DUAL CANONICAL BASES AND KAZHDAN-LUSZTIG POLYNOMIALS

JONATHAN BRUNDAN

Abstract. We derive a formula for the entries of the (unitriangular) transition matrices between the standard monomial and dual canonical bases of the irreducible polynomial representations of $U_q(\mathfrak{gl}_n)$ in terms of Kazhdan-Lusztig polynomials.

1. Introduction

In the last few years, there has been much interest in dual canonical bases associated to quantized enveloping algebras motivated by applications to representation theory: in many situations the basis of simple modules for the Grothendieck groups of various natural categories of modules in type A can be identified with the specialization at $q = 1$ of an appropriate dual canonical basis. For example, in [BK], we found just such an interpretation for dual canonical bases of the irreducible polynomial representations of $U_q(\mathfrak{gl}_n)$. This provided the incentive to revisit the extensive literature about these very special modules and their bases.

The main result of the article gives an explicit formula for the entries of the transition matrices between various standard monomial bases and the dual canonical basis of the irreducible polynomial representation parametrized by a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of d, in terms of the Kazhdan-Lusztig polynomials $P_{x,y}(t)$ associated to the symmetric group S_d. Using notation introduced later in the article, the polynomials arising as the entries of these matrices are of the form

$$(-q)^{\ell(y)-\ell(x)} \sum_{z \in \mathcal{D}_\nu \cap S_\nu x S_\mu} (-1)^{\ell(z)+\ell(y)} P_{zw_d, yw_d}(q^2)$$

for particular $x, y \in S_d$; see Theorem 26 and Remark 14. It is these polynomials which when evaluated at $q = 1$ compute composition multiplicities of the standard modules for the finite W-algebras/shifted Yangians studied in [BK]. We also show that all the coefficients of these polynomials are non-negative integers, by relating them to the dual canonical basis of the quantized coordinate algebra of the group of upper unitriangular matrices then appealing to results of Lusztig in that setting.

The basic strategy is as follows. Let $\mu = (\mu_1, \dots, \mu_l)$ be a composition having transpose partition equal to λ. Let $\mathcal{V}_n$ be the natural representation of $U_q(\mathfrak{gl}_n)$, over the field $\mathbb{Q}(q)$ where q is an indeterminate. By the Littlewood-Richardson

Address: Department of Mathematics, University of Oregon, Eugene, OR 97403.
E-mail: brundan@math.uoregon.edu.
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rule, the space of $U_q(\mathfrak{gl}_n)$-module homomorphisms

$$\xi_\mu : \bigwedge^{\mu_1}(\mathcal{V}_n) \otimes \cdots \otimes \bigwedge^{\mu_l}(\mathcal{V}_n) \to S^{\lambda_n}(\mathcal{V}_n) \otimes \cdots \otimes S^{\lambda_1}(\mathcal{V}_n)$$

is one dimensional, and the image of any non-zero such homomorphism ξ_μ is the irreducible $U_q(\mathfrak{gl}_\mathfrak{n})$-module $P^\lambda(\mathcal{V}_n)$ of highest weight λ. Now, the exterior and symmetric powers of $\mathcal{V}_n$ equipped with their natural monomial bases are based modules in the sense of [L, ch.27], so by dualizing Lusztig's construction of tensor product of based modules we obtain dual canonical bases for the above tensor products of exterior and symmetric powers. These bases have the remarkable property that the homomorphism ξ_μ (suitably normalized) maps dual canonical basis elements either to dual canonical basis elements or to zero. In this way, we obtain the dual canonical basis of $P^\lambda(\mathcal{V}_n)$ (= the upper global crystal base of Kashiwara) as the set of non-zero images of dual canonical basis elements of the tensor product of exterior powers under the map ξ_μ. Using this description, we are then able to relate dual canonical bases directly to Kazhdan-Lusztig polynomials using Schur-Weyl duality, following the algebraic approach initiated by Frenkel, Khovanov and Kirillov in [FKK].

In the main body of the article, we have also explained for completeness the dual argument, involving the homomorphism

$$\xi_\mu^* : S^{\lambda_1}(\mathcal{V}_n) \otimes \cdots \otimes S^{\lambda_n}(\mathcal{V}_n) \to \bigwedge^{\mu_l}(\mathcal{V}_n) \otimes \cdots \otimes \bigwedge^{\mu_1}(\mathcal{V}_n)$$

that is dual to the above map ξ_μ under certain natural pairings. The cokernel of ξ_μ^* gives another much-studied realization of the irreducible module $P^\lambda(\mathcal{V}_n)$. Again, it is the case that ξ_μ^* maps canonical basis elements either to canonical basis elements or to zero, which makes this point of view well-suited to relating the canonical basis of $P^\lambda(\mathcal{V}_n)$ (= the lower global crystal base) to the semi-standard basis of Dipper and James [DJ2]. In particular, we recover the explicit formula for the transition matrix between these bases in terms of Kazhdan-Lusztig polynomials obtained originally by Du [D1, D3] by a different method (involving the combinatorics of cells in the symmetric group). Along the way, we have included proofs of a number of related results about canonical and dual canonical bases which are known to experts but hard to find in the literature. In particular, in §6, we discuss in some detail the dual canonical basis of the quantized coordinate algebra of $m \times n$ matrices, in the spirit of the work of Berenstein and Zelevinsky [BZ]. This dual canonical basis also has a natural representation theoretic interpretation which does not seem to be widely known, in terms of certain blocks of the categories of Harish-Chandra bimodules associated to the Lie algebras $\mathfrak{gl}_d(\mathbb{C})$.

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2. Combinatorics

In this preliminary section, we gather together (almost) all of the combinatorial definitions needed later on. Let S_d denote the symmetric group acting on the left on the set $\{1, \dots, d\}$, with *basic transpositions* $s_1, \dots, s_{d-1}$, *length function* ℓ and *longest element* w_d. The following notation is quite standard:

- X_n denotes the integral weight lattice associated to the Lie algebra $\mathfrak{gl}_n$, that is, the abelian group $\mathbb{Z}^n$ with standard basis $\varepsilon_1, \dots, \varepsilon_n$ and inner product $(.,.)$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$;
- a choice of *simple roots* is given by $\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n$;
- $\geq$ is the corresponding *dominance ordering* on X_n defined by $\lambda \geq \mu$ if $(\lambda - \mu)$ is a sum of simple roots;
- Λ_n and Λ_n^+ denote the subsets of X_n consisting of all $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1, \dots, \lambda_n \geq 0$ and with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, respectively;
- for a weight $\lambda \in \Lambda_n$ with $|\lambda| := \lambda_1 + \cdots + \lambda_n = d$, S_λ denotes the parabolic subgroup $S_{\lambda_1} \times \cdots \times S_{\lambda_n}$ of S_d with longest element w_λ;
- $\mathcal{D}_\lambda$ is the set of all *minimal length* $S_\lambda \backslash S_d$-coset representatives.

Letting $I_n = \{1, \dots, n\}$, S_d also acts naturally on the right on the set of all *multi-indexes* $\alpha = (\alpha_1, \dots, \alpha_d) \in I_n^d$, so that $(\alpha \cdot x)_i = \alpha_{xi}$ for $\alpha \in I_n^d$ and $x \in S_d$. We write $\alpha \sim \beta$ if two multi-indexes $\alpha, \beta \in I_n^d$ lie in the same S_d-orbit. This is the case if and only if $\theta(\alpha) = \theta(\beta)$, where $\theta(\alpha) \in \Lambda_n$ denotes the *weight* of $\alpha \in I_n^d$ defined from $\theta(\alpha) = \sum_{i=1}^d \varepsilon_{\alpha_i}$. For $\lambda \in \Lambda_n$, let I_λ denote the set of all multi-indexes of weight λ. There is a bijection $d : I_\lambda \to \mathcal{D}_\lambda$ defined for $\alpha \in I_\lambda$ by letting $d(\alpha)$ be the unique element of $\mathcal{D}_\lambda$ such that $\alpha \cdot d(\alpha)^{-1}$ is a weakly increasing sequence.

Assume now that we are given weights $\mu \in \Lambda_m$ and $\nu \in \Lambda_n$ with $|\mu| = |\nu| = d$. The symmetric group S_d acts diagonally on the right on $I_\mu \times I_\nu$, and we let $(I_\mu \times I_\nu)/S_d$ denote the set of orbits. This set arises naturally in many different guises. Let us recall some of the most popular. The first involves $m \times n$ matrices $M = (m_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ with non-negative integer entries. Define the *row* and *column sums* of M to be the weights $\mathrm{ro}(M) = (\mu_1, \dots, \mu_m) \in \Lambda_m$ and $\mathrm{co}(M) = (\nu_1, \dots, \nu_n) \in \Lambda_n$ defined from

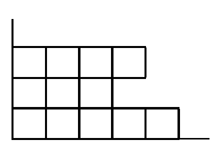
$$\mu_i = \sum_{j=1}^n m_{i,j} \qquad \text{and} \qquad \nu_j = \sum_{i=1}^m m_{i,j}.$$

Let $\Theta_{\mu,\nu}$ denote the set of all such matrices M with $\mathrm{ro}(M) = \mu$ and $\mathrm{co}(M) = \nu$. Given any pair $(\alpha, \beta) \in I_\mu \times I_\nu$, we obtain a matrix $M \in \Theta_{\mu,\nu}$ by letting

$$m_{i,j} = \#\{k = 1, \dots, d \mid \alpha_k = i, \beta_k = j\}.$$

This induces a bijective correspondence between the sets $(I_\mu \times I_\nu)/S_d$ and $\Theta_{\mu,\nu}$.

The second way is in terms of row standard tableaux of row shape μ and weight ν. To introduce these, we need the notion of the *row diagram* of a weight $\mu \in \Lambda_m$. This is the diagram drawn in the positive quadrant of the x-y plane consisting of μ_1 boxes in the first (bottom) row, ..., μ_m boxes in the mth row. For instance, if $\mu = (5, 3, 4)$ its row diagram is



A *tableau* of row shape μ and weight ν means a filling of the boxes of the row diagram of μ with integers, exactly ν_1 of which are equal to 1, ν_2 are equal to 2, ..., ν_n are equal to n. We sometimes use the notation $\sigma(A)$ for the *row shape* μ and $\theta(A)$ for the *weight* ν of the tableau A. Define an equivalence relation $\sim_{\mathrm{ro}}$ on the set of all such tableaux by declaring that $A \sim_{\mathrm{ro}} B$ if B can be obtained from A by permuting entries within rows. We say that A is *row standard* if its entries are *weakly increasing* along rows from left to right. Obviously, the row standard tableaux give a set of representatives for the $\sim_{\mathrm{ro}}$-equivalence classes. For instance

$$A = \begin{array}{|c|c|c|c|c}\hline 1&2&3&4& \multicolumn{1}{|c}{}\\ \hline 2&2&3&\multicolumn{1}{|c}{}&\\ \hline 1&1&2&4&\multicolumn{1}{c|}{4}\\ \hline\end{array} \tag{2.1}$$

is a row standard tableau of row shape $(5,3,4)$ and weight $(3,4,2,3)$. Let $\mathrm{Row}(\mu,\nu)$ denote the set of all row standard tableaux of row shape μ and weight ν. Given a tableau $A \in \mathrm{Row}(\mu,\nu)$, we obtain a matrix $M \in \Theta_{\mu,\nu}$ by defining $m_{i,j}$ to be the number of entries in the ith row of A that are equal to j. This defines a bijection $\mathrm{Row}(\mu,\nu) \to \Theta_{\mu,\nu}$, hence composing with the bijection in the previous paragraph we also obtain a bijection between $\mathrm{Row}(\mu,\nu)$ and the set $(I_\mu \times I_\nu)/S_d$. For example, with A as in (2.1), the corresponding matrix $M \in \Theta_{\mu,\nu}$ is the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and a representative $(\alpha,\beta) \in I_\mu \times I_\nu$ for the corresponding orbit is given by setting $\alpha = (3,3,3,3,2,2,2,1,1,1,1,1)$ and $\beta = (1,2,3,4,2,2,3,1,1,2,4,4)$.

The third way involves the set $\mathcal{D}^+_{\nu,\mu}$ of *maximal length* distinguished (S_ν, S_μ)-double coset representatives in the symmetric group S_d. We just explain how to define a bijection between $\mathrm{Row}(\mu,\nu)$ and $\mathcal{D}^+_{\nu,\mu}$. Given any tableau A of row shape μ and weight ν, define a sequence $\rho(A) \in I_\nu$ by *row reading* the entries of A along rows from left to right starting from the top row; for example, if A is as in (2.1) then $\rho(A)$ is the multi-index β from the end of the previous paragraph. Recalling the bijection $d : I_\nu \to D_\nu$ from the opening paragraph, the map $A \mapsto d(\rho(A))w_d$ defines a bijection between the set $\mathrm{Row}(\mu,\nu)$ and the set $D^+_{\nu,\mu}$. Moreover, $D_\nu \cap S_\nu d(\rho(A))w_d S_\mu = \{w_\nu d(\rho(B))w_d | B \sim_{\mathrm{ro}} A\}$. For a proof of a similar statement, see [DJ1, 1.7] or [Ma, 4.4].

There is a fourth way which is much more subtle than the ones discussed so far involving the Robinson-Schensted-Knuth correspondence; see [F, §4.1]. In this article, we actually only need a very special case of this fundamental bijection. To explain it, we must first introduce the notion of a column strict tableau. Suppose now that $\mu \in \Lambda_l$, $\nu \in \Lambda_n$ satisfy $|\mu| = |\nu| = d$. The mirror image of the row diagram of μ in the line $y = x$ gives the *column diagram* of μ. Thus, the column diagram has μ_1 boxes in the first (leftmost) column, $\dots$, μ_l boxes in the lth column. A tableau of *column shape* $\mu \in \Lambda_l$ and weight $\nu \in \Lambda_n$ means a filling of the boxes of the column diagram of μ with integers, exactly ν_j of which are equal to j for each $j = 1, \dots, n$. Call such a tableau *column strict* if its entries are *strictly increasing* along columns from bottom to top. Let $\mathrm{Col}(\mu,\nu)$ denote the set of all column strict tableaux of column shape μ and weight ν. Observe that the mirror image in the line $y = x$ of a tableau A of column shape μ defines a tableau A' of row shape μ. This is a useful trick for carrying over the earlier definitions to the present setting. For instance, we write $A \sim_{\mathrm{co}} B$ if $A' \sim_{\mathrm{ro}} B'$. The next definition breaks the symmetry: define the *column reading* $\gamma(A)$ to be the multi-index obtained by reading the entries of A along columns from top to bottom starting from the leftmost column. This is related to the row reading of A' by the equation $\gamma(A) = \rho(A') \cdot w_d$.

Assuming all parts of the composition μ are $\leq m$, let $\lambda = \mu' \in \Lambda^+_m$ be the *conjugate partition*, so λ_i is the number of boxes in the ith row of the column diagram of μ. Let $\mathrm{Dom}(\lambda, \nu)$ denote the familiar set of all *standard tableaux* of row shape λ and weight ν, that is, the tableaux in $\mathrm{Row}(\lambda, \nu)$ that are also column strict. (The unfamiliar symbol Dom here stands for "dominant" following the language used in [BK].) For a multi-index $\alpha \in I^d_n$, let $P(\alpha)$ denote the image of the word $\alpha_1\alpha_2 \cdots \alpha_d$ under the Robinson-Schensted correspondence; see e.g. [F, §4.1]. Thus, $P(\alpha)$ is the standard tableau $\varnothing \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_d$, where $\leftarrow$ denotes row insertion as in [F, §1.1]. Still writing $\lambda = \mu'$, define

$$\mathrm{Std}(\mu, \nu) = \{A \in \mathrm{Col}(\mu, \nu) \mid P(\gamma(A)) \text{ is of row shape } \lambda\}. \tag{2.2}$$

We refer to elements of $\mathrm{Std}(\mu, \nu)$ as *standard tableaux* of column shape μ and weight ν. In the special case μ is itself a partition, it is easy to see from the definition of the Robinson-Schensted map that $\mathrm{Std}(\mu, \nu)$ is the set of all tableaux in $\mathrm{Col}(\mu, \nu)$ that are also row standard, i.e. $\mathrm{Std}(\mu, \nu) = \mathrm{Dom}(\lambda, \nu)$. So the double meaning of the phrase "standard tableaux" is unambiguous. In general, by a result of Lascoux and Schützenberger [LS2], the *rectification map*

$$R : \mathrm{Std}(\mu, \nu) \to \mathrm{Dom}(\lambda, \nu), \qquad A \mapsto P(\gamma(A)) \tag{2.3}$$

is a bijection; see also [F, §A.5]. In the special case that μ is a partition, the map R is just the identity map. In general, R can be computed by repeatedly using *jeu de taquin* to permute adjacent columns of different lengths; see [LT, §4] for an example.

In proofs, we will use a rather different characterization of the set $\mathrm{Std}(\mu, \nu)$ and the rectification map in terms of crystals. To recall this, define a crystal $(I^d_n, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ in the sense of Kashiwara [K4] with underlying set I^d_n as follows. For $i = 1, \dots, n-1$, define the *i-signature* $(\sigma_1, \dots, \sigma_d)$ of $\alpha \in I^d_n$ by

$$\sigma_j = \begin{cases} + & \text{if } \alpha_j = i, \\ - & \text{if } \alpha_j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

From this the *reduced i-signature* is computed by successively replacing subsequences of the form $-+$ (possibly separated by 0's) in the signature with 0's until no $-$ appears to the left of a $+$. Let δ_j denote the d-tuple $(0, \dots, 0, 1, 0, \dots, 0)$ where 1 appears in the jth place. Now define

$$\tilde{e}_i(\alpha) := \begin{cases} \varnothing & \text{if there are no } -\text{'s in the reduced } i\text{-signature,} \\ \alpha - \delta_j & \text{if the leftmost } - \text{ is in position } j; \end{cases}$$

$$\tilde{f}_i(\alpha) := \begin{cases} \varnothing & \text{if there are no } +\text{'s in the reduced } i\text{-signature,} \\ \alpha + \delta_j & \text{if the rightmost } + \text{ is in position } j; \end{cases}$$

$$\varepsilon_i(\alpha) = \text{the total number of } -\text{'s in the reduced } i\text{-signature,}$$

$$\varphi_i(\alpha) = \text{the total number of } +\text{'s in the reduced } i\text{-signature.}$$

Recalling that $\theta(\alpha)$ denotes the weight of $\alpha \in I_n^d$, this completes the definition of the crystal $(I_n^d, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$. It is just the d-fold tensor product of the usual crystal associated to the natural $\mathfrak{gl}_n$-module, except that we have parametrized it from right to left rather than from left to right.

