + \cdots + p^d\lambda(d)$ be a p-adic expansion of λ, where each $\lambda(i)$ is p-restricted. Let $F : GL(n) \to GL(n)$ denote the Frobenius pth power map, and given a $GL(n)$-module V, let $V^{[j]}$ denote the jth Frobenius twist of V, with new action $g.v = F^j(g)v$ for $v \in V, g \in GL(n)$. Then, Steinberg's tensor product theorem [St], Theorem 41 tells us that $L_n(\lambda) \cong L_n(\lambda(0)) \otimes L_n(\lambda(1))^{[1]} \otimes \cdots \otimes L_n(\lambda(d))^{[d]}$. Let $\mathrm{soc}(\lambda)$ denote the socle of the restriction $\mathrm{res}^{GL(n)}_{GL(n-1)} L_n(\lambda)$. The following result is proved in [K2]:

5.1. **Theorem.** *Let $\lambda = \lambda(0)+p\lambda(1)+\cdots+p^d\lambda(d)$ be a p-adic expansion of $\lambda \in \Lambda^+(n)$. For all i, each composition factor of $\mathrm{soc}(\lambda(i))$ is p-restricted, and*

$$\mathrm{soc}(\lambda) \cong \mathrm{soc}(\lambda(0)) \otimes \mathrm{soc}(\lambda(1))^{[1]} \otimes \cdots \otimes \mathrm{soc}(\lambda(d))^{[d]}.$$

Now, observe that $\mathrm{res}^{GL(n)}_{GL(n-1)} L_n(\lambda)$ is semisimple if and only if $\dim \mathrm{soc}(\lambda) = \dim L_n(\lambda)$. Consequently, by the theorem, we see that $\mathrm{res}^{GL(n)}_{GL(n-1)} L_n(\lambda)$ is semisimple if and only if $\dim \mathrm{soc}(\lambda(i)) = \dim L_n(\lambda(i))$ for each i, which is if and only if each $\mathrm{res}^{GL(n)}_{GL(n-1)} L_n(\lambda(i))$ is semisimple. Moreover, in that case, the theorem shows that the composition factors of the restriction are the corresponding twisted tensor products of the composition factors of each $\mathrm{res}^{GL(n)}_{GL(n-1)} L_n(\lambda(i))$.

Hence, to prove the Main Theorem, it suffices to consider the case that λ is p-restricted. So, for the remainder of the section, we assume that $\lambda = (\lambda_1, \dots, \lambda_n)$ is a p-restricted dominant weight. Our first aim is to show that if λ is GJS, then $\mathrm{res}^{U(n)}_{U(n-1)} L_n(\lambda)$ is semisimple. We need a preliminary lemma:

5.2. **Lemma.** *Fix i with $1 \le i < n$ together with integers $x_i > 0$ and $x_k \ge 0$ for all $k \in (i..n)$. Let A be any subset of $(i..n)$ such that $k \in (i..n)$ and $x_k \neq 0$ imply $k \in A$. Then, the $F^{(x_i)}_{i,n} F^{(x_{i+1})}_{i+1,n} \dots F^{(x_{n-1})}_{n-1,n}$-coefficient of $F^{(x_i-1)}_{i,n} F^{(x_{i+1})}_{i+1,n} \dots F^{(x_{n-1})}_{n-1,n} S^\lambda_{i,n}(A)$ when written in terms of the PBW basis is $x_i \prod_{a \in A} \left(C^\lambda(i,a) + x_a\right)$.*

Proof. A routine induction on $(n-i)$ shows that for any subset B of $(i..n)$, the $F^{(x_i)}_{i,n} \dots F^{(x_{n-1})}_{n-1,n}$-coefficient of $F^{(x_i-1)}_{i,n} F^{(x_{i+1})}_{i+1,n} \dots F^{(x_{n-1})}_{n-1,n} F^B_{i,n}$ is $x_i \prod_{b\in B} x_b$. Consequently, by definition of $S^\lambda_{i,n}(A)$, the corresponding coefficient in $F^{(x_i-1)}_{i,n} F^{(x_{i+1})}_{i+1,n} \dots F^{(x_{n-1})}_{n-1,n} S^\lambda_{i,n}(A)$ is

$$x_i \sum_{B \subseteq (i..n)} H^\lambda_{i,n}(A,B) \prod_{b \in B} x_b.$$

Now note that if $B \subseteq A$, then $H^\lambda_{i,n}(A,B)$ is given by Lemma 4.1. If $B \not\subseteq A$, then $x_b = 0$ for some $b \in B$ by assumption. Consequently, the required coefficient equals

$$x_i \sum_{B \subseteq A} \prod_{a \in A \setminus B} C^\lambda(i,a) \prod_{b \in B} x_b$$

which factorizes to give the conclusion. $\square$

5.3. **Proposition.** *Suppose that $\lambda \in \Lambda^+(n)$ is a p-restricted GJS weight and $\mu \overset{\text{norm}}{\longleftarrow} \lambda$. Then, μ is allowable.*

Proof. Suppose that μ is not allowable, so that there are removable rows $1 \leq i < j < n$ such that $\mu_i < \lambda_i, \mu_j > \lambda_{j+1}$, with $B^\lambda(i,j) = 0$. Choose i, j so that the B-chain $i < j$ cannot be refined to a longer *proper* B-chain. The strategy is as follows. Let $y_k = \lambda_k - \mu_k$ for all $1 \leq k < n$ and $A = \{i < a < j+1 \,|\, C^\lambda(i,a) \neq 0\}$. We will consider the element

$$S := \underbrace{F_{1,n}^{(y_1)} \dots F_{i-1,n}^{(y_{i-1})}}_{S_1} \underbrace{F_{i,n}^{(y_i-1)} F_{i+1,n}^{(y_{i+1})} \dots F_{j,n}^{(y_j)}}_{S_2} \underbrace{F_{j+1,n}^{(y_{j+1}+1)} F_{j+2,n}^{(y_{j+2})} \dots F_{n-1,n}^{(y_{n-1})}}_{S_3} S_{i,j+1}^\lambda(A).$$

of $U^-(n)$. We wish to show that $Sf_\lambda = 0$ but that S has non-zero $F_{\mu,\lambda}$-coefficient when expanded in terms of the PBW basis for $U^-(n)$. This will show that μ is not normal by Lemma 3.5. For notational convenience we split S as a product $S_1 S_2 S_3 S_{i,j+1}^\lambda(A)$, where the S_i are the terms indicated above. If $j+1 = n$, then we take $S_3 = 1$.

