ELEMENTARY INVARIANTS FOR CENTRALIZERS OF NILPOTENT MATRICES

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Abstract. We construct an explicit set of algebraically independent generators for the center of the universal enveloping algebra of the centralizer of a nilpotent matrix in the general linear Lie algebra over a field of characteristic zero. In particular, this gives a new proof of the freeness of the center, a result first proved by Panyushev, Premet and Yakimova.

1. Introduction

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a composition of N such that either $\lambda_1 \geq \cdots \geq \lambda_n$ or $\lambda_1 \leq \cdots \leq \lambda_n$. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{gl}_N(F)$, where F is an algebraically closed field of characteristic zero. Let $e \in \mathfrak{g}$ be the nilpotent matrix consisting of Jordan blocks of sizes $\lambda_1, \dots, \lambda_n$ in order down the diagonal, and write $\mathfrak{g}_e$ for the centralizer of e in $\mathfrak{g}$. Panyushev, Premet and Yakimova [PPY] have recently proved that $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$, the algebra of invariants for the adjoint action of $\mathfrak{g}_e$ on the symmetric algebra $S(\mathfrak{g}_e)$, is a free polynomial algebra on N generators. Moreover, viewing $S(\mathfrak{g}_e)$ as a graded algebra as usual so $\mathfrak{g}_e$ is concentrated in degree one, they show that if $x_1, \dots, x_N$ are homogeneous generators for $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$ of degrees $d_1 \leq \cdots \leq d_N$, then the sequence $(d_1, \dots, d_N)$ of *invariant degrees* is equal to

$$
\begin{aligned}
&(\overbrace{1,\dots,1}^{\lambda_1\text{ 1's}},\overbrace{2,\dots,2}^{\lambda_2\text{ 2's}},\dots,\overbrace{n,\dots,n}^{\lambda_n\ n\text{'s}}) \qquad \text{if } \lambda_1 \geq \cdots \geq \lambda_n,\\
&(\underbrace{1,\dots,1}_{\lambda_n\text{ 1's}},\underbrace{2,\dots,2}_{\lambda_{n-1}\text{ 2's}},\dots,\underbrace{n,\dots,n}_{\lambda_1\ n\text{'s}}) \qquad \text{if } \lambda_1 \leq \cdots \leq \lambda_n.
\end{aligned}
$$

This is just one instance of the following conjecture formulated in this generality by Premet: *For any semisimple Lie algebra $\mathfrak{g}$ and any element $e \in \mathfrak{g}$ the invariant algebra $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$ is free.* In [PPY] this conjecture has already been verified in many other situations besides the type A case discussed here.

Returning to our special situation, let $Z(\mathfrak{g}_e)$ denote the center of the universal enveloping algebra $U(\mathfrak{g}_e)$. The standard filtration on $U(\mathfrak{g}_e)$ induces a filtration on the subalgebra $Z(\mathfrak{g}_e)$ such that the associated graded algebra $\operatorname{gr} Z(\mathfrak{g}_e)$ is canonically identified with $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$; see [D, 2.4.11]. We can lift the algebraically independent generators $x_1, \dots, x_N$ from $\operatorname{gr} Z(\mathfrak{g}_e)$ to $Z(\mathfrak{g}_e)$ to deduce (without resorting to Duflo's theorem [D, 10.4.5]) that $Z(\mathfrak{g}_e)$ is also a free polynomial algebra. The purpose of this note is to derive an explicit formula for a set $z_1, \dots, z_N$ of

2000 Mathematics Subject Classification: 17B35.

Research supported in part by NSF grant no. DMS-0139019.

algebraically independent generators for $Z(\mathfrak{g}_e)$, generalizing the well known set of generators of $Z(\mathfrak{g})$ itself (the special case $e = 0$) that arise from the Capelli identity. We call these the *elementary generators* for $Z(\mathfrak{g}_e)$. Passing back down to the associated graded algebra, one can easily obtain from them an explicit set of *elementary invariants* that generate $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$.

To formulate the main result precisely, we must first introduce some notation for elements of $\mathfrak{g}_e$. Let $e_{i,j}$ denote the ij-matrix unit in $\mathfrak{g}$. Draw a diagram with rows numbered $1, \dots, n$ from top to bottom and columns numbered $1, 2, \dots$ from left to right, consisting of λ_i boxes on the ith row in columns $1, \dots, \lambda_i$, for each $i = 1, \dots, n$. Write the numbers $1, \dots, N$ into the boxes along rows, and use the notation $\mathrm{row}(i)$ and $\mathrm{col}(i)$ for the row and column number of the box containing the entry i. For instance, if $\lambda = (4, 3, 2)$ then the diagram is

1	2	3	4
5	6	7	
8	9		

and the nilpotent matrix e of Jordan type λ is equal to $e_{1,2} + e_{2,3} + e_{3,4} + e_{5,6} + e_{6,7} + e_{8,9}$. For $1 \leq i, j \leq n$ and $\lambda_j - \min(\lambda_i, \lambda_j) \leq r < \lambda_j$, define

$$e_{i,j;r} := \sum_{\substack{1 \leq h,k \leq N \\ \mathrm{row}(h)=i, \mathrm{row}(k)=j \\ \mathrm{col}(k)-\mathrm{col}(h)=r}} e_{h,k}. \tag{1.1}$$

The vectors $\{e_{i,j;r} \mid 1 \leq i, j \leq n, \lambda_j - \min(\lambda_i, \lambda_j) \leq r < \lambda_j\}$ form a basis for $\mathfrak{g}_e$; see [BK2, Lemma 7.3]. Write $\mu \subseteq \lambda$ if $\mu = (\mu_1, \dots, \mu_n)$ is a composition with $0 \leq \mu_i \leq \lambda_i$ for each $i = 1, \dots, n$. Also let $|\mu| := \mu_1 + \cdots + \mu_n$ and $\ell(\mu)$ denote the number of non-zero parts of μ. Recall that $(d_1, \dots, d_N)$ are the invariant degrees as defined above. Given $0 \neq \mu \subseteq \lambda$ such that $\ell(\mu) = d_{|\mu|}$, suppose that the non-zero parts of μ are in the entries indexed by $1 \leq i_1 < \cdots < i_d \leq n$. Define the μth *column determinant*