In this paragraph, we write $\bigcup$ as shorthand for the union over all $\nu \in \Lambda_n$, and assume in addition that $m \leq n$. The row reading ρ resp. the column reading γ identifies the set $\bigcup \mathrm{Row}(\lambda, \nu)$ resp. $\bigcup \mathrm{Col}(\mu, \nu)$ with a subcrystal of I_n^d. This defines new crystals $(\bigcup \mathrm{Row}(\lambda, \nu), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ and $(\bigcup \mathrm{Col}(\mu, \nu), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$. It is well known that the map $A \mapsto P(\gamma(A))$ arising from the Robinson-Schensted correspondence commutes in the strict sense with the crystal operators $\tilde{e}_i, \tilde{f}_i$. Moreover, $\bigcup \mathrm{Dom}(\lambda, \nu)$ is a subcrystal of $\bigcup \mathrm{Row}(\lambda, \nu)$, indeed, it is precisely the connected component of $\bigcup \mathrm{Row}(\lambda, \nu)$ generated by the unique tableau $B \in \mathrm{Dom}(\lambda, \lambda)$, i.e. the tableau with all entries in its ith row equal to i. Since R necessarily maps the unique element $A \in \mathrm{Std}(\mu, \lambda)$ to this tableau B, we deduce that $\bigcup \mathrm{Std}(\mu, \nu)$ is the connected component of $\bigcup \mathrm{Col}(\mu, \nu)$ generated by A, and the rectification map $R : \bigcup \mathrm{Std}(\mu, \nu) \to \bigcup \mathrm{Dom}(\lambda, \nu)$ is an isomorphism of crystals. In this way, we obtain various different realizations of the usual highest weight crystal associated to the partition λ, one for each composition μ with $\mu' = \lambda$. The standard realization from [KN] is the one when μ is itself a partition.

Finally, we say a few words about the Bruhat ordering. Let $\leq$ denote the *opposite* of the usual Bruhat ordering on S_d, e.g. $w_d \leq 1$. This restricts to a partial ordering on the subset $\mathcal{D}^+_{\nu,\mu}$, for $\mu \in \Lambda_m, \nu \in \Lambda_n$ with $|\mu| = |\nu| = d$ as before. Hence using the above bijections, we get partial orderings also denoted $\leq$ on each of sets $(I_\mu \times I_\nu)/S_d, \Theta_{\mu,\nu}$ and $\mathrm{Row}(\mu, \nu)$. We want to record several equivalent ways of defining these partial orders directly; see [DJ1, 1.2] or [Ma, 3.8] for proofs of essentially the same statements, which are apparently due originally to Ehresmann. Suppose first that we are given tableaux A and B. Write $A \downarrow B$ if there exists an entry x in the ith row and an entry y in the jth row of A with $i < j$ and $x < y$ such that B is obtained from A by swapping the entries x and y. For example,

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 7&7\\\cline{1-2} 3&3&5\\\hline\end{array} \;\downarrow\; \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 7&7\\\cline{1-2} 3&5&5\\\hline\end{array} \;\downarrow\; \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 7&3\\\cline{1-2} 7&5&5\\\hline\end{array}$$

Then, $A \geq B$ in the Bruhat ordering on $\mathrm{Row}(\mu, \nu)$ if and only if there exist tableaux $C_1, \dots, C_r$ such that $A \sim_{\mathrm{ro}} C_1 \downarrow \cdots \downarrow C_r \sim_{\mathrm{ro}} B$. Given $A \in \mathrm{Row}(\mu, \nu)$, let $A_{\leq i}$ denote the tableau obtained from A by deleting all boxes in rows higher than the ith row, and let $A^{\leq j}$ denote the tableau obtained from A by deleting all boxes containing entries greater than j. The following are equivalent for $A, B \in \mathrm{Row}(\mu, \nu)$:

(i) $A \leq B$ in the Bruhat ordering on $\mathrm{Row}(\mu, \nu)$;
(ii) $\theta(A_{\leq i}) \leq \theta(B_{\leq i})$ in the dominance ordering on Λ_n for all $i = 1, \dots, m$ (recall θ denotes weight);
(iii) $\sigma(A^{\leq j}) \leq \sigma(B^{\leq j})$ in the dominance ordering on Λ_m for all $j = 1, \dots, n$ (recall σ denotes row shape).

From (ii) or (iii), one easily deduces the well known direct description of the Bruhat order on the set $\Theta_{\mu,\nu}$ itself: for $M, N \in \Theta_{\mu,\nu}$, we have that $M \leq N$ if and only if

$$\sum_{i=1}^{s}\sum_{j=1}^{t} m_{i,j} \leq \sum_{i=1}^{s}\sum_{j=1}^{t} n_{i,j}$$

for all $s = 1, \dots, m$ and $t = 1, \dots, n$.

We will also need the Bruhat ordering $\leq'$ on the set $\mathrm{Col}(\mu, \nu)$. This can be defined simply by $A \leq' B$ if $A' \geq B'$; equivalently, $d(\gamma(A)) \geq d(\gamma(B))$. In the special case that μ is a partition and $\lambda = \mu'$, we have now defined two partial orders $\leq'$ and $\leq$ on the set $\mathrm{Std}(\mu, \nu) = \mathrm{Dom}(\lambda, \nu)$, via its natural embeddings into $\mathrm{Col}(\mu, \nu)$ and $\mathrm{Row}(\lambda, \nu)$, respectively. The following lemma shows that these two partial orders coincide.

Lemma 1. *For $A, B \in \mathrm{Dom}(\lambda, \nu)$, we have that $A \leq B$ if and only if $A \leq' B$.*

Proof. Let $A, B \in \mathrm{Dom}(\lambda, \nu)$. Since A is standard, $\sigma((A')^{\leq j}) = \sigma((A^{\leq j})') = \sigma(A^{\leq j})'$, and similarly for B. By the third equivalent definition of the Bruhat ordering on $\mathrm{Row}(\lambda, \nu)$ above, we know that $A \leq B$ if and only if $\sigma(A^{\leq j}) \leq \sigma(B^{\leq j})$ for all $j = 1, \dots, n$. Since conjugation is order reversing on partitions, this is equivalent to $\sigma(A^{\leq j})' \geq \sigma(B^{\leq j})'$ for all $j = 1, \dots, n$, i.e. $\sigma((A')^{\leq j}) \geq \sigma((B')^{\leq j})$. This is the statement that $A' \geq B'$, hence $A \leq' B$. □

3. Quantized enveloping algebras

In this section, we recall the definition of the quantized enveloping algebra $\mathcal{U}_n = U_q(\mathfrak{gl}_n)$, following [L]. We will work over the field $\mathbb{Q}(q)$ where q is an indeterminate. An additive map $f : V \to W$ between $\mathbb{Q}(q)$-vectors spaces is called *antilinear* if $f(cv) = \bar{c} f(v)$ for all $c \in \mathbb{Q}(q), v \in V$, where $- : \mathbb{Q}(q) \to \mathbb{Q}(q)$ is the field automorphism with $\overline{q} = q^{-1}$. Also the quantum integer associated to $n \in \mathbb{N}$ is $[n] = (q^n - q^{-n})/(q - q^{-1})$ and the quantum factorial is $[n]! = [n][n-1] \cdots [2][1]$.

By definition, $\mathcal{U}_n$ is the $\mathbb{Q}(q)$-algebra on generators E_i, F_i $(i = 1, \dots, n-1)$ and K_i, K_i^{-1} $(i = 1, \dots, n)$ subject to relations

$$\begin{aligned}
K_i K_i^{-1} &= K_i^{-1} K_i = 1, & E_i E_j &= E_j E_i & &\text{if } |i-j| > 1,\\
K_i K_j &= K_j K_i, & E_i^2 E_j + E_j E_i^2 &= [2] E_i E_j E_i & &\text{if } |i-j| = 1,\\
K_i E_j K_i^{-1} &= q^{(\varepsilon_i, \varepsilon_j - \varepsilon_{j+1})} E_j, & F_i F_j &= F_j F_i & &\text{if } |i-j| > 1,\\
K_i F_j K_i^{-1} &= q^{(\varepsilon_i, \varepsilon_{j+1} - \varepsilon_j)} F_j, & F_i^2 F_j + F_j F_i^2 &= [2] F_i F_j F_i & &\text{if } |i-j| = 1,
\end{aligned}$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_{i,i+1} - K_{i+1,i}}{q - q^{-1}}.$$

Here, $K_{i,j}$ denotes $K_i K_j^{-1}$. In this article, we will always view $\mathcal{U}_n$ as a Hopf algebra with counit $\varepsilon : \mathcal{U}_n \to \mathbb{Q}(q)$ defined by $\varepsilon(E_i) = 0, \varepsilon(F_i) = 0$ and $\varepsilon(K_i) = 1$, and comultiplication $\Delta : \mathcal{U}_n \to \mathcal{U}_n \otimes \mathcal{U}_n$ defined by

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_{i+1,i}, \quad \Delta(F_i) = K_{i,i+1} \otimes F_i + F_i \otimes 1, \quad \Delta(K_i) = K_i \otimes K_i.$$

In the language of [K3], this comultiplication is adapted to taking tensor products of lower crystal bases at $q = 0$ and upper crystal bases at $q = \infty$. With only minor adjustments, it would also be perfectly possible to use throughout the article the comultiplication $\widetilde{\Delta} : \mathcal{U}_n \to \mathcal{U}_n \otimes \mathcal{U}_n$ from [L]. We just note for comparison that $\widetilde{\Delta}$ is defined by $\widetilde{\Delta} = (\tau \otimes \tau) \circ \Delta \circ \tau$, where $\tau : \mathcal{U}_n \to \mathcal{U}_n$ is the algebra antiautomorphism defined by $\tau(E_i) = F_i, \tau(F_i) = E_i$ and $\tau(K_i) = K_i$, and it is adapted to taking tensor products of lower crystal bases at $q = \infty$ and upper crystal bases at $q = 0$.

All $\mathcal{U}_n$-modules encountered in this article will be *polynomial representations*, meaning $\mathcal{U}_n$-modules V satisfying $V = \bigoplus_{\lambda \in \Lambda_n} V_\lambda$ where V_λ denotes the λ-*weight space*

$$V_\lambda = \{v \in V \mid K_i v = q^{(\lambda, \varepsilon_i)} v \text{ for all } i = 1, \dots, n\}.$$

Direct sums, tensor products and subquotients of polynomial representations are again polynomial. Moreover, the category of all polynomial representations of $\mathcal{U}_n$ is a *braided tensor category*, with braiding isomorphism $\mathcal{R}_{V,W} : V \otimes W \to W \otimes V$ defined like in [L, 32.1.5]. To review this definition in a little more detail, let $\Theta = \sum_{0 \leq \lambda \in X_n} \Theta_\lambda$ be the *quasi-R-matrix* defined as in [L, 4.1.2], but using our comultiplication Δ instead of the comultiplication $\widetilde{\Delta}$ used there. More precisely, $\Theta = (\tau \otimes \tau)(\widetilde{\Theta}^{-1})$ where $\widetilde{\Theta}$ is exactly Lusztig's quasi-R-matrix from [L, 4.1.2]. It is an element of a certain completion $(\mathcal{U}_n \otimes \mathcal{U}_n)^\wedge$ of the algebra $\mathcal{U}_n \otimes \mathcal{U}_n$, with $\Theta_0 = 1$ and $\Theta_\lambda \in \mathcal{U}_\lambda^+ \otimes \mathcal{U}_\lambda^-$ for each λ, where $\mathcal{U}_\lambda^+$ resp. $\mathcal{U}_\lambda^-$ denotes the $\pm\lambda$-weight space of the positive part $\mathcal{U}_n^+$ resp. the negative part $\mathcal{U}_n^-$ of $\mathcal{U}_n$. For polynomial representations V and W, all but finitely many Θ_λ act as zero on any given vector $v \otimes w \in V \otimes W$, by weight considerations. Hence it makes sense to view Θ as an invertible operator on $V \otimes W$. The braiding $\mathcal{R}_{V,W} : V \otimes W \to W \otimes V$ can now be defined to be the map $\mathcal{R}_{V,W} = \Theta \circ f \circ P$ where $f : W \otimes V \to W \otimes V$ is the map $w \otimes v \mapsto q^{(\lambda,\mu)} w \otimes v$ for v, w of weights λ, μ, respectively, and $P : V \otimes W \to W \otimes V$ is the permutation operator $v \otimes w \mapsto w \otimes v$. Suppose more generally that $V_1, \dots, V_d$ are all polynomial representations. For $1 \leq i < d$, let

$$\mathcal{R}_i : V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_d \to V_1 \otimes \cdots \otimes V_{i+1} \otimes V_i \otimes \cdots \otimes V_d$$

denote the $\mathcal{U}_n$-module isomorphism $\mathcal{R}_{V_i, V_{i+1}}$ acting on the ith and $(i+1)$th tensor positions. For a permutation $w \in S_d$, we obtain a well-defined map

$$\mathcal{R}_w : V_1 \otimes \cdots \otimes V_d \to V_{w^{-1}1} \otimes \cdots \otimes V_{w^{-1}d}$$

by setting $\mathcal{R}_w = \mathcal{R}_{i_1} \circ \cdots \circ \mathcal{R}_{i_\ell}$ if $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is a reduced expression for w.

In the remainder of the section, we want to discuss some properties of bar involutions. The bar involution on $\mathcal{U}_n$ is the unique antilinear automorphism such that $\overline{E_i} = E_i$, $\overline{F_i} = F_i$ and $\overline{K_i} = K_i^{-1}$. We say that a $\mathcal{U}_n$-module V possesses a *compatible bar involution* if it is equipped with an antilinear involution $- : V \to V$ such that $\overline{uv} = \overline{u}\,\overline{v}$ for each $u \in \mathcal{U}_n$ and $v \in V$. Suppose V and W are polynomial $\mathcal{U}_n$-modules with compatible bar involutions. Following [L, 27.3.1], there is a canonical way to define a compatible bar involution on the tensor product $V \otimes W$: given $v \in V$ and $w \in W$ we set

$$\overline{v \otimes w} = \Theta(\overline{v} \otimes \overline{w}). \tag{3.1}$$

More generally, given polynomial $\mathcal{U}_n$-modules $V_1, \dots, V_d$ each possessing a compatible bar involution, there is a compatible bar involution on $V_1 \otimes \cdots \otimes V_d$ defined as follows: pick any $1 \leq k < d$ then set

$$\overline{v_1 \otimes \cdots \otimes v_d} = \Theta((\overline{v_1 \otimes \cdots \otimes v_k}) \otimes (\overline{v_{k+1} \otimes \cdots \otimes v_d}))$$

where the bar involutions on the right hand side are defined inductively. By [L, 27.3.6], this definition is independent of the particular choice of k. Alternatively, in terms of the braiding, the bar involution on $V_1 \otimes \cdots \otimes V_d$ satisfies

$$\overline{v_1 \otimes \cdots \otimes v_d} = q^{-\sum_{i<j}(\lambda_i,\lambda_j)} \mathcal{R}_{w_d}(\overline{v_d} \otimes \cdots \otimes \overline{v_1}) \tag{3.2}$$

if v_i is of weight λ_i, recalling that w_d denotes the longest element of S_d. Also,

$$\overline{\mathcal{R}_w(v_1 \otimes \cdots \otimes v_d)} = \mathcal{R}_{w^{-1}}^{-1}(\overline{v_1 \otimes \cdots \otimes v_d}) \tag{3.3}$$

for any $w \in S_d$ and $v_i \in V_i$; the proof of this reduces easily to the case $\ell(w) = 1$ which follows using the identity $\Theta^{-1} = \overline{\Theta}$ from [L, 4.1.3].

We say that A is a *polynomial* $\mathcal{U}_n$*-algebra* if A is a polynomial $\mathcal{U}_n$-module and an associative algebra, with identity element 1_A and multiplication $\mu_A : A \otimes A \to A$, such that $u1_A = \varepsilon(u)1_A$ and $u(xx') = \mu_A(\Delta(u)(x \otimes x'))$ for each $u \in \mathcal{U}_n, x, x' \in A$. Given two polynomial $\mathcal{U}_n$-algebras A and B, the tensor product $A \otimes B$ is a polynomial $\mathcal{U}_n$-module; we make it into a polynomial $\mathcal{U}_n$-algebra by defining the multiplication $\mu_{A\otimes B} : A \otimes B \otimes A \otimes B \to A \otimes B$ from $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (\mathrm{id}_A \otimes \mathcal{R}_{B,A} \otimes \mathrm{id}_B)$. It is well known that this multiplication is associative. More generally, given polynomial $\mathcal{U}_n$-algebras $A_1, \dots, A_d$, we make the tensor product $A_1 \otimes \cdots \otimes A_d$ into a polynomial $\mathcal{U}_n$-algebra by iterating this construction. Explicitly, the multiplication is the map $(\mu_{A_1} \otimes \cdots \otimes \mu_{A_d}) \circ \mathcal{R}_w : A_1 \otimes \cdots \otimes A_d \otimes A_1 \otimes \cdots \otimes A_d \to A_1 \otimes \cdots \otimes A_d$ where $w : (1, 2, \dots, d, d+1, d+2, \dots, 2d) \mapsto (1, 3, \dots, 2d-1, 2, 4, \dots, 2d)$.