We first show that $\mathscr{B}^\lambda(i,j+1)$ is the disjoint union of $\mathscr{C}^\lambda(i,j+1)$ and $\{j\}$, and that *every* removable row k with $i < k < j$ is an element of A. First, note that $B^\lambda(i,j) = 0 \neq C^\lambda(i,j)$ as λ is p-restricted. Also, if k is not a removable row, then $C^\lambda(i,k) = B^\lambda(i,k)$. Thus, it suffices to show given any removable row k with $i < k < j$, that k is not an element of $\mathscr{C}^\lambda(i,j+1)$ or $\mathscr{B}^\lambda(i,j+1)$.

If $C^\lambda(i,k) = 0$ then $B^\lambda(k,j) = B^\lambda(i,j) - C^\lambda(i,k) = 0$. So the B-chain $i < j$ can be refined by Lemma 1.1, contradicting maximality. If $B^\lambda(i,k) = 0$, the B-chain $i < j$ can be refined, again by Lemma 1.1, to give a contradiction.

We have shown in particular that there is no injection from $\mathscr{B}^\lambda(i,j+1)$ to $\mathscr{C}^\lambda(i,j+1)$. So, $S_{i,j+1}^\lambda(A) f_\lambda = 0$ by [B1], Lemma 5.2. Hence, $Sf_\lambda = 0$.

Now we consider the $F_{\mu,\lambda}$-coefficient of S when expanded in terms of the PBW basis. Suppose first that $j+1 \neq n$. By definition, $S_{i,j+1}^\lambda(A)$ is a sum of monomials of the form $F_{i,j+1}^B H_{i,j+1}^\lambda(A,B)$. Now, $S_3 F_{i,j+1}^B = F_{i,n}^B F_{j+1,n}^{(y_{j+1})} \dots F_{n-1,n}^{(y_{n-1})}$ together with the term $F_{i,j+1}^B S_3$ which will not contribute to the $F_{\mu,\lambda}$-coefficient of S. Moreover, $H_{i,n}^\lambda(A,B)$ equals 0 if $B \not\subseteq (i..j+1)$ (by [B1], Lemma 4.6(ii)) and $H_{i,j+1}^\lambda(A,B)$ otherwise. Hence, the $F_{\mu,\lambda}$-coefficient of S equals the $F_{\mu,\lambda}$-coefficient of

$$S_1 S_2 S_{i,n}^\lambda(A) F_{j+1,n}^{(y_{j+1})} \dots F_{n-1,n}^{(y_{n-1})}$$

if $j+1 \neq n$. If $j+1 = n$, then by definition the $F_{\mu,\lambda}$-coefficient of S is the $F_{\mu,\lambda}$-coefficient of $S_1 S_2 S_{i,n}^\lambda(A)$.

In both cases, this coefficient is equal to the $F_{i,n}^{(y_i)} \dots F_{j,n}^{(y_j)}$-coefficient of $S_2 S_{i,n}^\lambda(A)$, which we computed in Lemma 5.2. It equals

$$y_i \prod_{a \in A} \left(C^\lambda(i,a) + y_a \right).$$

It just remains to show that this expression is non-zero. First note that $0 < y_i \leq \lambda_i - \lambda_{i+1}$ by assumption. So, $y_i \not\equiv 0 \ (p)$, as λ is p-restricted. Also, $C^\lambda(i,j) + y_j = B^\lambda(i,j) + y_j - (\lambda_j - \lambda_{j+1}) \neq 0$ as $B^\lambda(i,j) = 0$ and $y_j \not\equiv \lambda_j - \lambda_{j+1} \ (p)$ by assumption. So, if the expression vanishes, then some $C^\lambda(i,a) + y_a = 0$ for some $i < a < j$. Now, if $y_a = 0$, this is not the case by definition of A.

So, we may assume that $C^\lambda(i,k) + y_k = 0$ for some $i < k < j$ with $y_k > 0$, so that in particular k is a removable row. We now show that there is a B-chain from i to k, so that Lemma 1.1 gives a contradiction to maximality of the B-chain $i < j$. If

$y_k < \lambda_k - \lambda_{k+1}$, $C^\lambda(i,k) + y_k = 0$ implies that $R_i(\lambda)$ meets $R_k(\lambda)$, giving a B-chain from i to k. Finally, if $y_k = \lambda_k - \lambda_{k+1}$, then $C^\lambda(i,k) + y_k = B^\lambda(i,k) = 0$, so again there is a B-chain from i to k. □

5.4. **Corollary.** *If $\lambda \in \Lambda^+(n)$ is a p-restricted GJS weight, then $\operatorname{res}^{U(n)}_{U(n-1)} L_n(\lambda)$ is semisimple.*

Proof. We are going to use a consequence of the linkage principle, which says that $L_{n-1}(\nu)$ is a composition factor of $\Delta_{n-1}(\mu)$ only if ν and μ have the same residue content (that is, the p-residues in the diagrams $[\nu], [\mu]$ are in 1–1 correspondence). Indeed, it is proved in [J1], II.6.13 that $L_{n-1}(\nu)$ is a composition factor of $\Delta_{n-1}(\mu)$ only if ν and μ are conjugate under the action of the affine Weyl group W_a corresponding to $U(n-1)$. However it is well known (see e.g. [CL], section 4) that μ and ν are W_a-conjugate if and only if they have the same residue content.

Now, by Proposition 3.8, to prove the semisimplicity of the restriction, it suffices to show that every $f_{\mu,\lambda}$ for $\mu \overset{\text{norm}}{\longleftarrow} \lambda$ generates an irreducible $U(n-1)$-module. Suppose not. Then we can find weights μ and ν, both normal for λ, such that $L_{n-1}(\nu)$ is a composition factor of $U(n-1)f_{\mu,\lambda}$. That is, by Proposition 5.3 and the universality of standard modules, there are allowable weights μ and ν such that $L_{n-1}(\nu)$ is a composition factor of $\Delta_{n-1}(\mu)$.

From the linkage principle, ν and μ have the same residue content. By Lemma 1.2, λ either satisfies (JS1), when clearly no two distinct $\nu, \mu \longleftarrow \lambda$ have the same residue content, or (JS2). In the latter case, no two allowable weights even lie in the same level, so again cannot have the same residue content. □

We now want to prove the converse to Proposition 5.3 – that is, if λ is GJS and μ is allowable, then μ is normal.