$$\mathrm{cdet}(\mu) := \sum_{w \in S_d} \mathrm{sgn}(w) \tilde{e}_{i_{w1}, i_1; \mu_{i_1}-1} \tilde{e}_{i_{w2}, i_2; \mu_{i_2}-1} \cdots \tilde{e}_{i_{wd}, i_d; \mu_{i_d}-1}, \tag{1.2}$$

where $\tilde{e}_{i,j;r} := e_{i,j;r} - \delta_{r,0}\delta_{i,j}(i-1)\lambda_i$. We note by Lemma 3.8 below that all the $e_{i,j;r}$'s appearing on the right hand side of (1.2) necessarily satisfy the inequality $\lambda_j - \min(\lambda_i, \lambda_j) \leq r < \lambda_j$, so $\mathrm{cdet}(\mu)$ is well-defined. For $r = 1, \dots, N$, define

$$z_r := \sum_{\substack{\mu \subseteq \lambda \\ |\mu|=r, \ell(\mu)=d_r}} \mathrm{cdet}(\mu). \tag{1.3}$$

Main Theorem. *The elements $z_1, \dots, z_N$ are algebraically independent generators for $Z(\mathfrak{g}_e)$.*

In the situation that $\lambda_1 = \cdots = \lambda_n$, our Main Theorem was proved already by Molev [M], following Rais and Tauvel [RT] who established the freeness of $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$ in that case. Our proof for general λ follows the same strategy as Molev's proof, but we need to replace the truncated Yangians with their shifted analogs from

[BK2]. We have also included a self-contained proof of the freeness of $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$, although as we have said this was already established in [PPY]. Our approach to that is similar to the argument in [RT] and different from [PPY].

One final comment. In this introduction we have formulated the Main Theorem assuming either that $\lambda_1 \geq \cdots \geq \lambda_n$ or that $\lambda_1 \leq \cdots \leq \lambda_n$. Presumably most readers will prefer the former choice. However in the remainder of the article we will only actually prove the results in the latter situation, since that is the convention adopted in [BK2]–[BK4]. This is justified because the two formulations of the Main Theorem are simply equivalent, by an elementary argument involving twisting with an antiautomorphism of $U(\mathfrak{g})$ of the form $e_{i,j} \mapsto -e_{i',j'} + \delta_{i,j} c$.

The remainder of the article is organized as follows. In §2, we derive a new "quantum determinant" formula for the central elements of the shifted Yangians. In §3 we descend from there to the universal enveloping algebra $U(\mathfrak{g}_e)$ to prove that the elements z_r are indeed central. Then in §4 we prove the freeness of $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$ by restricting to a carefully chosen slice.

2. Shifted quantum determinants

The shifted Yangian $Y_n(\sigma)$ is defined in [BK2]. Here are some of the details. Let $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ *shift matrix*, that is, all its entries are non-negative integers and $s_{i,j} + s_{j,k} = s_{i,k}$ whenever $|i-j| + |j-k| = |i-k|$. Then $Y_n(\sigma)$ is the associative algebra over F defined by generators

$$\begin{aligned} &\{D_i^{(r)} \mid 1 \leq i \leq n, r > 0\}, \\ &\{E_i^{(r)} \mid 1 \leq i \leq n, r > s_{i,i+1}\}, \\ &\{F_i^{(r)} \mid 1 \leq i \leq n, r > s_{i+1,i}\} \end{aligned}$$

subject to certain relations. See [BK2, §2] for the full set.

For $1 \leq i < j \leq n$ and $r > s_{i,j}$, define elements $E_{i,j}^{(r)} \in Y_n(\sigma)$ recursively by

$$E_{i,i+1}^{(r)} := E_i^{(r)}, \qquad E_{i,j}^{(r)} := [E_{i,j-1}^{(r-s_{j-1,j})}, E_{j-1}^{(s_{j-1,j}+1)}]. \tag{2.1}$$

Similarly, for $1 \leq i < j \leq n$ and $r > s_{j,i}$ define elements $F_{i,j}^{(r)} \in Y_n(\sigma)$ by

$$F_{i,i+1}^{(r)} := F_i^{(r)}, \qquad F_{i,j}^{(r)} := [F_{j-1}^{(s_{j,j-1}+1)}, F_{i,j-1}^{(r-s_{j,j-1})}]. \tag{2.2}$$

As in [BK3, §2], we introduce a new set of generators for $Y_n(\sigma)$. For $1 \leq i < j \leq n$ define the power series $E_{i,j}(u), F_{i,j}(u) \in Y_n(\sigma)[[u^{-1}]]$ by

$$E_{i,j}(u) := \sum_{r > s_{i,j}} E_{i,j}^{(r)} u^{-r}, \qquad F_{i,j}(u) := \sum_{r > s_{j,i}} F_{i,j}^{(r)} u^{-r}, \tag{2.3}$$

and set $E_{i,i}(u) = F_{i,i}(u) = 1$ by convention. Also define

$$D_i(u) := \sum_{r \geq 0} D_i^{(r)} u^{-r} \in Y_n(\sigma)[[u^{-1}]],$$

for $1 \leq i \leq n$, where $D_i^{(0)} = 1$ by convention. Let $D(u)$ denote the $n \times n$ diagonal matrix with ii-entry $D_i(u)$ for $1 \leq i \leq n$, let $E(u)$ denote the $n \times n$ upper triangular matrix with ij-entry $E_{i,j}(u)$ for $1 \leq i \leq j \leq n$, and let $F(u)$ denote the $n \times n$ lower triangular matrix with ji-entry $F_{i,j}(u)$ for $1 \leq i \leq j \leq n$. Consider the product

$$T(u) = F(u)D(u)E(u)$$

of matrices with entries in $Y_n(\sigma)[[u^{-1}]]$. The ij-entry of the matrix $T(u)$ defines a power series

$$T_{i,j}(u) = \sum_{r \geq 0} T_{i,j}^{(r)} u^{-r} := \sum_{k=1}^{\min(i,j)} F_{k,i}(u)D_k(u)E_{k,j}(u) \tag{2.4}$$

for some new elements $T_{i,j}^{(r)} \in Y_n(\sigma)$. Note that $T_{i,j}^{(0)} = \delta_{i,j}$ and $T_{i,j}^{(r)} = 0$ for $0 < r \leq s_{i,j}$.