Lemma 2. *Suppose that $A_1, \dots, A_d$ are polynomial $\mathcal{U}_n$-algebras equipped with compatible bar involutions such that $\mu_{A_i}(\overline{x_i \otimes y_i}) = \overline{x_i y_i}$ for each i and $x_i, y_i \in A_i$. View the tensor product $A_1 \otimes \cdots \otimes A_d$ as a polynomial $\mathcal{U}_n$-algebra equipped with a compatible bar involution by the above constructions. Let $*$ denote the twisted multiplication on $A_1 \otimes \cdots \otimes A_d$ defined by the map $((\mu_{A_1} \circ \mathcal{R}_{A_1,A_1}) \otimes \cdots \otimes (\mu_{A_d} \circ \mathcal{R}_{A_d,A_d})) \circ \mathcal{R}_w$ where $w : (1, 2, \dots, d, d+1, d+2, \dots, 2d) \mapsto (1, 3, \dots, 2d-1, 2, 4, \dots, 2d)$. Then,*

$$\overline{(x_1 \otimes \cdots \otimes x_d)(y_1 \otimes \cdots \otimes y_d)} = q^{-(\lambda,\mu)}(\overline{y_1 \otimes \cdots \otimes y_d}) * (\overline{x_1 \otimes \cdots \otimes x_d})$$

for $x_i, y_i \in A_i$ such that $x_1 \otimes \cdots \otimes x_d$ is of weight λ and $y_1 \otimes \cdots \otimes y_d$ is of weight μ.

Proof. Using (3.3) and the definitions, we have that

$$\begin{aligned}
&\overline{(x_1 \otimes \cdots \otimes x_d)(y_1 \otimes \cdots \otimes y_d)} \\
&\quad = \overline{((\mu_{A_1} \otimes \cdots \otimes \mu_{A_d}) \circ \mathcal{R}_w)(x_1 \otimes \cdots \otimes x_d \otimes y_1 \otimes \cdots \otimes y_d)} \\
&\quad = ((\mu_{A_1} \otimes \cdots \otimes \mu_{A_d}) \circ \mathcal{R}_{w^{-1}}^{-1})(\overline{x_1 \otimes \cdots \otimes x_d \otimes y_1 \otimes \cdots \otimes y_d}) \\
&\quad = q^{-(\lambda,\mu)}((\mu_{A_1} \otimes \cdots \otimes \mu_{A_d}) \circ \mathcal{R}_{w^{-1}}^{-1} \circ \mathcal{R}_v)(\overline{y_1 \otimes \cdots \otimes y_d} \otimes \overline{x_1 \otimes \cdots \otimes x_d})
\end{aligned}$$

where w is as in the statement of the lemma and $v = (1\ d+1)(2\ d+2)\cdots(d\ 2d)$. Now the proof is completed by observing that $wvw^{-1} = (1\ 2)(3\ 4)\cdots(2d-1\ 2d)$ and then checking that lengths add correctly so that $\mathcal{R}_v = \mathcal{R}_{w^{-1}}\mathcal{R}_{wvw^{-1}}\mathcal{R}_w$. $\square$

4. Kazhdan-Lusztig polynomials

The next job is to review the definition of the parabolic Kazhdan-Lusztig polynomials associated to the symmetric group S_d. Let $\mathcal{H}_d$ denote the corresponding Hecke algebra. By definition, this is the $\mathbb{Q}(q)$-algebra with basis $\{H_x \mid x \in S_d\}$ and multiplication defined by the rules that $H_x H_y = H_{xy}$ if $\ell(xy) = \ell(x) + \ell(y)$, and

$$H_i^2 = 1 - (q - q^{-1})H_i, \tag{4.1}$$

where we write $H_i = H_{s_i}$ for short. Take any weight $\lambda \in \Lambda_n$ with $|\lambda| = d$. Corresponding to the parabolic subgroup S_λ of S_d, we have the parabolic subalgebra $\mathcal{H}_\lambda$ of $\mathcal{H}_d$ spanned by $\{H_x \mid x \in S_\lambda\}$. Let $1_{\mathcal{H}_\lambda}$ denote the one dimensional right $\mathcal{H}_\lambda$-module spanned by a vector 1_λ such that $1_\lambda H_i = q^{-1} 1_\lambda$ for each $H_i \in \mathcal{H}_\lambda$. Form the induced module

$$\mathcal{M}^\lambda = 1_{\mathcal{H}_\lambda} \otimes_{\mathcal{H}_\lambda} \mathcal{H}_d. \tag{4.2}$$

This has a natural basis $\{M_x \mid x \in \mathcal{D}_\lambda\}$ defined from $M_x = 1_\lambda \otimes H_x$. Now we can introduce the two families of parabolic Kazhdan-Lusztig polynomials, following [S, §3] closely. We need the bar involution on $\mathcal{H}_d$, that is, the unique antilinear automorphism of $\mathcal{H}_d$ such that $\overline{H_w} = H_{w^{-1}}^{-1}$ for each $w \in S_d$; in particular, $\overline{H_i} = H_i - (q - q^{-1})$. There is an induced bar involution on $\mathcal{M}^\lambda$, with $\overline{M_x} = 1_\lambda \otimes \overline{H_x}$ for each $x \in \mathcal{D}_\lambda$. By [S, 3.1,3.5], there are unique bar invariant elements $\underline{\tilde{M}}_x, \underline{M}_x \in \mathcal{M}^\lambda$ for each $x \in \mathcal{D}_\lambda$ such that

$$\underline{\tilde{M}}_x \in M_x + \sum_{y \in \mathcal{D}_\lambda} q^{-1}\mathbb{Z}[q^{-1}] M_y, \qquad \underline{M}_x \in M_x + \sum_{y \in \mathcal{D}_\lambda} q\mathbb{Z}[q] M_y.$$

In Soergel's notation, we have that

$$\underline{\tilde{M}}_y = \sum_{x \in \mathcal{D}_\lambda} (-1)^{\ell(x)+\ell(y)} n_{x,y}(q^{-1}) M_x, \qquad \underline{M}_y = \sum_{x \in \mathcal{D}_\lambda} m_{x,y}(q) M_x \tag{4.3}$$

for polynomials $n_{x,y}(q), m_{x,y}(q) \in \mathbb{Z}[q]$ which up to a shift are the usual parabolic Kazhdan-Lusztig polynomials of [KL, Deo]; see [S, 3.2] for the precise identification. Recalling that $\leq$ is the opposite of the usual Bruhat ordering on S_d, we have that $n_{x,x}(q) = m_{x,x}(q) = 1$ and $n_{x,y}(q) = m_{x,y}(q) = 0$ unless $x \geq y$.

We now want to review a completely different approach to the construction of these polynomials involving the quantized enveloping algebra $\mathcal{U}_n = U_q(\mathfrak{gl}_n)$ from §3 in place of the Hecke algebra $\mathcal{H}_d$. The coincidence here is well explained algebraically by Schur-Weyl duality, and that is the point of view we will take. The exposition in the remainder of the section is equivalent to that of [FKK], which we believe is the first place that this elementary approach appeared explicitly in the literature. There is also an older geometric explanation which relies on the local isomorphism between Schubert varieties and the varieties arising from representations of quivers in type A from [Ze]; see [GL]. To start with, let $\mathcal{V}_n$ denote the natural $\mathcal{U}_n$-module, that is, the polynomial representation on basis $\{v_i \mid i = 1, \dots, n\}$ with action defined by

$$K_i v_j = q^{(\varepsilon_i, \varepsilon_j)} v_j, \qquad E_i v_j = \delta_{i+1,j} v_i, \qquad F_i v_j = \delta_{i,j} v_{i+1}.$$

The tensor algebra $T(\mathcal{V}_n) = \bigoplus_{d \geq 0} T^d(\mathcal{V}_n)$ is a polynomial $\mathcal{U}_n$-algebra in the sense of §3. The $\mathcal{U}_n$-module $\mathcal{V}_n$ possesses compatible bar involution defined simply by

$\overline{v_i} = v_i$ for each $i = 1, \dots, n$. By the tensor product construction from §3, we get induced a compatible bar involution on each $T^d(\mathcal{V}_n)$, hence on the tensor algebra $T(\mathcal{V}_n)$ itself. The bar involution on $\mathcal{V}_n \otimes \mathcal{V}_n$ satisfies

$$\overline{v_i \otimes v_j} = \begin{cases} v_i \otimes v_j & \text{if } i \leq j, \\ v_i \otimes v_j + (q - q^{-1}) v_j \otimes v_i & \text{if } i > j. \end{cases} \tag{4.4}$$

This can be seen as follows: if $i \leq j$ all Θ_λ except for Θ_0 annihilate $v_i \otimes v_j$ by weight considerations hence $\overline{v_i \otimes v_j} = v_i \otimes v_j$ in these cases; then for $i > j$ one applies $F_{i-1} F_{i-2} \cdots F_j$ to both sides of the identity $\overline{v_j \otimes v_j} = v_j \otimes v_j$ to deduce the formula in these cases too.

Combining (4.4) with (3.2), one checks that the inverse braiding $\mathcal{R}^{-1}_{\mathcal{V}_n, \mathcal{V}_n}$ satisfies the quadratic relation (4.1). Hence, there is a well-defined *right action* of the Hecke algebra $\mathcal{H}_d$ on $T^d(\mathcal{V}_n)$ defined from $vH_w = \mathcal{R}_w^{-1}(v)$ for $v \in T^d(\mathcal{V}_n)$ and $w \in S_d$, making $T^d(\mathcal{V}_n)$ into a $(\mathcal{U}_n, \mathcal{H}_d)$-bimodule. To write this action of $\mathcal{H}_d$ down in a more familiar way in terms of generators, let $\alpha = (\alpha_1, \dots, \alpha_d) \in I_n^d$ be a multi-index as in §2. Define $M_\alpha = v_{\alpha_1} \otimes v_{\alpha_2} \otimes \cdots \otimes v_{\alpha_d}$, so that $\{M_\alpha \mid \alpha \in I_n^d\}$ is the standard basis for $T^d(\mathcal{V}_n)$. Then,

$$M_\alpha H_i = \begin{cases} M_{\alpha \cdot s_i} & \text{if } \alpha_i < \alpha_{i+1}, \\ q^{-1} M_\alpha & \text{if } \alpha_i = \alpha_{i+1}, \\ M_{\alpha \cdot s_i} - (q - q^{-1}) M_\alpha & \text{if } \alpha_i > \alpha_{i+1}, \end{cases} \tag{4.5}$$

for each $\alpha \in I_n^d$ and $i = 1, \dots, d-1$. We will also often work with the elements

$$M_\alpha^* = v_{\alpha_d} \otimes \cdots \otimes v_{\alpha_2} \otimes v_{\alpha_1} = M_{\alpha \cdot w_d}, \tag{4.6}$$

so $\{M_\alpha^* \,|\, \alpha \in I_n^d\}$ is the same basis as before but parametrized in the opposite way.

Now fix a weight $\lambda \in \Lambda_n$ with $|\lambda| = d$ and consider the λ-weight space $T^d_\lambda(\mathcal{V}_n)$ of $T^d(\mathcal{V}_n)$. It is well known, and easy to prove using (4.5) and the definition (4.2), that the map

$$\psi_\lambda : T^d_\lambda(\mathcal{V}_n) \to \mathcal{M}^\lambda, \qquad M_\alpha \mapsto M_{d(\alpha)}, \quad M_\alpha^* \mapsto M_{w_\lambda d(\alpha) w_d}$$

is an isomorphism of $\mathcal{H}_d$-modules. The key observation is that the restriction of the bar involution on $T^d(\mathcal{V}_n)$ to its λ-weight space agrees with the bar involution on $\mathcal{M}^\lambda$ under the isomorphism ψ_λ, i.e. for any $v \in T^d_\lambda(\mathcal{V}_n)$ we have that $\psi_\lambda(\overline{v}) = \overline{\psi_\lambda(v)}$. To see this, just note that if $\alpha \in I_\lambda$ has $\alpha_1 \leq \cdots \leq \alpha_d$, then M_α is bar invariant by weight considerations. Since $\psi_\lambda(M_\alpha) = M_1$ generates $\mathcal{M}^\lambda$ as an $\mathcal{H}_d$-module and M_1 is bar invariant too, it just remains to observe by (3.3) that $\overline{vh} = \overline{v}\,\overline{h}$ for any $v \in T^d_\lambda(\mathcal{V}_n)$ and $h \in \mathcal{H}_d$. We deduce comparing with the opening paragraph of the section that for every $\alpha \in I_\lambda$ there exist unique bar invariant elements L_α and L_α^* in $T^d_\lambda(\mathcal{V}_n)$ such that

$$L_\alpha \in M_\alpha + \sum_{\beta \in I_\lambda} q^{-1}\mathbb{Z}[q^{-1}] M_\beta, \quad L_\alpha^* \in M_\alpha^* + \sum_{\beta \in I_\lambda} q\mathbb{Z}[q] M_\beta^*.$$

Moreover, $\psi_\lambda(L_\alpha) = \underline{\tilde{M}}_{d(\alpha)}$ and $\psi_\lambda(L_\alpha^*) = \underline{M}_{w_\lambda d(\alpha) w_d}$. Let $l_{\alpha,\beta}(q) \in \mathbb{Z}[q^{-1}]$ and $l^*_{\alpha,\beta}(q) \in \mathbb{Z}[q]$ denote the coefficients defined from

$$L_\beta = \sum_{\alpha \in I_\lambda} l_{\alpha,\beta}(q) M_\alpha, \qquad L_\beta^* = \sum_{\alpha \in I_\lambda} l^*_{\alpha,\beta}(q) M_\alpha^*. \tag{4.7}$$

These are the same as the coefficients in (4.3), taking $x = d(\alpha), y = d(\beta)$ and $x = w_\lambda d(\alpha) w_d, y = w_\lambda d(\beta) w_d$, respectively. We have now constructed two new bases $\{L_\alpha \,|\, \alpha \in I_n^d\}$ and $\{L_\alpha^* \,|\, \alpha \in I_n^d\}$ for the tensor space $T^d(\mathcal{V}_n)$, which we call the *dual canonical* and the *canonical* bases. In Kashiwara's language, they are upper and lower global crystal bases, respectively.

Let us recall from [K3] the precise meaning of the previous sentence. Denote the $\mathbb{Z}[q,q^{-1}]$-submodule of $\mathcal{V}_n$ spanned by $v_1, \dots, v_n$ by $\mathscr{V}_n$; it is invariant under the action of Lusztig's integral form $\mathscr{U}_n$ for $\mathcal{U}_n$, i.e. the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\mathcal{U}_n$ generated by all $E_i^{(r)} = E_i^r/[r]!, F_i^{(r)} = F_i^r/[r]!$, $K_i^{\pm 1}$ and $\left[\begin{smallmatrix} K_i \\ r \end{smallmatrix}\right] = \prod_{s=1}^r \frac{K_i q^{1-s} - K_i^{-1} q^{s-1}}{q^s - q^{-s}}$. Taking tensor products over $\mathbb{Z}[q,q^{-1}]$, we obtain the $\mathbb{Z}[q,q^{-1}]$-lattice $T^d(\mathscr{V}_n)$ in $T^d(\mathcal{V}_n)$. Next, let $\mathbb{A}_0$ resp. $\mathbb{A}_\infty$ be the subring of $\mathbb{Q}(q)$ consisting of all rational functions having no pole at $q = 0$ resp. $q = \infty$, so $\mathbb{A}_\infty = \overline{\mathbb{A}_0}$. Let $T^d(\mathcal{V}_n)_0$ resp. $T^d(\mathcal{V}_n)_\infty$ be the $\mathbb{A}_0$- resp. $\mathbb{A}_\infty$-submodule of $T^d(\mathcal{V}_n)$ generated by the elements $\{M_\alpha^* \,|\, \alpha \in I_n^d\}$ resp. $\{M_\alpha \,|\, \alpha \in I_n^d\}$. Then, by Kashiwara's tensor product rules [K1, K2], $T^d(\mathcal{V}_n)_0$ resp. $T^d(\mathcal{V}_n)_\infty$ is a lower resp. upper crystal lattice at $q = 0$ resp. $q = \infty$, and the image of the basis $\{M_\alpha^* \mid \alpha \in I_n^d\}$ resp. $\{M_\alpha \mid \alpha \in I_n^d\}$ in $T^d(\mathcal{V}_n)_0 / q T^d(\mathcal{V}_n)_0$ resp. $T^d(\mathcal{V}_n)_\infty / q^{-1} T^d(\mathcal{V}_n)_\infty$ is a lower resp. upper crystal base at $q = 0$ resp. $q = \infty$. The actions of the lower resp. upper crystal operators on these crystal bases is described by the crystal $(I_n^d, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ from §2. Finally, the lower resp. upper global crystal base $\{L_\alpha^* \mid \alpha \in I_n^d\}$ resp. $\{L_\alpha \mid \alpha \in I_n^d\}$ is the unique lift of this local crystal base arising from the balanced triple $(\mathbb{Q} \otimes_{\mathbb{Z}} T^d(\mathscr{V}_n), T^d(\mathcal{V}_n)_0, \overline{T^d(\mathcal{V}_n)_0})$ resp. $(\mathbb{Q} \otimes_{\mathbb{Z}} T^d(\mathscr{V}_n), \overline{T^d(\mathcal{V}_n)_\infty}, T^d(\mathcal{V}_n)_\infty)$.

The only other thing we want to do in this section is to reprove the inversion formula for parabolic Kazhdan-Lusztig polynomials due originally to Douglass [Do] (see also [S, 3.9]) in terms of tensor space. The argument involves an important bilinear form $(.,.)$ on $T^d(\mathcal{V}_n)$ defined by setting $(M_\alpha, \overline{M_\beta^*}) = \delta_{\alpha,\beta}$ for each $\alpha, \beta \in I_n^d$. Recall that $\tau : \mathcal{U}_n \to \mathcal{U}_n$ is the antiautomorphism with $\tau(E_i) = F_i, \tau(F_i) = E_i$ and $\tau(K_i) = K_i$. Also let $\tau : \mathcal{H}_d \to \mathcal{H}_d$ be the antiautomorphism with $\tau(H_i) = H_{d-i}$.