5.5. **Theorem.** *If $\lambda \in \Lambda^+(n)$ is a p-restricted GJS weight, then*

$$\operatorname{res}^{U(n)}_{U(n-1)} L_n(\lambda) \cong \bigoplus_{\mu \text{ allowable}} L_{n-1}(\mu).$$

Proof. By Corollary 5.4 and Proposition 5.3, it suffices to show that if ν is allowable then $\nu \overset{\text{norm}}{\longleftarrow} \lambda$. We prove this by induction on level, the result being clear for level 0 when $\nu = (\lambda_1, \dots, \lambda_{n-1})$ [which is always normal as $f_{\nu,\lambda} = f_\lambda$]. So let ν be allowable in level l and suppose that the result has been proved for all smaller levels. Choose i minimal such that $\nu = (\mu_1, \dots, \mu_{i-1}, \mu_i - 1, \mu_{i+1}, \dots, \mu_{n-1})$ for some $\mu \longleftarrow \lambda$. By definition of allowable, μ is also allowable, so by induction, $\mu \overset{\text{norm}}{\longleftarrow} \lambda$.

Now, let $B = \mathscr{B}^{\mu,\lambda}(i,n), C = \mathscr{C}^\mu(i,n)$. We claim that $B = C$. For this, note first that $i \notin B$ as λ is p-restricted. Also, for $k > i$, if k is not a removable row for λ, then $B^{\mu,\lambda}(i,k) = C^\mu(i,k)$. So, it suffices to consider a removable row (for λ) $k > i$. Suppose first that $k > i$ is an element of C. Then, as ν is allowable and λ is GJS, $C^\mu(i,k) = 0$ implies that $\mu_k = \lambda_{k+1}$, for otherwise $R_i(\lambda)$ meets $R_k(\lambda)$ and hence $\mu_k = \lambda_{k+1}$, giving a contradiction. So $B^{\mu,\lambda}(i,k) = k - i + \mu_i - \lambda_{k+1} = k - i + \mu_i - \mu_k = C^\mu(i,k) = 0$, and $k \in B$. Conversely, suppose that $k > i$ is an element of B. We show that there is a B-chain from i to k. This is immediate if $\lambda_i = \mu_i$, since then $0 = B^{\mu,\lambda}(i,k) = B^\lambda(i,k)$. And if $\lambda_i \neq \mu_i$, then $B^{\mu,\lambda}(i,k) = 0$ implies that $R_i(\lambda)$ meets $R_k(\lambda)$, so again there is a

B-chain from i to k by definition of GJS. Consequently, as ν is allowable, $\mu_k = \lambda_{k+1}$, and $k \in C$ as required.

So, $B = C$ as claimed. Let $A = (i..n) \setminus C$. Proposition 4.5 (taking $\theta : C \to B$ to be the identity) implies that $S^{\mu}_{i,n}(A)f_{\mu,\lambda}$ is a non-zero element of $L_n(\lambda)$. We now show that it is a $U(n-1)$-high weight vector, so that $\nu \overset{\text{norm}}{\longleftarrow} \lambda$ as required. To prove this, we use Proposition 4.2 to see that it suffices to show that

$$S^{\mu}_{j,n}(A \cap (j..n))f_{\mu,\lambda} = 0$$

for all $i < j < n$ with $j \notin A$.

Suppose for a contradiction that there is some $j \in (i..n) \setminus A$ with $S^{\mu}_{j,n}(A')f_{\mu,\lambda} \neq 0$ where $A' = A \cap (j..n)$. Choose the greatest such j, so that Proposition 4.2 applied to A' shows that $S^{\mu}_{j,n}(A')f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector of weight $\nu' = (\mu_1, \dots, \mu_j - 1, \dots, \mu_{n-1})$. Then $\mu_j > \lambda_{j+1}$. On the other hand, $j \notin A$ implies $C^{\mu}(i,j) = 0$. So $R_i(\lambda)$ meets $R_j(\lambda)$. Hence, there is a B-chain from i to j which forces $\mu_j = \lambda_{j+1}$ since ν is allowable. The contradiction obtained completes the proof. □

Finally, we need to prove the converse to the Main Theorem.

5.6. **Theorem.** *Let $\lambda \in \Lambda^+(n)$ be a p-restricted weight that is not GJS. Then, the restriction of $L_n(\lambda)$ to $U(n-1)$ is not semisimple.*

Proof. The idea here is as follows: using the argument in the proof of Theorem 5.5 we will construct an operator $T \in U^-(n)$ such that Tf_λ is a non-zero $U(n-1)$-high weight vector of weight μ, where the $F_{\mu,\lambda}$-coefficient of T when written in terms of the PBW basis is zero. This will show that $L_n(\lambda)$ does not have semisimple restriction to $U(n-1)$ because of Proposition 3.8(iv). The argument is quite long, so we break it into a number of steps.

Step one. We can find rows $i < j$ such that $R_i(\lambda)$ meets $R_j(\lambda)$ but there is no B-chain from i to j. That is, we can find nodes (i,a) and (j,b) in $[\lambda]$ such that

(1) $1 \leq i < j < n, \lambda_{i+1} < a \leq \lambda_i$ and $\lambda_{j+1} < b \leq \lambda_j$;

(2) $\mathrm{res}_p(i,a) = \mathrm{res}_p(j,b)$;

(3) there is no B-chain from i to j.

We wish to choose (i,a) to be maximal with respect to the lexicographic ordering on $\mathbb{N} \times \mathbb{N}$ such that (1)–(3) hold for some (j,b). That is, for every node (i',a') with either $i' > i$ or $i' = i, a' > a$, there is no node (j',b') such that (1)–(3) hold for (i',a') and (j',b').

We now define $\mu \longleftarrow \lambda$ as follows. First, let $\mu_1 = \lambda_1, \dots, \mu_{i-1} = \lambda_{i-1}$ and $\mu_i = a-1, \mu_j = b$. The remaining μ_k for $i < k < n, k \neq j$ are defined by

$$\mu_k = \begin{cases} \lambda_{k+1} & \text{if there is a } B\text{-chain from } i \text{ to } k, \\ \lambda_{k+1} & \text{if } k > j \text{ and there is a } B\text{-chain from } j \text{ to } k, \\ \lambda_k & \text{otherwise.} \end{cases}$$

Suppose μ is in level l. Let $\mu(0) = (\lambda_1, \dots, \lambda_{n-1})$, and define $\mu(s)$ for $l \geq s > 0$ inductively by choosing k to be maximal such that the kth row of μ is different from the kth row of $\mu(s-1)$ and letting $\mu(s)$ equal $\mu(s-1)$ with one extra node removed from this kth row.