If the matrix σ is the zero matrix, we denote $Y_n(\sigma)$ simply by Y_n. The algebra Y_n is the Yangian associated to the Lie algebra $\mathfrak{gl}_n(F)$; see [MNO, §1] for its usual definition. In general, by [BK2, Corollary 2.2], there exists an injection $Y_n(\sigma) \hookrightarrow Y_n$ which sends the elements $D_i^{(r)}, E_i^{(r)}$, and $F_i^{(r)}$ in $Y_n(\sigma)$ to the elements with the same name in Y_n. However, this injection usually does not send *all* the elements $E_{i,j}^{(r)}, F_{i,j}^{(r)}$ and $T_{i,j}^{(r)}$ of $Y_n(\sigma)$ to the elements with the same name in Y_n. For the remainder of this section we will use this injection to identify $Y_n(\sigma)$ with a subalgebra of Y_n. To avoid confusion the elements $E_{i,j}^{(r)}, F_{i,j}^{(r)}$, and $T_{i,j}^{(r)}$ of $Y_n(\sigma)$ will be denoted ${}^\sigma E_{i,j}^{(r)}, {}^\sigma F_{i,j}^{(r)}$, and ${}^\sigma T_{i,j}^{(r)}$ respectively, while $E_{i,j}^{(r)}, F_{i,j}^{(r)}$, and $T_{i,j}^{(r)}$ will refer to the elements of Y_n. Similarly we write ${}^\sigma E_{i,j}(u), {}^\sigma F_{i,j}(u)$ and ${}^\sigma T_{i,j}(u)$ for the power series (2.3)–(2.4) computed in $Y_n(\sigma)[[u^{-1}]]$ to distinguish them from their counterparts in $Y_n[[u^{-1}]]$.

For an $n \times n$ matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ with entries in some associative (but not necessarily commutative) algebra, we define its *column determinant*

$$\operatorname{cdet} A := \sum_{w \in S_n} \operatorname{sgn}(w) a_{w1,1} a_{w2,2} \cdots a_{wn,n}. \tag{2.5}$$

For $1 \leq j \leq n$, we define a *left j-minor* of A to be a $j \times j$ submatrix of the form

$$\begin{pmatrix} a_{i_1,1} & a_{i_1,2} & \cdots & a_{i_1,j} \\ a_{i_2,1} & a_{i_2,2} & \cdots & a_{i_2,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_j,1} & a_{i_j,2} & \cdots & a_{i_j,j} \end{pmatrix}$$

for $1 \leq i_1 < i_2 < \cdots < i_j \leq n$. The following lemma is an easy exercise.

Lemma 2.1. *If for a fixed $1 \leq j \leq n$, the* cdet *of every left j-minor of an $n \times n$ matrix A with entries in some associative algebra is zero then* $\operatorname{cdet} A = 0$.

By [MNO, Theorem 2.10], it is known that the coefficients of the power series

$$C_n(u) := \operatorname{cdet} \begin{pmatrix} T_{1,1}(u) & T_{1,2}(u-1) & \cdots & T_{1,n}(u-n+1) \\ T_{2,1}(u) & T_{2,2}(u-1) & \cdots & T_{2,n}(u-n+1) \\ \vdots & \vdots & \ddots & \vdots \\ T_{n,1}(u) & T_{n,2}(u-1) & \cdots & T_{n,n}(u-n+1) \end{pmatrix} \tag{2.6}$$

belong to the center of Y_n. Define

$$
{}^\sigma C_n(u) := \operatorname{cdet}\begin{pmatrix} {}^\sigma T_{1,1}(u) & {}^\sigma T_{1,2}(u-1) & \cdots & {}^\sigma T_{1,n}(u-n+1) \\ {}^\sigma T_{2,1}(u) & {}^\sigma T_{2,2}(u-1) & \cdots & {}^\sigma T_{2,n}(u-n+1) \\ \vdots & \vdots & \ddots & \vdots \\ {}^\sigma T_{n,1}(u) & {}^\sigma T_{n,2}(u-1) & \cdots & {}^\sigma T_{n,n}(u-n+1) \end{pmatrix}. \tag{2.7}
$$

The goal in the remainder of the section is to prove the following theorem. Note this result is false without the assumption that σ is upper triangular.

Theorem 2.2. *Assuming that the shift matrix σ is upper triangular, i.e. $s_{i,j} = 0$ for $i > j$, we have that ${}^\sigma C_n(u) = C_n(u)$, equality in $Y_n[[u^{-1}]]$. In particular, the coefficients of the power series ${}^\sigma C_n(u)$ belong to the center of $Y_n(\sigma)$.*

For the proof, assume from now on that σ is upper triangular. For $0 \le j \le n$, let X_j be the $n \times n$ matrix whose first j columns are the same as the first j columns of the matrix in (2.6) and whose last $(n-j)$ columns are the same as the last $(n-j)$ columns of the matrix in (2.7). In this notation, the theorem asserts that $\operatorname{cdet} X_0 = \operatorname{cdet} X_n$. So we just need to check for each $j = 1, \dots, n$ that

$$
\operatorname{cdet} X_{j-1} = \operatorname{cdet} X_j. \tag{2.8}
$$

To see this, fix j and let $v := u - j + 1$ for short. Given a column vector $\vec{a}$ of height n, let $X(\vec{a})$ be the matrix obtained from X_j by replacing the jth column by $\vec{a}$. For $1 \le k \le j$, introduce the following column vectors:

$$
\vec{a} := \begin{pmatrix} {}^\sigma T_{1,j}(v) \\ {}^\sigma T_{2,j}(v) \\ \vdots \\ {}^\sigma T_{n,j}(v) \end{pmatrix}, \quad \vec{b}_k := \begin{pmatrix} T_{1,k}(v) \\ T_{2,k}(v) \\ \vdots \\ T_{n,k}(v) \end{pmatrix}, \quad \vec{c}_k := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ D_k(v) \\ F_{k,k+1}(v)D_k(v) \\ \vdots \\ F_{k,n}(v)D_k(v) \end{pmatrix}.
$$

Also define

$$
\begin{pmatrix} d_{1,k} \\ d_{2,k} \\ \vdots \\ d_{k-1,k} \end{pmatrix} := \begin{pmatrix} T_{1,1}(v) & T_{1,2}(v) & \cdots & T_{1,k-1}(v) \\ T_{2,1}(v) & T_{2,2}(v) & \cdots & T_{2,k-1}(v) \\ \vdots & \vdots & \ddots & \vdots \\ T_{k-1,1}(v) & T_{k-1,2}(v) & \cdots & T_{k-1,k-1}(v) \end{pmatrix}^{-1} \begin{pmatrix} T_{1,k}(v) \\ T_{2,k}(v) \\ \vdots \\ T_{k-1,k}(v) \end{pmatrix}
$$

and set $e_k := {}^\sigma E_{k,j}(v)$. In particular, $e_j = 1$.