Lemma 3. *The bilinear form $(.,.)$ is symmetric and $(uvh, w) = (v, \tau(u) w \tau(h))$ for all $u \in \mathcal{U}_n$, $h \in \mathcal{H}_d$ and $v, w \in T^d(\mathcal{V}_n)$.*

Proof. The second part is a routine direct check on generators. For the first part, we need to show that $(\overline{M_\beta^*}, M_\alpha) = \delta_{\alpha,\beta}$. By (3.2), $\overline{M_\beta^*} = q^{-\sum_{i<j}(\varepsilon_{\beta_i}, \varepsilon_{\beta_j})} M_\beta H_{w_d}^{-1}$. Since $\tau(H_{w_d}) = H_{w_d}$, we get that

$$\begin{aligned}(\overline{M_\beta^*}, M_\alpha) &= q^{-\sum_{i<j}(\varepsilon_{\beta_i}, \varepsilon_{\beta_j})} (M_\beta H_{w_d}^{-1}, M_\alpha) = q^{-\sum_{i<j}(\varepsilon_{\beta_i}, \varepsilon_{\beta_j})} (M_\beta, M_\alpha H_{w_d}^{-1}) \\ &= q^{\sum_{i<j}((\varepsilon_{\alpha_i}, \varepsilon_{\alpha_j}) - (\varepsilon_{\beta_i}, \varepsilon_{\beta_j}))} (M_\beta, \overline{M_\alpha^*}) = \delta_{\alpha,\beta}.\end{aligned}$$

□

Theorem 4. $(L_\alpha, L_\beta^*) = \delta_{\alpha,\beta}$.

Proof. Since L_β^* is bar invariant, we have by (4.7) that $L_\alpha = \sum_\gamma l_{\gamma,\alpha}(q) M_\gamma$, $L_\beta^* = \sum_\delta l_{\delta,\beta}^*(q^{-1}) \overline{M_\delta^*}$. Similarly, $L_\beta^* = \sum_\gamma l_{\gamma,\beta}^*(q) M_{\gamma \cdot w_d}$, $L_\alpha = \sum_\delta l_{\delta,\alpha}(q^{-1}) \overline{M_{\delta \cdot w_d}^*}$. Hence, by definition of the form,

$$(L_\alpha, L^*_\beta) = \sum_\gamma l_{\gamma,\alpha}(q) l^*_{\gamma,\beta}(q^{-1}) \equiv \delta_{\alpha,\beta} \pmod{q^{-1}\mathbb{Z}[q^{-1}]},$$

$$(L^*_\beta, L_\alpha) = \sum_\gamma l^*_{\gamma,\beta}(q) l_{\gamma,\alpha}(q^{-1}) \equiv \delta_{\alpha,\beta} \pmod{q\mathbb{Z}[q]}.$$

By Lemma 3, we know that $(L_\alpha, L^*_\beta) = (L^*_\beta, L_\alpha)$, so these two congruences together imply that $(L_\alpha, L^*_\beta) = \delta_{\alpha,\beta}$. □

Corollary 5. $\displaystyle\sum_{\gamma \in I^d_n} l_{\alpha,\gamma}(q) l^*_{\beta,\gamma}(q^{-1}) = \delta_{\alpha,\beta}.$

Remark 6. Let us explain the essential difference between the exposition here and that of [FKK]. In that paper, there are two different $\mathcal{U}_n$-module structures and two different bar involutions on the underlying vector space $T^d(\mathcal{V}_n)$. One of these is used to define the dual canonical basis, exactly as here. The other $\mathcal{U}_n$-module structure, which may be denoted $\widetilde{T}^d(\mathcal{V}_n)$, is defined using the comultiplication $\widetilde{\Delta}$ from §3, and its compatible bar involution is defined using the corresponding quasi-R-matrix $\widetilde{\Theta}$. Letting $\widetilde{M}_\alpha = v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_d} \in \widetilde{T}^d(\mathcal{V}_n)$, the canonical basis element $\widetilde{L}_\alpha$ is then the unique bar invariant element lying in $\widetilde{M}_\alpha + \sum_{\beta \in I^d_n} q^{-1}\mathbb{Z}[q^{-1}]\widetilde{M}_\beta$. To translate between this and the approach followed here, we note that there is a $\mathcal{U}_n$-module isomorphism $\widetilde{T}^d(\mathcal{V}_n) \to T^d(\mathcal{V}_n)$, $\widetilde{M}_\alpha \mapsto \overline{M^*_\alpha}$, $\widetilde{L}_\alpha \mapsto L^*_\alpha$.

5. Symmetric and exterior powers

In this section, we define canonical and dual canonical bases in tensor products of symmetric and exterior powers of $\mathcal{V}_n$, generalizing the canonical and dual canonical bases of tensor space from the previous section. We start with symmetric powers, then summarize the necessary changes for exterior powers at the end of the section. By definition, the *quantum symmetric algebra* $S(\mathcal{V}_n)$ is the quotient of $T(\mathcal{V}_n)$ by the two-sided ideal I generated by the elements

$$\{v_j \otimes v_i - q^{-1} v_i \otimes v_j \mid 1 \le i < j \le n\}. \tag{5.1}$$

Clearly, $I = \bigoplus_{d \ge 0} I_d$ where $I_d = I \cap T^d(\mathcal{V}_n)$. The *dth symmetric power* $S^d(\mathcal{V}_n)$ is the dth homogeneous component $T^d(\mathcal{V}_n)/I_d$ of $S(\mathcal{V}_n)$, so $S(\mathcal{V}_n) = \bigoplus_{d \ge 0} S^d(\mathcal{V}_n)$. One checks that I_2 is invariant both under the action of $\mathcal{U}_n$ and under the bar involution on $T^2(\mathcal{V}_n)$. Since I_2 generates I, it follows that *all* I_d are invariant under $\mathcal{U}_n$ and under the bar involution. Hence, $S^d(\mathcal{V}_n)$ is a $\mathcal{U}_n$-module quotient of $T^d(\mathcal{V}_n)$, and the bar involution on $T^d(\mathcal{V}_n)$ descends to give a compatible bar involution on $S^d(\mathcal{V}_n)$.

We also need the dual object, the *dth divided power* $\widetilde{S}^d(\mathcal{V}_n)$. To define this, let

$$X_d = \sum_{w \in S_d} q^{\ell(w_d) - \ell(w)} H_w \in \mathcal{H}_d.$$

Then, by definition, $\widetilde{S}^d(\mathcal{V}_n)$ is the $\mathcal{U}_n$-submodule $T^d(\mathcal{V}_n)X_d$ of $T^d(\mathcal{V}_n)$. It is well known that X_d is bar invariant, hence the bar involution on $T^d(\mathcal{V}_n)$ restricts to a well-defined compatible bar involution on $\widetilde{S}^d(\mathcal{V}_n)$, and also

$$H_i X_d = q^{-1} X_d = X_d H_i \tag{5.2}$$

for all i. Let $\iota : \widetilde{S}^d(\mathcal{V}_n) \hookrightarrow T^d(\mathcal{V}_n)$ be the inclusion and $\pi : T^d(\mathcal{V}_n) \twoheadrightarrow S^d(\mathcal{V}_n)$ be the quotient map. We claim that the bilinear form $(.,.)$ on $T^d(\mathcal{V}_n)$ induces a well defined pairing $(.,.) : S^d(\mathcal{V}_n) \times \widetilde{S}^d(\mathcal{V}_n) \to \mathbb{Q}(q)$ with $(\pi(v), w) = (v, \iota(w))$ for all $v \in T^d(\mathcal{V}_n)$ and $w \in \widetilde{S}^d(\mathcal{V}_n)$. To prove this, we need to show that $(\operatorname{Ker}\pi, \operatorname{Im}\iota) = 0$. By (5.1) and (4.5), $\operatorname{Ker}\pi$ is spanned by vectors of the form $v(H_i - q^{-1})$, while $\operatorname{Im}\iota$ is spanned by vectors of the form wX_d. Now Lemma 3 and (5.2) show that $(v(H_i - q^{-1}), wX_d) = (v, wX_d(H_{d-i} - q^{-1})) = 0$ proving the claim.

To define the standard bases for the spaces $S^d(\mathcal{V}_n)$ and $\widetilde{S}^d(\mathcal{V}_n)$, take $\alpha \in I_n^d$ with $\alpha_1 \leq \cdots \leq \alpha_d$. Define X_α to be the bar invariant element $\pi(M_\alpha)$ of $S^d(\mathcal{V}_n)$. Also, letting λ denote the weight $\theta(\alpha)$, set

$$X_\alpha^* = \sum_{\beta \sim \alpha} q^{\ell(\alpha,\beta)} M_\beta^* = \frac{1}{[\lambda_1]! \cdots [\lambda_n]!} M_\alpha X_d, \tag{5.3}$$

where $\ell(\alpha, \beta)$ denotes the length of the shortest element $w \in S_d$ with $\beta = \alpha \cdot w$. Note X_α^* belongs to $\widetilde{S}^d(\mathcal{V}_n)$ and it is bar invariant (indeed, it coincides with the canonical basis element L_α^*). Using (5.1) and (5.2) one checks easily that the vectors X_α and X_β^* for all weakly increasing $\alpha, \beta \in I_n^d$ span $S^d(\mathcal{V}_n)$ and $\widetilde{S}^d(\mathcal{V}_n)$, respectively. Finally, we have that

$$(X_\alpha, \overline{X_\beta^*}) = \sum_{\gamma \sim \beta} q^{-\ell(\beta,\gamma)} (M_\alpha, \overline{M_\gamma^*}) = \delta_{\alpha,\beta}.$$

This gives the linear independence needed to show that $\{X_\alpha | \alpha \in I_n^d, \alpha_1 \leq \cdots \leq \alpha_d\}$ is a basis for $S^d(\mathcal{V}_n)$ and $\{X_\alpha^* \mid \alpha \in I_n^d, \alpha_1 \leq \cdots \leq \alpha_d\}$ is a basis for $\widetilde{S}^d(\mathcal{V}_n)$.

Suppose more generally that $\mu \in \Lambda_m$ and $|\mu| = d$. Consider the $\mathcal{U}_n$-modules

$$S^\mu(\mathcal{V}_n) = S^{\mu_m}(\mathcal{V}_n) \otimes \cdots \otimes S^{\mu_1}(\mathcal{V}_n), \qquad \widetilde{S}^\mu(\mathcal{V}_n) = \widetilde{S}^{\mu_1}(\mathcal{V}_n) \otimes \cdots \otimes \widetilde{S}^{\mu_m}(\mathcal{V}_n).$$

Note $S^\mu(\mathcal{V}_n)$ is a quotient of $T^d(\mathcal{V}_n)$; we let $\pi : T^d(\mathcal{V}_n) \twoheadrightarrow S^\mu(\mathcal{V}_n)$ be the quotient homomorphism. Also, $\widetilde{S}^\mu(\mathcal{V}_n)$ is a submodule of $T^d(\mathcal{V}_n)$; we let $\iota : \widetilde{S}^\mu(\mathcal{V}_n) \hookrightarrow T^d(\mathcal{V}_n)$ be the inclusion. Since all $S^{\mu_i}(\mathcal{V}_n)$ and $\widetilde{S}^{\mu_j}(\mathcal{V}_n)$ possess compatible bar involutions, we get induced compatible bar involutions on $S^\mu(\mathcal{V}_n)$ and $\widetilde{S}^\mu(\mathcal{V}_n)$ by the general construction explained in §3. It is immediate from this construction that these bar involutions are consistent with the one on $T^d(\mathcal{V}_n)$ itself, i.e. the maps π and ι commute with the bar involutions. As before, the symmetric bilinear form $(.,.)$ on $T^d(\mathcal{V}_n)$ induces a well-defined pairing $(.,.) : S^\mu(\mathcal{V}_n) \times \widetilde{S}^\mu(\mathcal{V}_n) \to \mathbb{Q}(q)$ with $(\pi(v), w) = (v, \iota(w))$ for all $v \in T^d(\mathcal{V}_n)$ and $w \in \widetilde{S}^\mu(\mathcal{V}_n)$.

For each $\nu \in \Lambda_n$ with $|\nu| = d$, there are natural monomial bases for the ν-weight spaces $S^\mu_\nu(\mathcal{V}_n)$ and $\widetilde{S}^\mu_\nu(\mathcal{V}_n)$ of $S^\mu(\mathcal{V}_n)$ and $\widetilde{S}^\mu(\mathcal{V}_n)$, parametrized by the set $\operatorname{Row}(\mu, \nu)$ of row standard tableaux of row shape μ and weight ν from §2. To write these down, recall that $\rho(A)$ is the row reading of the tableau A. For $A \in \operatorname{Row}(\mu, \nu)$, let $M_A = \pi(M_{\rho(A)}) \in S^\mu(\mathcal{V}_n)$ and let $M_A^* \in \widetilde{S}^\mu(\mathcal{V}_n)$ be the unique element with

$$\iota(M_A^*) = \sum_{B \sim_{\mathrm{ro}} A} q^{\ell(A,B)} M^*_{\rho(B)}, \tag{5.4}$$

writing $\ell(A,B)$ for the minimal number of transpositions of neighbouring entries in the same row needed to get B from A. Then, the vectors $\{M_A \,|\, A \in \mathrm{Row}(\mu,\nu)\}$ give a basis for $S^\mu_\nu(\mathcal{V}_n)$ and the vectors $\{M_A^* \mid A \in \mathrm{Row}(\mu,\nu)\}$ give a basis for $\widetilde{S}^\mu_\nu(\mathcal{V}_n)$. Moreover, the pairing $(.,.)$ satisfies $(M_A, \overline{M_B^*}) = \delta_{A,B}$.

We can now introduce the canonical and dual canonical bases. Recalling the second equivalent definition of the Bruhat ordering on $\mathrm{Row}(\mu,\nu)$ from §2, one checks by weight considerations that the bar involutions on $S^\mu(\mathcal{V}_n)$, $\widetilde{S}^\mu(\mathcal{V}_n)$ satisfy

$$\overline{M_A} = M_A + (\text{a } \mathbb{Z}[q,q^{-1}]\text{-linear combination of } M_B\text{'s for } B < A),$$
$$\overline{M_A^*} = M_A^* + (\text{a } \mathbb{Z}[q,q^{-1}]\text{-linear combination of } M_B^*\text{'s for } B > A).$$

Hence by [D2, 1.2], we deduce that for every $A \in \mathrm{Row}(\mu,\nu)$ there are unique bar invariant elements $L_A \in S^\mu(\mathcal{V}_n)$ and $L_A^* \in \widetilde{S}^\mu(\mathcal{V}_n)$ such that

$$L_A \in M_A + \sum_{B \in \mathrm{Row}(\mu,\nu)} q^{-1}\mathbb{Z}[q^{-1}] M_B, \quad L_A^* \in M_A^* + \sum_{B \in \mathrm{Row}(\mu,\nu)} q\mathbb{Z}[q] M_B^*.$$

As before, we introduce notation for the coefficients:

$$L_B = \sum_{A \in \mathrm{Row}(\mu,\nu)} l_{A,B}(q) M_A, \qquad L_B^* = \sum_{A \in \mathrm{Row}(\mu,\nu)} l^*_{A,B}(q) M_A^*. \tag{5.5}$$

The polynomials $l_{A,B}(q) \in \mathbb{Z}[q^{-1}]$, $l^*_{A,B}(q) \in \mathbb{Z}[q]$ satisfy $l_{A,A}(q) = l^*_{A,A}(q) = 1$ and $l_{A,B}(q) = 0$ unless $A \leq B$, $l^*_{A,B}(q) = 0$ unless $A \geq B$. We have now constructed two new bases $\{L_A \mid A \in \mathrm{Row}(\mu,\nu)\}$ for $S^\mu_\nu(\mathcal{V}_n)$ and $\{L_A^* \mid A \in \mathrm{Row}(\mu,\nu)\}$ for $\widetilde{S}^\mu_\nu(\mathcal{V}_n)$, which we call the *dual canonical* and the *canonical* bases, respectively. They are upper and lower global crystal bases in the sense of [K3], the precise meaning of this phrase being just like in the previous section. We just note that the constructions just described can be carried out equally well over the ring $\mathbb{Z}[q,q^{-1}]$, to obtain the natural integral forms $S^\mu(\mathscr{V}_n)$ and $\widetilde{S}^\mu(\mathscr{V}_n)$. Thus, $S^\mu(\mathscr{V}_n)$ is the free $\mathbb{Z}[q,q^{-1}]$-module with basis given either by the M_A's or by the L_A's, $\widetilde{S}^\mu(\mathscr{V}_n)$ is the free $\mathbb{Z}[q,q^{-1}]$-module with basis given either by the M_A^*'s or by the L_A^*'s. Both are invariant under the action of Lusztig's $\mathbb{Z}[q,q^{-1}]$-form $\mathscr{U}_n$.

Theorem 7. *For $A \in \mathrm{Row}(\mu,\nu)$ we have that $\iota(L_A^*) = L^*_{\rho(A)}$ and $\pi(L_{\rho(A)}) = L_A$. Moreover, if $\alpha \in I_\nu$ is not equal to $\rho(A)$ for any $A \in \mathrm{Row}(\mu,\nu)$, then $\pi(L_\alpha) = 0$. Hence, $(L_A, L_B^*) = \delta_{A,B}$ for all $A, B \in \mathrm{Row}(\mu,\nu)$.*

Proof. Note for $A \in \mathrm{Row}(\mu,\nu)$ that $\iota(L_A^*)$ is bar invariant and it equals $M^*_{\rho(A)}$ plus a $q\mathbb{Z}[q]$-linear combination of M_β^*'s. Hence, $\iota(L_A^*) = L^*_{\rho(A)}$. Similarly, $\pi(L_{\rho(A)})$ is bar invariant and it equals M_A plus a $q^{-1}\mathbb{Z}[q^{-1}]$-linear combination of M_B's. Hence, it equals L_A. Moreover, if $\alpha \in I_\nu$ is not equal to $\rho(A)$ for any $A \in \mathrm{Row}(\mu,\nu)$, then $\pi(L_\alpha)$ is bar invariant and it is a $q^{-1}\mathbb{Z}[q^{-1}]$-linear combination of M_B's. Hence, it must be zero. Finally, for any $A, B \in \mathrm{Row}(\mu,\nu)$, we get that $(L_A, L_B^*) = (L_{\rho(A)}, L^*_{\rho(B)}) = \delta_{A,B}$, using Theorem 4. $\square$

Corollary 8. $\displaystyle\sum_{C\in \mathrm{Row}(\mu,\nu)} l_{A,C}(q) l^*_{B,C}(q^{-1}) = \delta_{A,B}$.