Step two. We wish to prove by induction on s that $\mu(s) \overset{\text{norm}}{\longleftarrow} \lambda$. This is clear for $s = 0$, so consider $\mu(s+1)$ assuming by induction that $\mu(s) \overset{\text{norm}}{\longleftarrow} \lambda$. The argument proceeds as in the proof of Theorem 5.5. For convenience, let $\nu = \mu(s+1), \gamma = \mu(s)$ and suppose ν is obtained from γ by removing one node from the kth row. Note that by the construction in step one, if $k < j$ then either $k = i$ or there is a B-chain from i to k, and if $k > j$ then there is a B-chain from either i or j to k.

Let $B = \mathscr{B}^{\gamma,\lambda}(k,n)$. We claim that $B \subseteq \mathscr{C}^{\gamma}(k,n)$. Take any $h \in B$. Then $h > k$ since λ is p-restricted. If h is not a removable row for λ, then $B^{\gamma,\lambda}(k,h) = C^{\gamma}(k,h)$ and $h \in \mathscr{C}^{\gamma}(k,n)$. So we may assume that h is a removable row for λ. We show first that there is a B-chain from k to h; this is immediate if $\lambda_k = \gamma_k$, since then $0 = B^{\gamma,\lambda}(k,h) = B^{\lambda}(k,h)$. If $\lambda_k \neq \gamma_k$, $B^{\gamma,\lambda}(k,h) = 0$ implies that $(k, \gamma_k + 1)$ has the same p-residue as $(h, \lambda_{h+1} + 1)$. In other words, $R_k(\lambda)$ meets $R_h(\lambda)$, so the required B-chain from k to h exists by the maximality assumption on (i,a). In particular, this shows that $h \neq j$ – for if $h = j$, then $k < j$ so there is a B-chain from i to k (or $i = k$) and from k to j, but by assumption there is no B-chain from i to j. Consequently, there is a B-chain from either i or j to k (or $k = i$ or j) and from k to h, so by the construction in step one, $\gamma_h = \lambda_{h+1}$. So, $B^{\gamma,\lambda}(k,h) = h-k+\gamma_k-\lambda_{h+1} = h-k+\gamma_k-\gamma_h = C^{\gamma}(k,h)$, and $h \in \mathscr{C}^{\gamma}(k,n)$. This proves the claim.

Set $A = (k..n) \setminus B$. Then, as $B \subseteq \mathscr{C}^{\gamma}(k,n)$, Proposition 4.5 (taking θ to be the identity and $C := B$) implies that $S^{\gamma}_{k,n}(A) f_{\gamma,\lambda}$ is a non-zero element of $L_n(\lambda)$. We now show that it is a $U(n-1)$-high weight vector, so that $\nu \xleftarrow{\text{norm}} \lambda$ as required. To prove this, we use Proposition 4.2 to see that it suffices to show that $S^{\gamma}_{h,n}(A \cap (h..n)) f_{\gamma,\lambda} = 0$ for all $k < h < n$ with $h \notin A$. Well, take the largest h with $k < h < n, h \notin A$ such that $S^{\gamma}_{h,n}(A') f_{\gamma,\lambda} \neq 0$ where $A' = A \cap (h..n)$. Then, Proposition 4.2 applied to A' shows that $S^{\gamma}_{h,n}(A') f_{\gamma,\lambda}$ is a non-zero $U(n-1)$-high weight vector of weight $\nu' = (\gamma_1, \dots, \gamma_{h-1}, \gamma_h - 1, \gamma_{h+1}, \dots, \gamma_{n-1})$, for some $h > k$. In particular, this means $h \in B$ must be a removable row for λ. But we proved in the previous paragraph that for any removable row $h \in B$ we have $\gamma_h = \lambda_{h+1}$ so $\nu' \longleftarrow \lambda$ is false, giving a contradiction.

Step three. Now let $\gamma = (\mu_1, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_{n-1})$ [that is, $\gamma = \mu(l-1)$]. By step two, $f_{\gamma,\lambda} \in L_n(\lambda)$. So there is an operator $S \in U^-(n)$ such that $S f_\lambda = f_{\gamma,\lambda}$. Moreover, we showed in step two that if $B = \mathscr{B}^{\gamma,\lambda}(i,n)$, $A = (i..n) \setminus B$, then $j \in A$ and $S^{\gamma}_{i,n}(A) S f_\lambda$ is a non-zero high weight vector, so a non-zero multiple of $f_{\mu,\lambda}$.

We now claim that the $F_{\mu,\lambda}$-coefficient of $T := S^{\gamma}_{i,n}(A) S$ when written in terms of the PBW basis is zero. This suffices to complete the proof of the theorem by Proposition 3.8(iv).

For the claim, we may write S as a linear combination of monomials in the PBW basis of the form $F^{(N_{i,i+1})}_{i,i+1} F^{(N_{i,i+2})}_{i,i+2} F^{(N_{i+1,i+2})}_{i+1,i+2} \dots F^{(N_{i,n})}_{i,n} \dots F^{(N_{n-1,n})}_{n-1,n}$ for non-negative integers $N_{h,k}$. By definition, $S^{\gamma}_{i,n}(A)$ equals $\displaystyle\sum_{B \subseteq (i..n)} H^{\gamma}_{i,n}(A,B) F^{B}_{i,n}$. So, no $F_{\mu,\lambda}$-coefficient arises from the PBW-expansion of each monomial in the product $S^{\gamma}_{i,n}(A) S$ except possibly in the case $B = \varnothing$. But by Lemma 4.1, $H^{\gamma}_{i,n}(A, \varnothing) = \displaystyle\prod_{d \in A} C^{\gamma}(i,d)$. As $j \in A$ and $C^{\gamma}(i,j) = j - i + \gamma_i - \gamma_j = j - i + a - b = \mathrm{res}_p(j,b) - \mathrm{res}_p(i,a) = 0$, this product is zero, proving the claim. $\square$

Theorem 5.5 and Theorem 5.6, together with the remarks after Theorem 5.1, complete the proof of the Main Theorem.

6 Application: completely splittable representations

We define the notion of a completely splittable representation of $GL(n)$ by analogy with the definition for symmetric groups – see [K4]. Recall that a ***standard Levi subgroup*** of $GL(n)$ is a subgroup of the form $GL(n_1) \times GL(n_2) \times \cdots \times GL(n_k)$, $n = n_1 + n_2 + \cdots + n_k$, embedded diagonally. A subgroup of $GL(n)$ is called a ***Levi subgroup*** if it is conjugate to a standard Levi subgroup. A rational irreducible $GL(n)$-module is called ***completely splittable*** if its restriction to any Levi subgroup is semisimple.

In this section, we will give a classification of all completely splittable representations. Of course, if $p = 0$, every rational irreducible $GL(n)$-module is completely splittable, so we exclude the case $p = 0$ for the remainder of this section. We begin with an elementary lemma.