Lemma 2.3. $\vec{a} = \displaystyle\sum_{k=1}^{j} \vec{c}_k e_k$.

Proof. In view of the assumption that σ is upper triangular, we have by (2.2)–(2.3) that ${}^\sigma F_{i,j}(v) = F_{i,j}(v)$ for all $1 \le i \le j \le n$. Now the lemma follows from the definition (2.4). □

Lemma 2.4. *For $1 \leq k \leq j$, we have that* $\vec{c}_k = \vec{b}_k - \displaystyle\sum_{l=1}^{k-1} \vec{b}_l d_{l,k}$.

Proof. Take $1 \leq i \leq n$ and consider the ith entry of the column vectors on either side of the equation. If $i \geq k$ then we need to show that $F_{k,i}(v)D_k(v) = T_{i,k}(v) - \sum_{l=1}^{k-1} T_{i,l}(v)d_{l,k}$, which is immediate from the identity [BK1, (5.4)]. If $i < k$ then we need to show that $0 = T_{i,k}(v) - \sum_{l=1}^{k-1} T_{i,l}(v)d_{l,k}$. To see this, note by the definition of $d_{l,k}$ that $\sum_{l=1}^{k-1} T_{i,l}(v)d_{l,k}$ is equal to the matrix product

$$\big(T_{i,1}(v) \cdots T_{i,k-1}(v)\big) \begin{pmatrix} T_{1,1}(v) & T_{1,2}(v) & \cdots & T_{1,k-1}(v) \\ T_{2,1}(v) & T_{2,2}(v) & \cdots & T_{2,k-1}(v) \\ \vdots & \vdots & \ddots & \vdots \\ T_{k-1,1}(v) & T_{k-1,2}(v) & \cdots & T_{k-1,k-1}(v) \end{pmatrix}^{-1} \begin{pmatrix} T_{1,k}(v) \\ T_{2,k}(v) \\ \vdots \\ T_{k-1,k}(v) \end{pmatrix}.$$

The left hand row vector is the ith row of the matrix being inverted, so this product does indeed equal $T_{i,k}(v)$. □

Lemma 2.5. *For any $1 \leq k \leq j-1$ and any f, we have that* $\operatorname{cdet} X(\vec{b}_k f) = 0$.

Proof. We apply Lemma 2.1. Take $1 \leq i_1 < \cdots < i_j \leq n$. The corresponding left j-minor of $X(\vec{b}_k f)$ is equal to

$$\begin{pmatrix} T_{i_1,1}(u) & T_{i_1,2}(u-1) & \cdots & T_{i_1,j-1}(u-j+2) & T_{i_1,k}(u-j+1)f \\ T_{i_2,1}(u) & T_{i_2,2}(u-1) & \cdots & T_{i_2,j-1}(u-j+2) & T_{i_2,k}(u-j+1)f \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{i_j,1}(u) & T_{i_j,2}(u-1) & \cdots & T_{i_j,j-1}(u-j+2) & T_{i_j,k}(u-j+1)f \end{pmatrix}.$$

The cdet of this matrix is zero by [BK1, (8.4)]. □

Now we can complete the proof of Theorem 2.2. Since cdet is linear in each column, we have by Lemmas 2.3-2.5 that

$$\operatorname{cdet} X(\vec{a}) = \sum_{k=1}^{j} \operatorname{cdet} X(\vec{b}_k e_k) - \sum_{k=1}^{j}\sum_{l=1}^{k-1} \operatorname{cdet} X(\vec{b}_l d_{l,k} e_k) = \operatorname{cdet} X(\vec{b}_j).$$

Since $X_{j-1} = X(\vec{a})$ and $X_j = X(\vec{b}_j)$, this verifies (2.8) hence the theorem.

3. The central elements z_r

For the remainder of the article, $\lambda = (\lambda_1, \dots, \lambda_n)$ denotes a fixed composition of N such that $\lambda_1 \leq \cdots \leq \lambda_n$ and $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ denotes the upper triangular shift matrix defined by $s_{i,j} := \lambda_j - \min(\lambda_i, \lambda_j)$. Let $\mathfrak{g} = \mathfrak{gl}_N(F)$ and $e \in \mathfrak{g}$ be the nilpotent matrix consisting Jordan blocks of sizes $\lambda_1, \dots, \lambda_n$ down the diagonal. Recall from the introduction that the centralizer $\mathfrak{g}_e$ of e in $\mathfrak{g}$ has basis

$$\{e_{i,j;r} \mid 1 \leq i,j \leq n, s_{i,j} \leq r < \lambda_j\} \tag{3.1}$$

where $e_{i,j;r}$ is the element defined by (1.1). We will view $\mathfrak{g}_e$ as a $\mathbb{Z}$-graded Lie algebra by declaring that the basis element $e_{i,j;r}$ is of degree r. There is an induced $\mathbb{Z}$-grading on the universal enveloping algebra $U(\mathfrak{g}_e)$.

In this section we are going to prove that the elements $z_1, \dots, z_N$ of $U(\mathfrak{g}_e)$ from (1.3) actually belong to the center $Z(\mathfrak{g}_e)$ of $U(\mathfrak{g}_e)$ by exploiting the relationship between $U(\mathfrak{g}_e)$ and the *finite W-algebra* $W(\lambda)$ associated to e. According to the definition followed here, $W(\lambda)$ is the quotient of the shifted Yangian $Y_n(\sigma)$ by the two-sided ideal generated by the elements $\left\{D_1^{(r)} \mid r > \lambda_1\right\}$. This is not the usual definition of the finite W-algebra, but it is equivalent to the usual definition thanks to the main result of [BK2]. The notation $T_{i,j}^{(r)}$ will from now on denote the canonical image in the quotient algebra $W(\lambda)$ of the element $T_{i,j}^{(r)} \in Y_n(\sigma)$ from (2.4) (which was also denoted ${}^\sigma T_{i,j}^{(r)}$ in the previous section). Recall that $T_{i,j}^{(r)} = 0$ for $0 < r < s_{i,j}$. In addition, now that we have passed to the quotient $W(\lambda)$, the following holds by [BK3, Theorem 3.5].