Corollary 9. $\displaystyle l_{A,B}(q) = \sum_{C\sim_{\mathrm{ro}} A} q^{-\ell(A,C)} l_{\rho(C),\rho(B)}(q), \quad l^*_{A,B}(q) = l^*_{\rho(A),\rho(B)}(q)$.

Remark 10. Using Corollary 9, the identification of the polynomials in (4.3) and (4.7), and [S, 2.6, 3.4], we obtain the following formulae relating the polynomials $l_{A,B}(q)$ and $l^*_{A,B}(q)$ directly to the original Kazhdan-Lusztig polynomials $P_{x,y}(t) \in \mathbb{Z}[t]$ from [KL]:

$$l_{A,B}(q) = q^{\ell(y)-\ell(x)} \sum_{z\in S_\nu x S_\mu} (-1)^{\ell(z)+\ell(y)} P_{zw_d, yw_d}(q^2),$$
$$l^*_{A,B}(q) = q^{\ell(y)-\ell(x)} P_{x,y}(q^{-2}),$$

where $x = d(\rho(A))w_d$ and $y = d(\rho(B))w_d$. Using these formulae, one sees that the inversion formula from Corollary 8 is the same as [D1, 1.3].

We now turn our attention to exterior powers. The proofs are all the same as the above proofs for symmetric powers, so we omit them. Note however that it is necessary throughout to interchange the roles of canonical and dual canonical bases. The *quantum exterior algebra* $\widetilde{\bigwedge}(\mathcal{V}_n)$ is the quotient of $T(\mathcal{V}_n)$ by the homogeneous two-sided ideal $J = \bigoplus_{d\geq 0} J_d$ generated by the elements

$$\{v_j \otimes v_i + q v_i \otimes v_j \mid 1 \leq i < j \leq n\} \cup \{v_i \otimes v_i \mid 1 \leq i \leq n\}.$$

Let $\widetilde{\bigwedge}^d(\mathcal{V}_n)$ be the dth homogeneous component $T^d(\mathcal{V}_n)/J_d$. It is a $\mathcal{U}_n$-module and inherits a compatible bar involution from the one on $T^d(\mathcal{V}_n)$. Although one usually calls $\widetilde{\bigwedge}^d(\mathcal{V}_n)$ the dth exterior power, we prefer here to reserve that name for the (isomorphic) dual object $\bigwedge^d(\mathcal{V}_n) = T^d(\mathcal{V}_n)Y_d$ where

$$Y_d = \sum_{w\in S_d} (-q)^{\ell(w)-\ell(w_d)} H_w \in \mathcal{H}_d.$$

Since Y_d is bar invariant, the bar involution on $T^d(\mathcal{V}_n)$ restricts to a compatible bar involution on $\bigwedge^d(\mathcal{V}_n)$. Recall also that $H_i Y_d = -qY_d = Y_d H_i$ for all $i = 1, \dots, d-1$. For $\alpha \in I^d_n$ with $\alpha_1 > \dots > \alpha_d$, let

$$Y_\alpha = \sum_{\beta\sim\alpha} (-q)^{-\ell(\alpha,\beta)} M_\beta = M^*_\alpha Y_d \in {\textstyle\bigwedge}^d(\mathcal{V}_n). \tag{5.6}$$

Also let Y^*_α be the image of M^*_α in the quotient $\widetilde{\bigwedge}^d(\mathcal{V}_n)$, Both Y_α and Y^*_α are bar invariant (indeed $Y_\alpha = L_\alpha$). The vectors $\{Y_\alpha \mid \alpha \in I^d_n, \alpha_1 > \dots > \alpha_d\}$ give a basis for $\bigwedge^d(\mathcal{V}_n)$ and the vectors $\{Y^*_\alpha \mid \alpha\in I^d_n, \alpha_1 > \dots > \alpha_d\}$ give a basis for $\widetilde{\bigwedge}^d(\mathcal{V}_n)$.

Now take $\mu \in \Lambda_l$ and $\nu \in \Lambda_n$ with $|\mu| = |\nu| = d$. Consider the $\mathcal{U}_n$-modules

$$\textstyle\bigwedge^\mu(\mathcal{V}_n) = \bigwedge^{\mu_1}(\mathcal{V}_n) \otimes \dots \otimes \bigwedge^{\mu_l}(\mathcal{V}_n), \qquad \widetilde{\bigwedge}^\mu(\mathcal{V}_n) = \widetilde{\bigwedge}^{\mu_l}(\mathcal{V}_n) \otimes \dots \otimes \widetilde{\bigwedge}^{\mu_1}(\mathcal{V}_n).$$

We write $\bigwedge^\mu_\nu(\mathcal{V}_n)$ and $\widetilde{\bigwedge}^\mu_\nu(\mathcal{V}_n)$ for the ν-weight spaces of these modules. Also let $\iota : \bigwedge^\mu(\mathcal{V}_n) \hookrightarrow T^d(\mathcal{V}_n)$ be the natural inclusion and $\pi : T^d(\mathcal{V}_n) \twoheadrightarrow \widetilde{\bigwedge}^\mu(\mathcal{V}_n)$ be the natural quotient map. There are compatible bar involutions on $\bigwedge^\mu(\mathcal{V}_n)$ and on $\widetilde{\bigwedge}^\mu(\mathcal{V}_n)$, consistent with the bar involution on $T^d(\mathcal{V}_n)$, and the form $(.,.)$ induces a pairing $(.,.) : \bigwedge^\mu(\mathcal{V}_n) \times \widetilde{\bigwedge}^\mu(\mathcal{V}_n) \to \mathbb{Q}(q)$. To define bases here, recall the definition of the set $\mathrm{Col}(\mu,\nu)$ and the column reading $\gamma(A)$ of a tableau of column shape μ from §2. For $A \in \mathrm{Col}(\mu,\nu)$, let N_A denote the unique element of $\bigwedge^\mu(\mathcal{V}_n)$ with

$$\iota(N_A) = \sum_{B \sim_{\mathrm{co}} A} (-q)^{-\ell'(A,B)} M_{\gamma(B)}, \tag{5.7}$$

where $\ell'(A,B)$ denotes $\ell(A',B')$. Let $N_A^* = \pi(M^*_{\gamma(A)})$. Then, $\{N_A \mid A \in \mathrm{Col}(\mu,\nu)\}$ is a basis for $\bigwedge^\mu_\nu(\mathcal{V}_n)$ and $\{N_A^* \mid A \in \mathrm{Col}(\mu,\nu)\}$ is a basis for $\widetilde{\bigwedge}^\mu_\nu(\mathcal{V}_n)$. Moreover, the pairing $(.,.)$ satisfies $(N_A, \overline{N_B^*}) = \delta_{A,B}$. We have that

$$\overline{N_A} = N_A + (\text{a } \mathbb{Z}[q,q^{-1}]\text{-linear combination of } N_B\text{'s for } B <' A),$$
$$\overline{N_A^*} = N_A^* + (\text{a } \mathbb{Z}[q,q^{-1}]\text{-linear combination of } N_B^*\text{'s for } B >' A).$$

Hence by [D2, 1.2] there are unique bar invariant elements $K_A \in \bigwedge^\mu(\mathcal{V}_n)$ and $K_A^* \in \widetilde{\bigwedge}^\mu(\mathcal{V}_n)$ for each $A \in \mathrm{Col}(\mu,\nu)$ such that

$$K_A \in N_A + \sum_{B \in \mathrm{Col}(\mu,\nu)} q^{-1}\mathbb{Z}[q^{-1}] N_B, \qquad K_A^* \in N_A^* + \sum_{B \in \mathrm{Col}(\mu,\nu)} q\mathbb{Z}[q] N_B^*.$$

We let

$$K_B = \sum_{A \in \mathrm{Col}(\mu,\nu)} k_{A,B}(q) N_A, \qquad K_B^* = \sum_{A \in \mathrm{Col}(\mu,\nu)} k^*_{A,B}(q) N_A^*. \tag{5.8}$$

Note $k_{A,B}(q) \in \mathbb{Z}[q^{-1}]$ and $k^*_{A,B}(q) \in \mathbb{Z}[q]$ satisfy $k_{A,A}(q) = k^*_{A,A}(q) = 1$ and $k_{A,B}(q) = 0$ unless $A \leq' B$, $k^*_{A,B}(q) = 0$ unless $A \geq' B$. We have now constructed bases $\{K_A \mid A \in \mathrm{Col}(\mu,\nu)\}$ for $\bigwedge^\mu_\nu(\mathcal{V}_n)$ and $\{K_A^* \mid A \in \mathrm{Col}(\mu,\nu)\}$ for $\widetilde{\bigwedge}^\mu_\nu(\mathcal{V}_n)$, which are the *dual canonical* (= upper global crystal) and *canonical* (= lower global crystal) bases, respectively. Finally, we note that the $\mathbb{Z}[q,q^{-1}]$-submodules of $\bigwedge^\mu(\mathcal{V}_n)$ and $\widetilde{\bigwedge}^\mu(\mathcal{V}_n)$ spanned by these bases give the natural integral forms $\bigwedge^\mu(\mathscr{V}_n)$ and $\widetilde{\bigwedge}^\mu(\mathscr{V}_n)$, which are invariant under the action of $\mathscr{U}_n$.

Theorem 11. *For $A \in \mathrm{Col}(\mu,\nu)$ we have that $\iota(K_A) = L_{\gamma(A)}$ and $\pi(L^*_{\gamma(A)}) = K_A^*$. Moreover, if $\alpha \in I_\nu$ is not equal to $\gamma(A)$ for any $A \in \mathrm{Col}(\mu,\nu)$, then $\pi(L^*_\alpha) = 0$. Hence, $(K_A, K_B^*) = \delta_{A,B}$ for all $A, B \in \mathrm{Col}(\mu,\nu)$.*

Corollary 12. $\displaystyle\sum_{C \in \mathrm{Col}(\mu,\nu)} k_{A,C}(q) k^*_{B,C}(q^{-1}) = \delta_{A,B}$.

Corollary 13. $k_{A,B}(q) = l_{\gamma(A),\gamma(B)}(q)$, $k^*_{A,B}(q) = \displaystyle\sum_{C \sim_{\mathrm{co}} A} (-q)^{\ell'(A,C)} l^*_{\gamma(C),\gamma(B)}(q)$.

Remark 14. Using Corollary 13 and [S, 2.6, 3.4] as before, we get that

$$k_{A,B}(q) = (-q)^{\ell(x)-\ell(y)} \sum_{z \in S_\nu} (-1)^{\ell(z)} P_{zx,y}(q^2),$$
$$k^*_{A,B}(q) = (-q)^{\ell(x)-\ell(y)} \sum_{z \in \mathcal{D}_\nu \cap S_\nu x S_\mu} (-1)^{\ell(z)+\ell(y)} P_{zw_d,yw_d}(q^{-2}),$$

where $x = d(\gamma(A))$ and $y = d(\gamma(B))$.

6. The quantized coordinate algebra

Consider now the tensor product $T^m(S(\mathcal{V}_n)) = S(\mathcal{V}_n) \otimes \cdots \otimes S(\mathcal{V}_n)$ of m copies of the symmetric algebra $S(\mathcal{V}_n)$. We obviously have that

$$T^m(S(\mathcal{V}_n)) = \bigoplus_{\substack{(\mu,\nu)\in\Lambda_m\times\Lambda_n \\ |\mu|=|\nu|}} S^\mu_\nu(\mathcal{V}_n)$$

so $T^m(S(\mathcal{V}_n))$ has standard basis $\{M_A \mid A \in \bigcup \mathrm{Row}(\mu,\nu)\}$ and dual canonical basis $\{L_A \mid A \in \bigcup \mathrm{Row}(\mu,\nu)\}$, where throughout the section $\bigcup$ denotes the union over all pairs $(\mu,\nu) \in \Lambda_m \times \Lambda_n$ with $|\mu| = |\nu|$. Because $S(\mathcal{V}_n)$ is a polynomial $\mathcal{U}_n$-algebra equipped with a compatible bar involution, $T^m(S(\mathcal{V}_n))$ also has a canonical algebra structure and a compatible bar involution, defined as at the end of §3. It is well known that this algebra coincides with the quantized coordinate algebra $\mathcal{O}_q(\mathcal{M}_{m,n})$ of the variety $\mathcal{M}_{m,n}$ of $m \times n$ matrices, that is, the $\mathbb{Q}(q)$-algebra on generators $\{x_{i,j} \mid i = 1,\dots,m, j = 1,\dots,n\}$ subject only to the relations

$$\begin{aligned}
x_{i,j}x_{k,l} &= x_{k,l}x_{i,j} && (i<k,\ j>l)\\
x_{i,j}x_{k,l} &= x_{k,l}x_{i,j} + (q-q^{-1})x_{k,j}x_{i,l} && (i<k,\ j<l)\\
x_{i,j}x_{k,j} &= q x_{k,j}x_{i,j} && (i<k)\\
x_{i,j}x_{i,l} &= q x_{i,l}x_{i,j} && (j<l)
\end{aligned}$$

for all $1 \le i,k \le m$ and $1 \le j,l \le n$. A proof is written down in [BZw, 4.2] (see also [Z]), but still we repeat the argument in Theorem 15 below since some of our choices are slightly different. First, we develop a little more combinatorial language. Recall from §2 that if $|\mu| = |\nu| = d$, then the set $\mathrm{Row}(\mu,\nu)$ is in canonical bijection with the set $(I_\mu \times I_\nu)/S_d$. We call elements $(\alpha,\beta) \in I_\mu \times I_\nu$ *double indexes.* For such a double index (α,β), introduce the monomial

$$M_{\alpha,\beta} := x_{\alpha_1,\beta_1}x_{\alpha_2,\beta_2}\cdots x_{\alpha_d,\beta_d} \in \mathcal{O}_q(\mathcal{M}_{m,n}).$$

We say that a double index (α,β) is *initial* if $\alpha_1 \ge \cdots \ge \alpha_d$ and $\beta_i \le \beta_{i+1}$ whenever $\alpha_i = \alpha_{i+1}$, and *terminal* if $\beta_1 \le \cdots \le \beta_d$ and $\alpha_i \ge \alpha_{i+1}$ whenever $\beta_i = \beta_{i+1}$. Let $(I_\mu \times I_\nu)^+$ and $(I_\mu \times I_\nu)^-$ denote the sets of all initial and terminal double indexes in $I_\mu \times I_\nu$, respectively. These give two distinguished choices of representatives for the orbits in $(I_\mu \times I_\nu)/S_d$. The canonical bijection $\mathrm{Row}(\mu,\nu) \to (I_\mu \times I_\nu)^+$ maps $A \in \mathrm{Row}(\mu,\nu)$ to the unique initial double index $(\alpha,\beta) \in (I_\mu \times I_\nu)^+$ with $\beta = \rho(A)$; see the end of the paragraph after (2.1) for an example.

Theorem 15. *There is an algebra isomorphism* $\psi : T^m(S(\mathcal{V}_n)) \to \mathcal{O}_q(\mathcal{M}_{m,n})$ *defined for any* $(\mu,\nu) \in \Lambda_m \times \Lambda_n$ *with* $|\mu| = |\nu|$ *and* $A \in \mathrm{Row}(\mu,\nu)$ *by* $\psi(M_A) = M_{\alpha,\beta}$*, where* $(\alpha,\beta) \in (I_\mu \times I_\nu)^+$ *is defined from* $\beta = \rho(A)$*. In particular, the monomials* $\{M_{\alpha,\beta} \mid (\alpha,\beta) \in \bigcup (I_\mu \times I_\nu)^+\}$ *form a basis for* $\mathcal{O}_q(\mathcal{M}_{m,n})$*.*

Proof. One checks relations using (3.2), (4.4) and (5.1) to see that there is a well-defined algebra homomorphism $\mathcal{O}_q(\mathcal{M}_{m,n}) \to T^m(S(\mathcal{V}_n)$ mapping the generator $x_{i,j}$ to $1 \otimes \cdots \otimes 1 \otimes v_j \otimes 1 \otimes \cdots \otimes 1$ (where v_j appears in the $(m+1-i)$th tensor position) for each $1 \leq i \leq m, 1 \leq j \leq n$. This maps $M_{\alpha,\beta}$ to M_A, hence it is an isomorphism since the vectors $\{M_A \mid A \in \bigcup \mathrm{Row}(\mu,\nu)\}$ are linearly independent and by the relations the monomials $\{M_{\alpha,\beta} \mid (\alpha,\beta) \in \bigcup (I_\mu \times I_\nu)^+\}$ span $\mathcal{O}_q(\mathcal{M}_{m,n})$. The map ψ is the inverse isomorphism. □

Let us view $\mathcal{O}_q(\mathcal{M}_{m,n})$ as an $X_m \times X_n$-graded algebra by declaring that the generator $x_{i,j}$ is of degree $(\varepsilon_i, \varepsilon_j) \in X_m \times X_n$. Thus,

$$\mathcal{O}_q(\mathcal{M}_{m,n}) = \bigoplus_{\substack{(\mu,\nu) \in \Lambda_m \times \Lambda_n \\ |\mu| = |\nu|}} \mathcal{O}_q(\mathcal{M}_{m,n})_{\mu,\nu}$$

where $\mathcal{O}_q(\mathcal{M}_{m,n})_{\mu,\nu}$ has basis $\{M_{\alpha,\beta} \mid (\alpha,\beta) \in (I_\mu \times I_\nu)^+\}$. From now on, we're going to *identify* $\mathcal{O}_q(\mathcal{M}_{m,n})_{\mu,\nu}$ with the ν-weight space $S^\mu_\nu(\mathcal{V}_n)$ of $S^\mu(\mathcal{V}_n)$ via the isomorphism ψ from Theorem 15. Thus, for $(\alpha,\beta) \in (I_\mu \times I_\nu)^+$, the monomial $M_{\alpha,\beta}$ is identified with M_A, where $A \in \mathrm{Row}(\mu,\nu)$ is defined from $\beta = \rho(A)$. The next result gives a direct description of the bar involution on $\mathcal{O}_q(\mathcal{M}_{m,n})$ arising from this identification.