6.1. **Lemma.** *Let L be a rational irreducible $GL(n)$-module. Then L is completely splittable if and only if $\operatorname{res}^{GL(n)}_{GL(m)} L$ is semisimple for all $m < n$ (the subgroup $GL(m)$ being embedded into the top left corner).*

Proof. We consider a standard Levi subgroup $H = GL(m) \times \overbrace{GL(1) \times \cdots \times GL(1)}^{(n-m) \text{ times}}$. Any (rational) irreducible H-module remains irreducible on restriction to $GL(m) < H$. So if L is completely splittable then $\operatorname{res}^{GL(n)}_{GL(m)} L$ is semisimple.

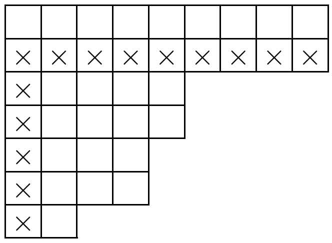
Conversely, assume this restriction is semisimple for all $m < n$. It suffices to prove that $\operatorname{res}^{GL(n)}_{H} L$ is semisimple for all *standard* Levi subgroups $H = GL(n_1) \times \cdots \times GL(n_k)$. Any subgroup $GL(n_j)$ appearing in this direct product is conjugate to the subgroup $GL(n_j)$ embedded into the top left corner. So by assumption the restriction to any such a subgroup is semisimple. Now the result follows from the following general fact, proved for example in [K4], 1.6: let G_1 and G_2 be groups, $G = G_1 \times G_2$, $\mathbb{F}$ be an algebraically closed field, and M be a finite dimensional $\mathbb{F}G$-module such that $\operatorname{res}^{G}_{G_i} M$ is semisimple for $i = 1, 2$. Then M is semisimple. (The assumption that the G_i are finite made in [K4], 1.6 was never used in the proof). $\square$

Now let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an arbitrary dominant weight for $GL(n)$. Let i be the smallest removable row for λ and let j be the largest removable row for λ – so $1 \le i \le j < n$. We define

$$\psi_n(\lambda) := j - i + \lambda_i - \lambda_{j+1}.$$

Note this is just $B^\lambda(i,j)$, but regarded now as an integer not as an element of $\mathbb{Z}/p\mathbb{Z}$. If $\lambda = c\delta = (c, c, \dots, c)$ then there are no removable rows and we put $\psi_n(\lambda) = 0$.

The easiest way to compute $\psi_n(\lambda)$ in practise is to first tensor with a power of det (which does not affect ψ_n) to ensure that $\lambda_n = 0$. In that case, we regard λ as a partition, and the number $\psi_n(\lambda)$ is the length of a particular hook in the diagram $[\lambda]$. In fact, providing $\lambda_n = 0$, then $\psi_n(\lambda) = \chi(\lambda')$ where χ is defined in [K4] and λ' is the transpose of the partition λ. For example, let $\lambda = (9, 9, 5, 5, 4, 4, 2, 0, \dots, 0) \in \Lambda^+(n)$ with $n \ge 8$. The boxes of the corresponding hook are marked with $\times$ in the following picture, and $\psi_n(\lambda) = 14$.



On the other hand, if $\lambda = (9, 9, 5, 5, 4, 4, 2)$ is a weight for $GL(7)$, one should first replace λ by $(7, 7, 3, 3, 2, 2, 0)$ and then see that $\psi_7(\lambda) = 11$.

The following theorem classifies all completely splittable representations:

6.2. **Theorem.** *Let $\lambda \in X^+(n)$ be an arbitrary dominant weight with p-adic expansion $\lambda = \lambda(0) + p\lambda(1) + \cdots + p^d\lambda(d)$. Then $L_n(\lambda)$ is completely splittable if and only if $\psi_n(\lambda(i)) \leq p$ for all $i = 0, 1, \ldots, d$.*

Proof. We note that Theorem 5.1 implies that a Steinberg tensor product is completely splittable if and only if all the terms involved are completely splittable. So to prove the theorem we need only consider the case that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is p-restricted.

We now proceed by induction on n. For $n = 1$ the result is trivial. Let $n > 1$. By tensoring with det if necessary, we assume that $\lambda_n = 0$. The result is obvious if $\lambda = 0$, so suppose also that $\lambda \neq 0$. Let i be the smallest removable row of λ, and let j be the largest removable row, so that $\psi_n(\lambda) = j - i + \lambda_i - \lambda_{j+1}$. Our assumption that $\lambda_n = 0$ implies that $\lambda_{j+1} = 0$.

Assume $\psi_n(\lambda) \leq p$. Then λ is GJS (as (JS1) holds). So, by the Main Theorem, $\operatorname{res}^{GL(n)}_{GL(n-1)} L_n(\lambda)$ is semisimple with composition factors $\{L_{n-1}(\mu) \mid \mu \text{ allowable for } \lambda\}$. If μ is allowable for λ, $\psi_{n-1}(\mu) \leq \psi_n(\lambda) + 1$ with equality if and only if $\mu_i \neq \lambda_i$ and $\mu_j \neq 0$. If $\psi_n(\lambda) = p$, then there is a B-chain from i to j, so such a μ would not be allowable. Consequently, $\psi_{n-1}(\mu) \leq p$ for all allowable μ. So, by the induction hypothesis, each $L_{n-1}(\mu)$ is completely splittable. Now it suffices to apply Lemma 6.1 to prove that $L_n(\lambda)$ is completely splittable as required.

In the other direction, assume that $\psi_n(\lambda) > p$. If $\operatorname{res}^{GL(n)}_{GL(n-1)} L_n(\lambda)$ is not semisimple, we are done. Otherwise λ is GJS, and

$$\operatorname{res}^{GL(n)}_{GL(n-1)} L_n(\lambda) \cong \bigoplus_{\mu \text{ allowable}} L_{n-1}(\mu)$$

(by the Main Theorem). So, by the inductive hypothesis and Lemma 6.1, it suffices to show that $\psi_{n-1}(\mu) > p$ for some μ allowable for λ. If $i = j$, then $p < \psi_n(\lambda) = \lambda_i$ which contradicts the p-restrictedness of λ. Otherwise, note that $\mu = (\lambda_1, \ldots, \lambda_{j-1}, 0, \ldots, 0) \in \Lambda^+(n-1)$ is allowable, and $\psi_{n-1}(\mu) = \psi_n(\lambda) - 1$. If $\psi_{n-1}(\mu) \leq p$ we must have $\psi_{n-1}(\mu) = p$ and $\psi_n(\lambda) = p + 1$. But then $R_i(\lambda)$ meets $R_j(\lambda)$, and there is no B-chain from i to j, which contradicts the fact that λ is GJS. $\square$

We call the weight $\lambda \in X^+(n)$ ***completely splittable*** if it satisfies the combinatorial condition in Theorem 6.2. Of course, the theorem shows that λ is completely splittable if and only if $L_n(\lambda)$ is completely splittable.