Lemma 3.1. *$T_{i,j}^{(r)} = 0$ for all $r > \lambda_j$.*

So the power series $T_{i,j}(u) := \sum_{r \geq 0} T_{i,j}^{(r)} u^{-r} \in W(\lambda)[[u^{-1}]]$ is actually a polynomial and $u^{\lambda_j} T_{i,j}(u)$ belongs to $W(\lambda)[u]$. Hence

$$\operatorname{cdet} \begin{pmatrix} u^{\lambda_1} T_{1,1}(u) & (u-1)^{\lambda_2} T_{1,2}(u-1) & \cdots & (u-n+1)^{\lambda_n} T_{1,n}(u-n+1) \\ u^{\lambda_1} T_{2,1}(u) & (u-1)^{\lambda_2} T_{2,2}(u-1) & \cdots & (u-n+1)^{\lambda_n} T_{2,n}(u-n+1) \\ \vdots & \vdots & \ddots & \vdots \\ u^{\lambda_1} T_{n,1}(u) & (u-1)^{\lambda_2} T_{n,2}(u-1) & \cdots & (u-n+1)^{\lambda_n} T_{n,n}(u-n+1) \end{pmatrix}$$

gives us a well-defined polynomial $Z(u) \in W(\lambda)[u]$. We have that

$$Z(u) = u^N + Z_1 u^{N-1} + \cdots + Z_{N-1} u + Z_N \tag{3.2}$$

for elements $Z_1, \dots, Z_N \in W(\lambda)$.

Lemma 3.2. *The elements $Z_1, \dots, Z_N$ belong to the center $Z(W(\lambda))$ of $W(\lambda)$.*

Proof. This follows from Theorem 2.2, because $Z(u)$ is equal to the canonical image of the power series from (2.7) multiplied by $u^{\lambda_1}(u-1)^{\lambda_2} \cdots (u-n+1)^{\lambda_n}$. □

We define a filtration $\mathrm{F}_0 W(\lambda) \subseteq \mathrm{F}_1 W(\lambda) \subseteq \cdots$ on $W(\lambda)$, which we call the *loop filtration*, by declaring that each generator $T_{i,j}^{(r+1)}$ is of filtered degree r. In other words, $\mathrm{F}_r W(\lambda)$ is the span of all monomials of the form $T_{i_1,j_1}^{(r_1+1)} \cdots T_{i_k,j_k}^{(r_k+1)}$ such that $r_1 + \cdots + r_k \leq r$. For an element $x \in \mathrm{F}_r W(\lambda)$, we write $\operatorname{gr}_r x$ for the canonical image of x in the rth graded component of the associated graded algebra $\operatorname{gr} W(\lambda)$. Applying the PBW theorem for $W(\lambda)$ from [BK3, Lemma 3.6], it follows that the loop filtration as defined here coincides with the filtration defined at the beginning of [BK4, §3]. So we can restate [BK4, Lemma 3.1] as follows.

Lemma 3.3. *There is a unique isomorphism of graded algebras*

$$\varphi : \operatorname{gr} W(\lambda) \xrightarrow{\sim} U(\mathfrak{g}_e)$$

such that $\varphi(\operatorname{gr}_r T_{i,j}^{(r+1)}) = (-1)^r e_{i,j;r}$ for all $1 \leq i, j \leq n$ and $s_{i,j} \leq r < \lambda_j$.

Let $(d_1, \dots, d_N)$ be the sequence of invariant degrees defined in the first paragraph of the introduction. Recall also the elements $z_1, \dots, z_N$ of $U(\mathfrak{g}_e)$ defined by the equation (1.3). The goal in the remainder of the section is to prove the following theorem.

Theorem 3.4. *For $r = 1, \dots, N$, the element $Z_r \in Z(W(\lambda))$ belongs to $\mathrm{F}_{r-d_r} W(\lambda)$ and $\varphi(\mathrm{gr}_{r-d_r} Z_r) = (-1)^{r-d_r} z_r$. In particular, the elements $z_1, \dots, z_N$ belong to the center $Z(\mathfrak{g}_e)$ of $U(\mathfrak{g}_e)$.*

To prove the theorem, we begin with several lemmas.

Lemma 3.5. *For $r = 1, \dots, N$, we have that*

$$Z_r = \sum_{\substack{\mu \subseteq \lambda \\ |\mu| = r}} \sum_{\nu \subseteq \mu} \left[\left(\prod_{i=1}^n (1-i)^{\mu_i - \nu_i} \binom{\lambda_i - \nu_i}{\lambda_i - \mu_i} \right) \times \left(\sum_{w \in S_n} \mathrm{sgn}(w) T_{w1,1}^{(\nu_1)} \cdots T_{wn,n}^{(\nu_n)} \right) \right].$$

Proof. Before we begin, we point out that when $i = 1$ the term $(1-i)^{\mu_i - \nu_i}$ in the product on the right hand side should be interpreted as 1 if $\nu_1 = \mu_1$ and as 0 otherwise. Write $\mathrm{coeff}_r\,(f(u))$ for the u^r-coefficient of a polynomial $f(u)$. By the definitions (2.5) and (3.2), we have that

$$\begin{aligned} Z_r &= \sum_{w \in S_n} \mathrm{sgn}(w)\, \mathrm{coeff}_{N-r} \left(u^{\lambda_1} T_{w1,1}(u) \times \cdots \times (u-n+1)^{\lambda_n} T_{wn,n}(u-n+1) \right) \\ &= \sum_{\substack{\mu \subseteq \lambda \\ |\mu| = r}} \sum_{w \in S_n} \mathrm{sgn}(w)\, \mathrm{coeff}_{\lambda_1 - \mu_1} (u^{\lambda_1} T_{w1,1}(u)) \times \cdots \times \\ &\qquad\qquad\qquad\qquad \mathrm{coeff}_{\lambda_n - \mu_n} ((u-n+1)^{\lambda_n} T_{wn,n}(u-n+1)). \end{aligned}$$