Theorem 16. *The bar involution on* $\mathcal{O}_q(\mathcal{M}_{m,n})$ *is the unique antilinear map such that* $\overline{x_{i,j}} = x_{i,j}$ *for all* $1 \leq i \leq m, 1 \leq j \leq n$ *and* $\overline{xy} = q^{(\mu,\bar\mu)-(\nu,\bar\nu)}\overline{y}\,\overline{x}$ *for all* $x \in \mathcal{O}_q(\mathcal{M}_{m,n})_{\mu,\nu}$ *and* $y \in \mathcal{O}_q(\mathcal{M}_{m,n})_{\bar\mu,\bar\nu}$*. Moreover, for* $(\alpha,\beta) \in (I_\mu \times I_\nu)^+$*, we have that* $\overline{M_{\alpha,\beta}} = M_{\alpha',\beta'}$ *where* $(\alpha',\beta') \in (I_\mu \times I_\nu)^-$ *is the unique terminal double index lying in the same* S_d*-orbit as* (α,β)*.*

Proof. Let $*$ be the twisted multiplication on $T^m(S(\mathcal{V}_n))$ from Lemma 2. One checks that the twisted multiplication on $S(\mathcal{V}_n)$ itself satisfies $x * y = q^{d\bar d}xy$ for $x \in S^d(\mathcal{V}_n)$ and $y \in S^{\bar d}(\mathcal{V}_n)$. Hence if $x_m \otimes \cdots \otimes x_1 \in S^{\mu_m}(\mathcal{V}_n) \otimes \cdots \otimes S^{\mu_1}(\mathcal{V}_n)$ is of weight ν and $y_m \otimes \cdots \otimes y_1 \in S^{\bar\mu_m}(\mathcal{V}_n) \otimes \cdots \otimes S^{\bar\mu_1}(\mathcal{V}_n)$ is of weight $\bar\nu$, we have that

$$\overline{(y_m \otimes \cdots \otimes y_1)} * \overline{(x_m \otimes \cdots \otimes x_1)} = q^{\mu_1\bar\mu_1 + \cdots + \mu_m\bar\mu_m}(\overline{y_m \otimes \cdots \otimes y_1})\,(\overline{x_m \otimes \cdots \otimes x_1}),$$

so by Lemma 2

$$\overline{(x_m \otimes \cdots \otimes x_1)(y_m \otimes \cdots \otimes y_1)} = q^{(\mu,\bar\mu)-(\nu,\bar\nu)}(\overline{y_m \otimes \cdots \otimes y_1})\,(\overline{x_m \otimes \cdots \otimes x_1}).$$

Clearly $\overline{x_{i,j}} = x_{i,j}$, so this proves the first statement of the lemma. The second can then be deduced by induction on d using the defining relations in $\mathcal{O}_q(\mathcal{M}_{m,n})$. □

Using Theorem 16, we can also give a direct characterization of the dual canonical basis $\{L_A \mid A \in \bigcup \mathrm{Row}(\mu,\nu)\}$ of $\mathcal{O}_q(\mathcal{M}_{m,n})$ arising from its identification with $T^m(S(\mathcal{V}_n))$. We often denote this basis instead by $\{L_{\alpha,\beta} \mid (\alpha,\beta) \in \bigcup (I_\mu \times I_\nu)^+\}$, where for each initial double index $(\alpha, \beta) \in (I_\mu \times I_\nu)^+$, $L_{\alpha,\beta}$ is the unique bar invariant element of $\mathcal{O}_q(\mathcal{M}_{m,n})$ with the property that

$$L_{\alpha,\beta} \in M_{\alpha,\beta} + \sum_{(\alpha',\beta') \in (I_\mu \times I_\nu)^+} q^{-1}\mathbb{Z}[q^1] M_{\alpha',\beta'}. \tag{6.1}$$

Applying the bar involution using Theorem 16, it is equally natural to parametrize this basis by terminal double indexes: it is the basis $\{L_{\alpha,\beta} \mid (\alpha, \beta) \in \bigcup (I_\mu \times I_\nu)^-\}$ where $L_{\alpha,\beta}$ is the unique bar invariant element of $\mathcal{O}_q(\mathcal{M}_{m,n})$ with

$$L_{\alpha,\beta} \in M_{\alpha,\beta} + \sum_{(\alpha',\beta') \in (I_\mu \times I_\nu)^-} q\mathbb{Z}[q] M_{\alpha',\beta'}, \tag{6.2}$$

for $(\alpha, \beta) \in (I_\mu \times I_\nu)^-$.

Remark 17. One finds this elementary approach to the definition of the dual canonical basis of $\mathcal{O}_q(\mathcal{M}_{n,n})$ already in work of Zhang [Zh]. Actually, Zhang uses an even simpler modified definition of the bar involution: his dual canonical basis is invariant instead under the antilinear algebra antiautomorphism $\varphi : \mathcal{O}_q(\mathcal{M}_{m,n}) \to \mathcal{O}_q(\mathcal{M}_{m,n})$ defined by $\varphi(x_{i,j}) = x_{i,j}$ for all $1 \leq i \leq m, 1 \leq j \leq n$. This is related to the bar involution defined here by the equation $\varphi(x) = q^{((\nu,\nu)-(\mu,\mu))/2}\overline{x}$ for $x \in \mathcal{O}_q(\mathcal{M}_{m,n})_{\mu,\nu}$. The dual canonical basis in [Zh] is equal to the dual canonical basis here up to multiplication by a power of q.

In the remainder of the section, we wish to record proofs of some further properties of this dual canonical basis, all of which are known but surprisingly hard to find explicitly in the literature. They were explained to me by Arkady Berenstein, who describes them as "folklore". First, to compensate for the asymmetry of our identification of $\mathcal{O}_q(\mathcal{M}_{m,n})$ with $\mathcal{T}^m(S(\mathcal{V}_n))$, there is an obvious duality between $m \times n$ matrices and $n \times m$ matrices: let $\tau : \mathcal{O}_q(\mathcal{M}_{m,n}) \to \mathcal{O}_q(\mathcal{M}_{n,m})$ be the antilinear algebra antiisomorphism defined on generators by $\tau(x_{i,j}) = x_{j,i}$, i.e.

$$\tau(M_{\alpha,\beta}) = M_{\beta \cdot w_d, \alpha \cdot w_d} \tag{6.3}$$

for $\alpha, \beta \in I_n^d$. Note if (α, β) is initial, then $(\beta \cdot w_d, \alpha \cdot w_d)$ is terminal. Moreover, by Theorem 16, we have that $\tau(\overline{x}) = \overline{\tau(x)}$ for all $x \in \mathcal{O}_q(\mathcal{M}_{m,n})$. Hence, for $(\alpha, \beta) \in (I_\mu \times I_\nu)^+$, $\tau(L_{\alpha,\beta})$ is bar invariant, and the definitions (6.1)–(6.2) now imply that

$$\tau(L_{\alpha,\beta}) = L_{\beta \cdot w_d, \alpha \cdot w_d}. \tag{6.4}$$

The equations (6.3)–(6.4) imply some symmetry in the transition matrices from (5.5). To write this down, define a bijection $\tau : \mathrm{Row}(\mu, \nu) \to \mathrm{Row}(\nu, \mu)$ by letting $\tau(A)$ be the unique element of $\mathrm{Row}(\nu, \mu)$ such that the number of entries on the ith row of $\tau(A)$ that equal j is the same as the number of entries on the jth row of A that equal i, for each $A \in \mathrm{Row}(\mu, \nu)$.

Lemma 18. $l_{A,B}(q) = l_{\tau(A),\tau(B)}(q), \quad l^*_{A,B}(q) = l^*_{\tau(A),\tau(B)}(q)$.

Proof. The first equalty is immediate from (6.3)–(6.4) and the definitions; the second then follows using the inversion formula from Corollary 8. $\square$

Next, we derive a closed formula for the dual canonical basis of $\mathcal{O}_q(\mathcal{M}_{2,n}) = S(\mathcal{V}_n) \otimes S(\mathcal{V}_n)$, i.e. the dual canonical basis elements L_A for row standard tableaux A with just *two rows*. Given $r, s \geq 0$ and integers $1 \leq a_1, \dots, a_r, b_1, \dots, b_s \leq n$ we will use the shorthand $M\binom{a_1 \cdots a_r}{b_1 \cdots b_s}$ resp. $L\binom{a_1 \cdots a_r}{b_1 \cdots b_s}$ for M_A resp. L_A, where A is the row standard tableau with entries $a_1, \dots, a_r$ on the top row and $b_1, \dots, b_s$ on the bottom row (arranged of course into weakly increasing order). For example, we have that $M\binom{a}{b} = x_{2,a} x_{1,b}$, and

$$L\binom{a}{b} = \begin{cases} M\binom{a}{b} - q^{-1} M\binom{b}{a} & \text{if } a > b, \\ M\binom{a}{b} & \text{if } a \leq b. \end{cases} \tag{6.5}$$

Lemma 19. *Let $1 \leq a_1, \dots, a_r, b_1, \dots, b_s, a, b \leq n$ such that $a > b$, $a_1 \leq \cdots \leq a_r$ and $b_1 \leq \cdots \leq b_s$. Assume that $a_i \notin \{b+1, \dots, a-1\}$ for each $i = 1, \dots, r$ and $b_j \notin \{b+1, \dots, a-1\}$ for each $j = 1, \dots, s$. Then,*

$$\begin{aligned} L\binom{a_1 \cdots a_r a}{b_1 \cdots b_r b} &= q^{\#\{i \,|\, a_i > a\} + \#\{j \,|\, b_j > a\}} L\binom{a_1 \cdots a_r}{b_1 \cdots b_s} L\binom{a}{b} \\ &= q^{\#\{i \,|\, a_i < b\} + \#\{j \,|\, b_j < b\}} L\binom{a}{b} L\binom{a_1 \cdots a_r}{b_1 \cdots b_s}. \end{aligned}$$

Proof. Let $\omega : \mathcal{O}_q(\mathcal{M}_{2,n}) \to \mathcal{O}_q(\mathcal{M}_{2,n})$ be the linear map defined by

$$\omega(M\binom{a_1 \cdots a_r}{b_1 \cdots b_s}) = M\binom{a_1 \cdots a_r a}{b_1 \cdots b_s b} - q^{-1 - \#\{i \,|\, a_i = a\} - \#\{j \,|\, b_j = b\}} M\binom{a_1 \cdots a_r b}{b_1 \cdots b_s a}$$

for any $r, s \geq 0$ and $1 \leq a_1, \dots, a_r, b_1, \dots, b_s \leq n$. Using the relations and (6.5), one checks that

$$x_{1,b_j} L\binom{a}{b} = \begin{cases} q^{-1} L\binom{a}{b} x_{1,b_j} & \text{if } b_j > a > b, \\ L\binom{a}{b} x_{1,b_j} & \text{if } b_j = a > b, \\ L\binom{a}{b} x_{1,b_j} & \text{if } a > b = b_j, \\ q L\binom{a}{b} x_{1,b_j} & \text{if } a > b > b_j. \end{cases}$$

Hence, recalling that $M\binom{a_1 \cdots a_r}{b_1 \cdots b_s} = x_{2,a_1} x_{2,a_2} \cdots x_{2,a_r} x_{1,b_1} x_{1,b_2} \cdots x_{1,b_s}$,

$$M\binom{a_1 \cdots a_r}{b_1 \cdots b_s} L\binom{a}{b} = q^{\#\{j \,|\, b_j < b\} - \#\{j \,|\, b_j > a\}} x_{2,a_1} \cdots x_{2,a_r} L\binom{a}{b} x_{1,b_1} \cdots x_{1,b_s}$$

Moreover,

$$\begin{aligned} x_{2,a_1} \cdots x_{2,a_r} L\binom{a}{b} x_{1,b_1} \cdots x_{1,b_s} &= x_{2,a_1} \cdots x_{2,a_r} (x_{2,a} x_{1,b} - q^{-1} x_{2,b} x_{1,a}) x_{1,b_1} \cdots x_{1,b_s} \\ &= q^{-\#\{i \,|\, a_i > a\} - \#\{j \,|\, b_j < b\}} M\binom{a_1 \cdots a_r a}{b_1 \cdots b_s b} - q^{-1 - \#\{i \,|\, a_i > b\} - \#\{j \,|\, b_j < a\}} M\binom{a_1 \cdots a_r b}{b_1 \cdots b_s a}. \end{aligned}$$

Hence $\omega(M\binom{a_1 \cdots a_r}{b_1 \cdots b_s}) = q^{\#\{i \,|\, a_i > a\} + \#\{j \,|\, b_j > a\}} M\binom{a_1 \cdots a_r}{b_1 \cdots b_s} L\binom{a}{b}$. A similar argument using instead the relations

$$L\binom{a}{b} x_{2,a_i} = \begin{cases} q x_{2,a_i} L\binom{a}{b} & \text{if } a_i > a > b, \\ x_{2,a_i} L\binom{a}{b} & \text{if } a_i = a > b, \\ x_{2,a_i} L\binom{a}{b} & \text{if } a > b = a_i, \\ q^{-1} x_{2,a_i} L\binom{a}{b} & \text{if } a > b > a_i \end{cases}$$

shows that $\omega(M\binom{a_1 \cdots a_r}{b_1 \cdots b_s}) = q^{\#\{i \,|\, a_i < b\} + \#\{j \,|\, b_j < b\}} L\binom{a}{b} M\binom{a_1 \cdots a_r}{b_1 \cdots b_s}$. Now by weight considerations we get that

$$\begin{aligned} x := \omega(L\binom{a_1 \cdots a_r}{b_1 \cdots b_s}) &= q^{\#\{i \,|\, a_i > a\} + \#\{j \,|\, b_j > a\}} L\binom{a_1 \cdots a_r}{b_1 \cdots b_s} L\binom{a}{b} \\ &= q^{\#\{i \,|\, a_i < b\} + \#\{j \,|\, b_j < b\}} L\binom{a}{b} L\binom{a_1 \cdots a_r}{b_1 \cdots b_s}. \end{aligned}$$

Using Theorem 16, we deduce that

$$\begin{aligned}\overline{x} &= q^{-\#\{i \mid a_i > a\} - \#\{j \mid b_j > a\} + r + s - (\varepsilon_a + \varepsilon_b, \varepsilon_{a_1} + \cdots + \varepsilon_{a_r} + \varepsilon_{b_1} + \cdots + \varepsilon_{b_s})} L\binom{a}{b} L\binom{a_1 \cdots a_r}{b_1 \cdots b_s} \\ &= q^{\#\{i \mid a_i < b\} + \#\{j \mid b_j < b\}} L\binom{a}{b} L\binom{a_1 \cdots a_r}{b_1 \cdots b_s} = x.\end{aligned}$$

Hence, x is bar invariant, and since it equals $M\binom{a_1 \cdots a_r a}{b_1 \cdots b_s b}$ plus a $q^{-1}\mathbb{Z}[q^{-1}]$-linear combination of other monomials, we have proved that $x = L\binom{a_1 \cdots a_r a}{b_1 \cdots b_s b}$. □

Theorem 20. *Let $r, s \geq 0$ and $t = \min(r, s)$. Suppose $1 \leq a_1, \dots, a_r, b_1, \dots, b_s \leq n$ satisfy the following property for all $i = 1, \dots, t$:*

> *If the set $\{a_j - b_k \mid i \leq j \leq r, i \leq k \leq s \text{ such that } a_j > b_k\}$ is non-empty, then $(a_i - b_i)$ is its smallest element.*

Then, up to multiplication by a power of q, the dual canonical basis element $L\binom{a_1 \cdots a_r}{b_1 \cdots b_s}$ is equal to

$$\prod_{\substack{1 \leq i \leq t \\ a_i > b_i}} (x_{2,a_i} x_{1,b_i} - q^{-1} x_{2,b_i} x_{1,a_i}) \prod_{\substack{1 \leq i \leq t \\ a_i \leq b_i}} x_{2,a_i} x_{1,b_i} \prod_{t < j \leq r} x_{2,a_j} \prod_{t < k \leq s} x_{1,b_k}$$

where the product is taken in any order. Every element of the dual canonical basis of $\mathcal{O}_q(\mathcal{M}_{2,n})$ can be obtained in this way.

Proof. Apply Lemma 19 and induction on t. The induction starts from the observation that if $a_i \leq b_j$ for *all* i, j then we have simply that $L\binom{a_1 \cdots a_r}{b_1 \cdots b_s} = M\binom{a_1 \cdots a_r}{b_1 \cdots b_s}$. □

Remark 21. Applying τ to Theorem 20, one also obtains a closed formula for the dual canonical basis of $\mathcal{O}_q(\mathcal{M}_{n,2})$, hence of the $U_q(\mathfrak{gl}_2)$-modules $S^\mu(\mathcal{V}_2)$ for all μ. As a special case, we recover the computation by Frenkel and Khovanov of the dual canonical basis of the $U_q(\mathfrak{gl}_2)$-module $T^d(\mathcal{V}_2)$; see [FKK, 3.1].

Remark 22. There is one other situation where it is possible to compute the canonical/dual canonical bases from §5 explicitly. In his thesis, Khovanov also computed the *canonical basis* of the $U_q(\mathfrak{gl}_2)$-module $T^d(\mathcal{V}_2)$, which is closely related to the parabolic Kazhdan-Lusztig polynomials studied by Lascoux and Schützenberger in [LS1]; see [FKK, 3.4]. The dual statement to this has been derived recently by Cheng, Wang and Zhang [CWZ, 6.17]; in particular, they give a closed formula for the canonical basis of $\bigwedge(\mathcal{V}_n) \otimes \bigwedge(\mathcal{V}_n)$, i.e. the canonical basis elements K_A^* for column strict tableaux A with just *two columns*.