In [MP], Mathieu and Papadopoulo have given a character formula for modules $L_n(\lambda)$ with λ completely splittable. As a consequence of their character formula, they were able to show that these modules are completely splittable, giving an alternative proof of the implication ($\Leftarrow$) in Theorem 6.2. Our methods also give the explicit character formula for completely splittable representations obtained in [MP]. In our notation, the result is as follows:

6.3. **Theorem.** *Let $\lambda \in \Lambda^+(n)$ be completely splittable. For $\mu \in \Lambda(n)$, the dimension of the μ-weight space of $L_n(\lambda)$ is equal to the number of standard λ-tableaux t of weight μ, satisfying the extra condition* (†) *that* shape$(t[m])$ *is allowable for* shape$(t[m+1])$ *for $m = n-1, n-2, \dots, 1$.*

Proof. Note that if t is a standard λ-tableau with shape$(t[m+1])$ completely splittable and shape$(t[m])$ is allowable for shape$(t[m+1])$ for some m, then shape$(t[m])$ is also completely splittable by the argument of Theorem 6.2 – so the condition (†) makes sense.

The argument now is the same as the proof of the standard basis theorem given in the appendix. Arguing as there, using the Main Theorem and Lemma 3.6, one can conclude that if $\mu(i)$ is normal for λ, then $L^i/L^{i-1} \cong L_{n-1}(\mu(i))$ (notation as in Lemma 3.6). Then, induction on n yields that the set of $F_t f_\lambda$ for all standard λ-tableaux t satisfying (†) is a basis for $L_n(\lambda)$. This stronger statement immediately implies the character formula as stated. □

We conclude by describing an equivalent condition to (†) in the p-restricted case. Fix $\lambda \in \Lambda^+(n)$ with λ both p-restricted and completely splittable. We define a ***p-hook*** to be a pair of nodes (i, j) and (i', j') in $[\lambda]$ with $i - j + p - 1 = i' - j'$. We refer to (i, j) as the top-right node in the p-hook, and (i', j') as the bottom-left node. Given a standard λ-tableau t, we say a p-hook is ***bad*** (in t) if

(1) the entry in the top-right node of the hook is strictly larger than the entry in the bottom-left node;

(2) no entry in the column containing the bottom-left node is equal to the entry in the top-right node.

We claim that if t is a standard λ-tableau, with λ p-restricted and completely splittable, then the condition (†) of Theorem 6.3 is equivalent to the condition that there are no bad p-hooks in t. To prove this, first observe, by Theorem 6.2 and the definition of $\psi_n(\lambda)$, that shape$(t[n-1])$ is allowable for λ if and only if there is no bad p-hook in t with top right entry equal to n. Now by induction, the condition (†) holds for $t[n-1]$ if and only if there is no bad p-hook in $t[n-1]$, which is if and only if there is no bad p-hook in t with top right entry at most $n-1$. The claim follows immediately from these two statements.

So, by the claim, the dimension of the μ-weight space of $L_n(\lambda)$ is equal to the number of standard λ-tableaux t of weight μ, containing no bad p-hooks.

6.4. **Example.** Let $p = 3, n = 4, \lambda = (3, 2, 1, 1), \mu = (2, 1, 2, 2)$. Note that $\psi_4(\lambda) = 3$, so λ is completely splittable. The standard λ-tableaux of weight μ are just the following:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 3 & \\ \cline{1-2} 3 & & \\ \cline{1-1} 4 & & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 & \\ \cline{1-2} 3 & & \\ \cline{1-1} 4 & & \\ \cline{1-1}\end{array}$$

Of these, the first contains a bad p-hook (namely, the nodes $(1,3),(2,2)$), and the second does not. Thus, the μ-weight space of $L_n(\lambda)$ is one dimensional.

Appendix: a short proof of the straightening rule

We were unable to find a suitable reference for the straightening rule used in section 2, so include here a short proof based on [J2], Satz II.6. We take our notation from (2.1). The standard basis theorem and the straightening rule will follow easily from the following lemma.

A.1 Lemma. *For $\lambda \in X^+(n)$, the module $\Delta_n(\lambda)$ is generated as a $U^-(n-1)$-module by*

$$\mathscr{G} := \left\{ F_{1,n}^{(a_1)} \dots F_{n-1,n}^{(a_{n-1})} e_\lambda \mid 0 \leq a_i \leq \lambda_i - \lambda_{i+1} \text{ for } i = 1, \dots, n-1 \right\}.$$

Proof. By the PBW basis,

$$\Delta_n(\lambda) = U^-(n)e_\lambda = \sum_{b_1,\dots,b_{n-1}\geq 0} U^-(n-1)F_{1,n}^{(b_1)} \dots F_{n-1,n}^{(b_{n-1})} e_\lambda.$$

So we need to show that $F_{1,n}^{(b_1)} \dots F_{n-1,n}^{(b_{n-1})} e_\lambda$ lies in the $U^-(n-1)$-submodule generated by $\mathscr{G}$, for any integers $b_1, \dots, b_{n-1} \geq 0$. To prove this, we use induction on the tuple $(b_1, \dots, b_{n-1})$, ordered lexicographically, the conclusion being immediate if $b_i \leq \lambda_i - \lambda_{i+1}$ for all i. So take a tuple $(b_1, \dots, b_{n-1})$ with $b_i > \lambda_i - \lambda_{i+1}$ for some i.

We need the commutator formula

$$F_{i,n}^{(b_i)} F_{i+1,n}^{(b_{i+1})} = F_{i+1,n}^{(b_i+b_{i+1})} F_{i,i+1}^{(b_i)} - \sum_{j=1}^{b_i} F_{i,i+1}^{(j)} F_{i,n}^{(b_i-j)} F_{i+1,n}^{(b_{i+1}+j)}$$

derived in the proof of [J2], Satz II.6. To prove this, observe that the subalgebra generated by $\{F_{i,i+1}^{(j)}, F_{i+1,n}^{(j)} \mid j \geq 0\}$ is a copy of $U^-(3)$, so it suffices to prove the following identity in $U^-(3,\mathbb{Z})$:

$$F_{2,3}^{(r+s)} F_{1,2}^{(r)} = \sum_{j=0}^{r} F_{1,2}^{(j)} F_{1,3}^{(r-j)} F_{2,3}^{(s+j)}.$$

For this, we work in the ring of formal power series $U^-(3,\mathbb{Z})[[u,v]]$. If $t = u, v$ or uv and $i > j$, $y_{i,j}(t)$ denotes $1 + tX_{j,i} + \dots + t^r \frac{X_{j,i}^r}{r!} + \dots$ in $U^-(3,\mathbb{Z})[[u,v]]$. By [St], Lemma 15, we have the identity $y_{2,3}(v)y_{1,2}(u) = y_{1,2}(u)y_{1,3}(uv)y_{2,3}(v)$ and the required formula follows by equating $u^r v^{r+s}$-coefficients.