Moreover for $i = 1, \dots, n$ we have that

$$\mathrm{coeff}_{\lambda_i - \mu_i} ((u-i+1)^{\lambda_i} T_{wi,i}(u-i+1)) = \sum_{\nu_i = 0}^{\mu_i} (1-i)^{\mu_i - \nu_i} \binom{\lambda_i - \nu_i}{\lambda_i - \mu_i} T_{wi,i}^{(\nu_i)}.$$

Substituting into the preceding formula for Z_r gives the conclusion. ☐

Lemma 3.6. *Suppose $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$ are compositions with $\nu \subseteq \mu$. We have that $|\nu| - \ell(\nu) \le |\mu| - \ell(\mu)$ with equality if and only if for each $i = 1, \dots, n$ either $\nu_i = \mu_i$ or $\nu_i = 0 = \mu_i - 1$.*

Proof. Obvious. ☐

Lemma 3.7. *For $r = 1, \dots, N$, we have that $d_r = \min\{\ell(\mu) \mid \mu \subseteq \lambda, |\mu| = r\}$.*

Proof. Set $d := d_r$ and $s := r - \lambda_n - \lambda_{n-1} - \cdots - \lambda_{n-d+2}$. By the definition of d_r, we have that $1 \le s \le \lambda_{n-d+1}$. The sum of the $(d-1)$ largest parts of λ is $\lambda_n + \lambda_{n-1} + \cdots + \lambda_{n-d+2}$, which is $< r$. Hence we cannot find $\mu \subseteq \lambda$ with $|\mu| = r$ and $\ell(\mu) < d$. On the other hand, $\mu := (0, \dots, 0, s, \lambda_{n-d+2}, \dots, \lambda_{n-1}, \lambda_n)$ is a composition with $\mu \subseteq \lambda$ with $|\mu| = r$ and $\ell(\mu) = d$. ☐

Lemma 3.8. *Given $0 \ne \mu \subseteq \lambda$ with $\ell(\mu) = d_{|\mu|}$, let $1 \le i_1 < \cdots < i_d \le n$ index the non-zero parts of μ. Then for any $w \in S_d$ and $j = 1, \dots, d$ we have that $\mu_{i_j} > \lambda_{i_j} - \min(\lambda_{i_{wj}}, \lambda_{i_j})$.*

Proof. If $wj \geq j$, this is clear since the right hand side of the inequality is zero. So suppose that $wj < j$, when the right hand side of the inequality equals $\lambda_{i_j} - \lambda_{i_{wj}}$. Assume for a contradiction that $\mu_{i_j} \leq \lambda_{i_j} - \lambda_{i_{wj}}$. Then we have that

$$|\mu| = \sum_{k=1}^{d} \mu_{i_k} \leq \left(\sum_{k=1}^{d} \lambda_{i_k}\right) - \lambda_{i_{wj}}.$$

Since $i_{wj} = i_k$ for some $k = 1, \dots, d$, this implies that there exists a composition $\nu \subseteq \lambda$ with $|\nu| = |\mu|$ and $\ell(\nu) = d - 1$. This contradicts Lemma 3.7. $\square$

Now we can prove the theorem. The term $T_{w1,1}^{(\nu_1)} \cdots T_{wn,n}^{(\nu_n)}$ in the expansion of Z_r from Lemma 3.5 belongs to $\mathrm{F}_{|\nu|-\ell(\nu)} W(\lambda)$. If $\nu \subseteq \mu \subseteq \lambda$ and $|\mu| = r$, Lemmas 3.6–3.7 imply that $|\nu| - \ell(\nu) \leq |\mu| - \ell(\mu) = r - \ell(\mu) \leq r - d_r$. This shows that Z_r belongs to $\mathrm{F}_{r-d_r} W(\lambda)$. Moreover, to compute $\mathrm{gr}_{r-d_r} Z_r$ we just need to consider the terms in the expansion of Z_r that have $\ell(\mu) = d_r$ and for each $i = 1, \dots, n$ either $\nu_i = \mu_i$ or $\nu_i = 0 = \mu_i - 1$. We deduce from Lemma 3.5 that

$$\mathrm{gr}_{r-d_r} Z_r = \sum_{\substack{\mu \subseteq \lambda \\ |\mu| = r, \ell(\mu) = d_r}} \sum_{w \in S_n} \mathrm{sgn}(w)\, \mathrm{gr}_{r-d_r} \left(\widetilde{T}_{w1,1}^{(\mu_1)} \cdots \widetilde{T}_{wn,n}^{(\mu_n)}\right)$$

where $\widetilde{T}_{i,j}^{(r)} := T_{i,j}^{(r)} + \delta_{i,j}\delta_{r,1}(1-i)\lambda_i$. Since $\widetilde{T}_{wi,i}^{(0)} = 0$ unless $wi = i$, we can further simplify this expression as follows. Let $d := d_r$ for short and given $\mu \subseteq \lambda$ with $\ell(\mu) = d$ define $1 \leq i_1 < \cdots < i_d \leq n$ so that $\mu_{i_1}, \dots, \mu_{i_d} \neq 0$. Then

$$\mathrm{gr}_{r-d} Z_r = \sum_{\substack{\mu \subseteq \lambda \\ |\mu| = r, \ell(\mu) = d}} \sum_{w \in S_d} \mathrm{sgn}(w) \left(\mathrm{gr}_{\mu_{i_1}-1} \widetilde{T}_{i_{w1},i_1}^{(\mu_{i_1})}\right) \cdots \left(\mathrm{gr}_{\mu_{i_d}-1} \widetilde{T}_{i_{wd},i_d}^{(\mu_{i_d})}\right).$$

Finally applying the isomorphism φ from Lemma 3.3, we get that

$$\varphi(\mathrm{gr}_{r-d} Z_r) = \sum_{\substack{\mu \subseteq \lambda \\ |\mu| = r, \ell(\mu) = d}} \sum_{w \in S_d} \mathrm{sgn}(w)(-1)^{\mu_{i_1}-1} \tilde{e}_{i_{w1},i_1;\mu_{i_1}-1} \cdots (-1)^{\mu_{i_d}-1} \tilde{e}_{i_{wd},i_d;\mu_{i_d}-1}$$

where $\tilde{e}_{i,j;r} := \mathrm{gr}_r \widetilde{T}_{i,j}^{(r+1)}$. The right hand side is $(-1)^{r-d} z_r$ according to the definitions in the introduction. Noting finally that, since Z_r is central in $W(\lambda)$ by Lemma 3.2, the element $\mathrm{gr}_{r-d} Z_r$ is central in $\mathrm{gr}\, W(\lambda)$, this completes the proof of Theorem 3.4.