Finally in this section, we want to make precise the relationship between the dual canonical basis of $\mathcal{O}_q(\mathcal{M}_{m,n})$ described here and the dual canonical basis of the quantized coordinate algebra $\mathcal{O}_q(\mathcal{T}_{m+n})$ of the group of all upper unitriangular $(m+n) \times (m+n)$-matrices[1]. Following [BZ], this is the $\mathbb{Q}(q)$-algebra on generators $\{t_{i,j} \mid 1 \leq i < j \leq m+n\}$ subject to the relations

$$t_{i,k} = \frac{t_{i,j} t_{j,k} - q^{-1} t_{j,k} t_{i,j}}{q - q^{-1}}$$

[1]Since completing this article, I have learnt of a preprint of Jakobsen and Zhang [JZh] which also makes this identification by similar arguments.

for $1 \leq i < j < k \leq m+n$ and

$$t_{i,j}t_{k,l} = t_{k,l}t_{i,j} \qquad (i<k, j>l) \text{ or } (i>l)$$
$$t_{i,j}t_{k,l} = t_{k,l}t_{i,j} + (q-q^{-1})t_{i,l}t_{k,j} \qquad (i<k<j<l)$$
$$t_{i,j}t_{k,j} = qt_{k,j}t_{i,j} \qquad (i<k)$$
$$t_{i,j}t_{i,l} = qt_{i,l}t_{i,j} \qquad (j<l)$$

for $1 \leq i < j \leq m+n$ and $1 \leq k < l \leq m+n$. We view $\mathcal{O}_q(\mathcal{T}_{m+n})$ as an X_{m+n}-graded algebra, by declaring that the generator $t_{i,j}$ is of weight $(\varepsilon_i - \varepsilon_j)$. In fact, by an observation of Drinfeld proved in [BZ], the algebra $\mathcal{O}_q(\mathcal{T}_{m+n})$ can be identified with the positive part $\mathcal{U}^+_{m+n}$ of the quantized enveloping algebra $U_q(\mathfrak{gl}_{m+n})$, so that $t_{i,i+1}$ is identified with E_i for each $i = 1, \dots, m+n-1$. Under this identification, the bar involution on $\mathcal{U}^+_{m+n}$ also defines a bar involution on $\mathcal{O}_q(\mathcal{T}_{m+n})$. Define a different antilinear involution $\sim$ of $\mathcal{O}_q(\mathcal{T}_{m+n})$ by setting $\widetilde{x} = q^{\frac{1}{2}(\mu,\mu)-\deg(\mu)}\sigma(\overline{x})$ for each x of weight μ. Here, $\sigma : \mathcal{O}_q(\mathcal{T}_{m+n}) \to \mathcal{O}_q(\mathcal{T}_{m+n})$ is the unique algebra antiautomorphism that fixes the generators $t_{i,i+1}$ for each $1 \leq i < m+n$, and for a weight $0 \leq \mu \in X_{m+n}$ its degree $\deg(\mu)$ is defined from $\deg(\varepsilon_i - \varepsilon_{i+1}) = 1$ and $\deg(\mu+\nu) = \deg(\mu) + \deg(\nu)$.

To define the dual canonical basis of $\mathcal{O}_q(\mathcal{T}_{m+n})$ following [LNT, §3.5], we must first introduce a PBW basis. Let J^d_{m+n} denote the set of all terminal double indexes $(\alpha, \beta) \in I^d_{m+n} \times I^d_{m+n}$, such that $\alpha_i < \beta_i$ for all $i = 1, \dots, d$. For $(\alpha,\beta) \in J^d_{m+n}$, define

$$E^*_{\alpha,\beta} = q^{\sum_{i=1}^{m+n}\nu_i(\nu_i-1)/2}\, t_{\alpha_1,\beta_1} \cdots t_{\alpha_d,\beta_d}$$

where $\nu = \theta(\beta) \in \Lambda_{m+n}$. This is *exactly* the PBW basis element denoted $\Phi(E^*(\mathbf{m}))$ in [LNT], parametrized by the multi-segment $\mathbf{m} = \sum_{i=1}^d [\alpha_i, \beta_i - 1]$. The elements $\{E^*_{\alpha,\beta} \mid (\alpha,\beta) \in \bigcup_{d\geq 0} J^d_{m+n}\}$ give a basis for $\mathcal{O}_q(\mathcal{T}_{m+n})$. By [LNT, 3.16], there is for $(\alpha,\beta) \in J^d_{m+n}$ a unique element $G^*_{\alpha,\beta} \in \mathcal{O}_q(\mathcal{T}_{m+n})$ such that $\widetilde{G^*_{\alpha,\beta}} = G^*_{\alpha,\beta}$ and

$$G^*_{\alpha,\beta} \in E^*_{\alpha,\beta} + \sum_{(\alpha',\beta') \in J^d_{m+n}} q\mathbb{Z}[q]E^*_{\alpha',\beta'}. \tag{6.6}$$

Moreover, the *dual canonical basis* of $\mathcal{O}_q(\mathcal{T}_{m+n})$ is $\{G^*_{\alpha,\beta} \mid (\alpha,\beta) \in \bigcup_{d\geq 0} J^d_{m+n}\}$; it is the basis dual to the canonical basis of $\mathcal{U}^+_{m+n}$ under a natural bilinear form normalized as in [LNT, §3.4].

Theorem 23. *There is an algebra monomorphism $\varphi : \mathcal{O}_q(\mathcal{M}_{m,n}) \to \mathcal{O}_q(\mathcal{T}_{m+n})$ such that $\varphi(x_{i,j}) = t_{i,j+m}$ for all $1 \leq i \leq m, 1 \leq j \leq n$. Moreover, given $\mu \in \Lambda_m$, $\nu \in \Lambda_n$ with $|\mu| = |\nu| = d$ and any $(\alpha,\beta) \in (I_\mu \times I_\nu)^-$, we have that*

$$\varphi(M_{\alpha,\beta}) = q^{-\sum_{i=1}^n \nu_i(\nu_i-1)/2}E^*_{\alpha,\beta'}, \qquad \varphi(L_{\alpha,\beta}) = q^{-\sum_{i=1}^n \nu_i(\nu_i-1)/2}G^*_{\alpha,\beta'},$$

where $\beta' = (\beta_1 + m, \dots, \beta_d + m)$.

Proof. It is clear from the relations that φ is a well-defined algebra homomorphism. Also, it sends $M_{\alpha,\beta}$ to $q^{-\sum_{i=1}^n \nu_i(\nu_i-1)/2}E^*_{\alpha,\beta'}$, hence it is injective. It just remains to show that it sends $L_{\alpha,\beta}$ to $q^{-\sum_{i=1}^n \nu_i(\nu_i-1)/2}G^*_{\alpha,\beta'}$. This follows easily comparing (6.2) and (6.6) as soon as we have checked that $\varphi(q^{\sum_{i=1}^n \nu_i(\nu_i-1)/2}L_{\alpha,\beta})$ is invariant under the antilinear involution $\sim$. One checks from the definition that $\widetilde{t_{i,j}} = t_{i,j}$ for all $1 \leq i < j \leq m+n$, and that $\widetilde{xy} = q^{(\mu,\nu)}\widetilde{y}\widetilde{x}$ for all x, y of weights μ, ν, respectively. Combining this with Theorem 16, it follows by induction on degree that

$$\varphi(\overline{x}) = q^{-\sum_{i=1}^{n}\nu_i(\nu_i-1)}\widetilde{\varphi(x)}$$

for any $x \in \mathcal{O}_q(\mathcal{M}_{m,n})_{\mu,\nu}$. Hence, $\widetilde{\varphi(L_{\alpha,\beta})} = q^{\sum_{i=1}^{n}\nu_i(\nu_i-1)}\varphi(L_{\alpha,\beta})$. □

Remark 24. This theorem means that one can appeal to the extensive literature on dual canonical bases of $\mathcal{O}_q(\mathcal{T}_{m+n})$ in order to obtain powerful results about the dual canonical basis of $\mathcal{O}_q(\mathcal{M}_{m,n})$ too. For example, by dualizing [L, 14.4.13(b)], it follows that the structure constants for multiplication in $\mathcal{O}_q(\mathcal{M}_{m,n})$ relative to the dual canonical basis in fact all lie in $\mathbb{N}[q, q^{-1}]$.

7. Polynomial representations

Assume throughout the section that $m \leq n$ and that $\mu \in \Lambda_l$ is a weight with $|\mu| = d$, such that the conjugate partition $\lambda = \mu'$ lies in Λ_m^+. Recall the definitions from §5 of the $\mathcal{U}_n$-modules $\bigwedge^\mu(\mathcal{V}_n)$ and $S^\lambda(\mathcal{V}_n)$. The ν-weight space of the first one has the two natural bases N_A and K_A parametrized by $\mathrm{Col}(\mu,\nu)$, while the ν-weight space of the second one has the two natural bases M_B and L_B parametrized by $\mathrm{Row}(\lambda,\nu)$. By the Littlewood-Richardson rule, the $\mathcal{U}_n$-module $\bigwedge^\mu(\mathcal{V}_n)$ resp. $S^\lambda(\mathcal{V}_n)$ has a composition factor of highest weight λ appearing with multiplicity one, and all the other composition factors are of highest weight $<\lambda$ resp. $>\lambda$ in the dominance ordering. Hence, the space $\mathrm{Hom}_{\mathcal{U}_n}(\bigwedge^\mu(\mathcal{V}_n), S^\lambda(\mathcal{V}_n))$ is one dimensional. We define $P^\lambda(\mathcal{V}_n)$ to be the image of any non-zero homomorphism $\bigwedge^\mu(\mathcal{V}_n) \to S^\lambda(\mathcal{V}_n)$. This is the well known realization of the irreducible polynomial representation of $\mathcal{U}_n$ of highest weight λ as a submodule of $S^\lambda(\mathcal{V}_n)$. We should note that since the spaces $\bigwedge^\mu(\mathcal{V}_n)$ are at least *isomorphic* for all μ with $\mu' = \lambda$, the one dimensionality of $\mathrm{Hom}_{\mathcal{U}_n}(\bigwedge^\mu(\mathcal{V}_n), S^\lambda(\mathcal{V}_n))$ implies that $P^\lambda(\mathcal{V}_n)$ is always the *same* subspace of $S^\lambda(\mathcal{V}_n)$, independent of the particular choice of μ.

Let us write down a canonical generator for the space $\mathrm{Hom}_{\mathcal{U}_n}(\bigwedge^\mu(\mathcal{V}_n), S^\lambda(\mathcal{V}_n))$. To do this, we identify $S^\lambda(\mathcal{V}_n)$ with a one-sided weight space of $\mathcal{O}_q(\mathcal{M}_{m,n})$ according to Theorem 15. Given $\beta \in I_n^d$ with $\beta_1 < \cdots < \beta_d$, define the *quantum flag minor*

$$D_\beta = \sum_{w \in S_d} (-q)^{\ell(w)} x_{w1,\beta_1} x_{w2,\beta_2} \cdots x_{wd,\beta_d} \tag{7.1}$$

$$= \sum_{w \in S_d} (-q)^{-\ell(w)} x_{d,\beta_{wd}} \cdots x_{2,\beta_{w2}} x_{1,\beta_{w1}}. \tag{7.2}$$

Recalling Theorem 16, it is immediate from this definition that D_β is bar invariant, hence it coincides with the dual canonical basis element $L_{\alpha,\beta}$ where $\alpha = (1, 2, \dots, d)$. Now for $A \in \mathrm{Col}(\mu,\nu)$, define

$$V_A := D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_d}, \tag{7.3}$$

where α_i denotes the multi-index obtained by reading the entries in the ith column of A from bottom to top. Thus, V_A is the product of the quantum flag minors corresponding to the columns of the tableau A. Clearly it belongs to the one-sided weight space $S^\lambda(\mathcal{V}_n)$ of $\mathcal{O}_q(\mathcal{M}_{m,n})$, so we can define a linear map

$$\xi_\mu : {\textstyle\bigwedge}^\mu(\mathcal{V}_n) \to S^\lambda(\mathcal{V}_n) \tag{7.4}$$

by setting $\xi_\mu(N_A) = V_A$ for all $A \in \mathrm{Col}(\mu,\nu)$. We note finally that if A is the unique element of $\mathrm{Col}(\mu,\lambda)$, so all entries on the ith row of A are equal to i, then

$$V_A = M_{R(A)} + (\text{a } \mathbb{Z}[q,q^{-1}]\text{-linear combination of } M_B\text{'s for } B < R(A)). \tag{7.5}$$

Of course, the rectification $R(A)$ in this case is just the tableau of row shape λ having all entries on its ith row equal to i. The proof of (7.5) is a straightforward consequence of the defining relations in $\mathcal{O}_q(\mathcal{M}_{m,n})$.

Lemma 25. *The map ξ_μ is a non-zero $\mathcal{U}_n$-module homomorphism.*

Proof. For each i, identify $T^{\mu_i}(\mathcal{V}_n)$ with a submodule of $\mathcal{O}_q(\mathcal{M}_{m,n})$ by identifying $v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_{\mu_i}}$ with $x_{\mu_i,\alpha_1} x_{\mu_i-1,\alpha_2} \cdots x_{1,\alpha_{\mu_i}}$. In this way, $T^d(\mathcal{V}_n) = T^{\mu_1}(\mathcal{V}_n) \otimes \cdots \otimes T^{\mu_l}(\mathcal{V}_n)$ is identified with a submodule of $\mathcal{O}_q(\mathcal{M}_{m,n})^{\otimes l}$. Let $A \in \mathrm{Col}(\mu,\nu)$ be a column strict tableau, and let α_i denote the multi-index obtained by reading the entries in the ith column of A from bottom to top. Comparing (5.6) with the right hand side of (7.2), the basis element $N_A = Y_{\alpha_1} \otimes \cdots \otimes Y_{\alpha_l}$ of $\bigwedge^\mu(\mathcal{V}_n) \subseteq T^d(\mathcal{V}_n)$ corresponds under this identification to the tensor product of quantum flag minors $D_{\alpha_1} \otimes \cdots \otimes D_{\alpha_l} \in \mathcal{O}_q(\mathcal{M}_{m,n})^{\otimes l}$. Since $\mathcal{O}_q(\mathcal{M}_{m,n})$ is a polynomial $\mathcal{U}_n$-algebra, multiplication defines a $\mathcal{U}_n$-module homomorphism $\mathcal{O}_q(\mathcal{M}_{m,n})^{\otimes l} \to \mathcal{O}_q(\mathcal{M}_{m,n})$ mapping N_A to V_A. Hence, ξ_μ is a $\mathcal{U}_n$-module homomorphism, and it is non-zero by (7.5). □

Theorem 26. *For any $\nu \in \Lambda_n$ and $A \in \mathrm{Col}(\mu,\nu)$, we have that*

$$\xi_\mu(K_A) = \begin{cases} L_{R(A)} & \text{if } A \in \mathrm{Std}(\mu,\nu), \\ 0 & \text{otherwise.} \end{cases}$$

The vectors $\{V_A \mid A \in \mathrm{Std}(\mu,\nu)\}$ and $\{L_B \mid B \in \mathrm{Dom}(\lambda,\nu)\}$ give natural bases for the ν-weight space $P^\lambda_\nu(\mathcal{V}_n)$ of $P^\lambda(\mathcal{V}_n)$. Moreover, for $A \in \mathrm{Col}(\mu,\nu)$, we have that

$$V_A = \sum_{B \in \mathrm{Std}(\mu,\nu)} k^*_{A,B}(q^{-1}) L_{R(B)}.$$

Proof. Recall the subring $\mathbb{A}_\infty$ of $\mathbb{Q}(q)$ from §4. Let $\bigwedge^\mu(\mathcal{V}_n)_\infty$ resp. $S^\lambda(\mathcal{V}_n)_\infty$ be the $\mathbb{A}_\infty$-submodule of $\bigwedge^\mu(\mathcal{V}_n)$ resp. $S^\lambda(\mathcal{V}_n)$ generated by all the N_A's resp. M_A's. It is an upper crystal lattice at $q=\infty$ in the sense of [K3], and the images of the N_A's resp. M_A's in $\bigwedge^\mu(\mathcal{V}_n)_\infty / q^{-1} \bigwedge^\mu(\mathcal{V}_n)_\infty$ resp. $S^\lambda(\mathcal{V}_n)_\infty / q^{-1} S^\lambda(\mathcal{V}_n)_\infty$ form an upper crystal base at $q=\infty$. The action of the upper crystal operators on this upper crystal base is described by the crystal $\bigcup \mathrm{Col}(\mu,\nu)$ resp. $\bigcup \mathrm{Row}(\lambda,\nu)$ from §2. Finally, $(\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge^\mu(\mathscr{V}_n), \overline{\bigwedge^\mu(\mathcal{V}_n)_\infty}, \bigwedge^\mu(\mathcal{V}_n)_\infty)$ resp. $(\mathbb{Q} \otimes_{\mathbb{Z}} S^\lambda(\mathscr{V}_n), \overline{S^\lambda(\mathcal{V}_n)_\infty}, S^\lambda(\mathcal{V}_n)_\infty)$ is a balanced triple, and the dual canonical basis of $\bigwedge^\mu(\mathcal{V}_n)$ resp. $S^\lambda(\mathcal{V}_n)$ is the corresponding lift of the upper crystal base. This puts us in the setup of [K3, §5].

Take any $\nu \in \Lambda_n$ and $A \in \mathrm{Col}(\mu,\nu)$ such that $\tilde{e}_i(A) = \varnothing$ for all i. Then, by [K3, 5.1.1], K_A is a non-zero highest weight vector in $\bigwedge^\mu(\mathcal{V}_n)$ of weight ν. Since all composition factors of $\bigwedge^\mu(\mathcal{V}_n)$ are of highest weight $\leq \lambda$, we have that $\nu \leq \lambda$. Since all composition factors of $S^\lambda(\mathcal{V}_n)$ are of highest weight $\geq \lambda$, we deduce that $\xi_\mu(K_A) = 0$ unless in fact $\nu = \lambda$. In that case, there is only one tableau $A \in \mathrm{Col}(\mu, \lambda)$, and so we must have that $K_A = N_A$. Since $\tilde{e}_i(R(A)) = \varnothing$ for all i too, we get by [K3, 5.1.1] once more that $L_{R(A)}$ is a highest weight vector in $S^\lambda(\mathcal{V}_n)$ of weight λ. Hence $\xi_\mu(K_A) = V_A = cL_{R(A)}$ for some non-zero scalar $c \in \mathbb{Q}(q)$. Since $L_{R(A)} = M_{R(A)}+$(a $q^{-1}\mathbb{Z}[q^{-1}]$-linear combination of M_B's for $B < R(A)$) we deduce from (7.5) that $c = 1$. Hence, $\xi_\mu(K_A) = L_{R(A)}$ in this special case.