Next note, by considering the action of $SL(2)$, that $F_{i,i+1}^{(b_i)}e_\lambda = 0$ as, by assumption, $b_i > \lambda_i - \lambda_{i+1}$. Combining this with the commutator formula from the previous paragraph, we deduce

$$F_{1,n}^{(b_1)} \cdots F_{n-1,n}^{(b_{n-1})} e_\lambda = -\sum_{j=1}^{b_i} F_{i,i+1}^{(j)} F_{1,n}^{(b_1)} \cdots F_{i,n}^{(b_i-j)} F_{i+1,n}^{(b_{i+1}+j)} \cdots F_{n-1,n}^{(b_{n-1})} e_\lambda.$$

All the terms on the right hand side involve tuples strictly lower than $(b_1, \dots, b_{n-1})$ in the lexicographic order, so lie in the $U^-(n-1)$-submodule generated by $\mathcal{G}$ by the induction hypothesis. □

As in (2.2), we write $\mu \longleftarrow \lambda$ if $\lambda = (\lambda_1, \dots, \lambda_n) \in X^+(n)$ and $\mu = (\mu_1, \dots, \mu_{n-1}) \in X^+(n-1)$ for some $n > 1$, satisfying the condition $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for $i = 1, \dots, n-1$. Recall given $\mu \longleftarrow \lambda$ that $F_{\mu,\lambda} := F_{1,n}^{(\lambda_1-\mu_1)} \cdots F_{n-1,n}^{(\lambda_{n-1}-\mu_{n-1})}$. In this notation, Lemma A.1 says that $\Delta_n(\lambda)$ is generated by $\{F_{\mu,\lambda}e_\lambda \mid \mu \longleftarrow \lambda\}$.

A.2 Proposition. *Take $\lambda \in X^+(n)$ and fix an ordering $\mu(1), \dots, \mu(N)$ of the weights $\{\mu \longleftarrow \lambda\}$ such that $\mu(i) > \mu(j)$ implies $i < j$. For $1 \leq i \leq N$, let Δ^i be the $U^-(n-1)$-module of $\Delta = \Delta_n(\lambda)$ generated by $\{F_{\mu(j),\lambda}e_\lambda \mid j \leq i\}$. Then,*

$$(0) = \Delta^0 \subset \Delta^1 \subset \cdots \subset \Delta^N = \Delta$$

is a filtration of Δ as a $U(n-1)$-module with $\Delta^i/\Delta^{i-1} \cong \Delta_{n-1}(\mu(i))$. Moreover, the image of $F_{\mu(i),\lambda}e_\lambda$ in Δ^i/Δ^{i-1} is a $U(n-1)$-high weight vector.

Proof. We first claim that each Δ^i is $U(n-1)$-stable and that the image of $F_{\mu(i),\lambda}e_\lambda$ in Δ^i/Δ^{i-1} is a (possibly zero) $U(n-1)$-high weight vector. We prove this by induction on i, so take i with $1 \leq i \leq N$ and assume by induction that Δ^{i-1} is $U(n-1)$-stable (which is obvious in the case $i = 1$). Our assumptions imply that $\mu(i)$ is maximal with respect to the dominance ordering amongst $\{\mu(j) \mid j \geq i\}$. But Δ/Δ^{i-1} is generated as a $U^-(n-1)$-module by $\{F_{\mu(j),\lambda}e_\lambda + \Delta^{i-1} \mid j \geq i\}$, so the image of $F_{\mu(i),\lambda}e_\lambda$ is either zero or a vector of maximal weight in Δ/Δ^{i-1}. The claim follows.

Now note that $\Delta^N = \Delta$ by Lemma A.1. So, we have constructed a filtration of Δ as a $U(n-1)$-module, where the factors are high weight modules. In particular, the dimension of Δ is at most $\sum_{\mu \longleftarrow \lambda} \dim \Delta_{n-1}(\mu)$, with equality if and only if Δ_i/Δ_{i-1} equals the universal high weight module $\Delta_{n-1}(\mu(i))$ for all i, which is what we want to prove. But now a calculation using Weyl's dimension formula shows that equality does indeed hold. □

We introduce some new notation, which is a variant on the Gelfand-Zetlin patterns of [GZ]. We write $\mu \subset \lambda$ if $\lambda = (\lambda_1, \dots, \lambda_m) \in X(m)$ and $\mu = (\mu_1, \dots, \mu_{m-1}) \in X(m-1)$ for some $m > 1$, satisfying the condition $\mu_i \leq \lambda_i$ for $i = 1, \dots, m-1$. For $\lambda \in X(n)$, a ***λ-chain*** is a chain $C = (C(1) \subset C(2) \subset \cdots \subset C(n))$ of weights with $C(n) = \lambda$ [so $C(i) \in X(i)$ for each i]. A ***weight chain*** is simply a λ-chain for some λ. The weight chain $(C(1) \subset \cdots \subset C(n))$ is ***standard*** if in fact $C(1) \longleftarrow C(2) \longleftarrow \dots \longleftarrow C(n)$ [so $C(i) \in X^+(i)$ for each i].

Given $\mu \subset \lambda$, with $\lambda \in X(m)$, define $F_{\mu,\lambda} := F_{1,m}^{(\lambda_1-\mu_1)} \cdots F_{m-1,m}^{(\lambda_{m-1}-\mu_{m-1})}$. Given a weight chain $C = (C(1) \subset \cdots \subset C(n))$, let $F_C := F_{C(1),C(2)}F_{C(2),C(3)} \cdots F_{C(n-1),C(n)}$. Observe that $\{F_C \mid \text{for fixed } \lambda \in X(n) \text{ and all } \lambda\text{-chains } C\}$ is precisely our usual PBW basis for $U^-(n)$ in this new notation.

A.3 Theorem (The standard basis theorem). *Fix $\lambda \in X^+(n)$. The set*

$$\{F_C e_\lambda \mid \text{for all standard } \lambda\text{-chains } C\}$$

is a basis for $\Delta_n(\lambda)$.