4. Proof of the main theorem

Now we consider the standard filtration on the universal enveloping algebra $U(\mathfrak{g}_e)$ and the induced filtration on the subalgebra $Z(\mathfrak{g}_e)$. By the PBW theorem, the associated graded algebra $\mathrm{gr}\, U(\mathfrak{g}_e)$ is identified with the symmetric algebra $S(\mathfrak{g}_e)$ (generated by $\mathfrak{g}_e$ in degree one). For $r = 1, \dots, N$, it is immediate from the definition (1.3) that the central element $z_r \in U(\mathfrak{g}_e)$ is of filtered degree d_r. Let

$x_r := \mathrm{gr}_{d_r} z_r \in S(\mathfrak{g}_e)^{\mathfrak{g}_e}$. Explicitly,

$$x_r = \sum_{\substack{\mu \subseteq \lambda \\ |\mu|=r,\, \ell(\mu)=d_r}} \sum_{w \in S_d} \mathrm{sgn}(w) e_{i_{w1},i_1;\mu_{i_1}-1} \cdots e_{i_{wd},i_d;\mu_{i_d}-1} \in S(\mathfrak{g}_e) \tag{4.1}$$

where as usual $1 \leq i_1 < i_2 \cdots < i_d \leq n$ denote the positions of the non-zero entries of μ.

Theorem 4.1. *The elements $x_1, \dots, x_N$ are algebraically independent generators for $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$.*

For the proof, let us from now on identify $S(\mathfrak{g}_e)$ with $F[\mathfrak{g}_e^*]$, the coordinate algebra of the affine variety $\mathfrak{g}_e^*$. Let

$$\{f_{i,j;r} \mid 1 \leq i,j \leq n, s_{i,j} \leq r < \lambda_j\} \tag{4.2}$$

be the basis for $\mathfrak{g}_e^*$ that is dual to the basis (3.1). By convention, we interpret $f_{i,j;r}$ as zero if $r < s_{i,j}$. The coadjoint action ad^* of $\mathfrak{g}_e$ on $\mathfrak{g}_e^*$ is given explicitly by the formula

$$(\mathrm{ad}^* e_{i,j;r})(f_{k,l;s}) = \delta_{j,l} f_{k,i;s-r} - \delta_{i,k} f_{j,l;s-r}. \tag{4.3}$$

The induced action of $\mathfrak{g}_e$ on $F[\mathfrak{g}_e^*]$ is defined by $(x \cdot \theta)(y) = -\theta((\mathrm{ad}^* x)(y))$ for $x \in \mathfrak{g}_e, y \in \mathfrak{g}_e^*, \theta \in F[\mathfrak{g}_e^*]$. It is for this action that the invariant subalgebra $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$ is identified with $F[\mathfrak{g}_e^*]^{\mathfrak{g}_e}$. Introduce the affine subspace

$$S := f + V \tag{4.4}$$

of $\mathfrak{g}_e^*$, where $f := f_{1,2;\lambda_2-1} + f_{2,3;\lambda_3-1} + \cdots + f_{n-1,n;\lambda_n-1}$ and V is the N-dimensional linear subspace spanned by the vectors $\{f_{n,i;r} \mid 1 \leq i \leq n, 0 \leq r < \lambda_i\}$. Let

$$\rho : F[\mathfrak{g}_e^*]^{\mathfrak{g}_e} \to F[S] \tag{4.5}$$

be the homomorphism defined by restricting functions from $\mathfrak{g}_e^*$ to the slice S.

Lemma 4.2. *The elements $\rho(x_1), \dots, \rho(x_N)$ are algebraically independent generators of $F[S]$.*

Proof. Take an arbitrary vector

$$v = f_{1,2;\lambda_2-1} + f_{2,3;\lambda_3-1} + \cdots + f_{n-1,n;\lambda_n-1} + \sum_{j=1}^{n} \sum_{t=0}^{\lambda_j-1} a_{j,t} f_{n,j;t} \in S.$$

Since $S \cong \mathbb{A}^N$, the algebra $F[S]$ is freely generated by the coordinate functions $p_{j,t} : v \mapsto a_{j,t}$ for $1 \leq j \leq n$ and $0 \leq t < \lambda_j - 1$. Also note for any $1 \leq i,j \leq n$ and $s_{i,j} \leq r < \lambda_j$ that

$$e_{i,j;r}(v) = \begin{cases} a_{j,r} & \text{if } i = n, \\ 1 & \text{if } j = i+1 \text{ and } r = \lambda_j - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Now fix $1 \leq r \leq N$. Let $d := d_r$ and $s := r - \lambda_n - \lambda_{n-1} - \cdots - \lambda_{n-d+2}$, so that $1 \leq s \leq \lambda_{n-d+1}$. We claim that $x_r(v) = (-1)^{d-1} a_{n-d+1,s-1}$, hence $\rho(x_r) = (-1)^{d-1} p_{n-d+1,s-1}$. Since every coordinate function $p_{j,t}$ arises in this way for a unique r, the lemma clearly follows from this claim.

To prove the claim, suppose we are given $w \in S_d$ and $\mu \subseteq \lambda$ such that $|\mu| = r$, the non-zero parts of μ are in positions $1 \leq i_1 < \cdots < i_d \leq n$, and the monomial $e_{i_{w1},i_1;\mu_{i_1}-1} \cdots e_{i_{wd},i_d;\mu_{i_d}-1}$ from the right hand side of (4.1) is non-zero on v. For at least one $j = 1, \dots, d$, we must have that $wj \geq j$, and for such a j the fact that $e_{i_{wj},i_j;\mu_{i_j}-1}(v) \neq 0$ implies by (4.6) that $wj = d$ and $i_d = n$. For all other $j \neq k \in \{1, \dots, d\}$, we have that $wk \neq d$ hence $i_{wk} \neq n$. But then the fact that $e_{i_{wk},i_k;\mu_{i_k}-1}(v) \neq 0$ implies by (4.6) that $i_k = i_{wk}+1$ and $\mu_{i_k} = \lambda_{i_k}$. So $wj = d$ and $wk = k-1$ for all $k \neq j$, which means that $w = (d\, d{-}1 \cdots 1)$ and $j = 1$. Moreover, $i_2 = i_1 + 1, i_3 = i_2 + 1, \dots, i_d = i_{d-1} + 1 = n$, which means that $(i_1, \dots, i_d) = (n-d+1, \dots, n-1, n)$. Hence, $\mu = (0, \dots, 0, s, \lambda_{n-d+2}, \dots, \lambda_{n-1}, \lambda_n)$. For this w and μ it is indeed the case that $e_{i_{w1},i_1;\mu_{i_1}-1} \cdots e_{i_{wd},i_1;\mu_{i_d}-1}(v) = a_{n-d+1,s-1}$ by (4.6) once more. Since $\ell(w) = d-1$ this and the definition (4.1) implies the claim. ∎