Now for the general case, the point is that there are two possibly different balanced triples in $P^\lambda(\mathcal{V}_n)$, one arising as a quotient of the balanced triple $(\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge^\mu(\mathscr{V}_n), \overline{\bigwedge^\mu(\mathcal{V}_n)_\infty}, \bigwedge^\mu(\mathcal{V}_n)_\infty)$, the other arising from the intersection with the balanced triple $(\mathbb{Q} \otimes_{\mathbb{Z}} S^\lambda(\mathscr{V}_n), \overline{S^\lambda(\mathcal{V}_n)_\infty}, S^\lambda(\mathcal{V}_n)_\infty)$. We have just checked in the previous paragraph that these two balanced triples agree on the highest weight space of the irreducible module $P^\lambda(\mathcal{V}_n)$. Hence by [K3, 5.2.2], they agree everywhere. This shows in particular that the map ξ_μ maps the upper crystal lattice $\bigwedge^\mu(\mathcal{V}_n)_\infty$ into $S^\lambda(\mathcal{V}_n)_\infty$, so we get an induced map $\bar{\xi}_\mu : \bigwedge^\mu(\mathcal{V}_n)_\infty/q^{-1}\bigwedge^\mu(\mathcal{V}_n)_\infty \to S^\lambda(\mathcal{V}_n)_\infty/q^{-1}S^\lambda(\mathcal{V}_n)_\infty$ commuting with the actions of the upper crystal operators. Moreover, the following diagram commutes

$$\begin{array}{ccc} \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge^\mu(\mathscr{V}_n) & \xrightarrow{\sim} & \bigwedge^\mu(\mathcal{V}_n)_\infty/q^{-1}\bigwedge^\mu(\mathcal{V}_n)_\infty \\ {\scriptstyle \xi_\mu}\Big\downarrow & & \Big\downarrow{\scriptstyle \bar{\xi}_\mu} \\ \mathbb{Q} \otimes_{\mathbb{Z}} S^\lambda(\mathscr{V}_n) & \xrightarrow{\sim} & S^\lambda(\mathcal{V}_n)_\infty/q^{-1}S^\lambda(\mathcal{V}_n)_\infty \end{array}$$

where the top and bottom maps are the canonical isomorphisms arising from the balanced triples. It now suffices to complete the proof of the first statement of the theorem to verify it at the level of local crystal bases. If $A \in \mathrm{Col}(\mu, \nu)$ satisfies $\tilde{e}_i(A) = \varnothing$ for all i, we are done by the previous paragraph. The general case follows by applying crystal operators, recalling the characterization of the set $\bigcup \mathrm{Std}(\mu, \nu)$ and the map R in terms of crystals from §2.

It follows immediately that $\{L_A \,|\, A \in \mathrm{Dom}(\lambda, \nu)\}$ is a basis for $P^\lambda_\nu(\mathcal{V}_n)$. By (5.8) and Corollary 12, we have for any $A \in \mathrm{Col}(\mu, \nu)$ that

$$N_A = \sum_{B \in \mathrm{Col}(\mu,\nu)} k^*_{A,B}(q^{-1}) K_B.$$

Applying the map ξ_μ, we get the formula for V_A. Finally unitriangularity of the transition matrix implies that $\{V_A \mid A \in \mathrm{Std}(\mu, \nu)\}$ is also a basis for $P^\lambda_\nu(\mathcal{V}_n)$. □

Remark 27. The basis $\{L_A \,|\, A \in \bigcup_\nu \mathrm{Dom}(\lambda, \nu)\}$ for $P^\lambda(\mathcal{V}_n)$ is Kashiwara's upper global crystal base (by the proof of Theorem 26) or Lusztig's dual canonical basis (by Remarks 30 and 31 below). It is the same basis independent of the choice of μ. On the other hand, the basis $\{V_A \mid A \in \bigcup_\nu \mathrm{Std}(\mu, \nu)\}$ definitely does depend on μ. Thus, we obtain a family of standard monomial bases for $P^\lambda(\mathcal{V}_n)$, one for each μ with $\mu' = \lambda$. These bases are not new; for instance, they were already constructed in [LT, 4.4] by a similar approach to the one here. In the case that μ is itself a partition, this basis is the q-analogue of the classical standard monomial basis. Note finally by the definition (7.3) and Remark 24 that the coefficients of the polynomials $k^*_{A,B}(q)$ appearing in Theorem 26 are non-negative integers.

Example 28. We list the polynomials $k^*_{A,B}(q)$ for $\mu = (3,2,2,1)$, $\nu = (2,2,2,1,1)$ and all $A, B \in \mathrm{Std}(\mu,\nu)$, i.e. part of the transition matrix from the standard monomial to the dual canonical basis of $P^\lambda(\mathcal{V}_n)$, where $\lambda = (4,3,1)$. We pick this example in order to point out that the AB-entry of this matrix is the same as the AB-entry of the matrix computed by Leclerc and Toffin in [LTo]; in particular [LTo] gives a simple algorithm to compute these polynomials.

$k^*_{A,B}(q)$	3 224 1135	3 225 1134	3 234 1125	3 235 1124	3 245 1123	4 225 1133	4 233 1125	4 235 1123	5 224 1133	5 233 1124	5 234 1123	4 335 1122	5 334 1122
3 224 1135	1	·	·	·	·	·	·	·	·	·	·	·	·
3 225 1134	q	1	·	·	·	·	·	·	·	·	·	·	·
3 234 1125	q	·	1	·	·	·	·	·	·	·	·	·	·
3 235 1124	q^2	q	q	1	·	·	·	·	·	·	·	·	·
3 245 1123	q	q^2	q^2	q	1	·	·	·	·	·	·	·	·
4 225 1133	q	q^2	·	·	·	1	·	·	·	·	·	·	·
4 233 1125	q	·	q^2	·	·	·	1	·	·	·	·	·	·
4 235 1123	$2q^2$	q^3	q^3	q^2	q	q	q	1	·	·	·	·	·
5 224 1133	q^2	·	·	·	·	q	·	·	1	·	·	·	·
5 233 1124	q^2	q	q^3	q^2	·	·	q	·	·	1	·	·	·
5 234 1123	$2q^3$	q^2	q^4	q^3	q^2	q^2	q^2	q	q	q	1	·	·
4 335 1122	q^3	·	·	·	q^2	q^4	q^2	q^3+q	·	·	·	1	·
5 334 1122	q^4	q^3	·	q^2	q^3	q^5	q^3	q^4+q^2	q^4	q^2	q^3+q	q	1

Now let us prepare to dualize. Recall the spaces $\widetilde{S}^\lambda(\mathcal{V}_n)$ and $\widetilde{\bigwedge}^\mu(\mathcal{V}_n)$ from §5, and the non-degenerate pairings $(.,.)$ between $S^\lambda(\mathcal{V}_n)$ and $\widetilde{S}^\lambda(\mathcal{V}_n)$ and between $\bigwedge^\mu(\mathcal{V}_n)$ and $\widetilde{\bigwedge}^\mu(\mathcal{V}_n)$. The space $\mathrm{Hom}_{\mathcal{U}_n}(\widetilde{S}^\lambda(\mathcal{V}_n), \widetilde{\bigwedge}^\mu(\mathcal{V}_n))$ is also one dimensional, and a canonical generator is given by the map

$$\xi^*_\mu : \widetilde{S}^\lambda(\mathcal{V}_n) \to \widetilde{\bigwedge}^\mu(\mathcal{V}_n) \tag{7.6}$$

that is dual to ξ_μ in the sense that $(v, \xi^*_\mu(w)) = (\xi_\mu(v), w)$ for all $v \in \bigwedge^\mu(\mathcal{V}_n), w \in \widetilde{S}^\lambda(\mathcal{V}_n)$. Define $\widetilde{P}^\lambda(\mathcal{V}_n)$ to be the cokernel of ξ^*_μ (or indeed any non-zero homomorphism $\widetilde{S}^\lambda(\mathcal{V}_n) \to \widetilde{\bigwedge}^\mu(\mathcal{V}_n)$). This is another realization of the irreducible polynomial representation of $\mathcal{U}_n$ as a quotient of $\widetilde{S}^\lambda(\mathcal{V}_n)$. It is always the *same* quotient of $\widetilde{S}^\lambda(\mathcal{V}_n)$ independent of the particular choice of μ. Actually in practise we will often view $\widetilde{P}^\lambda(\mathcal{V}_n)$ as a submodule of $\widetilde{\bigwedge}^\mu(\mathcal{V}_n)$ via the map ξ^*_μ, though of course this identification does depend on our fixed choice of μ. The pairing $(.,.)$ between

$S^\lambda(\mathcal{V}_n)$ and $\widetilde{S}^\lambda(\mathcal{V}_n)$ induces a well-defined non-degenerate pairing

$$(.,.) : P^\lambda(\mathcal{V}_n) \times \widetilde{P}^\lambda(\mathcal{V}_n) \to \mathbb{Q}(q). \tag{7.7}$$

Finally, for any $A \in \mathrm{Row}(\lambda,\nu)$, define $V_A^* = \xi_\mu^*(M_A^*) \in \widetilde{P}^\lambda(\mathcal{V}_n)$.

Theorem 29. *For any $\nu \in \Lambda_n$ and $A \in \mathrm{Row}(\lambda,\nu)$, we have that*

$$\xi_\mu^*(L_A^*) = \begin{cases} K^*_{R^{-1}(A)} & \text{if } A \in \mathrm{Dom}(\lambda,\nu), \\ 0 & \text{otherwise.} \end{cases}$$

The vectors $\{V_A^* \mid A \in \mathrm{Dom}(\lambda,\nu)\}$ and $\{K_B^* \mid B \in \mathrm{Std}(\mu,\nu)\}$ give two natural bases for the ν-weight space $\widetilde{P}^\lambda_\nu(\mathcal{V}_n)$ of $\widetilde{P}^\lambda(\mathcal{V}_n)$. Moreover, $(L_A, K_B^*) = \delta_{A,R(B)}$ for $A \in \mathrm{Dom}(\lambda,\nu)$, $B \in \mathrm{Std}(\mu,\nu)$. Finally, for any $A \in \mathrm{Row}(\lambda,\nu)$, we have that

$$V_A^* = \sum_{B \in \mathrm{Dom}(\mu,\nu)} l_{A,B}(q^{-1}) K^*_{R^{-1}(B)}.$$

Proof. For $A \in \mathrm{Col}(\mu,\nu)$, $B \in \mathrm{Row}(\lambda,\nu)$, $(K_A, \xi_\mu^*(L_B^*)) = (\xi_\mu(K_A), L_B^*)$, which by Theorems 7 and 26 is zero unless $A \in \mathrm{Std}(\mu,\nu)$ and $B = R(A)$. Now argue as in the last paragraph of the proof of Theorem 26 to get the remaining statements. □

Remark 30. Note that (unlike in Theorem 26) the particular choice of μ here is irrelevant: it only affects the parametrization of the bases not the bases themselves, so one may as well take $\mu = \lambda'$. Using Lusztig's results [L, 27.1.7,27.2.4] on filtrations of based modules, it is not hard to prove Theorem 29 directly, instead of by dualizing Theorem 26. This identifies the basis $\{K_A^* \mid A \in \bigcup_\nu \mathrm{Std}(\mu,\nu)\}$ for $\widetilde{P}^\lambda(\mathcal{V}_n)$ directly with the *canonical basis* in the sense of Lusztig, which is the *lower global crystal base* of Kashiwara (by Remarks 27 and 31). The basis $\{V_A^* \mid A \in \bigcup_\nu \mathrm{Dom}(\lambda,\nu)\}$ is the *semi-standard basis* of Dipper and James [DJ2].

Remark 31. We proved in Theorem 29 that the basis $\{L_A \mid A \in \bigcup_\nu \mathrm{Dom}(\lambda,\nu)\}$ for $P^\lambda(\mathcal{V}_n)$ is dual to the basis $\{K_A^* \mid A \in \bigcup_\nu \mathrm{Std}(\mu,\nu)\}$ for $\widetilde{P}^\lambda(\mathcal{V}_n)$ under the pairing $(.,.)$ from (7.7). We can give a more familiar definition of this pairing as follows. Let $A \in \mathrm{Col}(\mu,\lambda)$ be the tableau having all entries in its ith row equal to i. Then, $V_A = L_{R(A)}$ and $V^*_{R(A)} = K_A^*$ are the canonical highest weight vectors in $P^\lambda(\mathcal{V}_n)$ and $\widetilde{P}^\lambda(\mathcal{V}_n)$, respectively. By Theorem 29, we have that $(V_A, V^*_{R(A)}) = 1$. The pairing $(.,.)$ is characterized uniquely by this property and the fact from Lemma 3 that $(uv, w) = (v, \tau(u)w)$ for all $u \in \mathcal{U}_n, v \in P^\lambda(\mathcal{V}_n), w \in \widetilde{P}^\lambda(\mathcal{V}_n)$.

Remark 32. The constructions in this section actually yield bases for the $\mathbb{Z}[q,q^{-1}]$-forms $P^\lambda(\mathscr{V}_n)$ and $\widetilde{P}^\lambda(\mathscr{V}_n)$, meaning the image resp. cokernel of the restriction of the map ξ_μ resp. ξ_μ^* to $\bigwedge^\mu(\mathscr{V}_n)$ resp. $S^\lambda(\mathscr{V}_n)$. It is only here that the essential difference between the two constructions shows up: $\widetilde{P}^\lambda(\mathscr{V}_n)$ is the $\mathbb{Z}[q,q^{-1}]$-lattice in $\widetilde{P}^\lambda(\mathcal{V}_n)$ obtained by applying $\mathscr{U}_n$ to the canonical highest weight vector from Remark 31, and $P^\lambda(\mathscr{V}_n)$ is the dual lattice under the pairing $(.,.)$.

References

[BZ] A. Berenstein and A. Zelevinsky, String bases for quantum groups of type A_r, *Adv. Soviet Math.* **16** (1993), 51–89.

[BZw] A. Berenstein and S. Zwicknagl, Braided symmetric and exterior algebras; `math.QA/0504155`.

[BK] J. Brundan and A. Kleshchev, Representations of shifted Yangians; `math.RT/0508003`.

[CWZ] S.-J. Cheng, W. Wang and R. B. Zhang, A super duality and Kazhdan-Lusztig polynomials; `math.RT/0409016`.

[DJ1] R. Dipper and G. James, Representations of Hecke algebras of general linear groups, *Proc. London Math. Soc.* **52** (1986), 20–52.

[DJ2] R. Dipper and G. James, q-tensor space and q-Weyl modules, *Trans. Amer. Math. Soc.* **327** (1991), 251–282.

[Deo] V. Deodhar, On some geometric aspects of Bruhat orderings II: the parabolic analogue of Kazhdan-Lusztig polynomials, *J. Algebra* **111** (1987), 483–506.

[Do] J. M. Douglass, An inversion formula for relative Kazhdan-Lusztig polynomials, *Comm. Alg.* **18** (1990), 371–387.

[D1] J. Du, Canonical bases for irreducible representations of quantum GL_n, *Bull. London Math. Soc.* **24** (1992), 325–334.

[D2] J. Du, IC bases and quantum linear groups, *Proc. Symp. Pure Math.* **56** (1994), 135–148.

[D3] J. Du, Canonical bases for irreducible representations of quantum GL_n, II, *Proc. London Math. Soc.* **51** (1995), 461–470.

[FKK] I. B. Frenkel, M. Khovanov and A. A. Kirillov, Jr., Kazhdan-Lusztig polynomials and canonical basis, *Transform. Groups* **3** (1998), 321–336.

[F] W. Fulton, *Young tableaux*, LMS, 1997.

[GL] I. Grojnowski and G. Lusztig, On bases of irreducible representations of quantum GL_n, *Contemp. Math.* **139** (1992), 167–174.

[JZh] H. Jakobsen and H. Zhang, The exponential nature and positivity, preprint, 2005.

[K1] M. Kashiwara, Crystallizing the q-analogue of universal enveloping algebras, *Commun. Math. Phys.* **133** (1990), 249–260.

[K2] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, *Duke Math. J.* **63** (1991), 465–516.

[K3] M. Kashiwara, Global crystal bases of quantum groups, *Duke Math. J.* **69** (1993), 455–485.

[K4] M. Kashiwara, On crystal bases, *Proc. Canadian Math. Soc.* **16** (1995), 155–196.

[KN] M. Kashiwara and T. Nakashima, Crystal graphs for representations of the q-analogue of classical Lie algebras, *J. Algebra* **165** (1994), 295–345.

[KL] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

[LS1] A. Lascoux and M.-P. Schützenberger, Polynômes de Kazhdan et Lusztig pour les grassmanniennes, *Astérisque* **87–88** (1981), 249–266.

[LS2] A. Lascoux and M.-P. Schützenberger, Keys and standard bases, in: "Invariant theory and tableaux", D. Stanton ed., Springer, 1990.

[LNT] B. Leclerc, M. Nazarov and J.-Y. Thibon, Induced representations of affine Hecke algebras and canonical bases of quantum groups, in: "Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000)", pp. 115–153, *Progress in Math.* **210**, Birkhäuser, 2003.

[LT] B. Leclerc and J.-Y. Thibon, The Robinson-Schensted correspondence, crystal bases, and the quantum straightening at $q = 0$, *Electron. J. Combin.* **3** (1996), #11, 24 pp..

[LTo] B. Leclerc and P. Toffin, A simple algorithm for computing the global crystal basis of an irreducible $U_q(\mathfrak{sl}_n)$-module, *Internat. J. Algebra Comput.* **10** (2000), 191–208.

[L] G. Lusztig, *Introduction to quantum groups*, Birkhäuser, 1993.

[Ma] A. Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmmetric group*, University Lecture Series, vol. 15, AMS, 1999.

[S] W. Soergel, Kazhdan-Lusztig polynomials and a combinatoric for tilting modules, *Represent. Theory* **1** (1997), 83–114.

[Ze] A. Zelevinsky, Two remarks on graded nilpotent classes, *Russian Math. Surveys* **40** (1985), 249–250.

[Zh] H. Zhang, On dual canonical bases, *J. Phys. A* **37** (2004), 7879-7893.

[Z] R. B. Zhang, Howe duality and the quantum general linear group, *Proc. Amer. Math. Soc.* **131** (2003), 2681–2692.