Proof. Induction on n using Proposition A.2. □

We now want to prove the straightening rule, which describes how $F_C e_\lambda$ for an arbitrary monomial F_C in the PBW basis for $U^-(n)$ expands in terms of the standard basis for $\Delta_n(\lambda)$. There is a natural partial order on weight chains. Given chains $C = (C(1) \subset \cdots \subset C(n))$ and $D = (D(1) \subset \cdots \subset D(n))$, we write $C \leq D$ if either $C(i) = D(i)$ for all i, or $C(n) = D(n), \dots, C(i+1) = D(i+1)$ and $C(i) < D(i)$ for some i. We call this the ***dominance order*** on weight chains.

A.4 Theorem (The straightening rule). *Fix $\lambda \in X^+(n)$ and let $\Delta = \Delta_n(\lambda)$. Given an arbitrary λ-chain C, the vector $F_C e_\lambda$ can be expanded uniquely as a linear combination of basis elements $F_D e_\lambda$ for standard λ-chains D satisfying $D \geq C$ in the dominance order on λ-chains.*

Proof. Given $\mu \longleftarrow \lambda$, let Δ_μ be the $U^-(n-1)$-submodule of Δ generated by $\{F_{\nu,\lambda} e_\lambda \mid \nu \longleftarrow \lambda, \nu \geq \mu\}$ and let Δ'_μ be the $U^-(n-1)$-submodule of Δ generated by $\{F_{\nu,\lambda} e_\lambda \mid \nu \longleftarrow \lambda, \nu > \mu\}$. Choosing the ordering of $\{\mu \longleftarrow \lambda\}$ suitably in Proposition A.2, we see that Δ_μ is spanned by basis elements $F_D e_\lambda$ for all standard λ-chains $D = (D(1) \longleftarrow \dots \longleftarrow D(n))$ with $D(n-1) \geq \mu$.

As Δ is generated as a $U^-(n-1)$-module by $\{F_{\nu,\lambda} e_\lambda \mid \nu \longleftarrow \lambda\}$ by Proposition A.2, $F_{C(n-1),C(n)} e_\lambda$ lies in the sum of the modules Δ_ν for $\nu \longleftarrow \lambda, \nu \geq C(n-1)$. So, we can expand $F_C e_\lambda$ as a linear combination of terms $F_D e_\lambda$ for standard λ-chains D with $D(n-1) \geq C(n-1)$. If $D(n-1) \neq C(n-1)$, then $D > C$ as required.

So we need to consider terms in the expansion with $D(n-1) = C(n-1)$. This implies in particular that $C(n-1) \longleftarrow \lambda$, so letting $\mu = C(n-1)$, $F_C e_\lambda$ lies in Δ_μ and its image in $\Delta_\mu / \Delta'_\mu \cong \Delta_{n-1}(\mu)$ is the vector $F_{C'} e_\mu$ where e_μ is the image of $F_{\mu,\lambda} e_\lambda$ and C' is the μ-chain $(C(1) \subset \cdots \subset C(n-1))$. By induction on n, we can expand $F_{C'} e_\mu$ as a linear combination of terms $F_{D'} e_\mu$ for μ-chains D' satisfying $D' \geq C'$. Hence, $F_C e_\lambda$ can be expanded as required modulo Δ'_μ. But Δ'_μ is spanned by basis elements of the form $F_D e_\lambda$ where the chain $D = (D(1) \longleftarrow \dots \longleftarrow D(n))$ satisfies $D(n-1) > C(n-1)$, so again $D > C$, and the result follows. □

To obtain the standard basis theorem and straightening rule as described in (2.3), it just remains to translate between the notation of weight chains and the tableaux notation of (2.2). If $\lambda \in \Lambda^+(n)$, the map $t \mapsto (\mathrm{shape}(t[1]) \subset \mathrm{shape}(t[2]) \subset \cdots \subset \mathrm{shape}(t[n]))$ gives a injection from the set of row standard λ-tableau with entries in the ith row at least i into the set of λ-chains. It induces a bijection between standard λ-tableaux and standard λ-chains. Given this, it is straightforward to translate between the two notations.

References

[Bou] N. Bourbaki, *Groupes et algèbres de Lie*, chapitres IV-VI, Hermann, Paris, 1968.

[B1] J. W. Brundan, *Modular branching rules and the Mullineux map for Hecke algebras of type* **A**, Proc. London Math. Soc., to appear.

[B2] J. W. Brundan, *Lowering operators for $GL(n)$ and quantum $GL(n)$*, in: "Representations, cohomology and actions of finite groups (Seattle 1996)", Proc. Symposia Pure Math., to appear.

[CL] R. W. Carter and G. Lusztig, *On the modular representations of the general linear and symmetric groups*, Math. Z. **136** (1974), 193–242.

[GZ] I. M. Gelfand and M. L. Zetlin, *Finite dimensional representations of the group of unimodular matrices*, Dokl. Akad. Nauk. USSR **71** (1950), 825–828. [in Russian]

[G] J. A. Green, *Polynomial representations of GL_n*, Lecture Notes in Math., vol. 830, Springer-Verlag, 1980.

[J1] J. C. Jantzen, *Representations of algebraic groups*, Academic Press, Orlando, 1987.

[J2] J. C. Jantzen, *Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen*, Bonner Math. Schriften, No. 67, 1973.

[JS] J. C. Jantzen and G. M. Seitz, *On the representation theory of the symmetric groups*, Proc. London Math. Soc. **65** (1992), 475–504.

[K1] A. S. Kleshchev, *On restrictions of irreducible modular representations of semisimple algebraic groups and symmetric groups to some natural subgroups, I*, Proc. London Math. Soc. **69** (1994), 515–540.

[K2] A. S. Kleshchev, *On restrictions of irreducible modular representations of semisimple algebraic groups and symmetric groups to some natural subgroups, II*, Comm. Alg. **22** (1994), 6175–6208.

[K3] A. S. Kleshchev, *Branching rules for modular representations of symmetric groups, II*, J. reine angew. Math. **459** (1995), 163–212.

[K4] A. S. Kleshchev, *Completely splittable representations of symmetric groups*, J. Alg. **181** (1996), 584–592.

[L] G. Lusztig, *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras*, J. Am. Math. Soc. **3** (1990), 257–297.

[MP] O. Mathieu and G. Papadopoulo, *A character formula for a family of simple modular representations of GL_n*, preprint, 1997.

[St] R. Steinberg, *Lectures on Chevalley groups*, Yale University, 1967.