Lemma 4.3. *ρ is an isomorphism.*

Proof. Lemma 4.2 implies that ρ is surjective, so it just remains to prove that it is injective. Let $G := GL_N(F)$ acting naturally on $\mathfrak{g}$ by conjugation. Let G_e be the centralizer of e in G and identify $\mathfrak{g}_e$ with the Lie algebra of G_e, i.e. tangent space $T_\iota(G_e)$ to G_e at the identity element ι, as usual. Considering the coadjoint action Ad^* of G_e on $\mathfrak{g}_e^*$, we have that $F[\mathfrak{g}_e^*]^{G_e} = F[\mathfrak{g}_e^*]^{\mathfrak{g}_e}$. To prove that $\rho : F[\mathfrak{g}_e^*]^{G_e} \to F[S]$ is injective, it suffices to prove that $(\mathrm{Ad}^* G_e)(S)$ is dense in $\mathfrak{g}_e^*$, i.e. that the map $\phi : G_e \times S \to \mathfrak{g}_e^*, (g, x) \mapsto (\mathrm{Ad}^* g)(x)$ is dominant. This follows if we can check that its differential $d\phi_{(\iota,f)}$ at the point (ι, f) is surjective; see e.g. [S, Theorem 4.3.6(i)]. Identify the tangent spaces $T_f(S)$ and $T_f(\mathfrak{g}_e^*)$ with V and $\mathfrak{g}_e^*$. Then the differential $d\phi_{(\iota,f)} : \mathfrak{g}_e \oplus V \to \mathfrak{g}_e^*$ is given explicitly by the map $(x, v) \mapsto (\mathrm{ad}^* x)(f) + v$. We show that it is surjective by checking that every basis element $f_{i,j;r}$ from (4.2) belongs to its image.

To start with, it is easy to see each $f_{n,i;r}$ belongs to the image of $d\phi_{(\iota,f)}$, since each of these vectors belongs to V. Next, suppose that $1 \leq i \leq j < n$ and $0 \leq r < \lambda_i$. By (4.3), we have that

$$(\mathrm{ad}^* e_{i,j+1;\lambda_{j+1}-r-1})(f) = f_{j,i;r} - f_{j+1,i+1;\lambda_{i+1}-\lambda_{j+1}+r}.$$

Using this, we get that all $f_{j,i;r}$ with $i \leq j$ belong to the image of $d\phi_{(\iota,f)}$. Finally, suppose that $n \geq i > j \geq 1$ and $\lambda_i - \lambda_j \leq r < \lambda_i$. By (4.3) again, we have that

$$(\mathrm{ad}^* e_{i-1,j;\lambda_i-r-1})(f) = \begin{cases} -f_{j,i;r}, & \text{if } j = 1 \\ f_{j-1,i-1;\lambda_j-\lambda_i+r} - f_{j,i;r}, & \text{if } j > 1. \end{cases}$$

From this we see that all $f_{j,i;r}$'s with $i > j$ belong to the image of $d\phi_{(\iota,f)}$ too. This completes the proof. ∎

By Lemma 4.2, $\rho(x_1), \dots, \rho(x_N)$ are algebraically independent generators for $F[S]$. By Lemma 4.3, ρ is an isomorphism. Hence $x_1, \dots, x_N$ are algebraically independent generators of $F[\mathfrak{g}_e^*]^{\mathfrak{g}_e}$. This completes the proof of Theorem 4.1. Now we can deduce the Main Theorem from the introduction.

Corollary 4.4. *The elements $z_1, \dots, z_N$ are algebraically independent generators for $Z(\mathfrak{g}_e)$.*

Proof. It is obvious that $\operatorname{gr} Z(\mathfrak{g}_e) \subseteq S(\mathfrak{g}_e)^{\mathfrak{g}_e}$. We have observed already that $z_1, \dots, z_N \in Z(\mathfrak{g}_e)$ are of filtered degrees $d_1, \dots, d_N$ respectively, and by Theorem 4.1 the associated graded elements are algebraically independent generators for $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$. By a standard filtration argument (see e.g. the proof of [MNO, Theorem 2.13]), this is enough to deduce that $z_1, \dots, z_N$ are themselves algebraically independent generators for $Z(\mathfrak{g}_e)$. At the same time, we have reproved the well-known equality $\operatorname{gr} Z(\mathfrak{g}_e) = S(\mathfrak{g}_e)^{\mathfrak{g}_e}$. $\square$

To conclude the article, we give one application; see [BK3, Theorem 6.10], [PPY, Remark 2.1] and the footnote to [P, Question 5.1] for other proofs of this result. Recall the central elements $Z_1, \dots, Z_N$ of $W(\lambda)$ from Lemma 3.2.

Corollary 4.5. *The elements $Z_1, \dots, Z_N$ are algebraically independent generators for the center of $W(\lambda)$.*

Proof. The loop filtration on $W(\lambda)$ induces a filtration on $Z(W(\lambda))$. Clearly we have that $\operatorname{gr} Z(W(\lambda)) \subseteq Z(\operatorname{gr} W(\lambda))$. By Theorem 3.4, we know that $Z_1, \dots, Z_N \in Z(W(\lambda))$ are of filtered degrees $1 - d_1, \dots, N - d_N$ respectively, and by Corollary 4.4 the associated graded elements are algebraically independent generators for $Z(\mathfrak{g}_e)$. Hence $Z_1, \dots, Z_N$ are algebraically independent generators for $Z(W(\lambda))$. At the same time, we have proved that $\operatorname{gr} Z(W(\lambda)) = Z(\mathfrak{g}_e)$. $\square$

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